Subspaces with Normalized Tight Frame Wavelets in $\mathbf{R}$

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Abstract In this paper we investigate the subspaces of $L^2(\mathbb{R})$ which have normalized tight frame wavelets that are defined by set functions on some measurable subsets of $\mathbb{R}$ called Bessel sets. We show that a subspace admitting such a normalized tight frame wavelet falls into a class of subspaces called reducing subspaces. We also consider the subspaces of $L^2(\mathbb{R})$ that are generated by a Bessel set E in a special way. We present some results concerning the relation between a Bessel set E and the corresponding subspace of $L^2(\mathbb{R})$ which either has a normalized tight frame wavelet defined by the set function on E or is generated by E.

§1. Introduction and Preliminaries

The concept of frames first appeared in the late 40's and early 50's ([1], [10], [11]). There have been some renewed interests recently in frames due to the development of wavelet theory since wavelets and frames are tightly related ([4], [5], [6] and [9]). A sequence of elements $\{x_j\}$ in $\mathcal{H}$ is called a *Bessel sequence* if there exists a positive constant B such that $\sum_j |\langle x, x_j\rangle|^2 \leq B\|x\|^2$ for all $x \in \mathcal{H}$. If in addition, there exists a positive constant A such that $A\|x\|^2 \leq \sum_j |\langle x, x_j\rangle|^2$ for all $x \in \mathcal{H}$, then $\{x_j\}$ is called a frame for $\mathcal{H}$. The supremum of all such numbers A and the infimum of all such numbers B are called the *frame bounds* of the frame and denoted by A_0 and B_0 respectively. The frame is called a *tight* frame when $A_0 = B_0$ and is called a *normalized tight* frame when $A_0 = B_0 = 1$. The kind of frames that is related to wavelets and is of special interest to mathematicians is of the form $\{\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^jx - k) : \ j, k \in \mathbb{Z}\}$, where $\psi \in L^2(\mathbb{R})$. The function ψ is called a normalized tight frame wavelet (NTF wavelet) if this frame is a normalized tight frame (NTF). Similarly, ψ is called a tight frame wavelet (TF wavelet) if this frame is a tight frame. From an operator theoretic point of view, NTF wavelets are simply the frame vectors for the unitary system $\mathcal{U} = \{D^nT^l : \ l, n \in \mathbb{Z}\}$, where D and T are the unitary dilation and translation operators on $L^2(\mathbb{R})$ (see [7]) defined by $(Df)(x) = \sqrt{2}f(2x)$

and $(Tf)(x) = f(x-1)$ for any $f \in L^2(\mathbb{R})$. A characterization of the NTF wavelets is obtained in [8].

In this paper, we are mainly interested in the case when ψ is defined by $\widehat{\psi} = \frac{1}{\sqrt{2\pi}}\chi_E$, where E is a Lebesgue measurable set with finite measure. We will call E a *normalized tight frame wavelet set* (NTF wavelet set) when ψ is an NTF wavelet. A *tight frame wavelet set* (TF wavelet set) and a *frame wavelet set* are defined similarly. Likewise, E is called a *Bessel set* if $\{\psi_{j,k}(x) : \ j,k \in \mathbb{Z}\}$ is a Bessel sequence. All these concepts can be applied to the subspaces of $L^2(\mathbb{R})$ (to be formally defined in the next section) and many questions can be asked. In [2], it is shown that the subspaces of $L^2(\mathbb{R})$ that are invariant under the operators D and T (called *reducing subspaces*), are pertinent to the study of wavelets in subspace. It is shown there that such a subspace always has an *orthonormal wavelet*. It is natural to ask the question about the existence of NTF wavelets for an arbitrary subspace of $L^2(\mathbb{R})$. One may even try to characterize all the NTF wavelets for a given subspace of $L^2(\mathbb{R})$. As a first step, perhaps it will shed some light on the whole picture by looking at the NTF wavelets which are generated by some subsets of $\mathbb{R}$.

In this paper, we show that if X is a reducing subspace with an NTF wavelet ψ whose Fourier transform is $\frac{1}{\sqrt{2\pi}}\chi_E$, then E can be completely characterized (Theorem 1). On the other hand, we also show that if X admits an NTF wavelet ψ whose Fourier transform is $\frac{1}{\sqrt{2\pi}}\chi_E$ for a Bessel set E, then X must be a reducing subspace (Thoerem 3). Furthermore, we prove that a maximal reducing subspace exists in the subspace X_E *generated* (see Definition 3) by a Bessel set E and this maximal reducing subspace is completely characterized (Theorem 2).

We will use μ to denote the Lebesgue measure. The inner product of two functions f and g in $L^2(\mathbb{R})$ is denoted by $\langle f, g\rangle$. The Fourier transform of a function f in $L^2(\mathbb{R})$ is denoted by $\mathcal{F}(f)$ or simply $\widehat{f}$. Define $\widehat{T} = \mathcal{F}T\mathcal{F}^{-1}$, $\widehat{D} = \mathcal{F}D\mathcal{F}^{-1}$. We have $\widehat{D} = D^{-1} = D^*$ and $\widehat{T}(f) = e^{-is}f$ for $f \in L^2(\mathbb{R})$.

Let us introduce some set theoretic notations first.

Let E be a set. Define $\tau(E) = \bigcup_{n\in\mathbb{Z}} (E \cap [2n\pi, 2(n+1)\pi) - 2n\pi)$. If this is a disjoint union, we say that E is 2π-*translation equivalent* to $\tau(E)$, which is always a subset of $[0, 2\pi)$. If E and F are 2π-translation equivalent to the same subset in $[0, 2\pi)$ then we say E and F are 2π-*translation equivalent.*

This defines an equivalence relation and is denoted by $\overset{\tau}{\sim}$. It is clear that $\mu(E) \geq \mu(\tau(E))$. The equality holds if and only if $E \overset{\tau}{\sim} \tau(E)$. If $E \overset{\tau}{\sim} F$, then $\mu(E) = \mu(F)$. Two points $x, y \in E$ are said to be 2π-*translation equivalent in* E if $x - y = 2m\pi$ for some integer m. The 2π-*translation redundancy index* of a point x in E is the number of elements in its equivalence class. We write $E(\tau, k)$ for the set of all points in E with 2π-translation redundancy index k. Of course, $E(\tau, k)$ could be an empty set, a proper subset of E, or the set E itself. For $k \neq m$, $E(\tau, k) \cap E(\tau, m) = \emptyset$, so $E = E(\tau, \infty) \bigcup (\bigcup_{n\in\mathbb{N}} E(\tau, n))$ is a disjoint union.

Similarly, let E be a set and define $\delta(E) = \bigcup_{n\in\mathbb{Z}} 2^{-n}(E \bigcap ([-2^{n+1}\pi, -2^n\pi) \bigcup [2^n\pi, 2^{n+1}\pi))$. If this union is disjoint, we say that E is 2-*dilation equivalent* to $\delta(E)$, which is a subset of $[-2\pi, -\pi) \bigcup [\pi, 2\pi)$, the *Littlewood-Paley wavelet set*. If E and F are 2-dilation equivalent to the same subset in $[-2\pi, -\pi) \bigcup [\pi, 2\pi)$, then we say E and F are 2-*dilation equivalent*. This also defines an equivalence relation and is denoted by $\overset{\delta}{\sim}$. Two non-zero points $x, y \in E$ are said to be 2-*dilation equivalent* if $\log_2 \frac{x}{y} \in \mathbb{Z}$. The 2-*dilation redundancy index* of a point x in E is the number of elements in its equivalence class. The set of all points in E with 2-dilation redundancy index k is denoted by $E(\delta, k)$. For $k \neq m$, $E(\delta, k) \cap E(\delta, m) = \emptyset$ and $E = E(\delta, \infty) \bigcup (\bigcup_{n\in\mathbb{N}} E(\delta, n))$ is a disjoint union.

When E is a measurable set, one can prove that $E(\tau, m)$ and $E(\delta, m)$ are also measurable for any $m \geq 1$. Furthermore, each $E(\tau, m)$ can be decomposed into m disjoint measurable subsets $E^{(j)}(\tau, m)$ $(j = 1, 2, ..., m)$ such that each $E^{(j)}(\tau, m)$ is 2π-translation equivalent to a subset of $[0, 2\pi)$. Similarly, each $E(\delta, m)$ can be decomposed into m disjoint measurable subsets $E^{(j)}(\delta, m)$ $(j = 1, 2, ..., m)$ such that each $E^{(j)}(\delta, m)$ is 2-diation equivalent to a subset of $[-2\pi, -\pi) \cup [\pi, 2\pi)$. The proof of these facts is elementary and is left to our reader. Notice that the decompositions of $E(\tau, m)$ and $E(\delta, m)$ here are not unique, but we will choose one and stay with it throughout the paper.

Definition 1 *A measurable set E of finite measure is called a basic set if there exists $M > 0$ such that $\mu(E(\tau, m)) = \mu(E(\delta, m)) = 0$ for all $m > M$.*

Let X be a subspace of $L^2(\mathbb{R})$. A function $\psi \in X$ is an *orthonormal wavelet* for X if the set $\{D^n T^\ell \psi : n, \ell \in \mathbb{Z}\}$ forms an orthonormal basis for

X. A measurable set $E \subset \mathbb{R}$ is a *wavelet set* for X if the Fourier inverse transform of $\frac{1}{\sqrt{2\pi}}\chi_E$, namely $\mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}\chi_E)$, is an orthonormal wavelet for the subspace X. It is shown in [2] that X is a reducing subspace if and only if there exists a subset Ω of $\mathbb{R}$ with the property $\Omega = 2\Omega$ such that $\widehat{X} = L^2(\mathbb{R}) \cdot \chi_\Omega$, where $\widehat{X}$ is the set of all Fourier transforms of elements in X (which is itself a subspace of $L^2(\mathbb{R})$).

Definition 2 *Let X be a closed subspace of $L^2(\mathbb{R})$ and let ψ be a function in $L^2(\mathbb{R})$. Then ψ is called an NTF wavelet for X if $D^nT^\ell\psi \in X$ for each $n, \ell \in \mathbb{Z}$ and*

$$f \;=\; \sum_{n,\ell\in\mathbb{Z}} \langle f, D^nT^\ell\psi\rangle D^nT^\ell\psi, \forall f \in X. \tag{1}$$

A measurable set E is called an NTF wavelet set for X if the function ψ_E defined by $\widehat{\psi}_E = \frac{1}{\sqrt{2\pi}}\chi_E$ is an NTF wavelet for X.

Remark. The convergence in (1) is in norm topology of $L^2(\mathbb{R})$. The same is true for all other similar occasions unless otherwise specified.

Let E be a Lebesgue measurable set with finite measure. Let ψ be the function defined by $\widehat{\psi}(s) = \frac{1}{\sqrt{2\pi}}\chi_E$. Consider the formal sum

$$G_E(f) = \sum_{n,\ell\in\mathbb{Z}} \langle f, D^nT^\ell\psi\rangle D^nT^\ell\psi, \tag{2}$$

where f is any function in $L^2(\mathbb{R})$. If $G_E(f)$ converges (in $L^2(\mathbb{R})$ norm) for all $f \in L^2(\mathbb{R})$, then it defines a bounded linear operator on $L^2(\mathbb{R})$ (by Banach-Steinhaus Theorem). Apparently, when this is the case, E has to be a Bessel set. By definition, a frame wavelet set is always a Bessel set but not vice versa. The following proposition links the Bessel sets and basic sets.

Proposition 1 *Let E be a measurable set of finite measure. Then the following statements are equivalent.*

(i) E is a Bessel set.

(ii) E is a basic set.

(iii) G_E is a bounded linear operator.

The proof is omitted since (iii)$\Longrightarrow$ (i) is trivial, (i)$\Longrightarrow$(ii) and (ii) $\Longrightarrow$ (iii) can be found in Theorem 1 and its proof in [3].

Denote the formal sums

$$\sum_{\ell\in\mathbb{Z}}\langle\widehat{f},\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_E\rangle\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_E$$

and

$$\sum_{k,\ell\in\mathbb{Z}}\langle\widehat{f},\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_E\rangle\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_E$$

by $H_E^k(\widehat{f})$ and $H_E(\widehat{f})$ respectively. Notice that $A\|f\|^2\leq\langle G_E(f),f\rangle\leq B\|f\|^2$ is equivalent to $A\|\widehat{f}\|^2\leq\langle H_E(\widehat{f}),\widehat{f}\rangle\leq B\|\widehat{f}\|^2$.

Definition 3 *Let E be a Bessel set. Then $X_E=\{f\in L^2(\mathbb{R}):\ f=G_E(f)\}$ is called the subspace generated by E.*

Note: In general, the function ψ defined by $\hat{\psi}=\frac{1}{\sqrt{2\pi}}\chi_E$ may not belong to X_E so E may not be an NTF wavelet set for X_E. But if E is an NTF wavelet set for a closed subspace X, then $X=X_E$ by the definitions. What we are interested in is that under what conditions will E be an NTF wavelet set of X_E (or any other subspace X)? Furthermore, under what conditions will X_E be a reducing subspace? Some partial answers to these questions are obtained in this paper.

Let E be a Lebesgue measurable set in $\mathbb{R}$ with finite positive measure. For any $f\in L^2(\mathbb{R})$, let $\widehat{f}^k_{mj}$ be the $2^{k+1}\pi$ periodic extension of $\widehat{f}\cdot\chi_{2^kE^{(j)}(\tau,m)}$ over $\mathbb{R}$. The following proposition is the main tool we need in proving our results. Its proof can be found in [3].

Proposition 2 *Let E be a Bessel set (so it is also a basic set). Let M be as defined in Definition 1. Then $H_E^k(\widehat{f})$ converges to $\sum_{m=1}^M\sum_{j=1}^m\widehat{f}^k_{mj}\cdot\chi_{2^kE}$ and $H_E(\widehat{f})$ converges $\sum_{k\in\mathbb{Z}}\sum_{m=1}^M\sum_{j=1}^m\widehat{f}^k_{mj}\cdot\chi_{2^kE}$ for all $f\in L^2(\mathbb{R})$. Here the convergence is under the norm of $L^2(\mathbb{R})$.*

§2. Theorems and Proofs

We begin this section with the following theorem, which is a natural generalization of the characterization (Theorem 5.4, [7]) of normalized tight frame wavelet set for $L^2(\mathbb{R})$ to the reducing subspaces.

Theorem 1 *Let X be a reducing subspace so that $\widehat{X} = L^2(\mathbb{R}) \cdot \chi_\Omega$, where $\Omega = 2\Omega$ is a measurable subset in $\mathbb{R}$. Then a measurable set E is an NTF wavelet set for X if and only if $E = E(\delta, 1) = E(\tau, 1)$ and $\cup_{k\in\mathbb{Z}} 2^k E = \Omega$.*

Proof. Assume that E satisfies the conditions in Theorem 1. For any $f \in X$, we have $\widehat{f} = \sum_{n\in\mathbb{Z}} \widehat{f}\chi_{2^n E}$ since $\Omega = \bigcup_{n\in\mathbb{Z}} 2^n E$ and the union is a disjoint union. But by Proposition 2, $H_E(\widehat{f})$ also converges in $L^2(\mathbb{R})$ to $\sum_{n\in\mathbb{Z}} H_E^n(\widehat{f}) = \sum_{n\in\mathbb{Z}} \widehat{f}\chi_{2^n E}$ since $\mu(E(\tau, m)) = \mu(E(\delta, m)) = 0$ for all $m \geq 2$.

Now assume that E is an NTF wavelet set for X. Since X is a reducing subspace by the assumption, $\chi_E \in X = L^2(\mathbb{R}) \cdot \chi_\Omega$, where $\Omega = 2\Omega$. So $E \subseteq \Omega$ hence $E(\tau, m) \subseteq \Omega$. Therefore, $\widehat{f} = \chi_{E(\tau,m)} \in X$ and $\widehat{f} = H_E(\widehat{f})$. By Proposition (2), $H_E^0(\widehat{f}) = m\widehat{f}$. So $\|\widehat{f}\|^2 = \langle H_E(\widehat{f}, \widehat{f}\rangle \geq \langle H_E^0(\widehat{f}, \widehat{f}\rangle = m\|\widehat{f}\|^2$ Therefore $\mu(E(\tau, m)) = 0$ for all $m > 1$ and $E = E(\tau, 1)$. If $\mu(E(\delta, k)) > 0$ for some $k > 1$, we can find a subset F of E such that $\mu(F) > 0$ and $2^p F \subset E$ for some $p > 0$. A contradiction can be then derived in a similar way by choosing $\widehat{f} = \chi_F$. The fact that $\Omega = \bigcup_{n\in\mathbb{Z}} 2^n E$ is rather obvious. □

Theorem 2 *Let E be a Bessel set, then*

(i) If $E = E(\delta, 1)$, then X_E is a reducing subspace with $\widehat{X}_E = L^2(\mathbb{R})\chi_\Omega$, where $\Omega = \bigcup_{n\in\mathbb{Z}} 2^n E(\tau, 1)$.

(ii) If $E = E(\tau, 1)$, then X_E is a reducing subspace with $\widehat{X}_E = L^2(\mathbb{R})\chi_\Omega$, where $\Omega = \bigcup_{n\in\mathbb{Z}} 2^n E(\delta, 1)$.

(iii) There exists a maximal reducing subspace $Y \subset X_E$. Furthermore, $\widehat{Y} = L^2(\mathbb{R})\chi_\Omega$, where $\Omega = \bigcup_{n\in\mathbb{Z}} 2^n(E(\delta, 1) \cap E(\tau, 1))$.

Proof. (i) Let $\widehat{f} \in L^2(\mathbb{R}) \cdot \chi_\Omega$, and $F = E(\tau, 1)$. Since $\text{supp}(\widehat{f}) \subset \Omega = \cup_{n\in\mathbb{Z}} 2^n F$, $\widehat{f}_{mj}^k = 0$ for all $m \geq 2$. Therefore, by Proposition 2, $H_E(\widehat{f}) = \sum_{n\in\mathbb{Z}} \widehat{f} \cdot \chi_{2^n E} = \widehat{f}$. So $f \in X_E$ and it follows that $L^2(\mathbb{R}) \cdot \chi_\Omega \subseteq X_E$.

Now assume $f \in X_E$. By Proposition 2, we have $\widehat{f} = H_E(\widehat{f}) = \sum_{n\in\mathbb{Z}} \sum_{m=1}^{M} \sum_{j=1}^{m} \widehat{f}_{mj}^n \chi_{2^n E}$ where the summation converges in $L^2(\mathbb{R})$ norm hence converges almost everywhere. Let $s \in 2^n E(\tau, m)$ be any point such that the summation converges at all its $2^{n+1}\pi$ equivalent points (denoted by s_1, s_2, ..., s_m) in $2^n E(\tau, m)$, then we have $\widehat{f}(s_i) = \sum_{1\leq j\leq m} \widehat{f}(s_j)$ for any $1 \leq i \leq m$ by Proposition 1. This leads to $\widehat{f}(s_j) = 0$ if $m > 1$. Since s is arbitrary,

$\widehat{f} = 0$ on $2^n E(\tau, m)$ for any $n \in \mathbb{Z}$ and any $m > 1$. Therefore, $\text{supp}(\widehat{f}) \subset \cup_{n\in\mathbb{Z}} 2^n E(\tau, 1)$.

(ii) It is obvious that $L^2(\mathbb{R})\chi_\Omega \subset \widehat{X_E}$. Let $F = E \backslash E(\delta, 1)$ and $\Omega_1 = \cup_{n\in\mathbb{Z}} 2^n E(\delta, 1)$, $\Omega_2 = \cup_{n\in\mathbb{Z}} 2^n F$. For any $f \in X_E$, $\|\widehat{f}\|^2 = \|\widehat{f}\chi_{\Omega_1}\|^2 + \|\widehat{f}\chi_{\Omega_2}\|^2$. On the other hand, since $E = E(\tau, 1)$, we have $\widehat{f} = \sum_{n\in\mathbb{Z}} \widehat{f}\chi_{2^n E}$ by Proposition 2. It follows that $\|\widehat{f}\|^2 \geq \int_{\Omega_1} |\widehat{f}|^2 ds + 2\int_{\Omega_2} |\widehat{f}|^2 ds$. Thus, $\int_{\Omega_2} |\widehat{f}|^2 ds = 0$ hence the support of $\widehat{f}$ is contained in Ω_1.

(iii) It is obvious that the subspace defined by $\widehat{Y} = L^2(\mathbb{R})\chi_\Omega$ is a reducing subspace. We need only to show that if Y_1 is also a reducing subspace contained in X_E, then $Y_1 \subset Y$. Let $f \in Y_1$, then $\widehat{f} \in \widehat{Y_1}$ hence $|\widehat{f}| \in \widehat{Y_1}$. It follows that $|\widehat{f}| \in \widehat{X_E}$. By Proposition 1 and similar arguments in the above proofs, we get $\text{supp}(\widehat{f}) \subset \Omega$. Hence $\widehat{f} \in \widehat{Y}$. Details are left to our reader. □

Theorem 3 *Let X be a closed subspace of $L^2(\mathbb{R})$. The following statements are equivalent.*

(i) X is a reducing subspace.
(ii) $\widehat{X} = L^2(\mathbb{R}) \cdot \chi_\Omega$, where $\Omega = 2\Omega$ is a Lebesgue measurable set.
(iii) There exists a Bessel set E which is a wavelet set for X.
(iv) There exists a Bessel set E which is an NTF wavelet set for X.
(v) $X = X_E$ for a Bessel set E such that $E = E(\delta, 1) = E(\tau, 1)$.

Proof. (i)⇒(ii) is from Proposition 4.3 in [2]. (ii)⇒(iii) is Theorem 4.4 in [2]. (iii)⇒(iv) is trivial and (v)⇒(i) is implied by Theorem 2 above. So we only need to show that (iv)⇒(v). Notice that in general we have $\langle H_E(\widehat{f}), \widehat{f}\rangle \geq \langle H_E^0(\widehat{f}), \widehat{f}\rangle$ for any $f \in L^2(\mathbb{R})$. Let $\widehat{f} = \chi_E \in \widehat{X}$, we have $\mu(E) = \|\widehat{f}\|^2 = \langle H_E(\widehat{f}), \widehat{f}\rangle$. On the other hand, $H_E^0(f) = \sum_{m=1}^M m\chi_{E(\tau,m)}$ by Proposition 2. It follows that $\mu(E) \geq \sum_{m=1}^M \int_{\mathbb{R}} m\chi_{E(\tau,m)} ds = \sum_{m=1}^M m\mu(E(\tau, m))$, hence $\mu(E(\tau, m)) = 0$ for all $m > 1$. Similarly, one can show that $\mu(E(\delta, m)) = 0$ for all $m > 1$. $X = X_E$ since E is an NTF set for X as pointed out in the note after Definition 3. □

§3. Examples and Discussions

We now conclude this paper with some examples and discussions.

1. Let X be a closed reducing subspace of D, T. Then for any $0 < \alpha < 2\pi$, we can find an NTF wavelet set E for X such that $\mu(E) = \alpha$ as shown below.

Let $\widehat{X} = L^2(\mathbb{R}) \cdot \chi_\Omega$ where $\Omega = 2\Omega$. It is shown in [2] that there exists a measurable set S such that $S = S(\delta, 1)$, $\cup_{k\in\mathbb{Z}} 2^k S = \Omega$ and $S \overset{\tau}{\sim} [0, 2\pi]$. Let $n \in \mathbb{Z}$ be such that $\frac{\pi}{2^{n-1}} < min\{\alpha, 2\pi - \alpha\}$. For any $t \in [\frac{2\pi-\alpha}{2}, \pi]$, define $A = \bigcup_{0\neq l\in\mathbb{Z}} (S \cap ((-t, t) + 2l\pi))$ and $B = S \cap ((-t, -\frac{\pi}{2^n}) \bigcup (\frac{\pi}{2^n}, t))$. Let $C = S \setminus (A \bigcup B)$ and D be the subset of $(-\frac{\pi}{2^n}, -\frac{\pi}{2^{n+1}}) \cup (\frac{\pi}{2^{n+1}}, \frac{\pi}{2^n})$ that is 2-dilation equivalent to $A \cup B$. It can be verified that $S_t = C \cup D$ is an NTF wavelet set. Note that when $t = \frac{2\pi-\alpha}{2}$, the measure of S_t is greater than α. Whereas when $t = \pi$, the measure of S_t is less than $\frac{\pi}{2^{n-1}}$ which is less than α. Since the measure of S_t is a continuous function of t, the conclusion follows.

2. Let E be a wavelet set (for $L^2(\mathbb{R})$ or a reducing subspace). Let F be a measurable subset of E. Then by Theorem 1, F is a normalized tight frame wavelet set for the subspace X_F with $\widehat{X_F} = L^2(\mathbb{R}) \cdot \chi_\Omega$, where $\Omega = \bigcup_{n\in\mathbb{Z}} 2^n F$. In particular, any measurable subset in $[-2\pi, -\pi) \cup [\pi, 2\pi)$ with positive measure is such a set.

3. Any basic set E such that $E \neq E(\delta, 1)$ or $E \neq E(\tau, 1)$ is not an NTF wavelet set for X_E. This can be seen from Proposition 2.

4. Although Theorem 2 guarantees the existence of a maximal reducing subspace in X_E, it does not say when or whether X_E is itself a reducing subspace. The following example shows that in general X_E is not necessarily a reducing subspace. Let $E_1 = [2\pi, 2.75\pi)$, $E_2 = [4\pi, 4.75\pi)$, $E_3 = [4.75\pi, 5\pi)$, $E_4 = [10\pi, 10.75\pi)$, $E_5 = [10.75\pi, 11\pi)$ and $E = \cup_j E_j$. We have $E = E(\delta, 2)$, $E(\tau, 1) = \emptyset$, $E(\tau, 2) = E_3 \cup E_5$ and $E(\tau, 3) = E_1 \cup E_2 \cup E_4$. Define $\widehat{f}(s) = 2\chi_{E_1} - \chi_{E_2} - \chi_{E_4} + \chi_{E_3} - \chi_{E_5}$. $H_E(\widehat{f})(s) = \widehat{f}(s)$ by Proposition 1 so $\widehat{f}(s) \in \widehat{X_E}$, but $H_E(|\widehat{f}|)(s) \neq |\widehat{f}(s)|$. Thus, X_E cannot be a reducing subspace. Since if it were, then $|\widehat{f}(s)|$ would be in it as well.

5. Although we proved in Theorem 3 that any subspace admitting an NTF wavelet defined by a Bessel set is necessarily a reducing subspace, this is not true for general NTF wavelets. In fact, one can construct non-reducing subspaces of $L^2(\mathbb{R})$ that admit wavelets (hence NTF wavelets), see [2].

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