THE LOWER BOUNDS OF THE LENGTHS OF THICK KNOTS

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ABSTRACT

In this paper, we derive a formula that provides lower bounds on the minimal arc length required to tie a unit thickness knot in terms of the minimal crossing number of the knot. We prove that for any nontrivial knot with unit thickness, the minimal arc length is at least 24. This answers an open question (posted in [13]) in the negative: one cannot tie a knot using one foot-length of one-inch (diameter) rope. For knots with minimal crossing numbers up to 1850, our result yields larger lower bounds on the lengths of the knots than the known results.

Keywords: Knots, Links, Crossing Number, Thickness of Knots, Arc Length of Knots.

1. INTRODUCTION

The geometrical properties of a simple closed curve and the topological properties of a knot are quite different things. Interesting and hard questions arise when one tries to associate these properties. For instance, if the simple closed curve is a polygon, one could ask how many edges must this polygon have in order for it to have certain knot type [4]. Or if we know the knot type of a simple closed curve, what effect does it have on its total curvature [14]? In this paper, we are interested in the relation between the crossing number of a thick knot (a topological property) and its arc length (a geometric property). Loosely speaking, one can think of a thick knot as a physical knot tied by a rope and the radius of the rope can be regarded as the thickness of this knot. Mathematically, there are different ways to define the thickness of a knot. We refer our reader to [6], [11] and [13] for this. However, in this paper, we will adopt the so called disk thickness defined in [13]. Although the disk thickness is only defined for C^2 curves, it is shown in [6] that this thickness is equivalent to a thickness that is defined for C^1 curves. Using this definition, it is shown in [2,3] that the minimal crossing number of a knot of unit thickness is bounded by a constant times its length to the four third power. This four third power is also shown to be achievable for some knot families [5,9]. The four third power law implies that the arc length of a unit thickness knot is bounded below by a constant times its minimal crossing number raised to three fourth power. This constant is estimated to be at least 1.105 by the result obtained in [2].

Our question here is that if the radius of the rope is 1 unit, how many units of rope are needed to tie this knot? This is a challenging problem in knot theory. On the hand, it has been proven in [6] that within the set of all C^1 knot of unit thickness of any given knot type, there exists one that has the minimal arclength (called the *ropelength minimizer* in [6]). On the other hand, the numerical value of the minimum ropelength needed to tie any non-trivial knot is not known up to date. Naturally, we turn to find estimates for such minimum length. It is shown in [13] that the arc length of a knot with unit thickness is at least $5\pi \approx 15.71$. This lower bound has recently been improved to $2\pi(2+\sqrt{2}) \approx 21.45$ [6]. An open question asks whether one can tie a knot with one foot of one-inch diameter rope, i.e., if the minimum arc length of a knot with unit thickness is less than or equal to 24 [13]. Computer simulations suggest that this minimum arc length may be around 32 [15]. So the answer to this question is expected to be negative. Other experiments are also carried out to measure the amount of rope needed to tie a knot (not just the trefoil). In [10], various ropes are used to tie different knots. After a knot is tied, the ends of the rope are pulled tightly and the distance between the two ends are measured against their original distance (the total length of the rope). This gives an estimate of the amount of rope needed to tie this knot (with its ends open).

In this paper, we derive the following inequality

$$L(K) \geq \frac{1}{2}\left(17.334 + \sqrt{17.334^2 + 64\pi Cr(K)}\right), \tag{1}$$

where $L(K)$ is the arclength of the knot K (with unit thickness) and $Cr(K)$ is the minimal crossing number of K. This formula is comparable to and improves the following formula obtained in [2]:

$$L(K) \geq 4\sqrt{\pi Cr(K)} \left(= \frac{1}{2}\sqrt{64\pi Cr(K)}\right). \tag{2}$$

For $Cr(K) \leq 20$, (1) produces the following numerical results.

$Cr(K)$	$=$	3	4	5	6	7	8	9	10	11
$L(K)$	$>$	23.698	25.286	26.735	28.076	29.330	30.513	31.635	32.704	33.728

$Cr(K)$	$=$	12	13	14	15	16	17	18	19	20
$L(K)$	$>$	34.711	35.659	36.575	37.461	38.321	39.157	39.970	40.763	41.537

The lower bound $L(K) > 23.698$ is an improvement over the bound 21.45 obtained in [6]. When compared with $L(K) > 1.105(L(K))^{3/4}$, (1) provides larger lower bounds for $L(K)$ with $Cr(K)$ up to 1850 (and much better estimates when $Cr(K)$ is small since $L(K) > 1.105(L(K))^{3/4}$ is only intended for the asymptotic case). (1) is derived in Section 4. In Section 5, we use an elementary geometric approach to show that the minimum length required to tie a knot with unit thickness is larger than 24.

It is worthwhile to mention the counterpart of this question for polygons on the cubic lattice. Since there are only finitely many polygons of certain length on the cubic lattice, one may think at first (intuitively) that the minimum length of a knot on the cubic lattice may be easy to obtain. It turns out that this is not the case either. In fact, the only known (theoretical) result related to this question to date is that the minimum length for any knot on the cubic lattice is 24, and with this length one can only tie a trefoil [7, 8]. As the number of edges increases, the number of polygons on the cubic lattice increases exponentially and the situation quickly becomes very complicated. One can smooth a lattice knot with minimal length 24 by replacing its corners with quarter circles of suitable radius. The result is a smooth knot of thickness one half. It can be shown we can get a knot like this with length less than 20. This implies that we are sure that a trefoil can be tied with 40 units of rope if the knot is of unit thickness [10], without doing any experiment.

2. WEAKLY REDUCED KNOTS AND THEIR KNOT DIAGRAMS

The discussions in this section apply to any knot K that has a regular projection to a plane (or to a sphere under the spherical projection) with the exception that the projection may contain a few points such that each of them is the projection of a line segment on K. For any two points Y and Z on K, there are two arcs of K joining these two points. To distinguish them, we give K an orientation. In the case that K is given by a parametric equation, the orientation is naturally defined by the increasing direction of the parameter. The arc from Y to Z following this orientation is denoted by $\alpha(Y,Z)$ and the arc from Z to Y is denoted by $\alpha(Z,Y)$. We will assume from now on that an orientation of K is fixed. Let $P(A)$ be the projection of any subset A of K to the plane (A may be a point, an arc or K itself). If $Y, Z \in K$ and $P(Y) = P(Z)$, then $P(\alpha(Y,Z))$ is a loop (possibly with self intersections) in $P(K)$. Let $\overline{YZ}$ be the line segment joining Y and Z and consider the following two simple closed curves. One of them is $K_1 = \overline{YZ} \cup (K \backslash \alpha(Y,Z))$. The other one is $Q_1 = \overline{YZ} \cup \alpha(Y,Z)$. If Q_1 is a trivial knot and $\alpha(Y,Z)$ can be deformed to $\overline{YZ}$ via an ambient isotopy that is identity on $K \backslash \alpha(Y,Z)$, then K_1 is ambient isotopic to K. We will say that $\alpha(Y,Z)$ is *replaceable* by $\overline{YZ}$ (or simply replaceable) and that $P(\alpha(Y,Z))$ is a *removable loop* in $P(K)$. Replacing $\alpha(Y,Z)$ with $\overline{YZ}$ is called a *replacing operation.* A replacing operation reduces the total number of crossings in $P(K)$ without creating any new crossings or moving any remaining crossings (and their corresponding points on K). Figure 1 shows some replaceable loops in $P(K)$. Observe that if $P(\alpha(Y,Z))$ is a loop in $P(K)$ without self intersections such that all passes of $P(K)$ intersecting it overpass it (or all underpass it), then $\alpha(Y,Z)$ is replaceable by $\overline{YZ}$. To see this, one needs to consider $P(\alpha(Y,Z))$ in $P(K)$. A sequence of suitable Reidemeister moves can be carried out to reduce $P(\alpha(Y,Z))$ under the given condition. A case of this is shown in Figure 1(a).

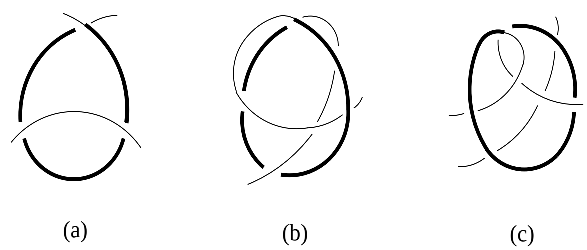

Figure 1: Examples of Removable Loops

Since a replacing operation reduces the total number of crossings of $P(K)$ by at least one, it can only be repeated finitely many times. When there is no more removable loops left, we come to a new knot K', which is ambient isotopic to K. The crossings of $P(K')$ and the corresponding points on K' projected to these crossings are subsets of $P(K)$ and K respectively. We call K' a *weakly reduced knot* of K and $P(K')$ a *weakly reduced* knot diagram of K. K' may not be obtained by just eliminating the original removable loops in $P(K)$ since removing a removable loop may create new ones. K' is not unique in general either since it depends on where the replacing process starts and which loops we choose to eliminate first. The number of crossings in $P(K')$ may not be the minimal crossing number of the knot type that K has either. This is why we call K' a weakly reduced knot of K.

A point in K is called a *strong under point* if it corresponds to an under crossing in $P(K')$. A *strong over point* in K is similarly defined. A *maximal weak underpass (overpass)* of $P(K)$ is a *pass* on $P(K)$ that corresponds to a *maximal underpass (overpass)* ([1]) on $P(K')$. Let K be a nontrivial knot and let K' be a weakly reduced knot of it. Start at any point $X \in K$ that is not projected to a crossing in $P(K)$ and travel along K following its orientation. Let Z_1 be the first strong over point and Z_2 be the last strong over point we meet before returning to X. $Z_1 \neq Z_2$ exist since K' has at least two maximal overpasses (see [1]).

Lemma 1. Assume that K is a nontrivial knot. Let $X \in K$ be a point that is not projected to a crossing in $P(K)$ and let Z_1, Z_2 be the first and last strong over points starting from X. Let Y_1 and Y_2 be the two strong under points corresponding to Z_1 and Z_2, then Y_1, $Y_2 \notin \alpha(Z_2, Z_1)$.

Proof. If $Y_1 \in \alpha(Z_2, Z_1)$, then $\alpha'(Y_1, Z_1)$ in K' corresponds to a removable loop since any pass of $P(K')$ intersecting it is over it by the definition of Z_1. The other case is similar. □

3. LEMMAS REGARDING KNOTS WITH UNIT THICKNESS

Throughout the rest of this paper, K stands for a knot of unit thickness, where the thickness is the one used in [6], i.e., the so called three-point thickness. Although the three-point thickness is defined for C^1 curves, we will restrict ourselves further to $C^{1,1}$ curves so that our results will apply to the ropelength minimizers. We will assume this condition in all the lemmas and theorems from now on. We will need some concepts and results from [6] and [13]. The arc length of $\alpha(Y, Z)$ is denoted by $\rho(Y, Z)$ so $\rho(Y, Z) \neq \rho(Z, Y)$ in general. Y and Z are called a *doubly critical pair* if $Y - Z$ is perpendicular to the tangent vectors $T(Y)$, $T(Z)$ of K at Y and Z. The Euclidean distance $d(Y, Z) = |Z - Y|$ between Y and Z is called a *doubly critical distance* if Y and Z are a doubly critical pair. Similarly, Y and Z are called a *singly critical pair* if $Y - Z$ is perpendicular to either $T(Y)$ or $T(Z)$. The corresponding $d(Y, Z)$ is then called a *singly critical distance*. If we let κ be the maximum curvature of K, then we have ([13]):

Lemma 2. $\min\{\frac{1}{\kappa}, \frac{d_0}{2}\} = 1$, where d_0 is the minimum of all doubly critical distances of K.

The following lemma can also be found in [6] and [13].

Lemma 3. Let Y and Z be a singly critical pair of K, then $d(Y, Z) \geq 2$. On the other hand, if $d(Y, Z) < 2$, then $\rho(Y, Z) < \pi$ or $\rho(Z, Y) < \pi$.

For any point $X \in \mathbf{R}^3$ and $r > 0$, $B_r(X)$ stands for the closed ball of radius r centered at X and $S_r(X)$ is the boundary of $B_r(X)$.

Lemma 4. If $X \in K$, then (1) $(B_r(X) \backslash S_r(X)) \cap K$ is a single arc of K for any $r \leq 2$; (2) any unit ball (not necessarily centered at a point on K) can only contain one single arc of

K; (3) if a unit ball is tangent to K at some point, then its interior does not contain any other point of K.

Proof. (1) If this is not the case, then $(B_2(X)\backslash S_2(X)) \cap K$ contains another arc ℓ of K that does not include X. Notice that the distance between ℓ and X is realized at a point $Y \in \ell$ that is inside $S_2(X)$. It follows that $X - Y$ must be perpendicular to $T(Y)$. Hence X and Y form a singly critical pair and $d(X,Y) \geq 2$ by Lemma 1. This is a contradiction since Y is inside $S_2(X)$. (2) and (3) follow directly from the definition of the thickness from [6]. $\square$

The following lemma is elementary and applies to any smooth or piecewise linear curve (not just to arcs on our knot K). The result is well known. For the sake of convenience, we choose to list it as a lemma. The two cases in the lemma are illustrated in Figure 2.

Lemma 5. Let $B_r(X_0)$ be the ball of radius r centered at X_0. Let Z_1 and Z_2 be any two points not contained in the interior of $B_r(X_0)$. Let $\ell(Z_1, Z_2)$ be a smooth or piecewise smooth curve joining Z_1, Z_2 that does not intersect $B_r(X_0)$ in its interior and let θ be the angle between $Z_1 - X_0$ and $Z_2 - X_0$. Let $\rho(Z_1, Z_2)$ be the arc length of $\ell(Z_1, Z_2)$, then we have

(a) If $\theta \leq \pi$, then $\rho(Z_1, Z_2) \geq r\theta$.

(b) If $\theta = \pi$, then

$$\begin{aligned}\rho(Z_1, Z_2) &\geq \pi r + \sqrt{d_1^2 - r^2} - r\cos^{-1}\frac{r}{d_1} + \sqrt{d_2^2 - r^2} - r\cos^{-1}\frac{r}{d_2} \\ &\geq \pi r + 2(\sqrt{d^2 - r^2} - r\cos^{-1}\frac{r}{d}),\end{aligned}$$

where $d_1 = d(X_0, Z_1)$, $d_2 = d(X_0, Z_2)$ and $d = (d_1 + d_2)/2$.

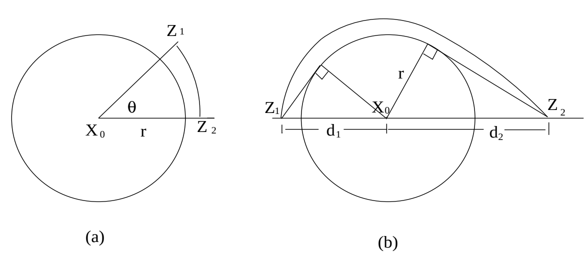

Figure 2: The minimum length of a curve going around a ball

While Lemma 5 is intuitive, the following lemma is a bit surprising.

Lemma 6. Let K be a knot and X, Y, Z be points of K such that $X \notin \alpha(Y, Z)$. Let θ be the angle between the vectors $Y - X$ and $Z - X$. Then $\rho(Y, Z) \geq 2\theta$.

Proof. Case 1. $\alpha(Y, Z) \subset B_2(X)$. Then $\alpha(X, Y)$ or $\alpha(Z, X) \subset B_2(X)$ by Lemma 4. Say $\alpha(X, Y) \subset B_2(X)$. Let $\gamma(t)$ be a parametric equation of K such that $\gamma(0) = X$ and

$\gamma(t_0) = Z$. By Lemma 3, the function $f(t) = |\gamma(t) - X|^2$ is a strictly increasing function for $t_0 > t > 0$. Thus we must have $d(X, Z) > d(X, Y)$. Furthermore, if we let Σ be the plane normal to K at X, then $\alpha(Y, Z)$ cannot intersect Σ by Lemma 3. In fact, $\alpha(Y, Z)$ lies in the side of Σ that $T(X)$ points to. So if X, Y and Z are colinear, then $Y - X$ and $Z - X$ point to the same direction hence $\theta = 0$. Thus in this case, there is nothing to prove. If X, Y and Z are not colinear, they determine a plane Σ_1. Let ℓ_1 be the line through X and Y and let ℓ_2 be the line through $(X + Y)/2$ that is perpendicular to ℓ_1. Let $O \in \ell_2$ such that O and Z are on the same side of ℓ_1 and $d(O, X) = d(O, Y) = 1$. So $X \in S_1(O)$ by this construction. Hence K is either tangent to $B_1(O)$ at X or enters $B_1(O)$ at a point near X. By Lemma 4 (2) and (3), $\alpha(Y, Z)$ will not intersect the interior of $B_1(O)$. Let θ_1 be the angle as marked in Figure 3 (where the page is in the plane Σ_1). Observe that $\theta_1 \geq 2\theta$. The result then follows by Lemma 5. If $\alpha(Z, X) \subset B_2(X)$, we can change the orientation of K and relabel Y and Z then apply the above proof.

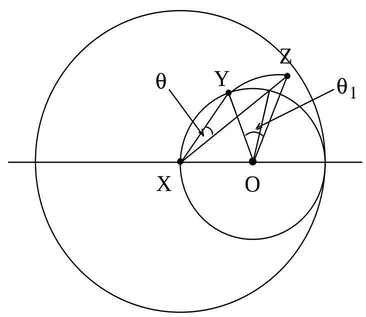

Figure 3: $\rho(Y, Z) \geq 2\theta$

Case 2. $\alpha(Y, Z)$ does not intersect the interior of $B_2(X)$. This follows directly from Lemma 5.

Case 3. There exists $W \in \alpha(Y, Z)$ such that $\alpha(Y, W) \subset B_2(X)$ or $\alpha(W, Z) \subset B_2(X)$. Say $\alpha(Y, W) \subset B_2(X)$. Let θ_1 be the angle between $Y - X$ and $W - X$, and let θ_2 be the angle between $W - X$ and $Z - X$. We have $\theta_1 + \theta_2 \geq \theta$. Since $\rho(W, Z) \geq 2\theta_2$ by Lemma 5 and $\rho(Y, W) \geq 2\theta_1$ by case 1, the result follows. The proof for the other case is identical. □

Proposition 1. Let K be a knot and P be a regular plane projection of K to a plane. Then there exists a weakly reduced knot K' of K such that any strong under-over point pair Y and Z in it must satisfy the condition $\rho(Y, Z) \geq 2\pi$ (and $\rho(Z, Y) \geq 2\pi$). The same conclusion holds for a spherical projection.

Proof. Start from a point on K and travel along K to search for the pairs Y, Z that satisfy the condition $P(Y) = P(Z)$ and $\rho(Y, Z) < 2\pi$. We will relabel Y, Z if $\rho(Z, Y) < 2\pi$. This way, $\alpha(Y, Z)$ is always the shorter arc but $P(Y)$ may be either an under crossing or an over crossing in $P(K)$. Once we find such a pair Y_1, Z_1, we will set it aside and continue our search for the next pair Y_2, Z_2 with the extra condition that Y_2 and Z_2 cannot be in $\alpha(Y_1, Z_1)$. The reason is that if Y_2 or Z_2 belongs to $\alpha(Y_1, Z_1)$, then once we replace $\alpha(Y_1, Z_1)$ by $\overline{Y_1 Z_1}$, the crossing $P(Y_2) = P(Z_2)$ would disappear. If it happens that $\alpha(Y_1, Z_1) \subset \alpha(Y_2, Z_2)$, then we will keep Y_2, Z_2 and discard Y_1, Z_1 from our collection for the same reason. We will renumber Y_2, Z_2 as Y_1, Z_1 in this case. In general, if we have found k such pairs (Y_1, Z_1), (Y_2, Z_2), ... , (Y_k, Z_k) such that $\rho(Y_j, Z_j) < 2\pi$ and $\alpha(Y_i, Z_i) \cap \alpha(Y_j, Z_j) = \emptyset$ for any $i \neq j$, we will go out to search for the next pair Y_{k+1}, Z_{k+1} with the extra condition that Y_{k+1}, Z_{k+1} are not contained in any $\alpha(Y_j, Z_j)$ $(1 \leq j \leq k)$. Once such a pair is found, we will check to see if $\alpha(Y_{k+1}, Z_{k+1})$ contains any of the existing $\alpha(Y_j, Z_j)$. If it does, then the contained ones will be discarded from our collection and we will renumber the remaining pairs. This process is repeated until we have no more such pairs left. Assume at the end, we have m such pairs. Keep in mind that if we now have a pair Y and $Z \in K$ such that $P(Y) = P(Z)$ and Y, Z are not a pair in our collection, then either $\rho(Y, Z) \geq 2\pi$ or one of Y, Z is contained in $\alpha(Y_j, Z_j)$ for some pair (Y_j, Z_j) from our collection. Renumber the pairs in our final collection by (Y_1,Z_1), (Y_2,Z_2), ... , (Y_m,Z_m) such that $d(Y_j, Z_j) \leq d(Y_{j+1}, Z_{j+1})$. For each $\alpha(Y_j, Z_j)$, we will define $M_j = \cup_{Y' \in \alpha(Y_j, Z_j)} \overline{Y'Z_j}$.

Claim 1. M_j does not intersect $K \backslash \alpha(Y_j, Z_j)$.

Proof of Claim 1. If this is not true, then there exists a point $X \in K \backslash \alpha(Y_j, Z_j)$ and a point $Y_0 \in \alpha(Y_j, Z_j)$ such that $X \in \overline{Y_0 Z_j}$. Since $X \notin \alpha(Y_j, Z_j)$, by Lemma 6, we have $\rho(Y_j, Z_j) \geq \rho(Y_0, Z_j) \geq 2\pi$. This contradicts the condition that $\rho(Y_j, Z_j) < 2\pi$.

Claim 2. If $m \geq k > j$, then M_k does not intersect $\overline{Y_j Z_j}$.

Proof of Claim 2. Let Σ be the plane determined by Y_j, Z_j and Y_k, which is the plane of the page shown in Figure 4. Let θ_1 and θ_2 be as marked in the figure. Observe that $\theta_1 + \theta_2 \geq \pi$ since $d(Y_k, Z_k) \geq d(Y_j, Z_j)$ (keep in mind that $Z_k - Y_k$ and $Z_j - Y_j$ are parallel). Assume that M_k intersects $\overline{Y_j Z_j}$ at some point X. Then there exists $Y_0 \in \alpha(Y_k, Z_k)$ such that $X \in \overline{Y_0 Z_k}$. Since Y_j and Z_j are not in $\alpha(Y_k, Z_k)$, Lemma 6 applies. If Y_0 is in region I as marked in Figure 4, then $\theta_1' > \theta_1$ and $\theta_2' > \theta_2$, hence

$$\rho(Y_k, Z_k) = \rho(Y_k, Y_0) + \rho(Y_0, Z_k) \geq 2(\theta_1' + \theta_2') > 2\pi,$$

which is a contradiction. On the other hand, if Y_0 is in region II, then $\angle Z_k Z_j Y_0 + \angle Y_0 Z_j Y_k = 2\pi - \angle Y_k Z_j Z_k > 2\pi - \theta_1 > \pi$, so we also have

$$\rho(Y_k, Z_k) = \rho(Y_k, Y_0) + \rho(Y_0, Z_k) > 2\pi.$$

The situation is similar if Y_0 is in region III. This finishes the proof of Claim 2.

We can now deform $\alpha(Y_1, Z_1)$ to $\overline{Y_1 Z_1}$ via M_1. Since this deformation does not affect the rest of K, we obtain a new knot K_1, which is ambient isotopic to K. We then deform $\alpha(Y_2, Z_2)$ to $\overline{Y_2 Z_2}$ via M_2. Since M_2 does not intersect $K_1 \backslash \alpha(Y_2, Z_2)$ by Claims 1 and 2, we obtain a new knot K_2 that is also ambient isotopic to K. This process can then be repeated until all $\alpha(Y_j, Z_j)$'s are replaced by $\overline{Y_j Z_j}$'s. The result is a new knot K_m that is ambient isotopic to K such that for each crossing $P(Y) = P(Z)$ in its projection, we must have $\rho(Y, Z) \geq 2\pi$. We can then work on K_m to remove any remaining replaceable loops until we reach a weakly reduced knot K', which has the properties we need. This finishes the proof for the plane projection case.

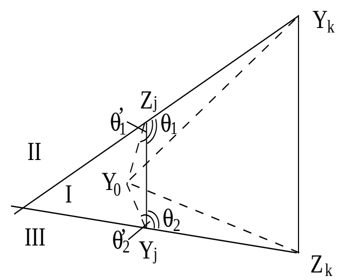

Figure 4: $\rho(Y_k, Z_k) \geq 2\theta$

The proof for the spherical projection is almost identical, except that (Y_j, Z_j)'s are ordered according to the ratio $\frac{r_j}{h_j}$, where $r_j = \min\{d(X,Y_j),d(X,Z_j)\}$, $h_j = \max\{d(X,Y_j), d(X,Z_j)\}$ and X is the center of the sphere on which K is projected to. We leave it to the reader to verify that if $\frac{r_j}{h_j} \geq \frac{r_i}{h_i}$, then we have $\theta_1 + \theta_2 \geq \pi$ (see Figure 5), which is what we need in proving Claim 2. □

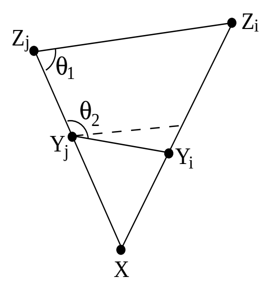

Figure 5: $\theta_1 + \theta_2 \geq \pi$

4. THE APPROACH BY MODIFIED AVERAGING CROSSING NUMBER

In this section, we will derive a formula that gives an upper bound for the minimum crossing number of a knot with unit thickness in terms of its length. Reversing this formula, we then get a lower bound for the length of the knot. Our result provides larger lower bounds for knots whose crossing numbers are not too big. Although this formula can certainly be improved for knots with larger minimal crossing numbers, we will not do so in this paper. Interestingly, when compared to the formula $L(K) \geq 1.105(Cr(K))^{\frac{3}{4}}$ (which can be obtained from the work of [2]), our formula provides larger lower bound even for $L(K)$ up to $Cr(K) = 1850$. So it is quite a powerful formula.

Let $\vec{v}$ be a unit vector and let $\Sigma_{\vec{v}}$ be the plane normal to $\vec{v}$ that passes through the origin. When we project two arcs ℓ' and ℓ'' to $\Sigma_{\vec{v}}$, they may intersect each other and we can count the number of crossings in the projection. We will use $c_{\vec{v}}(\ell', \ell'')$ to denote the number of crossings between ℓ' and ℓ'' under the projection to $\Sigma_{\vec{v}}$. By symmetry, we have $c_{\vec{v}}(\ell', \ell'') = c_{\vec{v}}(\ell'', \ell')$. The *averaging crossing number between ℓ' and ℓ''* is then defined as

$$\frac{1}{4\pi} \int_S c_{\vec{v}}(\ell', \ell'') d\mu, \tag{3}$$

where S is the unit sphere centered at the original point and μ is the measure of the area on S. We will use $a(\ell'', \ell')$ to denote the averaging crossing number between ℓ' and ℓ''. Similarly, we will use $c_{\vec{v}}(K)$ to denote the number of crossings in the projection of K to $\Sigma_{\vec{v}}$ (via direction $\vec{v}$). The *average crossing number of* K is defined as

$$a(K) = \frac{1}{4\pi} \int_S c_{\vec{v}}(K) d\mu. \tag{4}$$

Let n be a large positive integer and divide K into n arcs of equal length so that each arc is of length $L(K)/n$. Number them according to the orientation of K as ℓ_1, ℓ_2, ..., ℓ_n. We have

$$c_{\vec{v}}(K) = \frac{1}{2} \sum_{1 \leq i,j \leq n} c_{\vec{v}}(\ell_i, \ell_j), \tag{5}$$

hence

$$a(K) = \frac{1}{4\pi} \int_S c_{\vec{v}}(K) d\mu = \frac{1}{2} \sum_{1 \leq i,j \leq m} a(\ell_i, \ell_j). \tag{6}$$

The factor $\frac{1}{2}$ is in there because if a crossing is counted in $a(\ell_i, \ell_j)$, it will be counted again in $a(\ell_j, \ell_i)$. Let $Cr(K)$ be the minimum crossing number of K. Notice that this is the minimum of all possible projections of all knots that are of the same knot type as K. So we have $Cr(K) \leq c_{\vec{v}}(K)$ for any $\vec{v}$ hence $Cr(K) \leq a(K)$. We can actually improve this in the following way. By Proposition 1, if we only count the crossings $P(Y) = P(Z)$ in $P(K)$ that satisfy the conditions $\rho(Y, Z) \geq 2\pi$ and $\rho(Z, Y) \geq 2\pi$, we still get a number that is at least $Cr(K)$. The number of crossings counted this way is called the *modified* crossing number of K (in the projection to $\Sigma_{\vec{v}}$) and is denoted by $m_{\vec{v}}(K)$. Of course, we have $m_{\vec{v}}(K) \leq c_{\vec{v}}(K)$. The *modified average crossing number* of K, denoted by $m(K)$, is then defined by

$$m(K) = \frac{1}{4\pi} \int_S m_{\vec{v}}(K) d\mu. \tag{7}$$

The following inequality is comparable to (6):

$$m(K) \leq \frac{1}{2} \sum_{\rho(\ell_i, \ell_j) > 2\pi - 2\frac{L(K)}{n}} a(\ell_i, \ell_j) = \frac{1}{2} \sum_{1 \leq i \leq n} \sum_{J_i} a(\ell_i, \ell_j), \tag{8}$$

where J_i is the set of all j's satisfying $\rho(\ell_i, \ell_j) > 2\pi - 2L(K)/n$ or $\rho(\ell_j, \ell_i) > 2\pi - 2L(K)/n$. Here $\rho(\ell_i, \ell_j)$ means the length of arc on K joining ℓ_i and ℓ_j (following the given orientation).

Theorem 1. Let K be a knot with unit thickness and let γ be its arc length parameterized equation. Then we have

$$Cr(K) \leq m(K) \leq \frac{1}{4\pi} \int_K \int_{K_s} \frac{1}{|\gamma(t) - \gamma(s)|^2} dt ds, \tag{9}$$

where K_s is the set of points $\gamma(t)$ on K satisfying the condition $|t - s| \geq 2\pi$.

Proof. Let X be a point of ℓ_j and consider the ball $B_r(X)$. Notice that if ℓ_i and $\ell_j \cap B_r(X)$ intersect in their projection to $\Sigma_{\vec{v}}$, then $B_r(X)$ and ℓ_i must intersect in their projection to $\Sigma_{\vec{v}}$ as well. Let H be the set of all directions $\vec{v}$ such that $B_r(X)$ and ℓ_i intersect in their projection to $\Sigma_{\vec{v}}$ and let R be the distance between X and ℓ_i. Since the closer ℓ_i is to X, the larger the measure of H, we have $\mu(H) \leq 4r(L(K)/n + 2r)/R^2$. We leave the details of the proof of this formula to the reader. Keep in mind that $L(K)/n$ is the arc length of ℓ_i and that if the projections of ℓ_i and $B_r(X)$ intersect in the direction $\vec{v}$, then they intersect in the direction $-\vec{v}$ as well.

Now observe that if r is small enough, $c_{\vec{v}}(\ell_i, \ell_j \cap B_r(X)) \leq 1$ (unless the projection in the direction $\vec{v}$ is not regular). Thus we have

$$a(\ell_i, \ell_j \cap B_r(X)) = \frac{1}{4\pi}\int_S c_{\vec{v}}(\ell_i, \ell_j \cap B_r(X))d\mu \leq \frac{1}{4\pi}\int_H 1 d\mu \leq \frac{r(L(K)/n + 2r)}{\pi R^2}. \tag{10}$$

For k large enough, divide ℓ_j into k equal length pieces such that each piece is contained in a ball of radius $\frac{\rho_j}{2k}$ centered at a point on ℓ_j where $\rho_j = L(K)/n$ is the arc length of ℓ_j. Apply (10) to each of them and sum them up, we obtain

$$a(\ell_i, \ell_j) \leq \frac{L(K)(L(K)/n + L(K)/kn)}{2\pi n R_{ij}^2},$$

where R_{ij} is the minimum distance between ℓ_i and ℓ_j. Let $k \to \infty$, we obtain

$$a(\ell_i, \ell_j) \leq \frac{(L(K)/n)^2}{2\pi R_{ij}^2},$$

Substitute the above into (8) and let $n \to \infty$. The result follows. □

Remark. This estimate is consistent with the integral form of the average crossing number obtained in [12].

Theorem 2. Let K be a non-trivial knot with unit thickness, then

$$Cr(K) \leq \frac{1}{16\pi} L(K)(L(K) - 17.334). \tag{11}$$

Equivalently, we have

$$L(K) \geq \frac{1}{2}\left(17.334 + \sqrt{17.334^2 + 64\pi Cr(K)}\right). \tag{12}$$

Proof. For any $X = \gamma(s)$ on K, consider the spherical projection of K onto $S_2(X)$. For simplicity, we will assume that this projection is regular. If it is not, for any arbitrarily small number $\delta > 0$, there exists $X_\delta \in B_\delta(X)$ such that the spherical projection of K to $S_{2-\delta}(X_\delta)$ is regular and the following proof can be applied to this spherical projection. Once we let $\delta \to 0$ at the end, we will get the same result. The details are left to the reader. Let K' be a weakly reduced knot of K satisfying the conditions in Proposition 1 under this spherical projection. Let Z_1 be the first strong over point we meet starting from X and let Z_2 be the last strong over point we meet before returning to X. Thus

$X \in \alpha(Z_2, Z_1)$ and $\alpha(Z_2, Z_1)$ does not contain any other strong over points. Let Y_1, Y_2 be their corresponding strong under points. $\rho(Y_1, Z_1)$, $\rho(Y_2, Z_2)$, $\rho(Z_1, Y_1)$ and $\rho(Z_2, Y_2)$ are all greater than or equal to 2π. It follows that $d(Y_1, Z_1) \geq 2$ by Lemma 3. Since X, Y_1 and Z_1 are colinear and Y_1 is between X and Z_1, we must have $d(X, Z_1) > 2$. Similarly, we also have $d(X, Z_2) > 2$. This implies that $K \cap (B_2(X) \backslash S_2(X)) \subset \alpha(Z_2, Z_1)$. It follows from Lemma 1 that $Y_1, Y_2 \notin B_2(X) \backslash S_2(X)$. We further claim that $\rho(Z_1, Z_2) \geq 3\pi + \theta$ (so that $L(K) > 7\pi > 21.99$), where θ is the angle between $Y_1 - X$ and $Y_2 - X$. This is clearly the case if $\alpha(Z_1, Y_1) \cap \alpha(Y_2, Z_2) = \emptyset$, since we would have $\rho(Z_1, Z_2) > \rho(Z_1, Y_1) + \rho(Y_2, Z_2) \geq 4\pi$. If $\alpha(Z_1, Y_1) \cap \alpha(Y_2, Z_2) \neq \emptyset$, then we must have $Y_2 \in \alpha(Z_1, Y_1)$ and $Y_1 \in \alpha(Y_2, Z_2)$. Let $\theta_1 = \angle Z_1 Y_1 Y_2$ and $\theta_2 = \angle Y_1 Y_2 Z_2$, then $\theta_1 + \theta_2 = \pi + \theta$. Say $\theta_2 \geq \theta_1$, then $\theta_2 \geq (\pi + \theta)/2$. By Lemma 6, we have $\rho(Y_1, Z_2) > \pi + \theta$ hence $\rho(Z_1, Z_2) = \rho(Z_1, Y_1) + \rho(Y_1, Z_2) \geq 3\pi + \theta$.

Now, let $W_1 \in \alpha(Z_1, Z_2)$ be the point satisfying the condition $\rho(Z_1, W_1) = \pi + 2\sqrt{2} - 2 < \pi + 0.829$ and let $W_2 \in \alpha(Z_1, Z_2)$ be the point satisfying the condition $\rho(W_2, Z_2) = \pi + 2\sqrt{2} - 2$. It is clear that $\alpha(Z_1, W_1) \cap \alpha(W_2, Z_2) = \emptyset$. Without loss of generality, assume that $\gamma(0) = Z_1$, $\gamma(\pi + 2\sqrt{2} - 2) = W_1$, Y_1 is the original point and Z_1 is on the positive z-axis. Observe that for $0 \leq t \leq \pi$, $\gamma(t)$ cannot intersect the interior of $B_2(Y_1)$, or we would have $\rho(Z_1, Y_1) < 2\pi$ by Lemma 3, which would be a contradiction. Furthermore, for $0 < t \leq \pi$, the angle between $Z_1 - Y_1$ and $\gamma(t) - Y_1$ is less than or equal to $\frac{t}{2}$ by Lemma 6. Let Σ be the yz plane and let $C = \Sigma \cap B_2(Y_2)$, which is the circle of radius 2 centered at Y_1. Let $\gamma_1(t)$ be the arclength parameterized equation of C such that $\gamma_1(0) = (0, 0, 2)$. The equation of C can be written as $(0, 2\sin\frac{t}{2}, 2\cos\frac{t}{2})$. Point X is now $(0, 0, -(2 + h))$ where $h = d(X, Y_1) - 2 \geq 0$. The angle between $\gamma_1(t) - Y_1$ and $Z_1 - Y_1$ is $\frac{t}{2}$. It is now obvious that $|\gamma(t) - X| \geq |\gamma_1(t) - X|$. See Figure 6. Hence we have

$$\begin{aligned} & \int_0^\pi \frac{dt}{|\gamma(t) - X|^2} \leq \int_0^\pi \frac{dt}{|\gamma_1(t) - X|^2} \\ = & \int_0^\pi \frac{dt}{4 + (2+h)^2 + 4(2+h)\cos\frac{t}{2}} \leq \int_0^\pi \frac{dt}{8(1 + \cos\frac{t}{2})} = \frac{1}{4}. \end{aligned}$$

On the other hand, since $|\gamma(\pi) - Y_1| \geq 2$ and the angle between $Z_1 - X$ and $\gamma(\pi) - X$ is at most $\frac{\pi}{2}$, it follows that $|\gamma(\pi) - X| \geq 2\sqrt{2}$. So for any $\pi < t \leq \pi + 2\sqrt{2} - 2$, we have $|\gamma(t) - X| \geq 2\sqrt{2} + \pi - t$. This leads to

$$\int_\pi^{\pi + 2\sqrt{2} - 2} \frac{dt}{|\gamma(t) - X|^2} \leq \int_\pi^{\pi + 2\sqrt{2} - 2} \frac{dt}{(2\sqrt{2} + \pi - t)^2} = \int_2^{2\sqrt{2}} \frac{du}{u^2} = \frac{1}{4}(2 - \sqrt{2}).$$

Combining the two results above, we have

$$\int_{\alpha(Z_1, W_1)} \frac{dt}{|\gamma(t) - X|^2} \leq \frac{1}{4}(3 - \sqrt{2}).$$

Similarly, we also have

$$\int_{\alpha(W_2, Z_2)} \frac{dt}{|\gamma(t) - X|^2} \leq \frac{1}{4}(3 - \sqrt{2}).$$

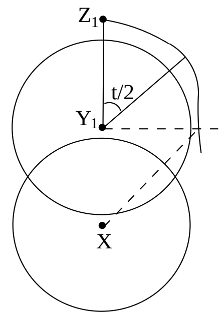

Figure 6: The distance from a point on $\alpha(Z_1, W_1)$ to X

Finally, we have

$$\begin{aligned}
& \frac{1}{4\pi}\int_{K_s}\frac{dt}{|\gamma(t)-X|^2} \\
= & \frac{1}{4\pi}\left(\int_{K_s\setminus\alpha(Z_1,W_1)\cup\alpha(W_2,Z_2)}+\int_{\alpha(Z_1,W_1)}+\int_{\alpha(W_2,Z_2)}\right)\frac{dt}{|\gamma(t)-X|^2} \\
\leq & \frac{1}{4\pi}\left(\frac{1}{4}(L(K)-6\pi-4\sqrt{2}+4)+\frac{1}{4}(6-2\sqrt{2})\right) \\
= & \frac{1}{16\pi}(L(K)+10-6(\pi+\sqrt{2}))\leq\frac{1}{16\pi}(L(K)-17.3348).
\end{aligned}$$

Substituting this into (9) and the result follows. □

Since the result of Theorem 2 falls short of getting the lower bound 24, we will reach that goal in the next section.

5. THE LENGTH OF A UNIT THICKNESS KNOT IS > 24

Theorem 3. Let K be a non-trivial knot with unit thickness, then $L(K) > 24$.

Proof. Let us assume the contrary, that is, K is a nontrvial knot but $L(K) \leq 24$. By Theorem 2, this implies that K can only be a trefoil. Let us continue the discussion from last section. For any $X = \gamma(s) \in K$, we know that $\rho(X, Z_1) \geq 2\pi$, $\rho(Z_2, X) \geq 2\pi$, $\rho(Z_1, Y_1) \geq 2\pi$, $\rho(Y_2, Z_2) \geq 2\pi$. $Y_2 \in \alpha(Z_1, Y_1)$ since otherwise we have $L(K) \geq 8\pi \approx 25.13$. We also know that $\rho(Z_1, Y_2) \geq \pi + \theta$ or $\rho(Y_1, Z_2) \geq \pi + \theta$. Combining these results, we see that $L(K) \geq 7\pi + \theta$, where θ is the angle between $Y_1 - X$ and $Y_2 - X$.

Claim 1. There exists $X \in K$ such that $\rho(X, Z_1) \leq 2\pi + 0.6$ and $\angle Z_1Y_1Y_2 > \pi/2$.

Proof. Assume the contrary. Then for any $X \in K$ such that $\angle Z_1Y_1Y_2 > \pi/2$ (if it is the case that $\angle Z_2Y_2Y_1 > \pi$, just change the orientation of K and use Z_2 as Z_1). Let $W_0 \in \alpha(X, Z_1)$ be such that $\rho(W_0, Z_1) = 0.6$. Since $\alpha(W_0, Z_1) \subset K_s$ and does not intersect the interior of $B_2(Y_1)$, the same technique used to prove Theorem 2 applies. We have

$$\int_{\alpha(W_0,Z_1)} \frac{dt}{|\gamma(t)-X|^2} \leq 0.1512/4.$$

Combining it with the result in Theorem 2, we get

$$\int_{K_s} \frac{dt}{|\gamma(t)-X|^2} \leq \frac{1}{16\pi}(L(K)-17.755).$$

Thus $Cr(K) \leq \frac{1}{16\pi}L(K)(L(K)-17.755)$. For $Cr(K)=3$, this yields $L(K) > 24.05$. So this case is impossible.

Claim 2. For the X we found in Claim 1, we have $d(X,Z_1) \leq 5.4$.

Proof. This is implied by Lemma 5(b). The details are left to the reader.

Claim 3. Let $W_1, W_2 \in \alpha(Z_1,Y_1)$ be such that $\rho(Z_1,W_1)=1.05$ and $\rho(W_2,Y_1)=1.05$. Then $\rho(W_1,W_2) < 2\pi$.

Proof. Since $\rho(Y_1,Z_2) > \pi$ by Claim 1 and Lemma 6, we have $\rho(Z_1,Y_1) = L(K) - \rho(X,Z_1) - \rho(Z_2,X) - \rho(Y_1,Z_2) \leq 24 - 5\pi < 2\pi + 2.01$. Hence $\rho(W_1,W_2) = \rho(Z_1,Y_1) - 2.1 < 2\pi$.

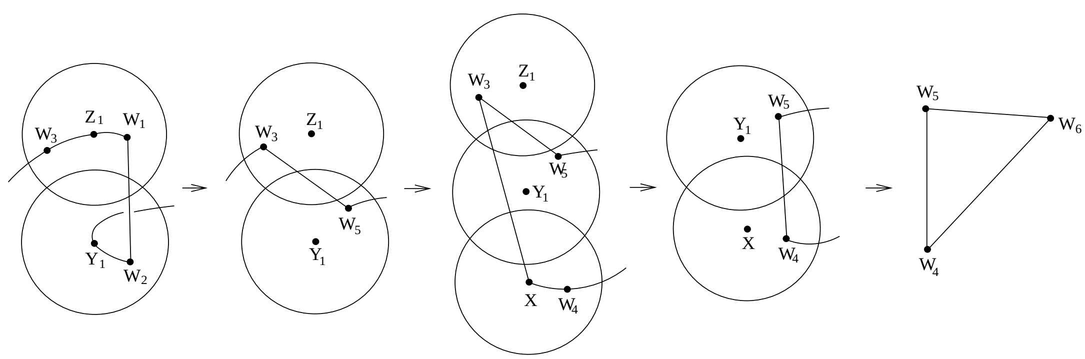

Figure 7: K is reduced to an unknot

Now, by the proof of Proposition 1, we see that $\alpha(W_1,W_2)$ is replaceable by $\overline{W_1W_2}$. A simple calculation shows that $\overline{W_1W_2} \subset B_2(Y_1) \cup B_2(Z_1)$ since $d(Y_1,Z_1) \leq 3.4$ by Claim 2. It is apparent that $(B_2(Z_1) \cap \alpha(X,W_1)) \cup \overline{W_1W_2} \cup (B_2(Y_1) \cap \alpha(W_2,Z_2))$ and $B_2(Y_1) \cup B_2(Z_1)$ form an unknotted ball-arc pair. Let $W_3 \in \alpha(X,Z_1)$ be such that $\rho(W_3,Z_1)=1$, $W_4 \in \alpha(Z_2,X)$ be such that $\rho(W_4,X)=1.05$ and $W_5 \in \alpha(Y_1,Z_2)$ be such that $\rho(Y_1,W_5)=1.05$. Since $B_2(Y_1) \cup B_2(Z_1)$ does not intersect the rest of K, replacing $\alpha(W_3,W_1)) \cup \overline{W_1W_2} \cup \alpha(W_2,W_5))$ by $\overline{W_3W_5}$ will not change the knot type of K. We see that $\alpha(X,W_3)$ is replaceable by $\overline{W_3X}$ by observing that the set $M' = \cup_{Y' \in \alpha(X,W_3)} \overline{Y'W_3}$ does not intersect $\overline{W_3W_5}$ and the rest of K, and then $\alpha(X,W_4) \cup \overline{W_3X} \cup \overline{W_3W_5}$ can be replaced by $\overline{W_4W_5}$ for the same reason above. Notice that we have $\rho(W_5,W_4) \leq 24 - 4\pi - 2.1 < 9.34$. Let $W_6 \in \alpha(W_5,W_4)$ be such that $\rho(W_6,W_4) = 2\pi - 0.01$, then $\alpha(W_6,W_4)$ is replaceable by $\overline{W_6W_4}$ (via deformation through the set $M'' = \cup_{Y' \in \alpha(W_6, W_4)} \overline{Y'W_4})$. It is now clear that $\alpha(W_5, W_6)$ is replaceable by $\overline{W_5W_6}$ so K is reduced (without changing its knot type) to $\overline{W_6W_4} \cup \overline{W_5W_4} \cup \overline{W_5W_6}$ hence K is a trivial knot. This process is illustrated in Figure 7. This contradiction shows that we must have have $L(K) > 24$. □

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