Wavelets with Frame Multiresolution Analysis

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Abstract

A frame multiresolution (FMRA for short) orthogonal wavelet is a single-function orthogonal wavelet such that the associated scaling space V_0 admits a normalized tight frame (under translations). In this paper, we prove that for any expansive matrix A with integer entries, there exist A-dilation FMRA orthogonal wavelets. FMRA orthogonal wavelets for some other expansive matrix with non integer entries are also discussed.

1. Introduction

Let A be a $d \times d$ real expansive matrix, i.e., a matrix with real entries whose eigenvalues are all of modules greater than one. Let $L^2(\mathbb{R}^d)$ be the set of all square Lebesgue integrable functions in $\mathbb{R}^d$ and let μ stand for the Lebesgue measure in $\mathbb{R}^d$. An *A-dilation wavelet* is a *single* function $\psi \in L^2(\mathbb{R}^d)$ such that the set

$$\{|\det A|^{\frac{n}{2}}\psi(A^n t - \ell) : n \in \mathbb{Z}, \ell \in \mathbb{Z}^d\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^d)$. One issue in the wavelet theory is how to construct A-dilation wavelets, especially the ones that are useful in applications. Although it is proven in [7] that single function A-dilation wavelets do exist for *every* expansive matrix A, finding the ones useful in applications is quite a different problem. The well-known and most widely applied A-dilation wavelets are the so called MRA wavelets. These are associated with (and constructed from) subspaces of $L^2(\mathbb{R}^d)$ with certain special properties [3]. One such property is the existence of an *orthonormal scaling function*. See next section for a detailed definition. Unfortunately, MRA A-dilation wavelets do not always exist. In the case that A is also a matrix with integer entries, it is known that an MRA A-dilation wavelet exists if and only if $|\det A| = 2$ [8, 10]. So, if A has integer entries and $|\det A| > 2$, then there are *no* MRA A-dilation wavelets (though there exist *multi*-MRA wavelets), even if $A = 2I$ when $d \geq 2$. For the matrices A with non-integer entries, it is not clear whether or when MRA A-dilation wavelets exist. In this paper, we will investigate those matrices A such that there exist MRA-like single function A-dilation wavelets. We will focus on a system similar, but less restrictive, to the MRA so that we can construct A-dilation wavelets that would share certain similar properties with the MRA A-dilation wavelets. Specifically, the A-dilation wavelets we construct will also have scaling functions which are "almost" orthonormal. The system is called a *frame multiresolution analysis (FMRA)*, which is a natural generalization of MRA and was introduced by Benedetto and Li ([4]). For more general multiresolution analysis and wavelets, please see the more recent work of Baggett [2]). In Section 3, we prove that FMRA A-dilation wavelets exist for all expansive matrices with integer entries. We also prove that FMRA A-dilation wavelets exist for some special non-integer expansive matrices. Some examples and discussions about FMRA wavelets for dilation matrices with irrational entries are presented in Section 4. For more related topics and works, we refer our reader to [1], [5] and [6].

2. The Definition of FMRA

A sequence $\{f_j : \ j \in \mathbb{N}\}$ of functions in a closed subspace X of $L^2(\mathbb{R}^d)$ is called a *normalized tight frame* for X if

$$\sum_j \langle f, f_j \rangle f_j = f, \quad \forall f \in X. \tag{1}$$

The convergence here is in norm and is *unconditional*, that is, the convergence and the limit will not change when the order in the sum is altered.

The Fourier transform $\mathcal{F}$ is defined by

$$(\mathcal{F}f)(s) \quad = \quad \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i(s \circ t)} f(t) d\mu, \tag{2}$$

for L^1-functions and can be extended to a unitary operator on $L^2(\mathbb{R}^d)$, where $s \circ t$ denotes the real inner product.

Definition 1 *A frame multiresolution analysis associated with a real expansive matrix A (or in short, an A-dilation FMRA) is a sequence* $\{V_j : \ j \in \mathbb{Z}\}$ *of closed subspaces of* $L^2(\mathbb{R}^d)$ *satisfying the following conditions:*

1. $V_j \subset V_{j+1}, \ \forall j \in \mathbb{Z}$;
2. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \ \overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$;
3. $f \in V_j$ *if and only if* $f(As) \in V_{j+1}, \ j \in \mathbb{Z}$;
4. *There exists a function* ϕ *in* V_0 *such that* $\{\phi(x-\ell) : \ell \in \mathbb{Z}^d\}$ *is a normalized tight frame for* V_0.

The function ϕ *in (4) is called a frame scaling function for the A-dilation FMRA. A function* ψ *in* $V_{-1} \ominus V_0$ *is called an A-dilation FMRA wavelet (or just an FMRA wavelet with the understanding that it is associated with the given expansive matrix A) if it is an A-dilation orthonormal wavelet for* $L^2(\mathbb{R}^d)$.

If, in the above definition, we replace "normalized tight frame" by "orthonormal basis" in (4), then we obtain the standard definition for a multiresolution analysis.

A measurable set $E \subset \mathbb{R}^d$ is called an *A-dilation wavelet set* if $\mathcal{F}^{-1}(\frac{1}{\sqrt{\mu(E)}}\chi_E)$ is an A-dilation wavelet. $E \subset \mathbb{R}^d$ is called an *A-dilation FMRA wavelet set* if $\mathcal{F}^{-1}(\frac{1}{\sqrt{\mu(E)}}\chi_E)$ is an A-dilation FMRA wavelet.

3. The Existence of FMRA Wavelets

In this section we prove that if the real expansive matrix A satisfies certain conditions then there exist A-dilation FMRA wavelets. This is done by showing that A-dilation FMRA wavelet sets exist under these conditions. The main results are stated in the two theorems below.

Theorem 1 *For every expansive matrix A with integer entries, there exists an A-dilation FMRA wavelet set.*

In the case that A does not necessarily have integer entries, we have

Theorem 2 *Let $A = diag(a_1, \ldots, a_d)$ be a diagonal expansive matrix such that $|a_i| \geq 2$ for some i. Then there exists an A-dilation FMRA wavelet set. In particular, if $d = 1$ and $A = a$, then there exists an a-dilation FMRA wavelet set for $L^2(\mathbb{R})$ if and only if $|a| \geq 2$.*

We need some preparations before proving these theorems. Two measurable sets E and F are $2\pi\mathbb{Z}^d$-*translation congruent* (or just translation congruent for short) modulo measure zero sets if there exist measurable partitions $\{E_\ell\}_{\ell\in\mathbb{Z}^d}$ of E and $\{F_\ell\}_{\ell\in\mathbb{Z}^d}$ of F (modulo null sets) such that $F_\ell = E_\ell + 2\ell\pi$. Analogously, two measurable sets G and H are A-*dilation congruent* if there exist measurable partitions $\{G_n\}_{n\in\mathbb{Z}}$ of G and $\{H_n\}_{n\in\mathbb{Z}}$ of H (modulo null sets) such that $G_n = A^n H_n$. A measurable set E is called *an A-dilation generator* of $\mathbb{R}^d$ if $\{A^m E : \ m \in \mathbb{Z}\}$ forms a partition of $\mathbb{R}^d$ (modulo null sets), and *a $2\pi\mathbb{Z}^d$-translation generator* (translation generator) of $\mathbb{R}^d$ if $\{E + 2\ell\pi : \ \ell \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$ (modulo null sets). Observe that all the A-dilation generators of $\mathbb{R}^d$ are A-dilation congruent to each other. Similarly all the translation generators are translation congruent to each other. So if we let Ω be the cube

$$\Omega = \{t = (t_1, \ ..., \ t_d) \in \mathbb{R}^d : \ \ -\pi \leq t_i < \pi, \ i = 1, \ ..., \ d\},$$

then a measurable subset F of $\mathbb{R}^d$ is a translation generator of $\mathbb{R}^d$ if and only if F and Ω are translation congruent. In this paper we will always use Ω to denote this cube.

Proposition 1 *Assume that A is expansive and $E \subset \mathbb{R}^d$. Then*

(i) E is an A-dilation wavelet set if and only if E is both a translation and an A^t-dilation generator of $\mathbb{R}^d$ (where A^t is the transpose of A).

(ii) E is an A-dilation FMRA wavelet set if and only if E is a wavelet set such that $E = A^t K \setminus K$ for some K with the property that $A^t K \supset K$ and K is translation congruent to a subset of Ω.

Proof. Part (i) is a well-known result and part (ii) follows from (i) and the definition of FMRA wavelet set, using the fact that $\{e^{i(\ell\circ x)}h(x) : \ell \in \mathbb{Z}^d\}$ is a normalized tight frame for $L^2(F)$ if and only if F is translation congruent to a subset of Ω. For the proof of this fact, cf [9]. □

Lemma 1 *Assume that A is an expansive matrix and there exists an A-dilation FMRA wavelet set E. Then $|\det A| \geq 2$.*

Notice that if A is expansive and has only integer entries, then $|\det A| \geq 2$ is automatic. So Lemma 1 gives a necessary condition for the existence of an A-dilation FMRA wavelet set for arbitrary expansive matrices.

Proof. Let K be the measurable set satisfying all the conditions in Proposition 2.1 (ii). Then $(2\pi)^d = \mu(E) = \mu(A^tK) - \mu(K) = (|\det A| - 1)\mu(K)$. Since K is translation congruent to a subset of Ω, it follows that $\mu(K) \leq (2\pi)^d$. Thus $|\det A| \geq 2$. □

For a subset $K \subset \mathbb{R}^d$, we use $\mathcal{T}(K)$ to denote the set $\cup_{\ell\in\mathbb{Z}^d}(K + 2\pi\ell)$. The following Lemma is from [8].

Lemma 2 *Suppose that G is a measurable subset of $\mathbb{R}^d$ and let $F = \Omega \cap \mathcal{T}(G)$. Then there exists a measurable subset K of G such that $G \setminus K$ and F are translation congruent and $K \cap \Omega = \emptyset$.*

Proposition 2 *If there exists a measurable set $K_0 \subset \mathbb{R}^d$ such that*

(i) the origin O is an interior point of K_0,

(ii) $A^tK_0 \supset K_0$,

(iii) K_0 is translation congruent to a subset of Ω,

(iv) $\cup_{\ell\in\mathbb{Z}^d}((A^tK_0 \setminus K_0) + 2\pi\ell) = \mathbb{R}^d$ (modulo a null set),

then there exists a subset K of K_0 satisfying conditions (i)-(iii) above such that $A^tK \setminus K$ is a translation generator of $\mathbb{R}^d$.

Proof. Let $F = \Omega$ and $G = A^tK_0 \setminus K_0$. Then by Lemma 2 and condition (iv), we can choose $F_0 \subset A^tK_0 \setminus K_0$ such that $F_0 \subset \Omega^c$ and that $(A^tK_0 \setminus K_0) \setminus F_0$ is translation congruent to Ω. Let $K_1 = K_0 \setminus (A^t)^{-1}F_0$. We claim that K_1 satisfies (i)–(iv). Since K_1 is a subset of K_0 , it is translation congruent to a subset of Ω, i.e. (iii) holds for K_1. Let $\mathcal{B}(O, r)$ be the open ball centered at the origin with radius r. Choose an r small enough so that $\mathcal{B}(O, r) \subset K_0$. Let $\delta > 0$ ($\delta < r$) be such that $(A^t)^{-1}\Omega^c \cap \mathcal{B}(O, \delta) = \emptyset$. Thus $\mathcal{B}(O, \delta) \subset K_1$ and (i) holds for K_1.

Since F_0 and K_0 are disjoint, it follows that

$$A^tK_1 = A^tK_0 \setminus F_0 \supset K_0 \setminus F_0 = K_0 \supset K_1.$$

So we have (ii) for K_1. Clearly

$$\begin{aligned} A^tK_1 \setminus K_1 &= (A^tK_0 \setminus F_0) \setminus (K_0 \setminus (A^t)^{-1}F_0) \\ &\supset (A^tK_0 \setminus F_0) \setminus K_0 = (A^tK_0 \setminus K_0) \setminus F_0. \end{aligned}$$

Therefore (iv) holds for K_1 since $(A^tK_0 \setminus K_0) \setminus F_0$ is a translation generator of $\mathbb{R}^d$. Furthermore,

$$\mu(F_0) = \mu(A^tK_0 \setminus K_0) - \mu(\Omega) = (|\det A| - 1)\mu(K_0) - (2\pi)^d,$$

hence

$$\begin{aligned}
\mu(K_1) &= \mu(K_0) - \mu((A^t)^{-1}F_0) \\
&= \mu(K_0) - [1 - \frac{1}{|\det A|}]\mu(K_0) + \frac{1}{|\det A|}(2\pi)^d \\
&= \frac{1}{|\det A|}[\mu(K_0) + (2\pi)^d].
\end{aligned}$$

Continuing the above argument, we obtain a decreasing sequence $\{K_n\}$ of measurable subsets and a sequence $\{F_n\}$ ($F_n \subset A^tK_n \setminus K_n$ and $F_n \subset \Omega^c$) such that

(a) $\mathcal{B}(O, \delta) \subset K_n$

(b) $A^tK_n \supset K_{n-1}$

(c) $(A^tK_n \setminus K_n) \setminus F_n$ is a translation generator of $\mathbb{R}^d$.

(d) $\mu(K_n) = \frac{1}{|\det A|}[\mu(K_{n-1}) + (2\pi)^d]$.

Let $K = \cap_{n=0}^{\infty} K_n$. We now claim that K satisfies conditions (i)-(iv). Due to (a) and (b), we only need to show that $AK \setminus K$ is a translation generator of $\mathbb{R}^d$. Since $\{K_n\}$ is decreasing, $\mu(K) = lim_{n\to\infty}\mu(K_n)$. By (d), we obtain $\mu(K) = \frac{1}{|\det A|}(\mu(K) + (2\pi)^d)$. So $\mu(K) = \frac{1}{|\det A|-1}(2\pi)^d$. Hence $\mu(A^tK \setminus K) = (|\det A| - 1)\mu(K) = (2\pi)^d$. Therefore, to prove that $A^tK \setminus K$ is a translation generator of $\mathbb{R}^d$, it suffices to show that the sets $\{(A^tK \setminus K) + 2\pi\ell : \ell \in \mathbb{Z}^d\}$ are disjoint. Assume, to the contrary, that there is a subset L of K with positive measure such that both L and $L + 2\ell\pi$ are contained in $A^tK \setminus K$ for some $\ell \in \mathbb{Z}^d$, and $L \cap (L + 2\ell\pi) \neq \emptyset$. Note that $A^tK \setminus K = A^tK \cap K^c = \cup_{n=0}^{\infty} A^tK \cap K_n^c$ and

$$K_0^c \subset K_1^c \subset \cdots \subset K_n^c \subset \cdots$$

where E^c denote the set complement of E in $\mathbb{R}^d$ for any subset E of $\mathbb{R}^d$. By considering a subset of L, we can assume that there exists $N > 0$ such that both L and $L + 2\ell\pi$ are contained in $A^tK \cap K_n^c$ for $n \geq N$. By the way F_n is chosen, either $L \cap F_n$ or $(L + 2\ell\pi) \cap F_n$ has positive measure. If $L \cap F_n$ has positive measure, then $(A^t)^{-1}(L \cap F_n) \cap K_{n+1} = \emptyset$. So $L \cap F_n \cap A^tK_{n+1} = \emptyset$. This is a contradiction since $L \cap F_n \subset L \subset A^t K \subset A^t K_{n+1}$. Similarly there would be a contradiction if $(L + 2\ell\pi) \cap F_n$ has a positive measure. Thus $A^t K \setminus K$ must be translation disjoint. □

Corollary 1 *Assume that A is an expansive matrix and there exists a set K_0 satisfying (i) to (iv) in Proposition 2. Then there exists an A-dilation FMRA wavelet set.*

Proof. Let K be as in Proposition 2. Then $A^t K \setminus K$ is a translation generator of $\mathbb{R}^d$, $A^t K \supset K$ and K is translation congruent to a subset of Ω. Since K contains O as an interior point and A is expansive, we have that $\cup_{n\in\mathbb{Z}}(A^t)^n K = \mathbb{R}^d$. This implies that $\cup_{n\in\mathbb{Z}}(A^t)^n(A^t K \setminus K) = \mathbb{R}^d \setminus \{O\}$. Since $\{(A^t)^n(A^t K \setminus K) : n \in \mathbb{Z}\}$ are disjoint sets, $A^t K \setminus K$ is an A^t-dilation generator of $\mathbb{R}^d$. Therefore, by Proposition 1, $A^t K \setminus K$ is an A-dilation FMRA wavelet set. □

We are now ready to prove the theorems.

Proof of Theorem 1. By Corollary 1, it suffices to prove that there exists a set $K_0 \subset \mathbb{R}^d$ satisfying the conditions in Proposition 2. By Lemma 2.3 in [8], there is a K_0 such that O is an interior point of K_0, K_0 is translation congruent to Ω and $A^t K_0 \supset K_0$. Therefore we only need to show that $\mathcal{T}(A^t K_0 \setminus K_0) = \mathbb{R}^d$. Notice that $A^t\mathbb{Z}^d \neq \mathbb{Z}^d$. Let $\ell_0 \notin A^t\mathbb{Z}^d$ and $y \in \mathbb{R}^d$. Since $\mathcal{T}(K_0) = \mathbb{R}^d$, it follows that

$$\cup_{\ell\in\mathbb{Z}^d}(A^t K_0 + 2\pi A^t \ell) = \cup_{\ell\in\mathbb{Z}^d}(A^t K_0 + 2\pi A^t \ell + 2\pi\ell_0) = \mathbb{R}^d.$$

Thus there exist $x_1, x_2 \in A^t K_0$ and $\ell_1, \ell_2 \in \mathbb{Z}^d$ such that

$$y = x_1 + 2\pi A^t \ell_1 = x_2 + 2\pi A^t \ell_2 + 2\pi\ell_0.$$

This implies that either $x_1 \notin K_0$ or $x_2 \notin K_0$ since K_0 is translation congruent to Ω and $\ell_0 \notin A^t\mathbb{Z}^d$. Therefore $y \in \mathcal{T}(A^t K_0 \setminus K_0)$. Since y is arbitrary in $\mathbb{R}^d$ (except for a measure zero subset), we have $\mathcal{T}(A^t K_0 \setminus K_0) = \mathbb{R}^d$, as desired. □

Proof of Theorem 2. We can assume that $a_1 \geq 2$. Choose $K_0 = \Omega$. Then

$$A^t K_0 \setminus K_0 \supset \{(x_1, ..., x_d) : \pi \leq |x_1| < 2\pi, -\pi \leq x_i < \pi, i = 2, ..., d\}.$$

Thus $\cup_{\ell\in\mathbb{Z}^d}((A^t K_0 \setminus K_0) + 2\pi\ell) = \mathbb{R}^d$. Clearly K_0 satisfies all the other requirements in Proposition 2. Therefore there exists an A-dilation FMRA wavelet set. □

4. Examples and Discussions

In this section we will give a few examples of A-dilation FMRA wavelet sets for some expansive matrices A with $|detA| > 2$, especially for some matrices with non-integer entries.

Example 1. FMRA set in $\mathbb{R}$ with $A = a > 2$. Let $\alpha \in (0, 2\pi(\frac{a-2}{a-1}))$. Define intervals E_n, F_n by

$$\begin{aligned} E_0 &= (2\pi - \alpha,\ 2\pi - \frac{\alpha}{a}); \\ E_n &= (2\pi \cdot \frac{a^{n+1}-1}{a^{n+1}-a^n} - \frac{\alpha}{a^n},\ 2\pi \cdot \frac{a^{n+1}-1}{a^{n+1}-a^n} - \frac{\alpha}{a^{n+1}}),\ n \in \mathbb{N}; \\ F_n &= \frac{1}{a}E_n. \end{aligned}$$

It is left to the reader to verify that the above defined sets have the following properties.

1. $F_n + 2\pi = E_{n+1}$;
2. $\{E_n : n \in \mathbb{N}\}$ is a family of disjoint subintervals in the interval $(\frac{2\pi}{a-1},\ \frac{2\pi a}{a-1})$.
3. $\{F_n\}$ is a family of disjoint sets which are also disjoint from the interval $(\frac{2\pi}{a-1},\ \frac{2\pi a}{a-1})$.

Define

$$E = (-\alpha,\ -\frac{\alpha}{a}) \cup ((\frac{2\pi}{a-1},\ \frac{2\pi a}{a-1}) \setminus \cup_{n=0}^{\infty} E_n) \cup (\cup_{n=0}^{\infty} F_n).$$

It is clear that the interval $(\frac{2\pi}{a-1}, \frac{2\pi a}{a-1})$ is a 2π-translation generator of $\mathbb{R}$. It contains subsets E_n and is disjoint from F_n and $(-\alpha, -\frac{1}{a}\alpha)$. Since $(-\alpha, -\frac{1}{a}\alpha) + 2\pi = E_0$ and $F_n + 2\pi = E_{n+1}$, E is a 2π-translation generator of $\mathbb{R}$.

The set $(-\alpha, -\frac{1}{a}\alpha) \cup (\frac{1}{a-1}2\pi, \frac{a}{a-1}2\pi)$ is an a-dilation generator of $\mathbb{R}$. Since $F_n = \frac{1}{a}E_n$ for each $n \in \mathbb{N} \cup \{0\}$, E is also an a-dilation generator of $\mathbb{R}$.

Let $K = \cup_{m=1}^{\infty} a^{-m}E$, then we have $E = aK \setminus K$ and $aK \supset K$. To show that E is an FMRA wavelet set, it suffices to show that K is 2π-translation congruent to a subset of $(-\pi, \pi)$ by Proposition 1. This follows from the fact that K is contained in the interval $(-\frac{1}{a}\alpha, \frac{2\pi}{a-1})$ with length less than 2π.

Note that this is a path-connected family of a-dilation FMRA wavelet sets with parameters λ and a.

Example 2. $A = \begin{pmatrix} 2 & 0 \\ 0 & a \end{pmatrix}$ where $a > 1$. Although the set K can be constructed from the set $K_0 = \Omega$ using the procedure described in the proof of Proposition 2, we have constructed one that is different. This construction can be generalized to matrices of the form $A = \begin{pmatrix} \lambda & 0 \\ 0 & a \end{pmatrix}$ with $a > 1$ and $\lambda \geq 2$. Define two sequences $\{c_n\}$ and $\{b_n\}$ such that $c_{-1} = 0$, $c_n = (1 - \frac{1}{2^{2n}})\frac{2\pi}{3} + \frac{\pi}{2^{2n+1}}$ if $n \geq 0$, $b_{-1} = \pi$ and $b_n = \frac{2\pi}{3} + \frac{\pi}{3 \cdot 2^{2n+2}}$ if $n \geq 0$. Notice that $\{c_n\}$ is increasing, $\{b_n\}$ is decreasing and

$$\lim_{n \to \infty} c_n = \lim_{n \to \infty} b_n = \frac{2\pi}{3}.$$

Let P_j be the rectangle with corners $(c_{j-1}, \frac{\pi}{a^{2j+1}})$, $(c_j, \frac{\pi}{a^{2j+1}})$, $(c_{j-1}, -\frac{\pi}{a^{2j+1}})$ and $(c_j, -\frac{\pi}{a^{2j+1}})$ where $j \geq 0$. Let Q_j be the rectangle with corners $(b_{j-1}, \frac{\pi}{a^{2j+2}})$, $(b_j, \frac{\pi}{a^{2j+2}})$, $(b_{j-1}, -\frac{\pi}{a^{2j+2}})$ and $(b_j, -\frac{\pi}{a^{2j+2}})$ where $j \geq 0$. Also, let $-P_j$, $-Q_j$ be the mirror images of P_j, Q_j through the y-axis respectively. Let $K = \cup_{j \geq 0}(P_j \cup Q_j \cup (-P_j) \cup (-Q_j))$. Then K is a subset of Ω containing O as an interior point. We leave the details to our reader to check that $K \subset A^t K$ and that $A^t K \setminus K$ is both a translation and a dilation generator of $\mathbb{R}^2$. Figures 1 and 2 illustrate the sets K and $A^t K \setminus K$.

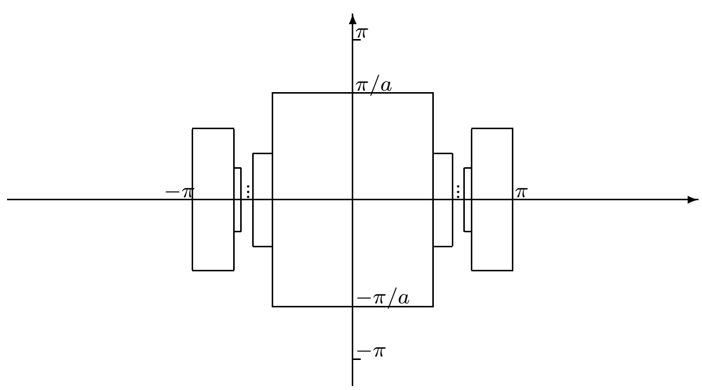

Figure 1: The set K.

Example 3. Let $A^t = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $K = \Omega$, then $A^t K \setminus K$ is an A^t-dilation generator of $\mathbb{R}^2$ and is translation congruent to Ω as shown

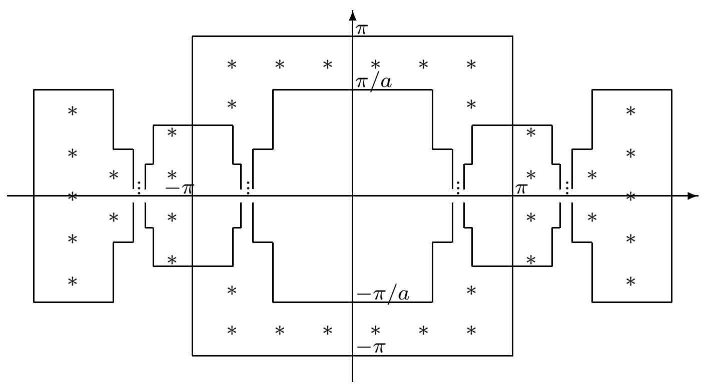

Figure 2: The set $A^K \setminus K$ (marked with $*$'s).

in Figure 3. Thus $A^tK \setminus K$ is an FMRA wavelet set. Notice that in this case $|\det A| = 2$. In general, if $|\det A| > 2$, then Ω cannot be the set K, since $\mu(A^t\Omega \setminus \Omega) > \mu(\Omega)$ hence $A^t\Omega \setminus \Omega$ cannot be translation congruent to Ω.

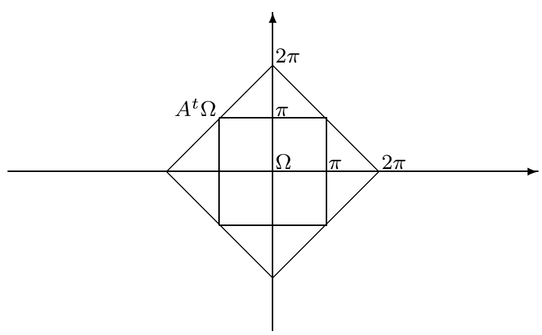

Figure 3: The set $A^t\Omega \setminus \Omega$.

Because of Theorem 1 and Theorem 2, we know that FMRA wavelet sets exist in the above examples before we constructed the set K. The following example shows a case not covered by the theorems.

Example 4. Let $A = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{pmatrix}$, then there exists an FMRA wavelet set $A^tK \setminus K$ where K is a subset of the set K_0 shown in Figure 4. One can verify that K_0 satisfies the conditions in Proposition 2 since $A^tK_0 \setminus K_0$ contains the set $(A^tK_0 \setminus K_0) \setminus \Omega$, which contains a subset that is translation congruent to Ω. The set $(A^tK_0 \setminus K_0) \cup \Omega$ is shown in Figure 5. Notice that $(A^tK_0 \setminus K_0) \setminus \Omega$ contains the union of the boxed areas outside Ω in the figure, which can totally cover Ω using proper translations. One may expect a more general result using the same approach. This is indeed the case. In fact, this example holds if $\sqrt{3}$ is replaced by any positive real number λ such that $2 \geq \lambda \geq (1+\sqrt{5})/2$. The details are left to our reader.

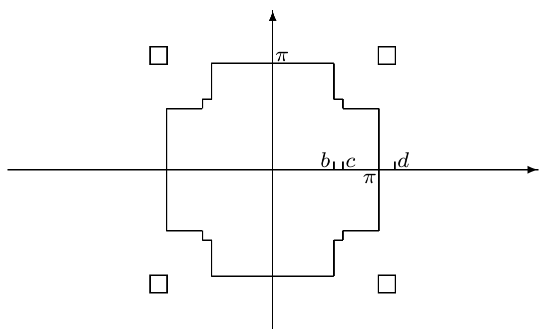

Figure 4: The set K_0, where $b = \frac{\pi}{\sqrt{3}}, c = \frac{2\pi}{3}, d = \frac{2\pi}{\sqrt{3}}$.

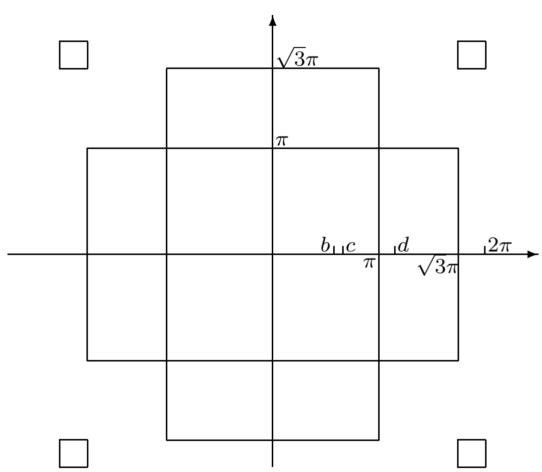

Figure 5: The set $(A^tK_0 \setminus K_0) \cup \Omega$.

We conclude this paper with the following remark. We know that a necessary condition for an expansive matrix A to have an A-dilation FMRA wavelet set is that $|\det A| \geq 2$. But is this also a sufficient condition? We cannot find a counter example. In general, the set K

is very difficult to construct. For instance, we are not even sure if the matrix $A = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$ has an A-dilation FMRA wavelet set. So this remains a challenging open question at this time.

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