UPPER BOUNDS ON LINKING NUMBERS OF THICK LINKS

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ABSTRACT

The maximum of the linking number between two lattice polygons of lengths n_1, n_2 (with $n_1 \leq n_2$) is proven to be of the order of $n_1(n_2)^{\frac{1}{3}}$. This result is generalized to smooth links of unit thickness. The result also implies that the writhe of a lattice knot K of length n is at most $26\, n^{4/3}/\pi$. In the second half of the paper examples are given to show that linking numbers of order $n_1(n_2)^{\frac{1}{3}}$ can be obtained when $n_1^3 \leq n_2$. When $n_1^3 < n_2$, it is further shown that the maximum of the linking number between these two polygons is bounded by cn_1^2 for some constant $c > 0$. Finally the maximal total linking number of lattice links with more than 2 components is generalized to k components.

Keywords: Knots, Links, Linking Number, Lattice Polygons, Lattice Links.

1. Introduction

The ϵ-neighborhood of a knot K is the set of all points in $\mathbb{R}^3$ that are within a distance $\leq \epsilon$ of K. A thickness of K can be defined as the supremum of ϵ such that for any $0 < \epsilon < t(K)$, the ϵ-neighbourhood of K is a solid torus which can be deformed into K via a strong deformation retract. In the case of C^2 smooth knots, a natural definition of thickness called the *disk thickness* is given in [10]. Various definitions of thicknesses are given and discussed in [6] for C^1 curves. All these definitions can be extended trivially to the case of links.

THe most fundamental question in the study of thick knots and links is the following: If K is a knot or a link of unit thickness, which knot or link types can K have? In this paper we shall address certain aspects of this question for links, and in particular bound the Gaussian linking number in terms of the total length of the thick link. This upper bound is similar in spirit to the upper bound on the crossing number of thick knots. It is known that the crossing number of a knot K of unit thickness is bounded by a constant times the length of K raised to the power 4/3. In particular, this result of G. Buck can be directly applied to polygonal knots in the cubic lattice, where it can be shown that the crossing number is bounded by $3.2\ell^{4/3}$ where ℓ is the length of the knot (which equals the number of steps in the lattice) [5]. It is also demonstrated in [5] that the power in this bound is sharp; that is, there are families of lattice knots in the cubic lattice with length $\ell \sim C^{3/4}$, where C is the crossing number of a member in the family. In addition, it is also known that the 4/3-power law is sharp for space curves [3,2]. These results can all be interpreted as bounds on the complexity of a given thick knot or link. In the case of knots, the knot complexity can be measured in a variety of ways [5,14], and crossing number is considered to be one such measure. In links, the Gaussian linking number is also one such natural measure of complexity, and in this paper we shall investigate bounds on it as a limit on the complexity of a given thick link with fixed length.

For links a natural measure of complexity, in addition to crossing number, is the (Gaussian) linking number. Since the linking number is bounded above by 1/2 of the crossing number of the link, one would expect that the asymptotic behavior of the linking number in terms of the total length of the link is similar to that of the crossing number. In this article we show that the asymptotic behaviour of the linking number of a link of unit thickness is consistent with the known results about the crossing number of a knot of unit thickness. However there are notable differences in the results obtained for linking numbers in the case when the components of the link have very different length.

In section 2, we establish an upper bound on the asymptotic behaviour of the linking number for lattice links. This result can be extended to smooth links with unit thickness; this is done in section 3, where we show that a link of unit thickness can be approximated by a lattice link in such a way that the number of edges in the lattice link is bounded linearly by the lengths of the curves in the link. We discuss some examples of lattice links in section 4, these show that the upper bound on the asymptotic behaviour of the linking number for lattice links is sharp up to the power. Section 5 deals with the case when the link contains more than two components and the case when one component of the link is extremely long compared to other components. Finally in section 6 we establish an upper bound on the asymptotic behaviour of the writhe of a knot using the main theorem in section 2 and some results on the writhe of a lattice knot from [12,13].

2. The case of lattice links

In this section we prove our main result; this is an upper bound on the linking number of lattice links of given length in the cubic lattice. The proof relies on Gauss' formula for linking number, and it proceeds by considering the implications of this formula for lattice links.

Theorem 1 *Let P_1 and P_2 be two polygons on the unit cubic lattice with length n_1 and n_2 respectively. Then the linking number $Lk(P_1, P_2)$ between P_1 and P_2 is bounded above by* $\min\{c_1 \cdot n_1 n_2^{\frac{1}{3}}, c_1 \cdot n_2 n_1^{\frac{1}{3}}\}$ *for some constant $c_1 > 0$.*

Proof. The linking number $Lk(P_1, P_2)$ can be written as the following integral (known as the Gauss' formula)

$$\frac{1}{4\pi}\int_{P_1}\int_{P_2}\frac{(x_1-x_2)\begin{vmatrix} dz_1 & dz_2 \\ dy_1 & dy_2 \end{vmatrix}+(y_1-y_2)\begin{vmatrix} dx_1 & dx_2 \\ dz_1 & dz_2 \end{vmatrix}+(z_1-z_2)\begin{vmatrix} dy_1 & dy_2 \\ dx_1 & dx_2 \end{vmatrix}}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}}, \tag{1}$$

where $(x_1, y_1, z_1) \in P_1$ and $(x_2, y_2, z_2) \in P_2$. Let e_1, e_2, ..., e_{n_1} be the edges of P_1, then the integral in (1) can be written as the summation of the following integrals $(1 \le i \le n_1)$

$$\frac{1}{4\pi}\int_{e_i}\int_{P_2}\frac{(x_1-x_2)\begin{vmatrix} dz_1 & dz_2 \\ dy_1 & dy_2 \end{vmatrix}+(y_1-y_2)\begin{vmatrix} dx_1 & dx_2 \\ dz_1 & dz_2 \end{vmatrix}+(z_1-z_2)\begin{vmatrix} dy_1 & dy_2 \\ dx_1 & dx_2 \end{vmatrix}}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}}, \tag{2}$$

which in turn can be written as a summation of the following integrals

$$\frac{1}{4\pi}\int_{e_i}\int_{\ell_{(m,n,p)}}\frac{(x_1-x_2)\begin{vmatrix} dz_1 & dz_2 \\ dy_1 & dy_2 \end{vmatrix}+(y_1-y_2)\begin{vmatrix} dx_1 & dx_2 \\ dz_1 & dz_2 \end{vmatrix}+(z_1-z_2)\begin{vmatrix} dy_1 & dy_2 \\ dx_1 & dx_2 \end{vmatrix}}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}}, \tag{3}$$

where $\ell_{(m,n,p)}$ stands for the edge of P_2 starting at point (m, n, p) (under the given orientation of P_2). For each fixed i, we wish to estimate (2). Since the integral (1) is invariant under transformations, without loss of generality, we can assume that e_i is the line segment joining $(0, 0, 0)$ and $(1, 0, 0)$, and $(m, n, p) \neq \{(0, 0, 0),\ (1, 0, 0)\}$ since P_1 and P_2 cannot intersect. If $\ell_{(m,n,p)}$ is parallel to e_i, it can be checked that the double integral in (3) is equal to zero. If $\ell_{(m,n,p)}$ is not parallel to e_i, then it joins the point (m, n, p) to one of $(m, n \pm 1, p)$ and $(m, n, p \pm 1)$. Say it joins (m, n, p) and $(m, n + 1, p)$. Since $dy_1 = 0$ and $dz_1 = 0$ on e_i and $dx_2 = 0$, $dz_2 = 0$ on $\ell_{(m,n,p)}$, integral (3) simplifies into

$$\frac{1}{4\pi}\int_0^1 dx_1 \int_n^{n+1} dy_2 \frac{-p}{((x_1-m)^2+y_2^2+p^2)^{\frac{3}{2}}}. \tag{4}$$

If $p = 0$, then the integral is zero and, if $p \neq 0$, then we have

$$4((x_1 - m)^2 + y_2^2 + p^2) \geq m^2 + n^2 + p^2,$$

for $0 \leq x_1 \leq 1$ and $n \leq y_2 \leq n + 1$. Thus, it follows that

$$\left| \frac{1}{4\pi} \int_0^1 dx_1 \int_n^{n+1} dy_2 \frac{-p}{((x_1 - m)^2 + y_2^2 + p^2)^{\frac{3}{2}}} \right| \leq \frac{1}{\pi} \frac{|p|}{(m^2 + n^2 + p^2)^{\frac{3}{2}}}. \tag{5}$$

The same inequality is obtained when $\ell_{(m,n,p)}$ joins (m, n, p) and $(m, n - 1, p)$. Similarly, the absolute value of integral (3) is bounded by

$$\frac{1}{\pi} \frac{|n|}{(m^2 + n^2 + p^2)^{\frac{3}{2}}} \tag{6}$$

if $\ell_{(m,n,p)}$ connects (m, n, p) and $(m, n, p \pm 1)$. Thus, the absolute value of integral (2) is bounded by

$$\frac{1}{\pi} \sum_{(m,n,p) \in S} \frac{\max\{|m|, |n|, |p|\}}{(m^2 + n^2 + p^2)^{\frac{3}{2}}}, \tag{7}$$

where S is the set of all vertices of P_2. We need to find a bound for (7). A lattice point (m, n, p) is said to be on the k*-th cube* (centered at $(0, 0, 0)$) if $\max\{|m|, |n|, |p|\} = k$. There are $(2k + 1)^3 - (2k - 1)^3 = 24k^2 + 2$ lattice points on the k-th cube. If (m, n, p) is on the $(k + j)$-th cube for any $j \geq 0$, then

$$\frac{\max\{|m|, |n|, |p|\}}{(m^2 + n^2 + p^2)^{\frac{3}{2}}} \leq \frac{1}{(k + j)^2} \leq \frac{1}{k^2}. \tag{8}$$

In order to obtain an upper bound for (7) we assume that P_2 has its vertices with indices as small as possible. Thus we assume that the vertices of P_2 fill all the lattice cubes with indices from 1 to some index $K_0 - 1$. All the remaining vertices of P_2 are on the lattice cube with index K_0. How many cubes can S fill? Since there are n_2 points in S and $(2K + 1)^3 - 1$ total points from the first to the K-th cubes, S can fill at most $K_0 \leq \frac{1}{2}(\sqrt[3]{n_2 + 1} + 1)$ cubes. This process then yields a bound for (7) of the form

$$\frac{1}{\pi} \sum_{1 \leq k \leq K_0} (24k^2 + 2) \cdot \frac{1}{k^2} \leq \frac{26}{\pi} n_2^{\frac{1}{3}}. \tag{9}$$

Since e_i is arbitrary, we see that the integral in (1) is bounded by $\frac{26}{\pi} n_1 n_2^{\frac{1}{3}}$. Interchanging the role of P_1 and P_2 in the above argument then yields the desired result. □

This completes the upper bound in the case of lattice links. Notice that if $n_1 = n_2 = n$, then a 4/3-power bound is obtained.

3. Linking numbers of smooth links of thickness one

In this section the results of section 2 will be extended to include the case of smooth links of thickness one. The length scale in the previous problem was set by the lattice, but here, the thickness of the smooth link will instead fill that role.

To keep the discussion here simple, we shall only use a definition of C^2-curves called *disk thickness*, thus restricting ourselves to C^2-curves. But our result still holds if other definitions of thicknesses are used. The disk thickness is also known as *curvature thickness* and was defined and studied in [10]. Let $\alpha(s)$ be the arc-length parametrized equation for a C^2 knot K, the disk (or curvature) thickness of K is denoted by $t_c(K)$ or $t_c(\alpha)$. It is shown in [10] that

$$t_c(\alpha) = \min\{1/\kappa(\alpha), d(\alpha)\},$$

where $\kappa(\alpha)$ is the maximum curvature of the curve α, and $d(\alpha)$ is the minimum separation between any two double critical points in α (a double critical pair $(\alpha(s), \alpha(t))$ is found when the chord between $\alpha(s)$ and $\alpha(t)$ is also normal to the tangent vectors of α at $\alpha(s)$ and $\alpha(t)$). Thus, for a curve of thickness 1, it follows that

$$1 = t_c(\alpha) \leq 1/\kappa(\alpha),$$

so that

$$\kappa(\alpha) \leq 1.$$

Thus, $|\alpha''| \leq 1$ since α is arc-length parametrized.

Let $B(X, r)$ denote the closed ball with center X and radius r. The r-neighbourhood of the curve α is given by $\bigcup_t B(\alpha(t), r)$. The r-neighbourhood of a curve of unit thickness is a solid torus which can be deformed into α via a strong deformation retract for all $r < 1$. It follows that $B(\alpha(t), r) \cap \alpha$ is a single simple arc by the definition of disk thickness [10].

Definition. Let $L = (L_1, L_2)$ be a two component smooth link with components of unit disk thickness. A polygonal link $K = (K_1, K_2)$ on some cubic lattice is called a lattice approximation of L if there exists an $r \in (0, 1)$ such that K is contained in the r-neighbourhood of L and there is an ambient isotopy that carries L to K which is the identity outside the 1-neighbourhood of L. □

We now show that a lattice approximation K of the link L with the same link-type as L can be constructed. This is done in the next lemma.

Lemma 1. Let $L = (L_1, L_2)$ be a two-component smooth link of unit thickness and let l_1, l_2 be the lengths of L_1, L_2 respectively. Then L can be approximated by a lattice link $K = (K_1, K_2)$ on the lattice $\mathbb{Z}^3/4$ (where $\mathbb{Z}^3/m$ is the cubic lattice with vertices of coordinates of the form p/m for any $p \in \mathbb{Z}$). Moreover, the numbers of edges in K_1 and K_2 are bounded by $14l_1$ and $14l_2$ respectively.

Proof. Let α be either the curve L_1 or L_2. As before, $\alpha(t)$ is an arc-length parametrization of the curve α. Start at a point $\alpha(t_1)$ in α. Consider the closed ball $B(\alpha(t_1), 1/2)$ which contains exactly one simple arc of α by the property of disk thickness. Let the endpoints of this arc be $\alpha(t_0)$ and $\alpha(t_2)$, with $t_0 < t_1 < t_2$. If this arc is replaced by the two line segments $(\alpha(t_0), \alpha(t_1))$ and $(\alpha(t_1), \alpha(t_2))$, then the newly obtained link is isotopic to L, by an isotopy which is an identity outside the 1-neighbourhood of L. The length of the arc in $B(\alpha(t_1), 1/2)$ is at least 1. Consider $B(\alpha(t_2), 1/2)$ next, which intersects α in an arc with endpoints $\alpha(t_1)$ and $\alpha(t_3)$, and repeat the arguments above. Continuing in this way, at most $2l$ balls are required to exhaust the length of α, where l is the length of α. The final ball will overlap $B(\alpha(t_1), 1/2)$, and so the final line-segment may have length less than 1/2. The result is a piecewise linear curve β which intersects α at its vertices $\alpha(t_0), \alpha(t_1), \alpha(t_2), \ldots, \alpha(t_m)$, where $m \leq 2l$. Moreover, P is ambient isotopic to α by an ambient isotopy which is the identity outside the 1-neighbourhood of α. Notice that the length of β is at most l.

Let e and f be two edges of P wich are joined at a vertex v on L, then the angle between e and f is greater than $150°$. This can be seen as follows: Let m be the intersection line of the plane normal to α at v and the plane spanned by e and f, then the angle between e and f is the sum of the angle ($\angle_1$) between e and m and the angle ($\angle_2$) between m and f. Consider a unit circle with e as a chord. Since the unit circle is a curve realizing the maximal allowed curvature of one, the angle between the radius of this circle and e is at most $\angle_1$. Since the length of e is at most 1/2, it follows that $\angle_1 \geq \cos^{-1}(1/4)$. The same argument applies to $\angle_2$. So the angle between e and f is greater than $2\cos^{-1}(1/4) > 150°$.

Continue now by placing L and its polygonal approximation $P = (P_1, P_2)$, constructed as above, into the lattice $\mathbb{Z}^3/4$ such that P does not intersect any edge in the lattice. Consider the set of elementary cubes in $\mathbb{Z}^3/4$ (of side length 1/4) which intersect P. It then follows that each elementary cube C intersecting P is contained in the 1/2-neighbourhood of L and $C \cap P$ falls into one of the following three cases:

(a) $C \cap P$ is a single arc that enters and exits C at two different faces.

(b) $C \cap P$ is a single arc that enters and exits C on a common face.

(c) $C \cap P$ consists of two separate line segments which must enter or exit C through one and only one face of C (this follows from the fact that the inclusive angles between line segments is at least $150°$).

Since each line segment in P has length at most $1/2$, and so can intersect at most 7 elementary cubes in $\mathbb{Z}^3/4$. Thus, P_1 intersects at most $14l_1$ and P_2 intersects at most $14l_2$ elementary cubes. Since P avoids intersections with edges and vertices in $\mathbb{Z}^3/4$, K can be constructed as follows. If the intersection of an elementary cube C with P is a single arc as in case (a) above, we will replace $C \cap P$ by line segments that join the center of C to the centers of its faces intersecting P, as shown in Figure 1.

Figure 1: *If the curve passes through opposite faces, it is replaced by a line segment joining the midpoints of the two opposite faces. If it passes through two adjacent faces, it is replaced by two line segments joining the midpoints of the faces via the midpoint of the cube.*

In case (b) above, say the simple curve emerges from two points on a common face F of the cube C_1. The curve must be part of two edges of P that is connected at one end inside C_1. F is also a face of a cube C_2, and C_2 must contain two separate line segments of P. In other word, case (b) is accompanied by case (c). This opposite of this is also true, case (c) is also accompanied by case (b). Therefore, in these cases, we can then simply replace that part of P contained in C_1 by the line segment on F joining the two points of P in F. The constraint on the inclusive angle between the line segments then requires that we obtain case (a).

All these constructions can be carried out inside the cubes intersected by the polygonal approximation to the link. Therefore, it induces an ambient isotopy that is the identity outside the 1-neighborhood of L. The result is a link $K = (K_1, K_2)$ in the dual lattice of $\mathbb{Z}^3/4$ with K_i containing no more than $14l_i$ edges. □

Theorem 2 *Let L be an embedding of a smooth link with two components L_1 and L_2 of length l_1 and l_2 respectively. If the link L has a thickness of one, then the linking number $Lk(L_1, L_2)$ between L_1 and L_2 is bounded above by* $\min\{c_2 \cdot l_1 l_2^{\frac{1}{3}}, c_2 \cdot l_2 l_1^{\frac{1}{3}}\}$ *for some constant $c_2 > 0$.*

Proof. Place L in $\mathbb{Z}^3/4$, and approximate it by a lattice link $K = (K_1, K_2)$ with components of lengths at most $14l_1$ and $14l_2$, using lemma 1. By theorem 1 the result follows. □

4. Examples and remarks

In this section we supply some examples and draw conclusions from them.

1. If $n_1 = n_2 = n$, then our result yields a bound of the form $\frac{26}{\pi}n^{\frac{4}{3}}$ for the linking number between lattice polygons P_1 and P_2. Since the linking number is bounded by the crossing number of P_1 and P_2 (as a two component knot), which is bounded by $an^{\frac{4}{3}}$ for some constant $a > 0$ as shown by G. Buck in [1], our result is consistent with the known bound on the crossing number of knots. The bound obtained in theorem 1 is sharp up to the powers; that is, there are examples of classes of lattice links which achieves the power-law behaviour of theorem 1. The following example is an example where P_1 and P_2 are of the same length, and the linking number is of the 4/3 power of the total length of the link.

Consider the link we can construct along the lines in figure 2. The two components are two identical polygons, but rotated from one another. While in the example each polygon consists of 3 layers with three square-shaped polygons glued into a spiral around an empty 3×3 square, this generalizes immediately to a case of n layers of n square-shaped polygons surrounding an empty $n \times n$ square. Stacking the layers together creates a polygon of length $O(n^3)$. The polygons can be linked by rotating one until it passes through the empty square of the other, and the linking number is $O(n^4)$. Thus, if l is the total length of the polygon, then $Lk = O(l^{4/3})$.

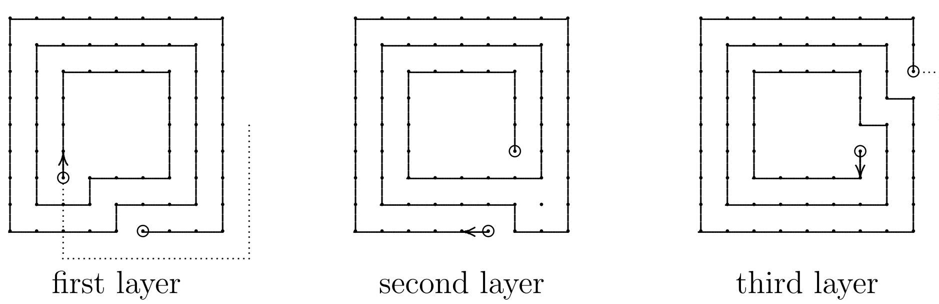

Figure 2: *By stacking these three layers together, a polygon can be created of length* $O(3^3)$*. Two copies of it can be linked by rotating one copy so that it passes through the empty square of the other. The polygon is closed by adding edges along the dotted lines. The linking number will be* 3^4.

2. The bound $cn^{4/3}$ on crossover number of knots of length n [1] strongly suggests the bounds on linking numbers obtained in this paper. However, the bound obtained here is quite different from the similar power-law bound obtained on the crossover number of lattice knots. For example, if $n_1 \gg n_2$, then the theorems here yields a bound of order $O(n_2 n_1^{1/3})$, which grows only in the 1/3-power of n_1, and which eventually becomes independent of n_1 if the length of n_2 is fixed as we will show in the next section. Below we demonstrate an example where the length of the two components are drastically different, and one obtains an upper bound of the order

$O(n_2 n_1^{1/3})$ as suggested by theorem 1.

Consider for example figure 3. Suppose that P_2 is a square of side length $n+1$ (and so $n_2 = 4n+4$). There are n^2 lattice points inside this square. If P_1 is a polygon that passes through these points in the same orientation, then figure 3 shows that P_1 can be realised with approximately $n_1 \sim 4n^3$ edges. Naturally, $Lk(P_1, P_2) = n^2$, and this is order $n_2 n_1^{1/3}$.

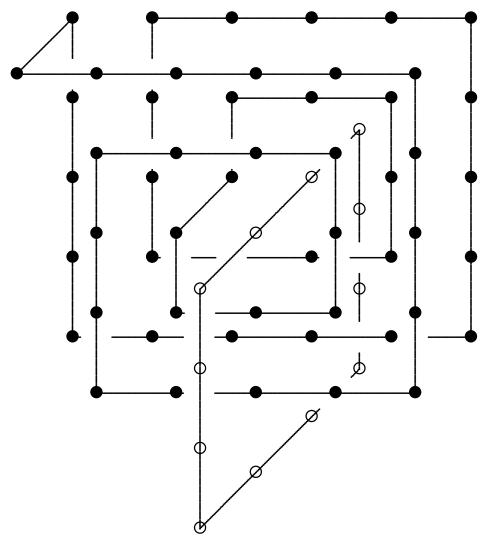

Figure 3: *If this example is generalised to a square P_2 of side length $(n+1)$, and P_1 with approximately $4n^3$ edges, then their linking number is n^2, and this is sharp in the bound $n_2 n_1^{1/3}$. Thus, any improvement to our bound can only occur in the coefficient $26/\pi$.*

5. Additional Results

In this section we consider two more cases: In the first case we consider a link with one relatively long component, and secondly, we consider links with more than two components.

If one component of a thick link $L = (L_1, L_2)$ (compared to the other component) has a relatively short length n_1, then the linking number of L ought be bounded in terms of n_1, regardless how long n_2 is. Observe that the bounds we obtained in sections 2 and 3 approach infinity as n_2 approaches infinity. This was still the case in the example in figure 3 despite the difference in the length between the two components. However when the difference in the length between the two components is even greater than the difference in the example in figure 3, the bound $O(n_2 n_1^{1/3})$ is no longer sharp. In such cases, we have the following theorem.

Theorem 3 *Let $P = (P_1, P_2)$ be a lattice link. Let n_1, n_2 be the lengths of P_1 and*

P_2 respectively. If $n_1^3 < n_2$, then $Lk(P_1, P_2) \leq c_3 n_1^2$ for some positive constant c_3. A similar result holds for a link $L = (L_1, L_2)$ of unit thickness.

Proof. Since the length of P_1 is n_1, there exists a cube C of side length $3n_1$ such that P_1 is contained inside C and each edge of P_1 is a distance of at least n_1 from the faces of C. Notice that if P_2 does not intersect C, then $Lk(P_1, P_2) = 0$, otherwise $P_2 \backslash C$ consists of open ended simple arcs (with their end points on the faces of C). There are at most $O(n_1^2)$ such open arcs β_i because there are only $54n_1^2 + 2$ lattice points on the faces of C. Complete each simple curve β_i into a closed curve by joining its endpoints with simple lattice path β_i' of length at most $9n_1$ which is entirely in the surface of the cube C. The resulting curve $\beta_i \cup \beta_i'$ has linking number 0 with P_1 since P_1 is entirely contained within C. We shall replace each β_i from P_2 by β_i' while orienting β_i' with the orientation induced by P_2. Let P_3 be the closed lattice curve obtained by replacing each β_i by β_i'. It may be the case that P_3 is not a simple closed curve, some of the β_i's may intersect one another, or even intersect some of the existing segments of P_2 in the faces of C. However, we have

$$\begin{aligned} Lk(P_1, P_2) &= \frac{1}{4\pi} \int_{P_1} \int_{P_2} \frac{(x_1 - x_2) \begin{vmatrix} dz_1 & dz_2 \\ dy_1 & dy_2 \end{vmatrix} + (y_1 - y_2) \begin{vmatrix} dx_1 & dx_2 \\ dz_1 & dz_2 \end{vmatrix} + (z_1 - z_2) \begin{vmatrix} dy_1 & dy_2 \\ dx_1 & dx_2 \end{vmatrix}}{((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)^{\frac{3}{2}}} \\ &= \frac{1}{4\pi} \int_{P_1} \int_{P_3} \frac{(x_1 - x_2) \begin{vmatrix} dz_1 & dz_2 \\ dy_1 & dy_2 \end{vmatrix} + (y_1 - y_2) \begin{vmatrix} dx_1 & dx_2 \\ dz_1 & dz_2 \end{vmatrix} + (z_1 - z_2) \begin{vmatrix} dy_1 & dy_2 \\ dx_1 & dx_2 \end{vmatrix}}{((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)^{\frac{3}{2}}}. \end{aligned}$$

This double integral can again be bounded using the techniques from theorem 1. For each e_i of P_1, consider the cube C' (which is completely contained in cube C) of side length $2n_1$ centered at one of the end points of the edge e_i. There are at most $O(n_1^3)$ edges of P_3 inside C' (these are the same as that of P_2) and there are at most $O(n_1^3)$ edges of P_3 outside of C'. For those edges in P_3 outside of C' the last term of equation (9) gives an upper bound of order $O(1/n_1^2)$ for each edge. Since there are at most $O(n_1^3)$ edges the contribution to the upper bound of the linking number is $O(n_1^3)/n_1^2 = O(n_1)$.

For those edges of P_3 inside C' the upper bound is again determined to be of the form $\sum_{1 \leq k \leq n_1} (24k^2 + 2) \cdot \frac{1}{k^2}$, as can be directly seen from equation (9). This is also an upper bound of order $O(n_1)$. Thus, it follows that the integral (2) (with P_2 replaced by P_3) is bounded above by $O(n_1)$. Since P_1 contains n_1 edges, the resulting upper bound has order $O(n_1^2)$, and this gives the desired result. In the case of smooth links of thickness 1 we can apply Lemma 1 again. □

Let us now consider the case of multiple component links. Let P be a lattice link with k components P_i of the same length. Assume that the total length of P is n and define the total linking number of P as $Lk(P) = \sum_{1 \leq i < j \leq k} |Lk(P_i, P_j)|$. The analysis of the bound on the linking number is similar to the proof of theorem 1. If we fix a component P_i and look at its linking with all the other components then we still get formula (7) with the exception that now S is the set of vertices consisting of all the components P_j excluding the component P_i.

This leads to the following formula as an upper bound of the linking of P_i with all the other components:

$$\frac{26}{\pi}\frac{n}{k}(n-\frac{n}{k})^{\frac{1}{3}} < \frac{26}{\pi}\frac{n^{\frac{4}{3}}}{k}.$$

Since there are k components we can multiply this by k to arrive at an upper bound of a constant times $n^{\frac{4}{3}}$ as expected. An example of a link with k components, rope length n and an upper bound of order $O(n^{4/3})$ is given in [3].

Suppose there are many relatively small components P_i and each component links j of the other components nontrivially. (An example of this would be medieval chain mail.) Now we arrive at a different upper bound. If the length of each component is $c = n/k$, then formula (7) becomes $\frac{26}{\pi}\frac{n}{k}(j \cdot \frac{n}{k})^{\frac{1}{3}} = \frac{26}{\pi}\frac{n}{k}(j \cdot c)^{\frac{1}{3}}$. Since there are k components we can multiply this by k to arrive at an upper bound of a constant times n. This says that the total linking number grows linearly with the total length (or the number of components) in chain mail.

6. The writhe of a lattice knot

The writhe of a lattice knot in the cubic lattice can be computed by averaging the linking number of the knot over four of its push-offs into non-antipodal octants [9]. In particular, let K be a knot on the cubic lattice of length n. Define the vectors $\mu_1 = (1/2, 1/2, 1/2)$, $\mu_2 = (-1/2, 1/2, 1/2)$, $\mu_3 = (1/2, -1/2, 1/2)$, $\mu_4 = (-1/2, -1/2, 1/2)$ in R^3. The push-off K_{μ_i} of the lattice knot K in the direction μ_i is defined by $K_{\mu_i} = \{x + \mu_i | x \in K\}$. It follows that $K \cap K_{\mu_i} = \emptyset$. The Lacher-Sumners theorem states that that the writhe of K, denoted by $Wr(K)$, is the average of the linking numbers $Lk(K, K_{\mu_i})$, that is $Wr(K) = 1/4 \sum_{1 \le i \le 4} Lk(K, K_{\mu_i})$ (for more on this, see reference [12,13]. Any link $K \cup K_{\mu_i}$ can be viewed as a lattice link on a subdivided cubic lattice of edge length 1/2. Moreover, the length of K is equal the length of K_{μ_i} (and each have length n). Thus, by theorem 1,

$$Lk(K, K_{\mu_i}) \le cn^{4/3}$$

for some constant c. This proves the following theorem.

Theorem 4 *Let K be a knot on the unit cubic lattice with length n. Then the writhe $Wr(K) \le cn^{4/3}$ for some constant c.*

This result is not unexpected. An upper bound of order $O(n^{4/3})$ has already been established for smooth knots, see reference [4].

References

[1] G. Buck and J. Simon, *Thickness and crossing number of knots*, Topology Appl. **91(3)** (1999), 245–257.

[2] G. Buck, *Four-thirds power law for knots and links*, Nature, **392** (1998), 238-239.

[3] J. Cantarella, R. B. Kusner and J. M. Sullivan, *Tight knot values deviate from linear relations*, Nature, **392** (1998), 237-238.

[4] J. Cantarella, D. Deturck and H. Gluck, *Upper bounds on the writhing of knots and the helicity of vector fields*, preprint.

[5] Y. Diao and C. Ernst, *The complexity of lattice knots*, Topology Appl. **90, no. 1-3** (1998), 1–9.

[6] Y. Diao, C. Ernst and E.J. Janse Van Rensburg, *Thicknesses of Knots*, Math. Proc. Camb. Phil. Soc., **126** (1999), 293-310.

[7] D. Kim and R. Kusner, *Torus Knots Extremising the Möbius Energy*, Experimental Math., **2** (1993), 1-9.

[8] R. B. Kusner and J. M. Sullivan, *Möbius Energies for Knots and Links, Surfaces and Submanifolds*, Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math., **2.1**, Amer. Math. Soc., Providence, RI (1997), 570–604.

[9] R.C. Lacher and D. W. Sumners, *Data Structures and Algorithms for the Computation of Topological Invariants of Entanglements: Link, Twist and Write*, Computer Simulations of Polymers, R.J. Roe (ed.), 365–373 (Prentice Hall, 1991).

[10] R. Litherland, J. Simon, O. Durumeric, and E. Rawdon. *Thickness of knots.* Topology and its Applications, **91(3)** (1999), 233–244.

[11] E. J. Janse van Rensburg and S. D. Promislow, *Minimal Knots in the Cubic Lattice*, J. Knot Theory and its Ram., **4** (1995) 115-130.

[12] E. J. Janse van Rensburg, D. W. Sumners, and S. G. Whittington, *The writhe of knots and links*, *Ideal Knots* (eds.A.Stasiak, V. Katritch and L.H. Kauffman), Series on Knots and Everything. World Scientific. **19** (1998), 70 - 87.

[13] E.J. Janse van Rensburg, E. Orlandini, D.W. Sumners, M.C. Tesi and S.G. Whittington, *The Writhe of Knots in the Cubic Lattice*, J. Knot Theory and its Ram., **6** (1996), 31-44.

[14] C. E. Soteros, D. S. Sumners, and S. G. Whittington, *Entanglement complexity of graphs in* Z^3, Math. Proc. Camb. Phil. Soc., **111** (1992), 75-91.