

# Frame Wavelet Sets in $\mathbb{R}^d$

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## ABSTRACT

In this paper, we discuss the characterization of frame wavelet sets. We extend some results obtained earlier in the one dimensional case. More specifically, we completely characterize tight frame wavelet sets in higher dimensions and obtain some necessary conditions and sufficient conditions for a set  $E$  to be a frame wavelet set in  $\mathbb{R}^d$ . Several examples are presented and compared with those in the one dimensional case. Using our results, one can easily construct various frame wavelet sets.

*Keywords:* Frame, Wavelet, Frame Wavelet, Frame Wavelet Set, Fourier Transform.

## §1. Introduction

A collection of elements  $\{x_j : j \in \mathcal{J}\}$  in a Hilbert space  $\mathcal{H}$  is called a *frame* if there exist constants  $a$  and  $b$ ,  $0 < a \leq b < \infty$ , such that

$$a\|f\|^2 \leq \sum_{j \in \mathcal{J}} |\langle f, x_j \rangle|^2 \leq b\|f\|^2, \quad \forall f \in \mathcal{H}. \quad (1)$$

Let  $a_0$  be the supremum of all such numbers  $a$  and  $b_0$  be the infimum of all such numbers  $b$ , then  $a_0$  and  $b_0$  are called the *frame bounds* of the frame

$\{x_j : j \in \mathcal{J}\}$ . When  $a_0 = b_0$  we say that the frame is *tight*. When  $a_0 = b_0 = 1$  we say the frame is *normalized tight*. Any orthonormal basis in a Hilbert space is a normalized tight frame but not vice versa. A function  $\psi \in L^2(\mathbb{R})$  is called a frame wavelet (in  $L^2(\mathbb{R})$ ) if the family of functions  $\{\psi_{n,\ell}(x) = 2^{\frac{n}{2}}\psi(2^n x - \ell) : n, \ell \in \mathbb{Z}\}$  is a frame on  $L^2(\mathbb{R})$ . Frame wavelets are the generalizations of wavelets in  $L^2(\mathbb{R})$ . See [7], [13] and [14] for some of the early works on frames and frame wavelets. For recent development and work on frame wavelets, see [1], [3], [4], [5], [6], [8], [9] and [11]. In the special case that the Fourier transform of  $\psi$  is  $\frac{1}{\sqrt{2\pi}}\chi_E$  for some Lebesgue measurable set  $E$  of finite measure, the study of the frame wavelet is reduced to the study of the set  $E$ , which is simpler in general. If we could characterize the set  $E$  that defines a frame wavelet in such a manner, then we would be able to find ways to construct frame wavelets and have a better understanding of frame wavelets in general [2]. Let  $\psi$  be defined by  $\hat{\psi} = \frac{1}{\sqrt{2\pi}}\chi_E$ , where  $\hat{\psi}$  is the Fourier transform of  $\psi$ . If  $\psi$  is a frame wavelet function, a tight frame wavelet function or a normalized tight frame wavelet function, the set  $E$  is called a *frame wavelet set*, a *tight frame wavelet set* or a *normalized tight frame wavelet set* accordingly. In this paper, we will call these sets *f*-sets, *t*-sets and *n*-sets respectively for short. Although normalized tight frame wavelets can be obtained from tight frame wavelets by rescaling, an *f*-set that defines a normalized tight frame wavelet cannot be obtained in such manner. That is, the characterization of *t*-sets is not equivalent to that of the *n*-sets. In [8], the question of how to characterize the *f*, *t* and *n*-sets in  $\mathbb{R}$  is raised and the characterization of the *n*-sets is obtained. In [1], the characterization of the *t*-sets is obtained, together with some necessary conditions and sufficient conditions for an *f*-set. All these were done in the one dimensional case.

In this paper, we will tackle the same problems in [1] in higher dimensions. As it turns out, we are able to extend all the results obtained in [1]. In section 2, we will introduce some necessary terms and concepts. The theorems are given in section 3 and the proofs of the theorems are given in section 5. In section 4, examples are given in various cases for comparison with the one dimensional cases.

## §2. Definitions

Let  $A$  be a  $d \times d$  real *invertible* matrix. It induces a unitary operator  $D_A$  acting on  $L^2(\mathbb{R}^d)$  defined by

$$(D_A f)(t) = |\det A|^{\frac{1}{2}} f(At), \forall f \in L^2(\mathbb{R}^d), t \in \mathbb{R}^d. \quad (2)$$

The matrix  $A$  is called *expansive* if all its eigenvalues have modulus greater than one. The operator  $D_A$  corresponding to a real expansive matrix  $A$  is called an  $A$ -*dilation* operator. In an analogous fashion, a vector  $s$  in  $\mathbb{R}^d$  induces a unitary *translation operator*  $T_s$  defined by

$$(T_s f)(t) = f(t - s), \forall f \in L^2(\mathbb{R}^d), t \in \mathbb{R}^d.$$

In this article we will only deal with expansive real matrices and translation operators  $T_\ell$  with  $\ell \in \mathbb{Z}^d$ .

Throughout this article, we will use  $\mathcal{F}$  to denote the Fourier-Plancherel transform on  $L^2(\mathbb{R}^d)$ . This is a unitary operator. If  $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ , then

$$(\mathcal{F}f)(s) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i(s \circ t)} f(t) dm, \quad (3)$$

where  $s \circ t$  denotes the real inner product. We also write  $\widehat{f}$  for  $\mathcal{F}f$ . For a subset  $X$  of  $L^2(\mathbb{R}^d)$ ,  $\widehat{X}$  is the set of the Fourier-Plancherel transforms of all elements in  $X$ . For a bounded linear operator  $S$  on  $L^2(\mathbb{R}^d)$ , we will denote  $\mathcal{F}S\mathcal{F}^{-1}$  by  $\widehat{S}$ . It is left to the reader to verify that we have  $\widehat{D}_A = D_{(A')^{-1}} = D_{A'}^{-1} = D_{A'}^*$  for any  $d \times d$  real invertible matrix  $A$  ( $A'$  is the transpose of  $A$ ) and  $\widehat{T}_\lambda f = e^{i(\lambda \circ s)} \cdot f$  for any  $\lambda \in \mathbb{R}^d$ . Furthermore, the following two conditions

$$a\|f\|^2 \leq \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}^d} |\langle f, D_A^n T^\ell \psi \rangle|^2 \leq b\|f\|^2 \quad (4)$$

and

$$a\|\widehat{f}\|^2 \leq \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}^d} |\langle \widehat{f}, \widehat{D}_A^n \widehat{T}^\ell \widehat{\psi} \rangle|^2 \leq b\|\widehat{f}\|^2 \quad (5)$$

are equivalent.

Let  $E$  be a Lebesgue measurable set of finite measure. The terms  $f$ -sets,  $t$ -sets and  $n$ -sets can be similarly defined as we did in the last section. For any  $\ell \in \mathbb{Z}^d$ , let  $I_\ell$  denote the  $d$ -cube  $[0, 1)^d + \ell$ . For any subset  $E$  of  $\mathbb{R}^d$ , define

$$\tau(E) = \bigcup_{\ell \in \mathbb{Z}^d} (E \cap 2\pi I_\ell - 2\pi\ell).$$

If the above is a *disjoint* union, we say that  $E$  is *translation equivalent* to  $\tau(E)$ , which is a subset of  $2\pi I_0$ , where  $I_0$  is the unit  $d$ -cube  $[0, 1)^d$ . If  $E$  and  $F$  are translation equivalent to the same subset in  $2\pi I_0$  then we say  $E$  and  $F$

are translation equivalent. This defines an equivalent relation and is denoted by  $\sim$ . Let  $\mu(\cdot)$  be the Lebesgue measure. It is clear that  $\mu(E) \geq \mu(\tau(E))$ . The equality holds if and only if  $E \sim \tau(E)$ . If  $E \sim F$ , then  $\mu(E) = \mu(F)$ . Two points  $x, y \in E$  are said to be *translation equivalent* if  $x - y = 2\pi\ell$  for some  $\ell \in \mathbb{Z}^d$ . The *translation redundancy index* of a point  $x$  in  $E$  is the number of elements in its equivalent class. We write  $E(\tau, k)$  for the set of all points in  $E$  with translation redundancy index  $k$ . In general,  $E(\tau, k)$  could be an empty set, a proper subset of  $E$ , or the set  $E$  itself. For  $k \neq m$ ,  $E(\tau, k) \cap E(\tau, m) = \emptyset$ , so

$$E = E(\tau, \infty) \bigcup \left( \bigcup_{n \in \mathbb{N}} E(\tau, n) \right).$$

**Lemma 1** *Let  $E$  be a Lebesgue measurable set in  $\mathbb{R}^d$ , then  $E(\tau, k)$  is measurable for each  $k$ . Furthermore,  $E(\tau, k)$  is a disjoint union of  $k$  measurable subsets  $E^{(j)}(\tau, k)$ ,  $j = 1, 2, \dots, k$  such that  $E^{(j)}(\tau, k) \sim \tau(E(\tau, k))$ . The same is true if  $k = \infty$ .*

The proof of the lemma is elementary and is left to the reader. An outline of the proof can be found in [1] in the one dimensional case. We need to point out that the partition of  $E(\tau, k)$  into the  $E^{(j)}(\tau, k)$ 's is not unique. However, throughout this paper, we will use the same partition to avoid any possible confusion. It can also be proved from the lemma that if  $E(\tau, \infty)$  is of finite measure, then it must have measure zero.

Similarly, two non-zero points  $x, y \in E$  are said to be  $A'$ -dilation equivalent if  $y = (A')^k x$  for some  $k \in \mathbb{Z}$ . The  *$A'$ -dilation redundancy index* of a point  $x$  in  $E$  is the number of elements in its equivalent class. The set of all points in  $E$  with  $A'$ -dilation redundancy index  $k$  is denoted by  $E(\delta, k)$ . For  $k \neq m$ ,  $(A')^j(E(\delta, k)) \cap E(\delta, m) = \emptyset$  for any  $j \in \mathbb{Z}$ . We have

$$E = E(\delta, \infty) \bigcup \left( \bigcup_{n \in \mathbb{N}} E(\delta, n) \right).$$

Similar to Lemma 1, we have the following lemma. The proof is again left to our reader.

**Lemma 2** *Let  $E$  be a Lebesgue measurable set in  $\mathbb{R}^d$ , then  $E(\delta, k)$  is measurable for each  $k$  and  $E(\delta, k)$  is a disjoint union of  $k$  measurable subsets  $E^{(j)}(\delta, k)$ ,  $j = 1, 2, \dots, k$  such that each point in  $E^{(j)}(\delta, k)$  is of  $A'$ -dilation redundancy index 1. The same is true if  $k = \infty$ .*

Again, we need to point out that the partition of  $E(\delta, k)$  into the  $E^{(j)}(\delta, k)$ 's is not unique but this will not affect our result as long as we stay with a (arbitrarily) chosen partition. We need also to point out that we will be working in the frequent domain and will be using the  $\widehat{D}_A$  operator, but  $\widehat{D}_A = (\widehat{D}_{A'})^{-1}$ , as we pointed out earlier. That is why  $A'$ -dilation is used here.

Now let  $E$  be a Lebesgue measurable set with finite measure and let  $\psi \in L^2(\mathbb{R}^d)$  be defined by  $\widehat{\psi} = (2\pi)^{-\frac{d}{2}}\chi_E$ . For any  $\widehat{f} \in L^2(\mathbb{R}^d)$ , let  $H_E\widehat{f}$  be the following formal summation:

$$H_E\widehat{f} = \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}^d} \langle \widehat{f}, \widehat{D}_A^n \widehat{T}^\ell \widehat{\psi} \rangle \widehat{D}_A^n \widehat{T}^\ell \widehat{\psi}. \quad (6)$$

Notice that if  $H_E\widehat{f}$  converges to a function in  $L^2(\mathbb{R}^d)$  under the  $L^2(\mathbb{R}^d)$  norm, then equation (5) is equivalent to

$$a\|\widehat{f}\|^2 \leq \langle H_E\widehat{f}, \widehat{f} \rangle \leq b\|\widehat{f}\|^2. \quad (7)$$

### §3. Main Theorems

We outline the main results obtained in this paper below.

**Theorem 1** *Let  $E$  be a Lebesgue measurable set with finite measure. Then the following statements are equivalent:*

(i)  $H_E$  defines a bounded linear operator in  $L^2(\mathbb{R}^d)$ , that is,  $H_E\widehat{f}$  converges in  $L^2(\mathbb{R}^d)$  for any  $\widehat{f} \in L^2(\mathbb{R}^d)$  and  $\|H_E\widehat{f}\| \leq b\|\widehat{f}\|$  for some constant  $b > 0$ .

(ii) There exists a constant  $c > 0$  such that

$$\sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}^d} |\langle \widehat{f}, \widehat{D}_A^n \widehat{T}^\ell (2\pi)^{-\frac{d}{2}}\chi_E \rangle|^2 \leq c\|\widehat{f}\|^2, \quad \forall \widehat{f} \in L^2(\mathbb{R}^d).$$

(iii) There exists a constant  $M > 0$  such that  $\mu(E(\delta, m)) = 0$  and  $\mu(E(\tau, m)) = 0$  for any  $m > M$ .

**Theorem 2** *Let  $E$  be a Lebesgue measurable set with finite measure. Then  $E$  is an  $f$ -set if (i)  $\cup_{n \in \mathbb{Z}} (A')^n E(\tau, 1) = \mathbb{R}^d$  and (ii) There exists  $M > 0$  such that  $\mu(E(\delta, m)) = 0$  and  $\mu(E(\tau, m)) = 0$  for any  $m > M$ .*

Furthermore, in this case, the corresponding frame has a lower bound at least 1, and an upper bound at most  $M^{\frac{5}{2}}$ .

**Theorem 3** Let  $E$  be a Lebesgue measurable set with finite measure. If  $E$  is an  $f$ -set, then (i)  $\cup_{n \in \mathbb{Z}} (A')^n E = \mathbb{R}^d$  and (ii) There exists  $M > 0$  such that  $\mu(E(\delta, m)) = 0$  and  $\mu(E(\tau, m)) = 0$  for any  $m > M$ .

**Theorem 4** Let  $E$  be a Lebesgue measurable set with finite measure. Then  $E$  is a  $t$ -set if and only if  $E = E(\tau, 1) = E(\delta, k)$  for some  $k \geq 1$  and  $\cup_{n \in \mathbb{Z}} (A')^n E = \mathbb{R}^d$ .

**Corollary 1** If  $E$  is a  $t$ -set, then the corresponding frame bound is an integer.

**Corollary 2**  $E$  is an  $n$ -set if and only if  $E = E(\tau, 1) = E(\delta, 1)$  and  $\cup_{n \in \mathbb{Z}} (A')^n E = \mathbb{R}^d$ .

**Theorem 5** If  $E = E(\tau, 1)$ ,  $\cup_{n \in \mathbb{Z}} (A')^n E(\tau, 1) = \mathbb{R}$  and there exist  $1 \leq k_1 \leq k_2$  such that  $\mu(E(\delta, m)) = 0$  for  $m < k_1$  and  $m > k_2$ ,  $\mu(E(\delta, k_1))\mu(E(\delta, k_2)) \neq 0$ , then  $E$  is an  $f$ -set whose corresponding frame has a lower bound  $k_1$  and an upper bound  $k_2$ .

**Theorem 6** If  $E = E(\delta, 1)$  and  $\mu(E(\tau, k)) \neq 0$  for some  $k > 1$ , then  $E$  is not an  $f$ -set.

#### §4. Examples and Discussions

Before we talk about the proofs of the theorems, we would like to look at several examples to help our reader to understand the theorems better. All examples are given in the case of  $d = 2$ , mainly for the sake of convenience in drawing pictures.

**Example 1.** Let  $A' = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ .  $A$  is expansive with  $\det A = 5$ . Let  $F_1$  be the square  $(-\pi, \pi)^2$  and let  $F_2 = A'(F_1)$ , the image of  $F_1$  under the mapping  $A'$ . It is easy to see that  $F_2$  is the square whose four corner points are  $(3\pi, \pi)$ ,  $(\pi, -3\pi)$ ,  $(-\pi, 3\pi)$  and  $(-3\pi, -\pi)$ . Let  $E = F_2 \setminus F_1$ . We have  $E = E(\delta, 1)$ ,  $\cup_{n \in \mathbb{Z}} (A')^n E = \mathbb{R}^2$ , and  $E = E(\tau, 4)$ .  $E^{(1)}(\tau, 4)$  is chosen and marked it in Figure 1. By Theorem 6,  $E$  is not a frame wavelet set.

**Example 2.** Let  $A$  be the same as that given in example 1 above. But this time we define  $E = F_1 \setminus (A')^{-1}(F_1)$  with  $F_1$  being the square defined above. Then  $E = E(\tau, 1) = E(\delta, 1)$  and  $\cup_{n \in \mathbb{Z}} (A')^n E = \mathbb{R}^2$ . By Corollary 2,  $E$  is a normalized tight frame wavelet set.

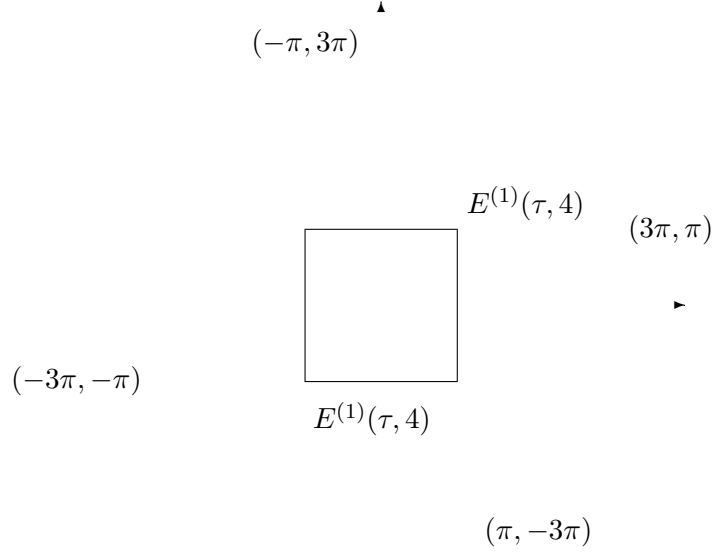


Figure 1

**Example 3.** Let  $A' = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ . As in example 1, let  $F_1$  be the square  $(-\pi, \pi)^2$  and let  $F_2 = A'(F_1)$ . Then  $F_2$  is the square whose four corner points are  $(\pi, 2\pi)$ ,  $(\pi, -2\pi)$ ,  $(-\pi, 2\pi)$  and  $(-\pi, -2\pi)$ . Let  $E = F_2 \setminus F_1$ . Then  $E = E(\delta, 1)$ ,  $\cup_{n \in \mathbb{Z}} (A')^n E = \mathbb{R}^2$ , and  $E = E(\tau, 1)$ . So  $E$  is a normalized tight frame wavelet set.

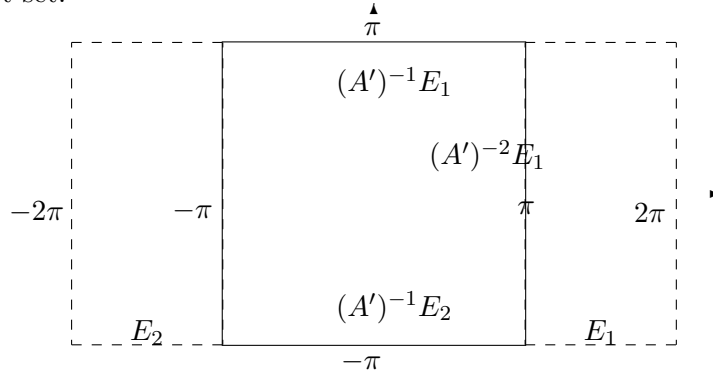


Figure 2:  $E = E_1 \cup E_2$ ,  $G = (A')^{-1}E \cup (A')^{-2}E_1$

**Example 4.** Let  $A$  and  $E$  be as defined in example 3. Notice that  $(A')^{-k}E$  is contained in  $(-\pi, \pi)^2$  for any  $k \geq 1$ . So if we let  $F = \cup_{1 \leq k \leq m} (A')^{-k}E$ , then  $F$  is a tight frame wavelet set with frame bound  $m$  by Theorem 4. Figure 2 shows a set  $G$  defined as the union of  $(A')^{-1}E$  and part of  $(A')^{-2}E$ . By Theorem 5,  $G$  is a frame wavelet set with lower frame bound 1 and upper

frame bound 2.

In the above examples, the square  $Q = (-\pi, \pi)^2$  is a proper subset of  $A'(Q)$ . This is not the case in general. As is shown in the following example.

**Example 5.** Let  $A' = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$ .  $A$  is an expansive matrix with  $\det(A) = -3$  and  $Q \not\subset A'(Q)$  as one can easily check. But if we let  $F$  be the square with corner points  $(\pi, 0)$ ,  $(0, \pi)$ ,  $(-\pi, 0)$  and  $(0, -\pi)$ , then  $F$  is a proper subset of  $A'(F)$ . Let  $E = F \setminus (A')^{-1}(F)$ . Then  $E$  is a normalized tight frame wavelet set. Figure 3 shows  $E$ ,  $A'(E)$  and  $(A')^2(E)$  and offers a hint how  $\mathbb{R}^2$  is divided into disjoint union of the sets  $(A')^k(E)$ .

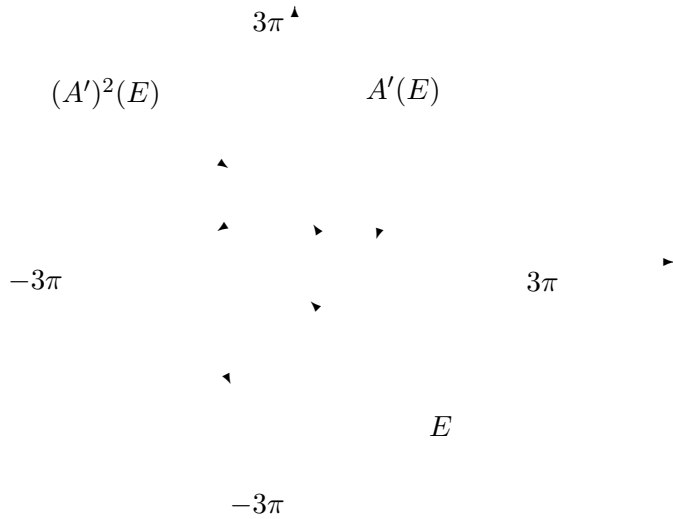


Figure 3:  $E$ ,  $A'(E)$  and  $(A')^2(E)$ .

## §5. Proofs of the Theorems

For the sake of convenience, we will drop the “hat” on the function  $f$  when dealing with  $H_E$ . Let  $f$  (which plays the same role as  $\hat{f}$  before) be in  $L^2(\mathbb{R}^d)$  and let  $E$  be a Lebesgue measurable set in  $\mathbb{R}^d$ . For any  $m \geq 1$  and  $1 \leq j \leq m$ , a function in  $L^2(\mathbb{R}^d)$  with support in  $E^{(j)}(\tau, m)$  can be extended periodically to  $\mathbb{R}^d$  so that it is of period  $2\pi$  in each of its variables. We will call this extension a  $2\pi$  periodical extension. The result is a function which is square integrable over any compact subset of  $\mathbb{R}^d$ . Substituting  $s = (A')^k u$  (where  $u, s$  are column vectors) in  $f(s) \cdot \chi_{(A')^k E^{(j)}(\tau, m)}(s)$ , we



obtain a function (denoted by  $g(u)$ ) whose support is in  $E^{(j)}(\tau, m)$ . We can then extend it to  $L^2(\mathbb{R}^d)$  periodically of period  $2\pi$ . Let  $G(u)$  be this  $2\pi$  periodical extension of  $g(u)$  and define  $f_{mj}^k(s) = G((A')^k s)$ . In particular,  $f_{mj}^0$  is just the  $2\pi$  periodical extension of  $f \cdot \chi_{E^{(j)}(\tau, m)}$  over  $\mathbb{R}^d$ . For  $k \in \mathbb{Z}$ , we define

$$H_E^k f = \sum_{\ell \in \mathbb{Z}} \langle f, \widehat{D}_A^k \widehat{T}^\ell (2\pi)^{-\frac{d}{2}} \chi_E \rangle \widehat{D}_A^k \widehat{T}^\ell (2\pi)^{-\frac{d}{2}} \chi_E. \quad (8)$$

We will only discuss the convergence of the sum  $H_E^k f$  under the  $L^2(\mathbb{R}^d)$  norm in this paper unless it is otherwise stated.

The following example is an effort to help our reader to understand the definition of the function  $f_{mj}^k$ . Let  $E = [\pi, 2\pi) \times [\pi, 2\pi) \cup [\pi, 2\pi) \times [3\pi, 4\pi)$  so that  $E = E(\tau, 2)$ . Let  $E^{(1)}(\tau, 2) = [\pi, 2\pi) \times [\pi, 2\pi)$ ,  $E^{(2)}(\tau, 2) = [\pi, 2\pi) \times [3\pi, 4\pi)$ . Let  $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$ .

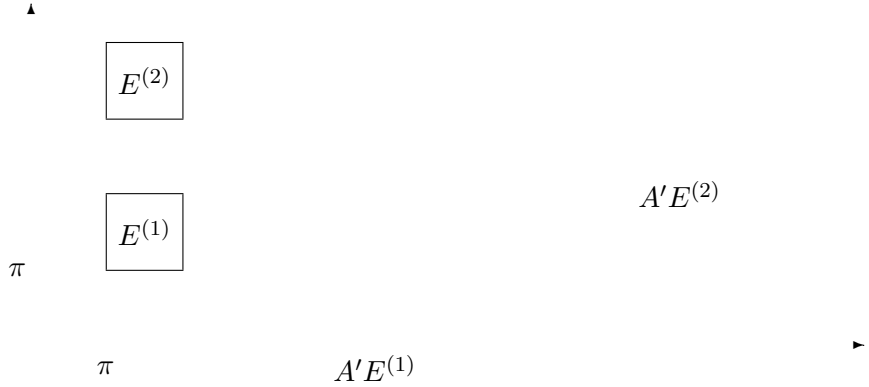


Figure 4:  $E$  and  $A'E$ .

Figure 4 shows the images of  $E^{(1)}(\tau, 2)$  and  $E^{(2)}(\tau, 2)$  under the mapping  $A'$ . Let  $f = \chi_F$  where  $F = [3\pi, 6\pi) \times [0, \pi)$ . Then  $f_{mj}^k = 0$  for all  $m \neq 2$  since  $E(\tau, m)$  is empty.  $f_{2,2}^k = 0$  since the support of  $f$  does not intersect  $(A')^k E^{(2)}(\tau, 2)$  for all  $k$ .  $f_{2,1}^k = 0$  for all  $k \neq 1$  for the same reason. The substitution  $s = A'u$  gives  $g(u) = \chi_{E'}$  where  $E'$  is the upper triangle in  $E^{(1)}(\tau, 2)$ , as one can check. Figure 5 shows the support of the  $2\pi$  extension  $G(u)$  of  $g(u)$  and Figure 6 shows the support of  $f_{2,1}^1$ .

The following elementary result will be needed later so we stated it here without proof. Let  $f$  be a  $2\pi$  periodical function that is square integrable

over  $[0, 2\pi]^d$ , then for any sets  $E, G$  that are Lebesgue measurable and translation equivalent (that means  $E = E(\tau, 1)$ ,  $G = G(\tau, 1)$  and  $\tau(E) = \tau(G)$ ), we have

$$\langle f, \chi_E \rangle = \langle f, \chi_G \rangle = \langle f, \chi_{\tau(E)} \rangle. \quad (9)$$



Figure 5: The Support of  $G(u)$



Figure 6: The Support of  $f_{2,1}^1$

**Lemma 3** *Let  $E$  be a Lebesgue measurable set in  $\mathbb{R}^d$  with finite measure.*

*The following statements are equivalent:*

- (i)  $f = H_E^0 f$  for all  $f \in L^2(\mathbb{R}^d)$  with  $\text{supp}(f) \subset E$ .
- (ii)  $E = E(\tau, 1)$ .

**Proof.** (ii) $\Rightarrow$ (i). Since  $E = E(\tau, 1)$ ,  $E$  is translation equivalent to  $\tau(E)$ . Let  $F = [0, 2\pi)^d \setminus \tau(E)$  and  $G = F \cup E$ . Then  $G$  is translation equivalent to  $[0, 2\pi)^d$  and  $\{(2\pi)^{-\frac{d}{2}} e^{-i\ell \cdot s} \cdot \chi_G : \ell \in \mathbb{Z}^d\}$  is an orthonormal basis for  $L^2(G)$ , so  $f = H_G^0 f$ . Multiplying both side of this by  $\chi_E$  yields the desired result.

(i) $\Rightarrow$ (ii). Assume that  $E$  is a set which satisfies (i) but not (ii). Then  $\mu(E(\tau, k)) > 0$  for some  $k > 1$  where  $\mu$  is the Lebesgue measure.

Define  $g$  by  $g(s) = \chi_{E^{(1)}(\tau, k)}(s) - \chi_{E^{(2)}(\tau, k)}(s)$  so that  $\text{supp}(g) \subset E$ . By equation (9), we get  $H_E^0 g = 0$ . This contradicts (i).  $\square$

**Lemma 4** *Let  $E, F$  be Lebesgue measurable sets of finite measure such that  $\tau(E(\tau, k)) \cap \tau(F) = \emptyset$  for some natural number  $k$ . Then for any  $f \in L^2(\mathbb{R}^d)$  and any  $m \geq 1$ , we have*

$$\sum_{\ell \in \mathbb{Z}^d} \langle f(s), (2\pi)^{-\frac{d}{2}} e^{-i\ell \cdot s} \cdot \chi_{E(\tau, k)} \rangle (2\pi)^{-\frac{d}{2}} e^{-i\ell \cdot s} \chi_{F(\tau, m)} = 0 \quad (10)$$

under the  $L^2(\mathbb{R}^d)$  norm. Consequently,

$$\sum_{\ell \in \mathbb{Z}^d} \langle f(s), (2\pi)^{-\frac{d}{2}} e^{-i\ell \cdot s} \cdot \chi_{E(\tau, k)} \rangle (2\pi)^{-\frac{d}{2}} e^{-i\ell \cdot s} \quad (11)$$

converges to 0 pointwise for almost all  $s \in \mathbb{R}^d$  that are not translation equivalent to any point in  $E(\tau, k)$ .

**Proof.** For any  $1 \leq j \leq k$  and  $1 \leq n \leq m$ , consider  $G = E^{(j)}(\tau, k) \cup F^{(n)}(\tau, m)$ . We have  $G = G(\tau, 1)$ . By Lemma 3,

$$f \chi_{F^{(n)}(\tau, m)} = H_{F^{(n)}(\tau, m)}^0 f, \quad f \chi_G = H_G^0 f. \quad (12)$$

Multiplying both sides of the second equation above by  $\chi_{F^{(n)}(\tau, m)}$ , then subtracting the first equation from it yields

$$\sum_{\ell \in \mathbb{Z}^d} \langle f(s), (2\pi)^{-\frac{d}{2}} e^{-i\ell \cdot s} \cdot \chi_{E^{(j)}(\tau, k)} \rangle (2\pi)^{-\frac{d}{2}} e^{-i\ell \cdot s} \cdot \chi_{F^{(n)}(\tau, m)} = 0.$$

The result then follows.  $\square$

**Lemma 5** *Let  $E$  be a Lebesgue measurable set in  $\mathbb{R}^d$  with finite positive measure. Let  $f \in L^2(\mathbb{R}^d)$ . Then*

$$H_{E(\tau, m)}^k f = \sum_{j=1}^m f_{mj}^k \cdot \chi_{(A')^k E(\tau, m)}.$$

**Proof.** For  $k = 0$ , we have

$$\sum_{j=1}^m f_{mj}^0 \cdot \chi_{E(\tau,m)} = \sum_{j=1}^m f_{mj}^0 \left( \sum_{i=1}^m \chi_{E^{(i)}(\tau,m)} \right) = \sum_{i,j=1}^m f_{mj}^0 \cdot \chi_{E^{(i)}(\tau,m)}. \quad (13)$$

By Lemma 3,

$$\begin{aligned} & f_{mj}^0 \cdot \chi_{E^{(i)}(\tau,m)} \\ = & \sum_{\ell \in \mathbb{Z}^d} \langle f_{mj}^0, (2\pi)^{-\frac{d}{2}} e^{-i\ell \circ s} \cdot \chi_{E^{(i)}(\tau,m)} \rangle (2\pi)^{-\frac{d}{2}} e^{-i\ell \circ s} \chi_{E^{(i)}(\tau,m)}. \end{aligned} \quad (14)$$

On the other hand,

$$\begin{aligned} & \langle f_{mj}^0, (2\pi)^{-\frac{d}{2}} e^{-i\ell \circ s} \cdot \chi_{E^{(i)}(\tau,m)} \rangle \\ = & \langle f_{mj}^0, (2\pi)^{-\frac{d}{2}} e^{-i\ell \circ s} \cdot \chi_{E^{(j)}(\tau,m)} \rangle \\ = & \langle f, (2\pi)^{-\frac{d}{2}} e^{-i\ell \circ s} \cdot \chi_{E^{(j)}(\tau,m)} \rangle \end{aligned}$$

by equation (9). Combining this with (14) and the result for  $k = 0$  follows. If  $k \neq 0$ , then substitute  $s$  by  $(A')^k u$  and apply the above result.  $\square$

For the sake of convenience, we denote  $\langle f(s), (2\pi)^{-\frac{d}{2}} |\det A|^{-\frac{k}{2}} e^{-i\ell \circ (A')^{-k} s} \chi_{(A')^k E} \rangle$  by  $a_{k\ell}$ . To avoid confusion, we have to keep in mind that  $a_{k\ell}$  depends on  $E$  and  $f$  under discussion.

**Lemma 6** *Let  $E$  be a Lebesgue measurable set in  $\mathbb{R}^d$  with finite positive measure. The following statements are equivalent:*

- (i) *There exists a constant  $a > 0$  such that for any  $k \in \mathbb{Z}$ ,  $\sum_{\ell \in \mathbb{Z}^d} |a_{k\ell}|^2 \leq a \|f \chi_{(A')^k E}\|^2$  for all  $f \in L^2(\mathbb{R}^d)$ .*
- (ii) *There exists  $M > 0$  such that  $\mu(E(\tau, m)) = 0$  for all  $m \geq M$ .*

**Proof.** The following is an outline for the case  $k = 0$ . If  $k \neq 0$ , a substitution  $s = (A')^k u$  reduces it to the case  $k = 0$ .

(ii)  $\Rightarrow$  (i) By Lemmas 4 and 5,  $H_E^0 f = \sum_{m=1}^M \sum_{j=1}^m f_{mj}^0 \cdot \chi_{E(\tau,m)}$  where the convergence is under the  $L^2(\mathbb{R}^d)$  norm. By (9), we have

$$\int_{\mathbb{R}} |f_{mj}^0 \chi_{E(\tau,m)}|^2 ds = \int_{E(\tau,m)} |f_{mj}^0|^2 ds = m \int_{E^{(j)}(\tau,m)} |f|^2 ds. \quad (15)$$

It follows that  $\int_{E(\tau,m)} |\sum_{j=1}^m f_{mj}^0|^2 ds \leq m^2 \int_{E(\tau,m)} |f|^2 ds$ . And

$$\|H_E^0 f\|^2 \leq M \sum_{m=1}^M m^2 \int_{E(\tau,m)} |f|^2 ds \leq M^3 \int_E |f|^2 ds = M^3 \|f \chi_E\|^2.$$

So

$$\sum_{\ell \in \mathbb{Z}^d} |a_{0\ell}|^2 = \langle H_E^0 f, f \chi_E \rangle \leq \|H_E^0 f\| \cdot \|f \chi_E\| \leq M^{\frac{3}{2}} \|f \chi_E\|^2.$$

(i) $\Rightarrow$ (ii) Assume this is not true, then (i) holds for some  $E$  that does not satisfy (ii). Thus,  $\mu(E(\tau, m_0)) > 0$  for some  $m_0 > a$ . Let  $f = \chi_{E(\tau, m_0)} \in L^2(\mathbb{R}^d)$ , then  $\|f\|^2 = \mu(E(\tau, m_0))$ . By Lemma 5,

$$\langle f, H_E^0 f \rangle = \langle f, m_0 f \rangle = m_0 \mu(E(\tau, m_0)) = m_0 \|f\|^2.$$

This contradicts the assumption that  $a < m_0$ .  $\square$

We are now ready to outline the proofs of the theorems.

**Proof of Theorem 1.** (i)  $\Rightarrow$  (ii) This is obvious from  $\|\langle f, g \rangle\| \leq \|f\| \|g\|$ .

(iii)  $\Rightarrow$  (i) By Lemma 6 and its proof,  $\int_{\mathbb{R}^d} |H_E^k f|^2 ds \leq M^3 \|f \cdot \chi_{(A')^k E}\|^2$ . Notice that  $\sum_{k \in \mathbb{Z}} |H_E^k f|$  converges pointwise since for each  $s \in \mathbb{R}$ , there are at most  $M$  nonzero terms. Note that  $\sum_{j \in \mathbb{Z}} \chi_{(A')^j E} \leq M$  by the given condition. Since the support of  $|H_E^k f|$  is in  $(A')^k E$ , for any  $L_1, L_2 > 0$ , we have

$$\begin{aligned} & \int_{\mathbb{R}} \left( \sum_{-L_1 \leq k \leq L_2} |H_E^k f| \right)^2 ds \\ & \leq \sum_{-L_1 \leq p, q \leq L_2} \int_{(A')^p E \cap (A')^q E} |H_E^p f| \cdot |H_E^q f| ds \\ & \leq \frac{1}{2} \sum_{-L_1 \leq p, q \leq L_2} \left( \int_{(A')^p E \cap (A')^q E} |H_E^p f|^2 ds + \int_{(A')^p E \cap (A')^q E} |H_E^q f|^2 ds \right) \\ & = \sum_{-L_1 \leq p \leq L_2} \int_{(A')^p E} |H_E^p f|^2 \sum_{-L_1 \leq q \leq L_2} \chi_{(A')^q E} ds \\ & \leq M \sum_{-L_1 \leq p \leq L_2} \int_{(A')^p E} |H_E^p f|^2 ds \\ & = M \sum_{-L_1 \leq p \leq L_2} \int_{\mathbb{R}} |H_E^p f|^2 ds \\ & \leq M^4 \sum_{-L_1 \leq p \leq L_2} \int_{(A')^p E} |f|^2 ds \\ & = M^4 \int_{\mathbb{R}} |f|^2 \sum_{-L_1 \leq p \leq L_2} \chi_{(A')^p E} ds \\ & \leq M^5 \int_{\mathbb{R}} |f|^2 ds = M^5 \|f\|^2. \end{aligned} \tag{16}$$

Therefore,  $\int_{\mathbb{R}} (\sum_{k \in \mathbb{Z}} |H_E^k f|)^2 ds \leq M^5 \|f\|^2$  by Fatou's lemma. This also leads to

$$\lim_{K_1, K_2 \rightarrow \infty} \int_{\mathbb{R}} \left( \sum_{k \leq -K_1, k \geq K_2} |H_E^k f| \right)^2 ds = 0. \quad (17)$$

That is,  $\sum_{k \in \mathbb{Z}} |H_E^k f|$  (hence  $\sum_{k \in \mathbb{Z}} H_E^k f$  as well) converges in  $L^2(\mathbb{R})$ .

We can then show that  $H_E f$  converges to  $\sum_{k \in \mathbb{Z}} H_E^k f$  in  $L^2(\mathbb{R})$ . The proof is long and is omitted. For a complete (similar) proof, please refer to [1].

(ii) $\Rightarrow$ (iii) If there exists  $m_0 > c$  such that  $\mu(E(\tau, m_0)) > 0$ , then we will derive a contradiction the same way as we did in the proof of Lemma 6, since  $\sum_{\ell \in \mathbb{Z}^d} |a_{0\ell}|^2 \leq \sum_{k \in \mathbb{Z}, \ell \in \mathbb{Z}} |a_{k\ell}|^2 \leq c \|f\|^2$ . So  $\mu(E(\tau, m)) = 0$  for all  $m > c$ . Now, if  $\mu(E(\delta, m_0)) > 0$  for some  $m_0 > c$ , (this includes the case  $m_0 = \infty$ .) then there exists a subset  $F$  of  $E$ , such that  $(A')^{k_j} F \subset E$  for some  $q > c$  integers  $k_{q-1} > k_{q-2} > \dots > k_0 = 0$ . The proof of this is elementary and is left to our reader. A complete proof in the one dimensional case can be found in [1]. Define  $f = \chi_F$ . By Lemma 6,  $H_E^k f$  converges for each  $k$ . In particular, for  $k = 0, -k_1, \dots, -k_{q-1}$ , we have  $H_E^k f = \sum_{m=1}^M \sum_{j=1}^m f_{mj}^k \chi_{(A')^k E} \geq f$ . It then follows that  $\sum_{k, \ell \in \mathbb{Z}} |a_{k\ell}|^2 \geq \sum_{j=-(q-1)}^0 \sum_{\ell \in \mathbb{Z}} |a_{k_j \ell}|^2 = \sum_{j=-(q-1)}^0 \langle H_E^{-k_j} f, f \rangle \geq q \|f\|^2 > c \|f\|^2$ . This contradicts the assumption.  $\square$

Notice that under the conditions given in (iii) of Theorem 1, we have the following decomposition of  $H_E f$ :

$$H_E f = H_{E(\tau,1)} f + H_{E(\tau,2)} f + \dots + H_{E(\tau,m)} f. \quad (18)$$

by Lemma 4 and Theorem 1.

**Proof of Theorem 2.** By (16), we have  $|\langle H_E f, f \rangle| \leq M^{\frac{5}{2}} \|f\|^2$ . On the other hand,  $\sum_{j=1}^m f_{mj}^k \cdot \chi_{(A')^k E(\tau, m)} = \sum_{\ell \in \mathbb{Z}^d} a_{k\ell} \widehat{D}_A^k \widehat{T}^\ell (\sqrt{2\pi})^{-d} \chi_{E(\tau, m)}$  (with  $a_{k\ell}$  depending on  $E(\tau, m)$ , not  $E$ ), hence  $\langle \sum_{j=1}^m f_{mj}^k \cdot \chi_{(A')^k E(\tau, m)}, f \rangle = \sum_{\ell \in \mathbb{Z}} |a_{k\ell}|^2 \geq 0$ . It follows that

$$\begin{aligned} \langle H_E f, f \rangle &= \sum_{k \in \mathbb{Z}} \sum_{m=1}^M \sum_{j=1}^m \langle f_{mj}^k \cdot \chi_{(A')^k E(\tau, m)}, f \rangle \\ &\geq \sum_{k \in \mathbb{Z}} \langle f \cdot \chi_{(A')^k E(\tau, 1)}, f \rangle \\ &= \int_{\mathbb{R}} |f|^2 \left( \sum_{k \in \mathbb{Z}} \chi_{(A')^k E(\tau, 1)} \right) ds \geq \|f\|^2 \end{aligned}$$

since  $\sum_{k \in \mathbb{Z}} \chi_{(A')^k E(\tau, 1)} \geq 1$  by the given condition.  $\square$

**Proof of Theorem 3.** This is obvious from Theorem 1.  $\square$

**Proof of Theorem 4.** If  $E = E(\delta, k) = E(\tau, 1)$  for some  $k \geq 1$  and  $\cup_{n \in \mathbb{Z}} (A')^n E = \mathbb{R}^d$ , then

$$H_E f = \sum_{n \in \mathbb{Z}} H_E^n f = \sum_{n \in \mathbb{Z}} f \chi_{(A')^n E}$$

and  $\sum_{n \in \mathbb{Z}} \chi_{(A')^n E} = k$ . So for any  $f \in L^2(\mathbb{R}^d)$ , we have  $\langle H_E f, f \rangle = \langle \sum_{n \in \mathbb{Z}} f \chi_{(A')^n E}, f \rangle = \int_{\mathbb{R}^d} |f|^2 \sum_{n \in \mathbb{Z}} \chi_{(A')^n E} ds = k \|f\|^2$ .

Now assume that  $E$  is a tight frame wavelet set but  $\mu(E(\tau, m_0)) > 0$  for some  $m_0 > 1$ . Let  $g = \chi_{E^{(1)}(\tau, m_0)}$ ,  $h = \chi_{E^{(2)}(\tau, m_0)}$  and  $f_1 = g + h$ ,  $f_2 = g - h$ .  $\langle H_E(f_1), f_1 \rangle = \langle H_E(f_2), f_2 \rangle$  since  $E$  is a tight frame wavelet set and  $\|f_1\| = \|f_2\|$ . We leave it to our reader to check that  $\langle H_E g, h \rangle$  and  $\langle H_E h, g \rangle$  are both positive. This then easily leads to  $\langle H_E(f_1), f_1 \rangle \neq \langle H_E(f_2), f_2 \rangle$ , which is a contradiction. So  $E = E(\tau, 1)$ . Finally, assume that  $\mu(E(\delta, k_1)) \neq 0$  and  $\mu(E(\delta, k_2)) \neq 0$  for some  $k_1 \neq k_2$ . Then for  $f_1 = \chi_{E(\delta, k_1)}$  and  $f_2 = \chi_{E(\delta, k_2)}$ ,  $H_E(f_1) = k_1 f_1$  and  $H_E(f_2) = k_2 f_2$ . This leads to  $\langle H_E(f_1), f_1 \rangle = k_1 \|f_1\|^2$ ,  $\langle H_E(f_2), f_2 \rangle = k_2 \|f_2\|^2$ . So  $E$  is not a tight frame wavelet set by definition. Again, we get a contradiction.  $\square$

The corollaries of the theorems now follow trivially and the proofs are omitted. Using the ideas in the proofs of Theorem 2 and Theorem 4, it is then not hard to prove Theorem 5. For Theorem 6, one needs only to show a counterexample. One such example is  $f = \chi_{E^{(1)}(\tau, k)} - \chi_{E^{(2)}(\tau, k)}$ . We leave the details here to our reader.

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