The s-elementary Frame Wavelets are Path Connected

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ABSTRACT

An s-elementary frame wavelet is a function $\psi \in L^2(\mathbb{R})$ which is a frame wavelet and is defined by a Lebesgue measurable set $E \subset \mathbb{R}$ such that $\hat{\psi} = \frac{1}{\sqrt{2\pi}}\chi_E$. In this paper we prove that the family of s-elementary frame wavelets is a path-connected set in the $L^2(\mathbb{R})$-norm. This result also holds for s-elementary A-dilation frame wavelets in $L^2(\mathbb{R}^d)$ in general. We also consider the uniform path-connectivity of the sets of frame wavelets, normalized tight frame wavelets and s-elementary frame wavelets. We prove that none of these sets is uniformly path-connected.

Keywords: Frames, Wavelets, Frame Wavelets, Frame Wavelet Sets, Fourier Transform.

1. Introduction

Let $L^2(\mathbb{R})$ be the set of Lebesgue square integrable functions on $\mathbb{R}$. The Fourier transform for $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t)dt.$$

This is denoted by $\mathcal{F}f$ or $\widehat{f}$. It is known that $\mathcal{F}$ can be uniquely extended to a unitary operator on $L^2(\mathbb{R})$. Let D and T be the dilation and translation operators on $L^2(\mathbb{R})$, namely $(Df)(x) = \sqrt{2}f(2x)$ and $(Tf)(x) = f(x-1)$ for any $f \in L^2(\mathbb{R})$. We will use $\widehat{D}, \widehat{T}$ for the product $\mathcal{F}D\mathcal{F}^{-1}$ and $\mathcal{F}T\mathcal{F}^{-1}$. It is known that $\widehat{D} = D^{-1}$ and $\widehat{T}f(t) = e^{-ist}f(t)$ ([3]). A function $\psi \in L^2(\mathbb{R})$ is called a *frame wavelet* for $L^2(\mathbb{R})$ if there exist two positive constants $0 < a \leq b$ such that for any $f \in L^2(\mathbb{R})$,

$$a\|f\|^2 \leq \sum_{n,\ell\in\mathbb{Z}} |\langle f, D^n T^\ell \psi\rangle|^2 \leq b\|f\|^2. \tag{1}$$

If one can choose $a = b$ in (1), then ψ is called a *tight frame wavelet*. Furthermore, if $a = b = 1$, then ψ is called a *normalized tight frame wavelet*. Let E be a Lebesgue measurable set of finite measure and χ_E be the corresponding characteristic function. If the function $\psi_E \in L^2(\mathbb{R})$ defined by $\widehat{\psi_E} = \frac{1}{\sqrt{2\pi}}\chi_E$ is a frame wavelet, a tight frame wavelet or a normalized tight frame wavelet for $L^2(\mathbb{R})$, then the set E is called a *frame wavelet set*, a *tight frame wavelet set* or a *normalized tight frame wavelet set* for $L^2(\mathbb{R})$ respectively. The corresponding function ψ_E is called an *s-elementary*, a *tight s-elementary* or a *normalized tight s-elementary* frame wavelet. The name *s-elementary* is borrowed from [3, 7], where a wavelet whose Fourier transform is of the form $\frac{1}{\sqrt{2\pi}}\chi_E$ is called an *s-elementary wavelet*.

The topological property of various families of wavelets is an interesting topic in the study of wavelet theory. In [3], a question about the path-connectedness of the set of all orthonormal wavelets was raised. In fact, discussions on such issues can be traced a few years back before the publication of [3]. One can ask similar questions on the families of normalized tight frame wavelets and frame wavelets. These turn out to be very hard questions and all remain unsolved at this time. However, it is proved that the family of s-elementary (orthonormal) wavelets is path-connected in [7] and it is later shown that the set of all MRA-wavelets is also path-connected [6,8]. In [2], the authors proved that the set of normalized tight s-elementary frame wavelets is path-connected.

In this paper, we are mainly concerned with the path-connectedness of the set of s-elementary frame wavelets. Showing the path-connectedness of the set of s-elementary frame wavelets can potentially lend a helping hand in proving the path-connectivity of the set of all frame wavelets, since one would only need to show that any frame wavelet is path-connected to an s-elementary frame wavelet. Since the set of normalized tight s-elementary frame wavelets is path-connected, it seems plausible that the set of s-elementary frame wavelets may also be path-connected. However, proving it is not trivial. The reason is that the proof in [2] relies on the characterization of the normalized tight frame wavelet sets (which is given in [1]). Yet, the characterization of frame wavelet sets is still an open question at this time. So, it is somewhat surprising that we are able to use the partial results about frame wavelet sets developed in [1] to prove that the set of s-elementary frame wavelets is indeed path-connected. This result can be generalized (by using a similar argument and some results from [2]) to s-elementary frame wavelets in higher dimensional cases with arbitrary expansive matrix dilations. This is done in Section 3. In the last section, we discuss the uniform path-connectivity of the sets of frame wavelets, normalized tight frame wavelets and s-elementary frame wavelets. We prove that none of these sets is uniformly path-connected.

2. Basic Concepts and Lemmas

Throughout this paper, we only deal with subsets of $\mathbb{R}$ that are Lebesgue measurable. Thus, in all lemmas and theorems, it is understood that all sets involved are Lebesgue measurable. Most definitions and the proofs of the lemmas in this section can be found in [1] and [2]. Please refer to these two papers for the details.

Let E be a Lebesgue measurable set in $\mathbb{R}$. A point $x \in E$ is said to have a dilation index $\delta_E(x) = k$ if there are exactly k points in the set $E \cap (\cup_{n\in\mathbb{Z}} 2^n x)$. For each fixed natural number k, the set $E(\delta, k) = \{x \in E : \ \delta_E(x) = k\}$ is Lebesgue measurable. Furthermore, each $E(\delta, k)$ is a disjoint union of k measurable sets $\{E^j(\delta, k) : 1 \leq j \leq k\}$ such that $\delta_{E^j(\delta,k)}(x) = 1$ for each point $x \in E^j(\delta, k)$. Similarly a point $x \in E$ is said to have a 2π-translation index $\tau_E(x) = k$ if there are exactly k points in the set $E \cap (\cup_{n\in\mathbb{Z}}(2\pi n + x))$. For each fixed natural number k, the set $E(\tau, k) = \{x \in E : \ \tau_E(x) = k\}$ is Lebesgue measurable and is a disjoint union of k measurable sets $\{E^j(\tau, k) : 1 \leq j \leq k\}$ such that $\tau_{E^j(\tau,k)}(x) = 1$ for each point $x \in E^j(\tau, k)$. If there is a number M such that $E(\delta, k)$ and $E(\tau, k)$ are null sets for all $k > M$, the set E is called a *basic set.*

Assume further that E is of finite measure. For any $f \in L^2(\mathbb{R})$. Define:

$$(H_E f)(s) = \sum_{n,\ell\in\mathbb{Z}} \langle f, \widehat{D}^n \widehat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_E \rangle \widehat{D}^n \widehat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_E(s). \tag{2}$$

A set E is called a *Bessel set* if $H_E f$ converges in norm unconditionally for each $f \in L^2(\mathbb{R})$ and $\langle H_E f, f\rangle \leq B\|f\|^2$ for some constant $B > 0$. Theorem 1 of [1] implies the following lemma.

Lemma 1 *A set E is Bessel if and only if it is a basic set. Moreover, if $\mu(E(\delta,m)) = \mu(E(\tau,m)) = 0$ for all $m > M$ (where μ is the Lebesgue measure), then $\langle H_E f, f\rangle \leq M^{5/2}\|f\|^2$ for any $f \in L^2(\mathbb{R})$.*

On the other hand, the same argument used in the proof of Theorem 2 in [1] leads us to the following lemma.

Lemma 2 *Let E be a basic set. Assume that $\Omega = \cup_{k\in\mathbb{Z}} 2^k E(\tau,1) = \cup_{k\in\mathbb{Z}} 2^k E$. Then*

$$\langle H_E f, f\rangle \geq \|f\|^2, \forall f \in L^2(\mathbb{R}), \mathrm{supp}(f) \subset \Omega.$$

Lemma 3 below is obtained by using Lemma 1 and Lemma 2.

Lemma 3 *Let E be a basic set and $E(\tau,m) = E(\delta,m) = \emptyset, \forall m > M$. Let F be a measurable set such that $E \subset \cup_{k\in\mathbb{Z}} 2^k F$ and $F = F(\tau,1)$. Then*

$$\langle H_E f, f\rangle \leq M^{5/2}\langle H_F f, f\rangle, \quad \forall f \in L^2(\mathbb{R}).$$

Proof. Define $\Omega = \cup_{k\in\mathbb{Z}} 2^k F$ and $\Omega_1 = \mathbb{R}\backslash\Omega$. Let $f \in L^2(\mathbb{R})$. Denote $f_1 = f\chi_\Omega$ and $f_2 = f\chi_{\Omega_1}$. Then we have $H_E f_2 = H_F f_2 = 0$ and $\langle H_E f_1, f_2\rangle = \langle H_F f_1, f_2\rangle = 0$. Hence $\langle H_E f, f\rangle = \langle H_E f_1, f_1\rangle \leq M^{5/2}\|f_1\|^2$ by Lemma 1 and $\langle H_F f, f\rangle = \langle H_F f_1, f_1\rangle \geq \|f_1\|^2$ by Lemma 2. The result follows. $\Box$

For any $E \subset \mathbb{R}$, let $\tau(E) = \bigcup_{\mathbb{Z}}(E + 2k\pi)$. Be careful not to confuse $\tau(E)$ with $\tau_E(x)$, the translation index of x in E. We say that two sets E and F are 2π-translation disjoint if $\tau(E) \cap \tau(F) = \emptyset$. The following lemma is obtained from Lemma 5 of [1].

Lemma 4 . *If E and F are 2π-translation disjoint basic sets, then*

$$H_{E\cup F} f \;=\; H_E f + H_F f, \forall f \in L^2(\mathbb{R}).$$

It is well-known that if $\psi = \psi_E$, then (1) is equivalent to

$$a\|f\|^2 \leq \sum_{n,\ell\in\mathbb{Z}} |\langle f, \widehat{D}^n \widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_E\rangle|^2 \leq b\|f\|^2. \tag{3}$$

Combining this with (2), we get

Lemma 5 . *For $\psi = \psi_E$, (1) is equivalent to*

$$a\|f\|^2 \le \langle H_E f, f\rangle \le b\|f\|^2, \quad \forall f \in L^2(\mathbb{R}). \tag{4}$$

Finally, we will need the following lemma in proving our main theorem in the next section. This lemma is the one dimensional case of Theorem 4 in [2].

Lemma 6 *The family of normalized tight s-elementary frame wavelets is path-connected under the $L^2(\mathbb{R})$ norm.*

3. The Main Theorem and its Proof

In this section we prove our main result.

Theorem 1 . *The family of s-elementary frame wavelets is path-connected under the $L^2(\mathbb{R})$ norm.*

Proof. We will prove that for a given frame wavelet set E, there is a continuous path of the form χ_{W_t} connecting χ_E to χ_F, where each W_t is a frame wavelet set and F is a normalized tight frame set. This implies that each s-elementary frame wavelet is connected by a continuous path (of s-elementary frame wavelets) to a normalized tight s-elementary frame wavelet. This in turn implies the theorem by Lemma 6.

Let E be a frame wavelet set and ψ_E be the corresponding s-elementary frame wavelet. E is a Bessel set hence a basic set by Lemma 1. So there is a number M such that $E(\tau, m) = E(\delta, m) = \emptyset, \forall m > M$. Thus we can choose $B = M^{5/2}$ in (1) by Lemma 1. Let $a > 0$ be the lower frame bound of ψ_E. We have $a\|f\|^2 \le \langle H_E f, f\rangle \le M^{5/2}\|f\|^2$ for all $f \in L^2(\mathbb{R})$ by Lemma 5. Let m_0 be a positive integer large enough so that $M/2^{m_0} < 1/4$. Let

$$F = [-\frac{2\pi}{2^{m_0+1}}, -\frac{\pi}{2^{m_0+1}}) \cup [\frac{\pi}{2^{m_0+1}}, \frac{2\pi}{2^{m_0+1}}).$$

By Corollary 3 of [1], the set F is a normalized tight frame set. It is left to the reader to verify that $E \cup F$ is a basic set and every measurable subset of $E \cup F$ is a basic set.

For any $s \in E$, there is a unique integer $k(s)$ such that $s/2^{k(s)} \in F$. Thus $h(s) = s/2^{k(s)}$ defines a mapping from E to F. We leave it to our reader to prove that the image of each measurable subset in E under h is measurable. Furthermore, if E' is a subset of $E \cap \mathbb{R} \backslash [-\pi, \pi]$, then $\mu(h(E')) < \frac{1}{2^{m_0+1}} \mu(E')$. Define

$$
\begin{aligned}
F_t^0 &= [-\frac{2\pi}{2^{m_0+1}}, -\frac{(2-t)\pi}{2^{m_0+1}}] \cup [\frac{\pi}{2^{m_0+1}}, \frac{(1+t)\pi}{2^{m_0+1}}] \\
F_t^1 &= h(\tau(F_t^0) \cap (E \backslash F_t^0)), \\
F_t^2 &= h(\tau(F_t^1) \cap (E \backslash F_t^1)), \\
&\cdots \\
F_t^n &= h(\tau(F_t^{n-1}) \cap (E \backslash F_t^{n-1})), \\
&\cdots \\
F_t &= \bigcup_{k \geq 0} F_t^k, t \in [0, 1].
\end{aligned}
$$

Notice that the set F_t is a measurable subset of $E \cup F$, hence it is a basic set. Let $E_t = \tau(F_t) \cap E$. It is clear that any point in $\tau(E_t)$ must be in $\tau(F_t)$ hence cannot be in $\tau(E \backslash E_t)$. So the sets E_t and $E \backslash E_t$ are 2π-translation disjoint. By Lemma 4 we have

$$H_E f = H_{E_t} f + H_{E \backslash E_t} f.$$

Hence

$$\langle H_E f, f \rangle = \langle H_{E_t} f, f \rangle + \langle H_{E \backslash E_t} f, f \rangle \geq a \|f\|^2. \tag{5}$$

Similarly,

$$H_{F_t \cup (E \backslash E_t)} f = H_{F_t} f + H_{E \backslash E_t} f$$

since F_t and $E \backslash E_t$ are also 2π-translation disjoint. It follows that

$$\langle H_{F_t \cup (E \backslash E_t)} f, f \rangle = \langle H_{F_t} f, f \rangle + \langle H_{E \backslash E_t} f, f \rangle. \tag{6}$$

Notice that $F_t = F_t(\tau, 1)$ since $F_t \subset F$ and $F = F(\tau, 1)$. Let $x \in E_t = E \cap \tau(F_t)$. If $x \notin F_t$, then $x \in \tau(F_t^n) \cup (E \backslash F_t^n)$ for some $n \geq 0$. So $h(x) \in F_t^{n+1} \subset F_t$. Hence we have

$$E_t \subset \bigcup_{k \in \mathbb{Z}} 2^k F_t. \tag{7}$$

By Lemma 3 we have

$$\langle H_{F_t} f, f \rangle \geq M^{-\frac{5}{2}} \langle H_{E_t} f, f \rangle. \tag{8}$$

Now define $W_t = F_t \cup (E \backslash E_t)$. Since $W_t \subset F \cup E$, it is a basic set. By Lemma 1, there is a positive number B (independent of t) such that

$$\langle H_{W_t} f, f\rangle \leq B\|f\|^2, \quad \forall f \in L^2(\mathbb{R}). \tag{9}$$

On the other hand, (5), (6) and (8) imply that

$$\begin{aligned}\langle H_{W_t} f, f\rangle &= \langle H_{F_t} f, f\rangle + \langle H_{E\backslash E_t} f, f\rangle \\ &\geq M^{-\frac{5}{2}}\langle H_{E_t} f, f\rangle + \langle H_{E\backslash E_t} f, f\rangle \\ &> M^{-\frac{5}{2}}(\langle H_{E_t} f, f\rangle + \langle H_{E\backslash E_t} f, f\rangle) \\ &\geq aM^{-\frac{5}{2}}\|f\|^2.\end{aligned}$$

Therefore, W_t is a frame wavelet set for each $t \in [0,1]$. It is easy to verify that $W_0 = E$ and $W_1 = F$.

To complete the proof of Theorem 1 we need to show that the mapping $t \to \chi_{W_t}$ is continuous in norm. We will achieve this in a few steps.

Step 1: We first show that the mapping $t \to \chi_{F_t}$ is continuous in norm. For $0 \leq t \leq 1$, we have $\mu(F_t^0) \leq \pi/2^{m_0}$. By the property of E, for a point $s \in F_t^0$, the set $\{s + 2k\pi : k \in \mathbb{Z}\} \cap E$ has at most M points. This implies that

$$\mu(\tau(F_t^0) \cap (E\backslash F_t^0)) \quad \leq \quad M\mu(F_t^0). \tag{10}$$

Since $\tau(F_t^0) \cap (E\backslash F_t^0) \subset \mathbb{R}\backslash[-\pi,\pi]$, it follows from (10) that

$$\begin{aligned}\mu(F_t^1) &\leq \frac{1}{2^{m_0+1}}\mu(\tau(F_t^0) \cap (E\backslash F_t^0)) \\ &\leq \frac{M}{2^{m_0+1}}\mu(F_t^0) \leq \frac{1}{4}\mu(F_t^0).\end{aligned}$$

By induction, we have

$$\mu(F_t^n) \leq \frac{M}{2^{m_0+1}}\mu(F_t^{n-1}) \leq \frac{1}{4^n}\mu(F_t^0).$$

Therefore, the convergence of $\chi_{\cup_{0\leq k\leq n} F_t^k}$ to χ_{F_t} is uniform with respect to $t \in [0,1]$ hence it suffices to prove that the mapping $t \to \chi_{F_t^n}$ is continuous in norm for each n. We prove this by induction. Clearly, the mapping $t \to \chi_{F_t^0}$ is continuous. Assume that it is true for n. We will show that it is true for $n+1$. For this purpose, we write $K\Delta L = (K\backslash L) \cup (L\backslash K)$ for any sets K and L, and let $D_t^n = \tau(F_t^n) \cap (E\backslash F_t^n)$. For any $t, t' \in [0,1]$, we claim that $D_t^n \Delta D_{t'}^n \subset \tau(F_t^n \Delta F_{t'}^n) \cap E$. Let $s \in D_t^n \Delta D_{t'}^n$. We can assume that $s \in D_t^n \backslash D_{t'}^n$. Then there is an integer k such that $s + 2k\pi \in F_t^n$. However $s \notin F_t^n$. It follows that $k \neq 0$. Thus $s \notin F_{t'}^n$, for otherwise we would have both s and $s + 2k\pi \in F_{t'}^n \cup F_t^n \subset F \subset [-\pi, \pi)$ which is impossible since $k \neq 0$. Therefore $s \in E \backslash F_{t'}^n$. Since $s \notin D_{t'}^n = \tau(F_{t'}^n) \cap (E \backslash F_{t'}^n)$, it follows that $s \notin \tau(F_{t'}^n)$. Hence $s + 2k\pi \in F_t^n \Delta F_{t'}^n$ and therefore $s \in \tau(F_t^n \Delta F_{t'}^n) \cap E$, as expected.

We now have

$$F_t^{n+1} \Delta F_{t'}^{n+1} \subset h(D_t^n \Delta D_{t'}^n) \subset h(\tau(F_t^n \Delta F_{t'}^n) \cap E). \tag{11}$$

Therefore,

$$\begin{aligned} \mu(F_t^{n+1} \Delta F_{t'}^{n+1}) &\leq \mu(h((F_t^n \Delta F_{t'}^n)^+ \cap E)) \\ &\leq \frac{M}{2^{m_0+1}} \mu(F_t^n \Delta F_{t'}^n). \end{aligned} \tag{12}$$

(12) implies that the mapping $t \to \chi_{F_t^{n+1}}$ is continuous since the mapping $t \to \chi_{F_t^n}$ is. This completes the proof that the mapping $t \to \chi_{F_t^n}$ is continuous in norm for all n. Hence the mapping $t \to \chi_{F_t}$ is continuous, as claimed.

Step 2: We now show that the mapping $t \to \chi_{E_t}$ is also continuous. In fact, this follows from the inclusion $E_t \Delta E_{t'} \subset \tau(F_t \Delta F_{t'}) \cap E$, which implies that

$$\mu(E_t \Delta E_{t'}) \leq \mu(\tau(F_t \Delta F_{t'}) \cap E) \leq M \mu(F_t \Delta F_{t'}).$$

Step 3: Finally, the continuity of $t \to \chi_{W_t}$ follows from the continuity of the mappings $t \to \chi_{F_t}$ and $t \to \chi_{E \backslash E_t}$ and the fact that $F_t \cap (E \backslash E_t) = \emptyset$. This completes our proof of Theorem 1. $\square$

In [1], it is shown that the frame bound of an s-elementary tight frame wavelet is a positive integer. If we use S_f to denote the set of all s-elementary frame wavelets, $S_f(j)$ to denote the set of all s-elementary tight frame wavelets of frame bound $j \geq 1$ (so $S_f(1)$ is the set of all s-elementary normalized tight frame wavelets), then it is not hard to see that $S_f(j)$ and $S_f(k)$ are not path-connected in the set of all s-elementary tight frame wavelets if $j \neq k$. For $j \neq 1$, it remains unclear whether $S_f(j)$ is path-connected. This situation is illustrated in the following figure.

We now point out that Theorem 1 is also valid for higher dimensional cases. Let A be a $d \times d$ matrix *expansive* matrix, that is, all eigenvalues

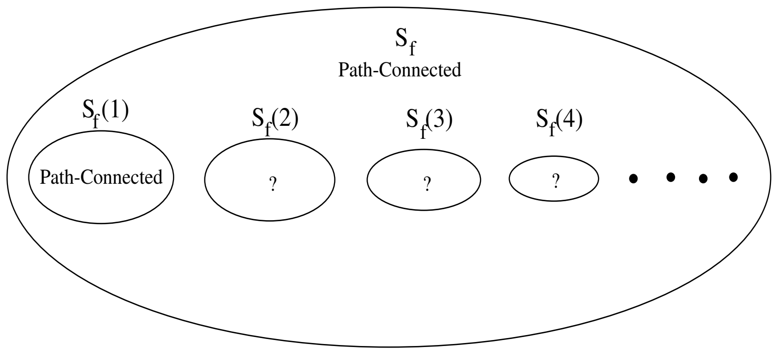

Figure 1: The illustration of the path-connectedness of s-elementary frame wavelets.

of A have norm greater than 1. Let D_A be the unitary operator defined by $D_A f(s) = |\det A|^{\frac{1}{2}} f(As)$ and T_ℓ be the unitary operator defined by $T_\ell f(s) = f(s-\ell)$, where $f \in L^2(\mathbb{R}^d)$ and $\ell \in \mathbb{Z}^d$. A function $\psi \in L^2(\mathbb{R}^d)$ is called an A-dilation frame wavelet for $L^2(\mathbb{R}^d)$ if there exist two positive constants $0 < a \leq b$ such that for any $f \in L^2(\mathbb{R}^d)$,

$$a\|f\|^2 \leq \sum_{n\in\mathbb{Z},\ell\in\mathbb{Z}^d} |\langle f, D_A^n T^\ell \psi\rangle|^2 \leq b\|f\|^2. \tag{13}$$

The Fourier-Plancherel transform $\mathcal{F}$ on $L^2(\mathbb{R}^d)$ is a unitary operator such that for $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$,

$$(\mathcal{F}f)(s) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i(s\circ t)} f(t) dm,$$

where $s \circ t$ denotes the real inner product. If the function $\psi_E \in L^2(\mathbb{R}^d)$ defined by $\widehat{\psi} = \frac{1}{\sqrt{2\pi}}\chi_E$ for some measurable set E in $\mathbb{R}^d$ is a frame wavelet for $L^2(\mathbb{R}^d)$, then the function ψ_E is called an *s-elementary A-dilation frame wavelet*. We have

Theorem 2 *The family of s-elementary A-dilation frame wavelets is path-connected in $L^2(\mathbb{R}^d)$ norm.*

4. Local Commutant and Uniform Connectivity

Let ψ be a fixed orthonormal wavelet. The local commutant [3] at ψ is the set:

$$C_\psi(D,T) = \{A \in B(L^2(\mathbb{R})) : AD^nT^m\psi = D^nT^mA\psi\}.$$

For each frame wavelet η, there is a unique operator $U_\eta \in C_\psi(D,T)$ such that $U_\eta\psi = \eta$, U_η^* is injective and has closed range. Moreover, η is an orthonormal wavelet if and only if U_η is unitary, while η is a normalized tight frame wavelet if and only if U_η^* is an isometry ([5]).

Two frame wavelets η_0 and η_1 are said to be *uniformly path-connected* if there is a path of frame wavelets $\{\eta_t : t \in [0,1]\}$ such that U_{η_t} is a continuous path in the operator norm (and hence $\{\eta_t : t \in [0,1]\}$ is a continuous path in L^2-norm). The uniform connectivity for certain classes of wavelets is related to the interpolation theory of wavelets and was investigated in several papers (cf. [3], [4]). Since uniform path-connectedness is stronger than the ordinary path-connectedness, it is natural to ask that whether S_f (or $S_f(1)$) is uniformly path-connected. We will prove that this is not the case. In fact, we will prove that the set of frame wavelets is not uniformly path-connected either. We need the following simple lemma.

Lemma 7 . *Let U be a unitary operator. If V is an isometry such that $\|U - V\| < 1$, then it must be unitary.*

Proof. Write $V = U + (V - U) = U(I + U^*(V - U))$. Since $\|U^*(V - U)\| \leq \|V - U\| < 1$, it follows that $(I + U^*(V - U))$ is invertible. Thus V is invertible and hence it is unitary. □

Theorem 3 . *None of the following sets is uniformly path-connected:*

(i) The set of all frame wavelets;

(ii) The set of all normalized tight frame wavelets;

(iii) The set of all s-elementary frame wavelets.

Proof. We will only prove that set (i) is not uniformly path-connected. The other two cases are similar. Let η_0 be a Riesz wavelet (i.e, $\{D^nT^\ell\eta : n.\ell \in \mathbb{Z}\}$ is a Riesz basis for $L^2(\mathbb{R})$) and η_1 be a frame wavelet which is not a Riesz wavelet. We claim that η_0 and η_1 can never be uniformly path-connected. In fact, if there exist $\{\eta_t : t \in [0,1]\}$ such that $\{U_{\eta_t}\}$ is a continuous path in the operator norm. Write $U(t) = U_{\eta_t}$ and $S(t) = U(t)U(t)^*$. Then it is obvious that $S(t)$ is also continuous in the operator norm. Since η_t is a frame wavelet, it follows that $S(t)$ (which is refereed as the *frame operator* in the literature, cf. [5]) is invertible for all t. By the continuity of the inverse operation, we have that $S(t)^{-1/2}$ must be continuous. From the polar decomposition of $U(t)$, we have that $V(t) = U(t)^*S(t)^{-1/2}$ is an isometry for each t. Therefore

$V(t)$ is a continuous path (in the operator norm) consisting of isometries. Since η_0 is Riesz wavelet and η is not, we have that $V(0)$ is unitary and but $V(1)$ is an isometry which is not unitary. This implies that $V(t)$ is a continuous path connecting a unitary and a non-unitary isometry, which contradicts with Lemma 7. $\square$

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