The Weighted Riesz-Galerkin Method for Elliptic Boundary Value Problems on Unbounded Domains

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Recently Babuška-Oh introduced the Method of Auxiliary Mapping (MAM) which efficiently handles elliptic boundary value problems containing singularities. In this paper, the Weighted Riesz-Galerkin Method (WRGM) is investigated by introducing special weight functions. Together with this method, MAM is modified to yield highly accurate finite element solutions to general elliptic boundary value problems on the exterior of bounded domains at low cost.

1. INTRODUCTION

In this paper, we introduce a new approach to deal with a general elliptic boundary value problem of the form

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial u(x)}{\partial x_i}) + b(x)u(x) = f(x)$$

on an unbounded domain Ω which is an exterior of a bounded domain.

Over the years, related to the Finite Element Method (FEM)([9],[29]), much work has been done on unbounded domain problems ([1], [9], [11], [12], [14], [15], [22],[28]). Several numerical methods for these problems were suggested. The following approaches are the most typical: truncating the unbounded part of the domain and introducing an artificial boundary condition on the resulting artificial boundary; coupling boundary element method with FEM ([19]); and using infinite elements ([6], [7], [8], [31]).

The basic idea of these approaches is to divide the given unbounded domain Ω into two parts: the bounded part $\Omega_c = \{x \in \Omega : |x| \leq c\}$ and the unbounded part $\Omega_\infty = \Omega \backslash \Omega_c$. In ([16], [17], [18], [20], [22]), under the assumption that $f(x) = 0$ on Ω_∞, artificial boundary conditions were set up for the artificial boundaries of the remaining bounded

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domain Ω_c. Thus, these approaches are not applicable unless $f(x)$ has compact support. Moreover, this is impractical if the support of $f(x)$ is very large. In ([6], [7], [8], [31]), Ω_∞ is partitioned into a finite number of infinite elements incorporated with the meshes on Ω_c. Then the special decay shape functions are constructed for those infinite elements. Thus, the implementation of this method in a FEM code leads to an alteration of the structure of the standard FEM code.

The new method introduced in this paper, similar to the infinite element approach, does not have these difficulties. First, we introduce an auxiliary mapping that can transform Ω_∞ onto a bounded domain $\hat{\Omega}_\infty$, the unit ball. Then we apply the standard FEM to $\Omega_c \cup \hat{\Omega}_\infty$ (like one point compactification of Ω for FEM).

The novelty of our method is that no artificial boundaries are created and no alteration of any existing FEM code is required for its implementation. The numerical examples show that our method is effective in handling unbounded domain problems. Moreover, the method yields highly accurate numerical solutions at low cost.

This paper is organized as follows: In section 2, a cut-off weight function, which makes the weighted Sobolev space being the usual one on the bounded sub-domain Ω_c, is introduced. Also, the Weighted Riesz-Galerkin Method (WRGM) is introduced and the existence of a solution of the related variational problem is proved. In section 3, an auxiliary mapping is constructed to deal with the unbounded subdomain Ω_∞. The new approach dealing with unbounded domain problems is described in the framework of the p-version of FEM. In section 4, the efficiency of the method is tested with various types of elliptic boundary value problems on the exterior bounded domains for the cases when supports of $f(x)$ are not bounded.

For a clearer presentation of the method, proofs of technical lemmas, used in theory development and numerical experiments, were placed in appendix. The method is also applicable to elasticity problems on unbounded domains([26]).

2. THE WEIGHTED RIESZ-GALERKIN METHOD

Throughout this paper $\Omega \subset \mathbf{R}^n$ denotes an unbounded domain which is the exterior of a closed bounded domain enclosed by a simple closed curve Γ(see, Fig. 1). $|x| = (x_1^2 + ... + x_n^2)^{\frac{1}{2}}$, $x \in \Omega$. Then $g : \Omega \longrightarrow [0, 1]$, is a smooth cut off function satisfying the following properties:

(i) $g(x) = 0$ on $\Omega_c = \{x \in \Omega : |x| < c\}$,

(ii) $g(x) = 1$ on $x \in \Omega \setminus \Omega_{c+b} = \{\Omega : |x| > c + b\}, b > 0.$

b is selected so that $|\nabla g|$ is as small as possible. For convenience of coding, the construction of a specific cut-off function is given in appendix 1.

DEFINITION 2.1. Let the space $H^k(\Omega; \mu, g)$, with $k \geq 0$ an integer, be the Banach space of all functions $u(x)$ such that

$$\|u\|^2_{H^k(\Omega;\mu,g)} = \int_\Omega e^{-2\mu g(x)|x|} \sum_{0\leq|\alpha|\leq k} [D^\alpha u(x)]^2 dx < \infty,$$

where α is the multi-index, that is, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in (\mathbf{N}_0^+)^n$, $\mathbf{N}_0^+$:= the set of non-negative integers. $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $D^\alpha = \dfrac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} ... (\partial x_n)^{\alpha_n}}$. Here μ is a nonnegative real number, which is $\ll 1$ and will be determined later. For $\mu = 0$, one gets the usual Sobolev space. In particular, $H^0(\Omega;\mu,g) = L^2(\Omega;\mu,g)$.

It is convenient to employ the operators ∇, div, and Δ. These are defined by $\nabla u = (\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n})^T$, where T means transpose, div(f) = $\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}$, where $f = (f_1, ..., f_n)^T$, and $\Delta u = \text{div}\,(\nabla u)$.

2.1. Weighted Residue Method

Let us consider a general second order elliptic boundary value problem,

$$\begin{aligned} -\sum \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial u(x)}{\partial x_i}) + b(x)u(x) &= f(x) \text{ in } \Omega, &(1)\\ u(x) &= 0 \text{ on } \Gamma, &(2) \end{aligned}$$

where $f \in L^2(\Omega;\mu,g)$, $0 < \alpha \le b(x) \le \beta$ for all $x \in \Omega$, and the coefficient matrix is bounded, symmetric and positive definite at each point $x \in \Omega$:

$$\begin{aligned} a_{ij}(x) &= a_{ji}(x), &(3)\\ \sum_{i,j=1}^n a_{ij}(x)\eta_i\eta_j &\ge \alpha\sum_{i=1}^n \eta_i^2, &(4)\\ \sum_{i,j=1}^n a_{ij}(x)\eta_j\xi_i &\le \beta\Big(\sum_{i=1}^n \xi_i^2\Big)^{1/2}\Big(\sum_{j=1}^n \eta_j^2\Big)^{1/2}, &(5) \end{aligned}$$

for all n-tuples of real numbers $(\eta_1, ..., \eta_n), (\xi_1, ..., \xi_n)$. Here the constants $\alpha > 0$ and $\beta > 0$ are independent of x.

First of all, we prove that the elliptic problem (1)-(2) is well posed. For brevity, we assume that the coefficients, a_{ij}, b, are constants, $\Omega \subset \mathbf{R}^3$, and $\partial\Omega$ is Lipschitz continuous. However, arguments for dimension two and variable coefficients are similar. In what follows, under these assumptions, the followings are proved:

(i) For each $f \in L^2(\Omega;\mu,g)$, this problem has a unique solution u;
(ii) The mapping $f \to u$ is continuous with respect to $L^2(\Omega;\mu,g)$-norm.

Let $F \in L^2(\mathbf{R}^3;\mu,g)$ be an extension of $f \in L^2(\Omega;\mu,g)$ such that

$$\|F\|_{L^2(\mathbf{R}^3;\mu,g)} \le C_1\|f\|_{L^2(\Omega;\mu,g)} \tag{6}$$

(see, Chapter 3 of [23] for details). Then the elliptic problem, $-\Delta w + w = F$ in $\mathbf{R}^3$, has a solution of the form

$$w(x) = \int_{\mathbf{R}^3} \frac{e^{-|x-\xi|}}{4\pi|x-\xi|}F(\xi)d\xi.$$

Now, the following lemma shows that the mapping $F \to w$ is continuous with respect to $L^2(\mathbf{R}^3;\mu,g)$-norm. A detailed proof of this technical lemma is shown in appendix.

LEMMA 2.1. For some constant C_2,

$$\|w\|_{L^2(\mathbf{R}^3;\mu,g)} \le C_2\|F\|_{L^2(\mathbf{R}^3;\mu,g)}. \tag{7}$$

By applying Sobolev imbedding theorem, Lemma 2.1 and the well known estimates for bounded domains, we have

$$\begin{aligned} \|w|_{\partial\Omega_1}\|_{H^{3/2}(\partial\Omega_1)} &\leq C_3\|w\|_{H^2(\Omega_1)}, &(8)\\ \|w\|_{H^2(\Omega_1)} &\leq C_4[\|F\|_{L^2(\Omega_2)}+\|w\|_{L^2(\Omega_2)}] \leq C_5\|F\|_{L^2(\mathbf{R}^3;\mu,g)}, &(9)\end{aligned}$$

where $\Omega_1 = \mathbf{R}^3\backslash\Omega$ and Ω_2 is a ball containing Ω_1. Since $\partial\Omega_1 = \partial\Omega$, by (8) and (9), one gets

$$\|w|_{\partial\Omega}\|_{H^{3/2}(\partial\Omega)} \leq C_3C_5\|F\|_{L^2(\mathbf{R}^3;\mu,g)}. \tag{10}$$

Now let v be the solution of

$$\begin{aligned} -\Delta v + v &= 0 \text{ in } \Omega,\\ v &= -w|_{\partial\Omega} \text{ on } \partial\Omega.\end{aligned}$$

Then we have

$$\|v\|_{H^2(\Omega)} \leq C_6\|w|_{\partial\Omega}\|_{H^{3/2}(\partial\Omega)}. \tag{11}$$

Let $u = v + w$. Then it follows from (6) ,(10), (11) and Lemma 2.1 that

$$\begin{aligned} \|u\|_{L^2(\Omega;\mu,g)} &\leq \|v\|_{L^2(\Omega;\mu,g)} + \|w\|_{L^2(\Omega;\mu,g)}\\ &\leq (C_3C_5C_6+C_2)\|F\|_{L^2(\mathbf{R}^3;\mu,g)}\\ &\leq (C_3C_5C_6+C_2)C_1\|f\|_{L^2(\Omega;\mu,g)}. &(12)\end{aligned}$$

Moreover, u is a solution of $-\Delta u + u = f$ in Ω and satisfies the homogeneous Dirichlet Boundary condition, which, together with (12), implies the well-posedness of (1)-(2).

Therefore, (1)-(2) has a unique solution in the weighted Sobolev space even when $f(x) = 1$. However, for practical applications, this paper is concerned with the case when f decays as $|x| \to \infty$.

Next, we define the **Weighted Residue Method** corresponding to (1)-(2) as follows: Find $u \in V$ such that, for all $v \in V$,

$$\int_\Omega \Big\{-\text{div}\Big([a_{ij}(x)]\cdot\nabla_x u(x)\Big) + b(x)u(x) - f(x)\Big\}\, v(x)e^{-2\mu g(x)|x|}dx = 0, \tag{13}$$

where $V = H_0^1(\Omega;\mu,g) = \{v \in H^1(\Omega;\mu,g) : v = 0 \text{ along } \Gamma\}$ (see, Fig. 1 for Γ).

Applying the divergence theorem, one gets

$$\begin{aligned} &- \int_\Omega \text{div}\Big([a_{ij}(x)]\cdot\nabla u(x)\Big)v(x)e^{-2\mu g(x)|x|}dx\\ &= \int_\Omega (\nabla u(x))^T\cdot[a_{ij}(x)]\cdot\nabla(v(x)e^{-2\mu g(x)|x|})dx\end{aligned}$$

$$
\begin{aligned}
&-\int_{\partial\Omega}\nu^T\cdot[a_{ij}(x)]\cdot(\nabla u(x))v(x)e^{-2\mu g(x)|x|}ds\\
=&\int_{\Omega}e^{-2\mu g(x)|x|}(\nabla u(x))^T\cdot[a_{ij}(x)]\cdot(\nabla v(x))dx\\
&+\int_{\Omega}v(x)(\nabla u(x))^T\cdot[a_{ij}(x)]\cdot(\nabla e^{-2\mu g(x)|x|})dx\\
&-\int_{\partial\Omega}\nu^T\cdot[a_{ij}(x)]\cdot(\nabla u(x))v(x)e^{-2\mu g(x)|x|}ds,
\end{aligned}
$$

where $\nu(x)=(\nu_1(x),...,\nu_n(x))^T$ is the outward normal vector to the boundary Γ. Thus, the Weighted Residue Method can be restated as follows: Find $u\in V$ such that

$$\mathcal{A}(u,v)=\mathcal{F}(v),\qquad \text{for all } v\in V, \tag{14}$$

where

$$
\begin{aligned}
\mathcal{A}(u,v) &= \int_{\Omega}\Big\{e^{-2\mu g(x)|x|}(\nabla u(x))^T\cdot[a_{ij}(x)]\cdot(\nabla v(x))\\
&+ v(x)(\nabla u(x))^T\cdot[a_{ij}(x)]\cdot(\nabla e^{-2\mu g(x)|x|})\\
&+ e^{-2\mu g(x)|x|}b(x)u(x)v(x)\Big\}\,dx, \qquad (15)\\
\mathcal{F}(v) &= \int_{\Omega}e^{-2\mu g(x)|x|}f(x)v(x)dx. \qquad (16)
\end{aligned}
$$

Since $g(x)=0$ for $|x|<c$, the variational formulation on Ω_c is the standard one. That is, (15)-(16) can be rewritten as follows:

$$\int_{\Omega_c}[(\nabla u(x))^T\cdot[a_{ij}(x)]\cdot(\nabla v(x))+b(x)u(x)v(x)]dx=\int_{\Omega_c}f(x)v(x)dx,$$

which is the standard variational formulation of (1)-(2).

The weight function is 1 when $|x|<c$ and $e^{-2\mu|x|}$ when $|x|>c+b$ respectively and hence the gradient of the weight function is independent of $\nabla g(x)$. However, the gradient vector of the weight function for $|x|\in[c,c+b]$ depends on $\nabla g(x)$ as follow:

$$
\begin{aligned}
\nabla e^{-2\mu g(x)|x|} &= -2\mu e^{-2\mu g(x)|x|}\nabla(g(x)|x|)\\
&= -2\mu e^{-2\mu g(x)|x|}(g(x)\nabla|x|+|x|\nabla g(x))\\
&= -2\mu e^{-2\mu g(x)|x|}\{h(|x|-c)\nabla|x|+|x|\frac{dh}{dt}(|x|-c)\nabla|x|\}\\
&= -2\mu e^{-2\mu g(x)|x|}\{h(|x|-c)+|x|\frac{dh}{dt}(|x|-c)\}\nabla|x|\\
&= -2\mu e^{-2\mu g(x)|x|}H(x)\nabla|x|, \qquad (17)
\end{aligned}
$$

where $H(x)=h(|x|-c)+|x|\dfrac{dh}{dt}(|x|-c)$ and $h(t)$ is defined by (A.1).

The maximum value of the smooth function $H(x)$ will play a key roll in the proof of the coercivity of the non-symmetric bilinear form (15). Thus, for simplicity of notation, we let

$$\rho(a;b;c)=\max\{h(t)+(t+c)\frac{dh(t)}{dt}:0\le t\le b\}, \tag{18}$$

where the constants a, b, c are shown in appendix 1.

Remark. (1) It is possible to make the maximum value $\rho(a;b;c)$ small by selecting the constants a, b, and c properly. However, an optimal choice of the three constants will depend on the choice of μ and will be discussed with numerical examples in section 4.

(2) The radial directional cut-off function h can be planted at any place in the outside of Ω_c by defining $g(x) = h(|x| - \bar{c}), \bar{c} > c$. The choice of $\bar{c} > c$ leads to $e^{-2\mu g(x)|x|} = 1$ for $|x| \leq \bar{c}$. Thus, the damping effect caused by the weight function may be reduced. However, this choice implies $\rho(a;b;\bar{c}) > \rho(a;b;c)$, which could yield a slower convergence (see Theorem 2.2 below). Actually, the numerical experiments show that $\bar{c} = c$ is optimal.

2.2. The Coercivity constant

In what follows, we prove that if we choose μ, ρ so that $0 \leq \mu\rho < \dfrac{\alpha}{\beta}$, then the variational problem (14) has a unique solution. Recall α and β are the positive constants given in (4)-(5).

LEMMA 2.2. For $u, v \in H^1(\Omega;\mu,g)$, we have the following:

(i) $\mathcal{A}(u,u) \geq (\alpha - \mu\beta\rho) \parallel u \parallel^2_{H^1(\Omega;\mu,g)}$

(ii) $|\mathcal{A}(u,v)| \leq [\beta + 2\mu\rho\beta] \parallel u \parallel_{H^1(\Omega;\mu,g)} \parallel v \parallel_{H^1(\Omega;\mu,g)}$

(iii) $|\mathcal{F}(v)| \leq \parallel f \parallel_{L_2(\Omega;\mu,g)} \parallel v \parallel_{H^1(\Omega;\mu,g)}$

Proof. (i) First, from the ellipticity condition (4), we have

$$\begin{aligned}
& \int_\Omega e^{-2\mu g(x)|x|} (\nabla u(x))^T \cdot [a_{ij}(x)] \cdot (\nabla u(x)) dx + \int_\Omega e^{-2\mu g(x)|x|} b(x)(u(x))^2 dx \\
\geq\ & \int_\Omega (e^{-\mu g(x)|x|} (\nabla u(x))^T \cdot [a_{ij}(x)] \cdot (e^{-\mu g(x)|x|} \nabla u(x)) dx + \alpha \|u\|^2_{L^2(\Omega;\mu,g)} \\
\geq\ & \alpha \int_\Omega [e^{-\mu g(x)|x|} \nabla u(x)] \cdot [e^{-\mu g(x)|x|} \nabla u(x)] dx + \alpha \|u\|^2_{L^2(\Omega;\mu,g)} \\
=\ & \alpha \|u\|^2_{H^1(\Omega;\mu,g)}.
\end{aligned} \tag{19}$$

Second, by (5), (17), and Hölder's inequality, we obtain the following inequality.

$$\begin{aligned}
& \Big| \int_\Omega u(x)(\nabla u(x))^T \cdot [a_{ij}] \cdot (\nabla e^{-2\mu g(x)|x|}) dx \Big| \\
=\ & \Big| \int_\Omega -2\mu H(x) u(x) e^{-2\mu g(x)|x|} \Big((\nabla u(x))^T \cdot [a_{ij}] \cdot (\nabla |x|) \Big) dx \Big| \\
\leq\ & 2\mu\rho \int_\Omega \Big| \sum_{i,j=1}^n a_{ij} \left(\frac{\partial u}{\partial x_i} u(x) e^{-2\mu g(x)|x|} \right) \left(\frac{x_j}{|x|} \right) \Big| dx \\
\leq\ & 2\mu\beta\rho \int_\Omega |u(x)| e^{-\mu g(x)|x|} \left\{ \sum_{i=1}^n \left(\frac{\partial u(x)}{\partial x_i} e^{-\mu g(x)|x|} \right)^2 \right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^n \left(\frac{x_i}{|x|} \right)^2 \right\}^{\frac{1}{2}} dx \\
\leq\ & 2\mu\beta\rho \parallel u(x) \parallel_{L^2(\Omega;\mu,g)} |u(x)|_{H^1(\Omega;\mu,g)} \\
\leq\ & \mu\beta\rho \parallel u \parallel^2_{H^1(\Omega;\mu,g)}\ .
\end{aligned} \tag{20}$$

Now, from (15), (19), and (20), we have

$$\begin{aligned}\mathcal{A}(u,u) &\geq \alpha \parallel u \parallel^2_{H^1(\Omega;\mu,g)} - \mu\beta\rho \parallel u \parallel^2_{H^1(\Omega;\mu,g)} \\ &\geq (\alpha - \mu\beta\rho) \parallel u \parallel^2_{H^1(\Omega;\mu,g)} .\end{aligned}$$

(ii) Let

$$\mathcal{B}(u,v) = \int_\Omega e^{-2\mu g(x)|x|}\{\nabla u(x)\cdot[a_{ij}(x)]\cdot(\nabla v(x))^T + b(x)u(x)v(x)\}dx.$$

Then, by the Cauchy-Schwarz inequality

$$\begin{aligned}|\mathcal{B}(u,v)| &\leq \mathcal{B}(u,u)^{\frac{1}{2}}\mathcal{B}(v,v)^{\frac{1}{2}} \\ &\leq \Big[\int_\Omega e^{-2\mu g(x)|x|}(\beta\nabla u(x)\cdot\nabla u(x) + \beta u(x)^2)dx\Big]^{\frac{1}{2}} \\ &\quad \Big[\int_\Omega e^{-2\mu g(x)|x|}(\beta\nabla v(x)\cdot\nabla v(x) + \beta v(x)^2)dx\Big]^{\frac{1}{2}} \\ &\leq \beta \parallel u \parallel_{H^1(\Omega;\mu,g)}\parallel v \parallel_{H^1(\Omega;\mu,g)} .\end{aligned}$$

By an argument similar to (20), we have

$$\begin{aligned}&|\int_\Omega v(x)(\nabla u(x))^T\cdot[a_{ij}(x)]\cdot(\nabla e^{-2\mu g(x)|x|})dx| \\ \leq\ & 2\mu\rho\beta\Big[\int_\Omega(|v(x)|e^{-\mu g(x)|x|})^2dx\Big]^{\frac{1}{2}}\Big[\int_\Omega\sum_{i=1}^n\left(\frac{\partial u(x)}{\partial x_i}e^{-\mu g(x)|x|}\right)^2dx\Big]^{\frac{1}{2}} \\ \leq\ & 2\mu\rho\beta \parallel v \parallel_{H^1(\Omega;\mu,g)}\parallel u \parallel_{H^1(\Omega;\mu,g)} .\end{aligned}$$

Thus, we have

$$|\mathcal{A}(u,v)| \leq (\beta + 2\mu\rho\beta) \parallel v \parallel_{H^1(\Omega;\mu,g)}\parallel u \parallel_{H^1(\Omega;\mu,g)}$$

(iii) Finally, we have

$$\begin{aligned}|\mathcal{F}(v)| &\leq \int_\Omega |e^{-\mu g(x)|x|}f(x)||e^{-\mu g(x)|x|}v(x)|dx \\ &\leq \parallel f \parallel_{L_2(\Omega;\mu,g)}\parallel v \parallel_{H^1(\Omega;\mu,g)} \qquad \text{(Hölder inequality).}\end{aligned}$$

■

Now, by applying Lemma 2.2 to the Lax-Milgram Theorem([9]), we have the following existence and uniqueness of the solution of (14).

THEOREM 2.1. Suppose the constants μ, a, b, c are properly selected so that $0 \leq \mu\rho < \frac{\alpha}{\beta}$. Then the variational problem (14) has a unique solution $u_{ex}(x)$ in $H^1_0(\Omega;\mu,g)$.

Next, the Weighted Riesz-Galerkin (approximation) Method(WRGM) of the variational problem (14) is the following: Given a finite dimensional subspace $S \subset V$, find $u_p(x) \in S$ such that

$$\mathcal{A}(u_p,v) = \mathcal{F}(v),\ \text{ for all } v \in S, \tag{21}$$

where $\mathcal{A}(.,.)$ and $\mathcal{F}(.)$ are the bilinear form and the linear functional in (15) and (16), respectively.

By applying Céa's Lemma ([9]), we have the following.

THEOREM 2.2. Under the hypotheses of Theorem 2.1, the discrete problem (21) has a unique solution. Moreover, we have

$$\|u_{ex} - u_p\|_{H^1(\Omega;\mu,g)} \leq \frac{\beta(1+2\mu\rho)}{\alpha - \rho\mu\beta} \parallel u_{ex} - v \parallel_{H^1(\Omega;\mu,g)}, \text{ for all } v \in S.$$

3. THE METHOD OF AUXILIARY MAPPING

In this section, the Method of Auxiliary Mapping, introduced by Babuška and Oh ([2],[21], [24], [25]), is modified so that it can handle unbounded domain problems. The essential components of the method are as follows:

i. The given unbounded domain is divided into two regions; a bounded region Ω_c and an unbounded region $\Omega_\infty = \Omega \setminus \bar{\Omega}_c$.

ii. The unbounded region Ω_∞ is mapped to another bounded region $\hat{\Omega}_\infty$ (a unit ball) by an auxiliary mapping φ.

iii. The standard FEM is applied to the bounded region, $\hat{\Omega}_\infty \cup \Omega_c$ which has no artificial boundaries. In triangulating $\hat{\Omega}_\infty \cup \Omega_c$ for the Finite Element mesh, the nodes along $\Omega_c \cap \Omega_\infty$ and the corresponding nodes on $\hat{\Omega}_\infty$ share the same node numbers. Thus, two regions are treated as one connected domain in the computation process. Moreover, the transformed bilinear form $\hat{\mathcal{A}}(\cdot,\cdot)$, given in Lemma 3.2, is used for computing local stiffness matrices of the elements in the transformed domain $\hat{\Omega}_\infty$.

The novelty of our method is to obtain an accurate numerical solution without introducing any artificial boundaries.

3.1. The Auxiliary Mapping

Let S and $\hat{S}$ be subsets of $\mathbf{R}^n$. Let $\varphi : S \rightarrow \hat{S}$ be a bijective transformation. The Jacobian of φ is denoted by $J(\varphi)$ and $|J(\varphi)|$ denotes its determinant. φ will be selected so that $|J(\varphi)| > 0$ for all $x \in S$.

The shift of a function $u : S \rightarrow \mathbf{R}$ onto $\hat{S}$, through the bijective mapping φ, is denoted by $\hat{u} := u \circ \varphi^{-1}$. Similarly, we denote $J_{ij}(\varphi) \circ \varphi^{-1}$ by $\widehat{J_{ij}(\varphi)}$, where $J_{ij}(\varphi)$ is the (i, j) component of the Jacobian of φ.

Let $(x_1, x_2, ..., x_n)$ and $(\xi_1, \xi_2, ..., \xi_n)$ denote the coordinates of $x \in S$ and $\xi = \varphi(x) \in \hat{S}$, respectively. Let

$$\nabla_\xi = (\frac{\partial}{\partial \xi_1}, ..., \frac{\partial}{\partial \xi_n})^T \quad , \qquad \nabla_x = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})^T,$$
$$|\xi| = (\xi_1^2 + ... + \xi_n^2)^{\frac{1}{2}} \quad , \qquad |x| = (x_1^2 + ... + x_n^2)^{\frac{1}{2}}.$$

Then, by applying the chain rule to $u(x) = \hat{u}(\varphi(x))$, we have

$$(\nabla_x u(x)) \circ \varphi^{-1}(\xi) = \left[J(\varphi) \cdot (\nabla_\xi \hat{u} \circ \varphi(x))\right] \circ \varphi^{-1}(\xi) = \widehat{J(\varphi)} \cdot \nabla_\xi \hat{u}(\xi). \qquad (22)$$

Thus, the following change of variables is obtained:

$$\int_S (\nabla_x u)^T \cdot [a_{ij}(x)] \cdot (\nabla_x v) dx = \int_{\hat{S}} |J(\varphi^{-1})| (\nabla_\xi \hat{u})^T \widehat{J(\varphi)}^T \cdot [\hat{a}_{ij}(\xi)] \cdot \widehat{J(\varphi)} (\nabla_\xi \hat{v}) d\xi,$$

which implies the following lemma.

LEMMA 3.1. Let $[q_{ij}^{(n)}(\xi)] = |J(\varphi^{-1})| \widehat{J(\varphi)}^T \cdot [\hat{a}_{ij}(\xi)] \cdot \widehat{J(\varphi)}$ and suppose u and v are in $H^1(S;\mu,g)$. Then

$$\int_S e^{-2\mu g(x)|x|} (\nabla_x u)^T \cdot [a_{ij}(x)] \cdot (\nabla_x v) dx = \int_{\hat{S}} e^{-2\mu \hat{g}(\xi)|\varphi^{-1}(\xi)|} (\nabla_\xi \hat{u})^T \cdot [q_{ij}^{(n)}(\xi)] \cdot (\nabla_\xi \hat{v}) d\xi,$$

where $\hat{g}(\xi)|\varphi^{-1}(\xi)| = (g \circ \varphi^{-1})(\xi)(\varphi_1^{-1}(\xi)^2 + ... + \varphi_n^{-1}(\xi)^2)^{1/2}$, and φ_i^{-1} is the ith component function of φ^{-1}.

Let $\hat{\Omega}_\infty = \{\xi \in \mathbf{R}^n : |\xi| \leq 1\}$ and $\Omega_\infty = \Omega \cap \{x \in \mathbf{R}^n : |x| \geq c\}$. Then the auxiliary mappings $\varphi_{(n)} : \Omega_\infty \to \hat{\Omega}_\infty$ is defined by

$$\varphi_{(n)}(x) = \frac{c(x_1, x_2, ..., x_{n-1}, -x_n)}{|x|^2}, \qquad c > 0. \tag{23}$$

By using Lemma 3.1, we derive the change of variables, by the auxiliary mapping $\varphi_{(n)}$, that gives the transformed bilinear form $\hat{\mathcal{A}}(\cdot,\cdot)$ and the transformed linear functional $\hat{\mathcal{F}}$ on $\hat{\Omega}_\infty$ corresponding to (15)-(16). The proof of this lemma can be found in appendix 3.

LEMMA 3.2. Let $|J(\varphi_{(n)}^{-1})| \widehat{J(\varphi_{(n)})}^T \cdot [\hat{a}_{ij}(\xi)] \cdot \widehat{J(\varphi_{(n)})} = [q_{ij}^{(n)}]$. Then, for $u, v \in H^1(\Omega;\mu,g)$, we have $\mathcal{A}(u,v)|_{\Omega_\infty} = \hat{\mathcal{A}}(\hat{u},\hat{v})|_{\hat{\Omega}_\infty}$ and $\mathcal{F}(v)|_{\Omega_\infty} = \hat{\mathcal{F}}(\hat{v})|_{\hat{\Omega}_\infty}$, where

$$\begin{aligned} \mathcal{A}(u,v)|_{\Omega_\infty} &:= \int_{\Omega_\infty} \Big\{ e^{-2\mu g(x)|x|} (\nabla_x u(x))^T \cdot [a_{ij}(x)] \cdot (\nabla_x v(x)) \\ &+ \; v(x) (\nabla_x u(x))^T \cdot [a_{ij}(x)] \cdot \Big(\nabla_x e^{-2\mu g(x)|x|} \Big) \Big\} dx, \end{aligned} \tag{24}$$

$$\begin{aligned} \hat{\mathcal{A}}(\hat{u},\hat{v})|_{\hat{\Omega}_\infty} &:= \int_{\hat{\Omega}_\infty} e^{-2\mu \hat{g}(\xi) c/|\xi|} \\ &\Big\{ (\nabla_\xi \hat{u})^T \cdot [q_{ij}^{(n)}] \cdot (\nabla_\xi \hat{v}) + \hat{v}(\xi) W(\xi) (\nabla_\xi \hat{u})^T \cdot [q_{ij}^{(n)}] \cdot (\xi) \Big\} d\xi \end{aligned} \tag{25}$$

$$\mathcal{F}(v)|_{\Omega_\infty} := \int_{\Omega_\infty} e^{-2\mu g(x)|x|} f(x) v(x) dx, \tag{26}$$

$$\hat{\mathcal{F}}(\hat{v})|_{\hat{\Omega}_\infty} := \int_{\hat{\Omega}_\infty} |J(\varphi_\infty)| e^{-2\mu \hat{g}(\xi) c/|\xi|} \hat{f}(\xi) \hat{v}(\xi) d\xi. \tag{27}$$

Here, $W(\xi) = \dfrac{2\mu c}{|\xi|^4} \Big[c \dfrac{dh}{dt} (\dfrac{c}{|\xi|} - c) + \hat{g}(\xi)|\xi| \Big]$ and $h(t)$ is defined by (A.1).

For the implementation purpose, the transformed coefficients $q_{ij}^{(n)}$ of this Lemma are computed in a specific form in appendix 4.

3.2. Constructions of Finite Element Spaces

For brevity, in what follows, we develop our method in case of dimension two. The three dimensional case will be dealt in ([27]).

First of all, let us select a positive constant c and divide Ω into two parts; Ω_c and Ω_∞, where

$$\Omega_c = \{x \in \Omega : |x| \le c\}; \qquad \Omega_\infty = \{x \in \Omega : |x| \ge c\}.$$

The constant c is determined by the size of the bounded region on which an accurate numerical solution is desirable. Unlike the conventional methods, an accuracy of this method is practically independent of the size of c.

3.2A: Mesh Generation. Suppose $\Delta_c = \{E_k : k = 1, 2, ..., N(\Delta_c)\}$ represents a specific mesh on Ω_c. Suppose $z_1, z_2, ..., z_d$ be all those nodes in the mesh Δ_c which lie on the circle $\Omega_c \cap \Omega_\infty$ (Fig. 1) and their polar coordinates are $(c, \theta_1), ..., (c, \theta_d)$. Now divide Ω_∞ into infinite triangular elements $T^\infty_{N(\Delta_c)+j}, j = 1, ..., N(\Delta_\infty)$, by the rays connecting the origin and all of nodes $z_1, ..., z_d$ as shown in Fig. 1. The specific mesh on Ω_∞ constructed in this way is denoted by Δ_∞. Then $\Delta = \Delta_c \cup \Delta_\infty$ is a specific mesh for the unbounded domain Ω.

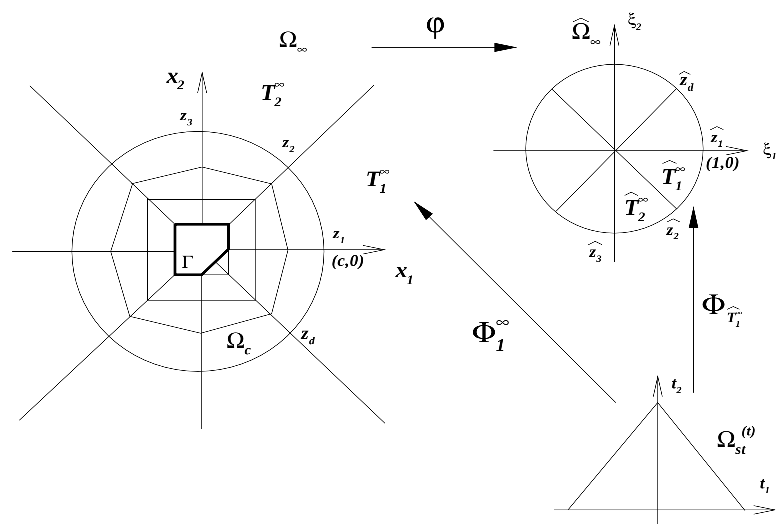

FIG. 1. The Scheme of Triangulation $\Delta := \Delta_c \cup \Delta_\infty$, the infinite domain Ω_∞, the mapped finite domain $\hat{\Omega}_\infty$, and the singular elemental mapping Φ_1^∞.

3.2B: Construction of Elemental Mappings. For sake of the convenience of notations in elemental mappings and the implementation of the new method, we introduce a fictitious node z_∞ (which is an extra point in one point compactification, $\Omega^* = \Omega \cup \{z_\infty\}$, of the infinite domain Ω). Since it was assumed that the source function $f(x)$ and the solution $u(x)$ decay, it is natural to assume that all of the functions we consider are zero at z_∞. The construction of a finite element space of this section is similar to constructing a finite element space on the one point compactification, $\Omega^* = \Omega \cup \{z_\infty\}$, of Ω by imposing a nodal constraint at z_∞. However, the compactification of Ω will not be constructed. In what follows, it is implemented and interpreted in the context of a conventional finite element space.

By the inversion mapping φ defined by (23), the nodes z_k on $\Omega_c \cap \Omega_\infty$ are mapped to nodes on the unit circle, $\hat{z}_k = (1, -\theta_k), k = 1, ..., d$, and we define $\varphi(z_\infty) := (0, 0)$.

An infinite triangular element $T_n^\infty := (z_{n+1} \to z_n \to z_\infty)$ is mapped to a finite triangular element $\hat{T}_n^\infty := (\hat{z}_{n+1} \to \hat{z}_n \to (0, 0))$, whose first side is a circular arc (see, T_1^∞ and $\hat{T}_1^\infty$ of Fig. 1). Suppose $\Phi_{\hat{T}_n^\infty}$ is the blending type elemental mapping (see chapter 6 of [30]) from the reference element $\Omega_{st}^{(t)}$ onto a curved triangular element $\hat{T}_n^\infty$, then

$$\Phi_n^\infty = \varphi^{-1} \circ \Phi_{\hat{T}_n^\infty} : \Omega_{st}^{(t)} \longrightarrow T_n^\infty \tag{28}$$

is called a **singular elemental mapping**.

Let $\mathcal{M}$ be the vector of elemental mappings assigned to the elements in $\Delta = \Delta_c \cup \Delta_\infty$ by the following rule:

- Assign the conventional elemental mappings Φ_n to the elements $E_n \subset \Omega_c$;
- Assign the singular elemental mappings Φ_n^∞ defined by (28) to the elements $T_n^\infty \subset \Omega_\infty$.

3.2C: Degree of Basis Functions. Suppose $\Omega_{st}^{(*)}$ represents either the reference triangular element $\Omega_{st}^{(t)}$ or the reference quadrilateral element $\Omega_{st}^{(q)}$ and let $\mathcal{P}_p(\Omega_{st}^{(*)})$ be the space of polynomials of degree $\leq p$ defined on $\Omega_{st}^{(*)}$.

Then **the finite element space**, denoted by $S^p(\Omega, \Delta, \mathcal{M})$, is the set of all functions u defined on Ω such that

- $u \circ \Phi_n \in \mathcal{P}_p(\Omega_{st}^{(*)})$ for each element $E_n \in \Delta_c$;
- $u \circ \Phi_n^\infty \in \mathcal{P}_p(\Omega_{st}^{(t)})$ for each element $T_n^\infty \in \Delta_\infty$.

Let us note that each member of $S^p(\Omega, \Delta, \mathcal{M})$ is in $H^1(\Omega, \mu, g)$ except the nodal basis function corresponding to the fictitious node z_∞. However, because of the zero-nodal-constraint at z_∞, we may claim that $S^p(\Omega, \Delta, \mathcal{M}) \subset H^1(\Omega; \mu, g)$.

Since the singular elemental mappings Φ_n^∞ are designed to agree with the conventional elemental mappings along the common sides on $\Omega_c \cap \Omega_\infty$, $S^p(\Omega, \Delta, \mathcal{M})$ is **" exactly conforming"** ([30]). In other words, each member of this finite element space is continuous in order to ensure good approximation properties.

3.2D: The p-Version of FEM. As usual, the finite element solution u_{fe} is the projection of the exact solution into $S^p(\Omega, \Delta, \mathcal{M})$, with respect to a proper inner product. The dimension of the vector space $S^p(\Omega, \Delta, \mathcal{M})$ is called the **Number of Degrees of Freedom (DOF).**

In **the p-Version of the Finite Element Method**([3],[4],[30]), to obtain the desired accuracy, the mesh Δ of the domain Ω is fixed and only the degree p of the basis polynomials is increased.

3.2E. Computation of local stiffness matrices and local load vectors for infinite elements T^{∞}. In the master element approach, the computations of local stiffness matrices and local load vectors for infinite triangular elements T^{∞} in $\Omega^*_{\infty} = \Omega \cup \{z_{\infty}\}$ can not be computed by a conventional FE code because infinite elements are not standard finite elements.

However, this problem is circumvented by computing the transformed bilinear form $\hat{\mathcal{A}}|_{\hat{\Omega}^*_{\infty}}(\cdot,\cdot)$ and the transformed linear functional $\hat{\mathcal{F}}|_{\hat{\Omega}^*_{\infty}}(\cdot)$ given in Lemma 3.2, on the mapped triangular elements $\hat{T}^{\infty}$ together with the elemental mapping

$$\hat{\Phi}_{\hat{T}^{\infty}} : \Omega^{(t)}_{st} \longrightarrow \hat{T}^{\infty},$$

where $\hat{\Phi}_{\hat{T}^{\infty}} = \varphi \circ \Phi_{T^{\infty}}$ is an elemental mapping of blending type and $\Phi_{T^{\infty}}$ is defined by (28).

In other words, the local stiffness matrices and local load vectors are computed by the following rule:

- Use $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{F}(\cdot,\cdot)$ for the elements E in the bounded subdomain Ω_c.
- Use $\hat{\mathcal{A}}|_{\hat{\Omega}^*_{\infty}}(\cdot,\cdot)$ and $\hat{\mathcal{F}}|_{\hat{\Omega}^*_{\infty}}(\cdot)$ on the elements $\hat{T}_{\infty}$ in $\hat{\Omega}^*_{\infty}$, for the infinite tetrahedral elements T_{∞} in Ω^*_{∞}.

On Ω_c, this method uses the conventional FEM incorporated with the standard bilinear form. Thus, stiffness matrices and load vectors for the elements E in Ω_c can be computed by any existing finite element code without alteration. However, for those infinite tetrahedral elements, the transformed non-symmetric bilinear form $\hat{\mathcal{A}}(\cdot,\cdot)$ and the transformed linear functional $\hat{\mathcal{F}}$ of Lemma 3.2, is employed. Thus, this method can also be implemented on any existing finite element code. The only draw back of the method is that the local stiffness matrices corresponding to infinite triangular elements T^{∞} are non-symmetric.

In summary, our method is to apply a conventional p-version of FEM on the bounded domain $\Omega_c \cup_{\varphi_{\infty}} \hat{\Omega}^*_{\infty}$ with the two phase bilinear form $\mathcal{A}|_{\Omega_c}(\cdot,\cdot) \cup_{\varphi_{\infty}} \hat{\mathcal{A}}|_{\hat{\Omega}^*_{\infty}}(\cdot,\cdot)$. and the two phase linear functional $\mathcal{F}|_{\Omega_c}(\cdot) \cup_{\varphi_{\infty}} \hat{\mathcal{F}}|_{\hat{\Omega}^*_{\infty}}(\cdot)$.

4. NUMERICAL RESULTS

In this section, we make some comparisons between our method and domain truncating method. Various numerical examples are given in order to demonstrate an effectiveness of our method. Throughout this section, Ω denotes the unbounded domain in Fig. 2.

$\mathcal{U}(u) = \mathcal{A}(u,u)/2$ is the strain energy of u and $\|u\|_E = \sqrt{\mathcal{U}(u)}$ is the energy norm of u. Since $g(x) = 0$ on Ω_c, $\mathcal{A}(u,u)$ has the weight $e^{-\mu g(x)|x|} = 1$ on Ω_c. It is known ([24],[30]) that $\|u_{fe} - u_{ex}\|^2_E = |\mathcal{U}(u_{fe}) - \mathcal{U}(u_{ex})|$, provided that all boundary conditions are either homogeneous Dirichlet or arbitrary traction boundary conditions. In this section, by the **Relative Error(%) in Energy Norm**, we mean

$$100 \cdot \left[\frac{|\{\mathcal{U}(u_{ex}|_{\Omega_c}) - \mathcal{U}(u_{fe}|_{\Omega_c})\}|}{\mathcal{U}(u_{ex}|_{\Omega_c})} \right]^{1/2}. \tag{29}$$

EXAMPLE 4.1. Let $u(x_1,x_2) = |x|^{-1}, |x| = \sqrt{x_1^2 + x_2^2}$. Then u solves

$$-\triangle u(x) + u(x) = f(x) \quad \text{in } \Omega \tag{30}$$

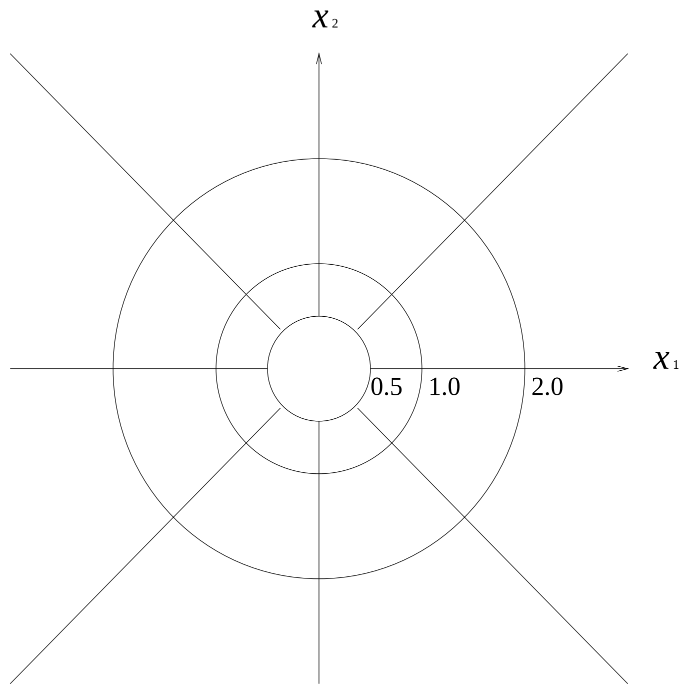

FIG. 2. The scheme of unbounded domain Ω and Triangulation of Ω

$$\frac{\partial u(x)}{\partial n} \;=\; g_N(x) \qquad \text{on } \Gamma$$

where $f(x) = -|x|^{-3} + |x|^{-1}$, and $g_N(x) = |x|^{-2}$.

f is not square integrable, however $\int_\Omega |f|^2 e^{-2g(x)|x|} dx < \infty$ and hence $f \in L_2(\Omega; \mu, g)$.

Theorem 2.2 shows that the rate of convergence of our method depends on those constants μ, ρ. Since $\alpha = \beta = 1$ for this example, ρ and μ can be selected freely so that the stability constant $\alpha - \rho\mu\beta = 1 - \rho\mu \approx 1$.

Throughout this section, $a = b = 8$ for the cut off function constructed in appendix. The damping parameter b relaxes the gradient of the cut-off function defined by (A.1).

From this example, we have the followings:

(i) From Theorem 2.2, we prefer to choose μ as small as possible. Actually,Table 1 shows that the relative errors in energy norm is getting smaller as μ is smaller and smaller.

(ii) However, if μ is small, the damping weight function $e^{-2\mu\hat{g}(\xi)c/|\xi|}$ of $\hat{\mathcal{A}}(\cdot,\cdot)$ and $\hat{\mathcal{F}}(\cdot,\cdot)$ of Lemma 3.2 becomes large. Hence, as one can see from the last column of Table 1, the overall accuracy deteriorates when μ becomes too small.

(iii) It is also desirable to make ρ as small as possible. However, ρ can not be smaller than 2.0. In this section, the cutoff function $g(x)$ is selected so that $\rho \leq 2.8$.

(iv) Theorem 2.2 states that the convergence is slow for a larger constant μ. Numerical results of Table 1 support the theory.

The relative errors in Table 1 are depicted in Fig. 3 in log-log scale. Let us note that the domain truncating method is not applicable to this problem because the support of $f(x)$ is unbounded.

TABLE 1
The Relative Errors in Energy Norm (%) of FE solutions of $-\Delta u + u = f$ by WRGM when the true solution is $u_{ex} = |x|^{-1}$.

p-deg	DOF	$\mu = 0.5$	$\mu = 0.1$	$\mu = 0.04$	$\mu = 0.01$	$\mu = 0.004$	$\mu = 0.001$
1	16	57.01	55.02	53.36	50.95	49.67	48.22
2	48	24.13	24.08	24.01	23.90	23.84	23.78
3	80	7.90	7.72	7.66	7.62	7.60	7.59
4	128	2.62	2.19	2.15	2.11	2.09	2.10
5	192	1.71	0.82	0.72	0.68	0.69	0.74
6	272	1.65	0.60	0.41	0.31	0.30	0.29
7	368	1.72	0.66	0.38	0.19	0.05	0.19
8	480	1.30	0.64	0.38	0.17	0.03	0.16

The second example is to show the convergence rates with respect to various choices of μ when the true solution decays much more slowly.

EXAMPLE 4.2. Let $u(x_1, x_2) = |x|^{-1/2}, |x| = \sqrt{x_1^2 + x_2^2}$. Then u solves

$$\begin{aligned} -\triangle u(x) + u(x) &= f(x) \quad \text{in } \Omega, \\ \frac{\partial u(x)}{\partial n} &= g_N(x) \quad \text{on } \Gamma, \end{aligned} \tag{31}$$

where $f(x) = -0.25|x|^{-2.5} + |x|^{-1/2}$, and $g_N(x) = 0.5|x|^{-3/2}$.

Since the true solution of this example decays much slower than that of the first example, the weighted norm of u is larger than the previous one. Hence, as one can see from Table 2, the convergence of FE solutions are slower than the first one.

TABLE 2
The Relative Errors in Energy Norm (%) of FE solutions of $-\Delta u + u = f$ by WRGM, when the true solution is $u_{ex} = |x|^{-1/2}$.

p-deg	DOF	$\mu = 0.1$	$\mu = 0.07$	$\mu = 0.05$	$\mu = 0.01$	$\mu = 0.001$
1	16	47.09	54.94	63.76	119.31	282.64
2	48	13.8	17.16	21.81	44.68	52.66
3	80	5.66	7.48	9.36	16.93	26.69
4	128	2.02	2.77	4.12	13.25	34.81
5	192	0.34	1.29	2.34	7.31	15.89
6	272	0.95	0.81	0.63	3.26	9.67
7	368	0.56	0.21	0.29	3.42	7.76
8	480	0.55	0.33	0.38	1.82	2.70

The third example is to apply our method to general elliptic partial differential equation.

EXAMPLE 4.3. Let $u(x_1, x_2) = \dfrac{x_1}{x_1^2 + x_2^2} (= \dfrac{\cos\theta}{r})$. Then u solves

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u(x)}{\partial x_i}) + u(x) = f(x) \text{ in } \Omega$$

TABLE 3

The Relative Errors in Energy Norm (%) of the FE solutions of $-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i})+ u = f$ obtained by WRGM, when the true solution is $u_{ex} = \cos\theta/|x|$.

p-deg	DOF	$\mu = 0.1$	$\mu = 0.05$	$\mu = 0.01$	$\mu = 0.001$	$\mu = 0.0001$
1	16	11.85	11.96	12.24	12.55	12.55
2	56	6.73	6.73	6.73	6.73	6.73
3	104	0.69	0.69	0.69	0.69	0.69
4	176	0.36	0.36	0.35	0.35	0.35
5	272	0.073	0.069	0.070	0.071	0.071
6	392	0.030	0.022	0.012	0.0091	0.0089
7	536	0.035	0.021	0.0059	0.0019	0.0017
8	704	0.041	0.020	0.0052	0.00072	0.00031

$$\sum_{i,j=1}^{2} a_{ij}\nu_j \frac{\partial u}{\partial x_i} \;=\; g_N(x) \text{ on } \Gamma,$$

where

$$\begin{aligned}
a_{11} &= 5, a_{12} = a_{21} = 1, a_{22} = 3,\\
f(x) &= -\frac{4x_1^3 - 4x_2^3 - 12x_1x_2^2 + 12x_1^2x_2}{(x_1^2+x_2^2)^3} + \frac{x_1}{(x_1^2+x_2^2)}\\
g_N(x) &= [(5\cos\theta + \sin\theta)\cos 2\theta + (\cos\theta + 3\sin\theta)\sin 2\theta]/r^2,\\
(\nu_1,\nu_2) &= -(\cos\theta, \sin\theta).
\end{aligned}$$

The eigenvalues of the coefficient matrix are $4 \pm \sqrt{2}$. Hence, $\alpha = \min\{1, 4-\sqrt{2}\}$ and $\beta = 4+\sqrt{2}$. If $a = 8.0$, $b = 8.0$, $c = 2.0$, then $\rho = \max\{|\lambda(t)| : 0 \le t \le b, \lambda(t) = h(t) + (t+c)\frac{dh}{dt}(t)\}$, is ≤ 2.8. Thus, the coercivity constant becomes

$$\alpha - \rho\mu\beta = 1 - (2.8)(\mu)(4+\sqrt{2}) \ge 0.9985$$

when $\mu = 0.0001$. Thus, the upper bound $\dfrac{\beta(1+2\mu\rho)}{\alpha - \rho\mu\beta}$ of the error stated in Theorem 2.2 is 5.425. Therefore, the convergence is expected to be fast as it is shown in Table 3.

Remark. From these three examples, we have the following conclusion about the dependence of accuracy on the size of μ: *if the right side f decays fast (examples 4.1 and 4.3), a very small damping parameter μ can be selected to have a highly accurate solution. However, if the right side function f decays slowly (example 4.2), the choice of small μ does not yield the best solution.* It is because the norm of the true solution becomes very large when μ is small and f decays slowly. Actually, no weight is necessary if f has a compact support. In this case, our method yields highly accurate finite element solutions without introducing the weight functions.

The last example is different from our model problem. Now the Laplace equation, $-\Delta u = 0$, is considered so that our method can be compared with the domain truncating method in which the sommerfeld boundary condition is imposed along the artificial boundary Γ_A. It was shown in section 5 of ([2]) that if $|f|^2|x|^4$ is integrable, μ is allowed to be zero. In the following example, no weight functions are used.

TABLE 4
The Relative Errors in Energy norm (%) of the FE solutions of $\Delta u = 0$ obtained by WRGM ("Mapping") as well as by the Domain Truncation Method ("Domain-Cut"), when the true solution is $u_{ex} = 2\cos 2\theta/|x|^2$.

	c = 2			*c = 4*		
p-deg	DOF	Mapping	Domain-Cut	DOF	Mapping	Domain-Cut
1	16	50.40	50.22	32	50.58	50.58
2	48	15.89	13.80	88	15.94	15.82
3	80	7.03	3.90	144	7.06	6.76
4	128	2.30	7.74	224	2.30	1.12
5	192	0.64	8.06	328	0.64	1.91
6	272	0.15	8.08	456	0.16	2.01
7	368	0.02	8.08	608	0.04	2.01
8	480	0.03	8.08	784	0.01	2.01

EXAMPLE 4.4. Let $u(x_1, x_2) = 2\cos 2\theta/r^2$. Then u solves

$$\Delta u = 0 \text{ in } \Omega, \tag{32}$$

$$\frac{\partial u}{\partial n} = \frac{4\cos 2\theta}{r^3} \text{ along } \Gamma. \tag{33}$$

Let $\Omega_\infty = \Omega \backslash \{(r,\theta) : r \le c\}$ and let $\partial\overline{\Omega}_\infty = \Gamma_A$. Then the Domain Truncating Method is solving (32)-(33) on Ω_c by imposing the following Sommerfeld boundary condition along the artificial boundary Γ_A.

$$\frac{\partial u}{\partial r} + \frac{1}{r}u = 0 \text{ along } \Gamma_A. \tag{34}$$

In this example, we consider the case when c is either 2 or 4. Since $\dfrac{\partial u}{\partial r} + \dfrac{1}{r}u = \dfrac{2\cos 2\theta}{r^3}$, imposing sommerfeld-type artificial boundary condition (34) assumes a large amount of error when c is small. As it is shown in Table 2 and Fig. 4, the error is decreasing as r is getting bigger, which implies a longer computing time. However, our method is virtually independent on the size of the bounded domain Ω_c. Table 4 shows that the errors corresponding to $c = 2$ are about the same as the errors corresponding to $c = 4$.

APPENDIX

In this section, a cut-off function $g(x)$ used in this paper is constructed. Next, we prove Lemma 2.1 and Lemma 3.2. Finally, the coefficients of the transformed bilinear form are given in a specific form for the implementation of the proposed method.

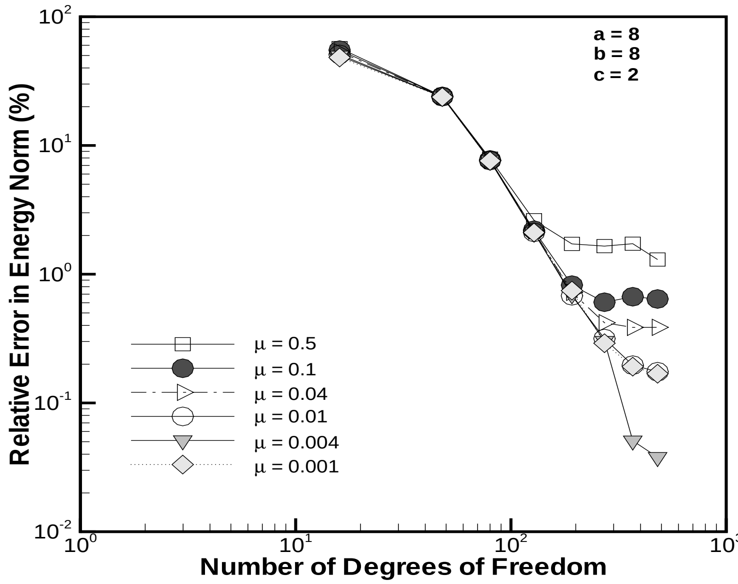

FIG. 3. The Relative Errors in Energy Norm % of FE solutions of $-\Delta u + u = f$, obtained by WRGM with various weights, when the true solution is $u_{ex} = 1/|x|$.

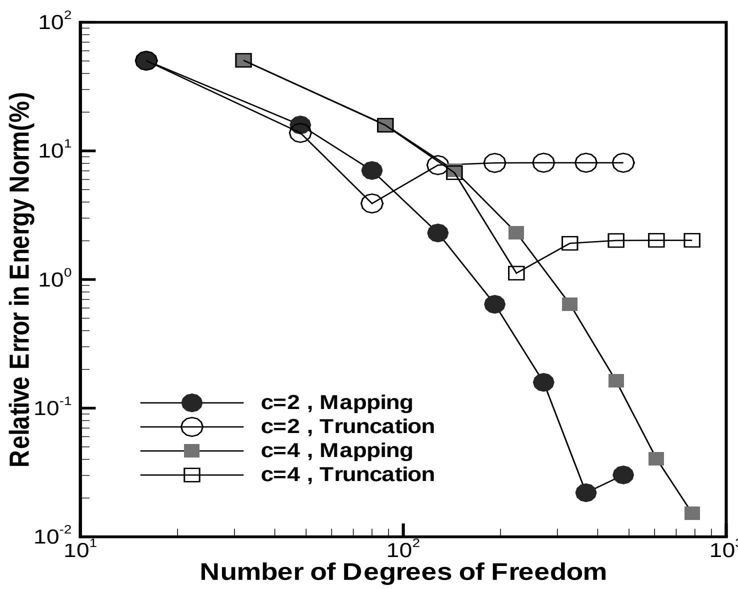

FIG. 4. The Relative Error in Energy Norm (%) of FE solutions of $\Delta u = 0$, obtained by WRGM (“Mapping") and obtained by Domain Truncation (“Truncation").

A.1. CONSTRUCTION OF CUT-OFF FUNCTION.

Let $q(t)$ be defined by

$$q(t) = \begin{cases} e^{-a/t^2} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

where a is a positive constant. Then the function $q(t)$ is smooth. Let

$$h(t) = \frac{q(t)}{(q(t) + q(b-t))}. \tag{A.1}$$

Then $h(t)$ is also smooth, and $0 \leq h(t) \leq 1$,

$$h(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \\ 1 & \text{if } b \leq t. \end{cases}$$

Set

$$g(x; a, b, c) = h(|x| - c), \tag{A.2}$$

where a, b, c are positive constants and $|x| = (x_1^2 + ... + x_n^2)^{\frac{1}{2}}$, $x \in \Omega$. Then g is a smooth function satisfying the following properties:

(i) $0 \leq g(x; a, b, c) \leq 1$ for $x \in \Omega$,
(ii) $g(x; a, b, c) = 0$ for $x \in \Omega_c$,
(iii) $g(x; a, b, c) = 1$ for $x \in \Omega \setminus \Omega_{c+b}$,

where $\Omega_\gamma = \{x : |x| \leq \gamma\} \cap \Omega$ for $\gamma = c, c + b$. In case there is no confusion, we simply denote this cut-off function by $g(x)$. We are interested in the constants a, b for which the size of the gradient vector of g becomes small.

A.2. PROOF OF LEMMA 2.1

Proof. In order to prove the inequality (7), let $\bar{F}(x) = e^{-\mu g(x)|x|}F(x)$ and $\bar{w}(x) = e^{-\mu g(x)|x|}w(x)$. Then by the definition 2.1

$$\|F\|_{L^2(\mathbf{R}^3,\mu,g)} = \|\bar{F}\|_{L^2(\mathbf{R}^3)} \text{ and } \|w\|_{L^2(\mathbf{R}^3,\mu,g)} = \|\bar{w}\|_{L^2(\mathbf{R}^3)}. \tag{A.3}$$

Let

$$K(x, \xi) = e^{-\mu g(x)|x|} \frac{e^{-|x-\xi|}}{4\pi|x - \xi|} e^{\mu g(\xi)|\xi|}.$$

Then it follows from (**??**) that

$$\bar{w}(x) = \int_{\mathbf{R}^3} K(x, \xi)\bar{F}(\xi)d\xi. \tag{A.4}$$

Let

$$\Omega_B^x = \{x \in \mathbf{R}^3 : |x| \leq c + b\} \text{ and } \Omega_\infty^x = \{x \in \mathbf{R}^3 : |x| > c + b\}. \tag{A.5}$$

Since $|x - \xi| \geq ||x| - |\xi||$ and $0 \leq g(x) \leq 1$,

$$|K(x, \xi)| \leq \frac{e^{-|x-\xi|}}{4\pi|x - \xi|} e^{\mu|\xi|} \leq \frac{e^{-|x|}e^{(1+\mu)|\xi|}}{|x - \xi|}, \tag{A.6}$$

$$|K(x, \xi)| \leq \frac{e^{-|x-\xi|}}{4\pi|x - \xi|} e^{\mu|\xi|} \leq \frac{e^{|x|}e^{(\mu-1)|\xi|}}{|x - \xi|}. \tag{A.7}$$

Since $e^{-|x-\xi|} = e^{-\mu|x-\xi|}e^{(\mu-1)|x-\xi|}$ and $g(x) = 1$ on Ω_∞^x,

$$\begin{aligned} |K(x,\xi)| &= \frac{1}{4\pi|x-\xi|}e^{-\mu(|x|-|\xi|)}e^{-\mu|x-\xi|}e^{(\mu-1)|x-\xi|} \\ &\leq \frac{1}{|x-\xi|}e^{-\mu(|x|-|\xi|)}e^{-\mu(|\xi|-|x|)}e^{(\mu-1)|x-\xi|} \\ &= \frac{1}{|x-\xi|}e^{(\mu-1)|x-\xi|} \quad \text{for } (x,\xi) \in \Omega_\infty^x \times \Omega_\infty^\xi. \end{aligned} \tag{A.8}$$

Since $0 < \mu << 1$, by applying (A.6) on $\Omega_\infty^x \times \Omega_B^\xi$, (A.7) on $\Omega_B^x \times \Omega_\infty^\xi$, and (A.8) on $\Omega_\infty^x \times \Omega_\infty^\xi$, respectively, one can show that

$$\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}|K(x,\xi)|d\xi \leq C \quad \text{and} \quad \sup_{\xi\in\mathbf{R}^3}\int_{\mathbf{R}^3}|K(x,\xi)|dx \leq C$$

for some constant C. Hence Generalized Young's Inequality (page 9 of [13]) implies

$$\|\bar{w}\|_{L^2(\mathbf{R}^3)} \leq C\|\bar{F}\|_{L^2(\mathbf{R}^3)}. \tag{A.9}$$

Now, together with (A.3), this proves the lemma. ■

A.3. PROOF OF LEMMA 3.2

Proof. Since $g(x) = h(|x| - c)$, its shifted function under the mapping $\varphi_{(n)}$ is

$$\begin{aligned} \hat{g}(\xi) &= (g \circ \varphi_{(n)}^{-1})(\xi) = h(\frac{c}{|\xi|} - c) \\ &= \frac{q(c/|\xi| - c)}{q(c/|\xi| - c) + q(b - (c/|\xi| - c))}. \end{aligned}$$

Thus, we have

$$\hat{g}(\xi) = \begin{cases} 1 & \text{if } (\frac{c}{|\xi|} - c) \geq b \quad (\text{or } |\xi| \leq \frac{c}{b+c}), \\ 0 & \text{if } (\frac{c}{|\xi|} - c) \leq 0 \quad (\text{or } |\xi| \geq 1), \\ \dfrac{e^{-a/t^2}}{e^{-a/t^2} + e^{-a/(b-t)^2}} & \text{otherwise,} \end{cases}$$

where $t = c/|\xi| - c$.

For $|\xi| \in (c/(b+c), 1)$, we have

$$\nabla_\xi \hat{g}(\xi) = \frac{dh}{dt}(\frac{c}{|\xi|} - c)\nabla(\frac{c}{|\xi|}) = \frac{dh}{dt}(\frac{c}{|\xi|} - c)(-\frac{c}{|\xi|^2})\nabla_\xi|\xi|.$$

Thus,

$$\begin{aligned} \nabla_\xi e^{-2\mu\hat{g}(\xi)c/|\xi|} &= (-2\mu)e^{-2\mu\hat{g}(\xi)c/|\xi|}\left\{(\nabla_\xi\hat{g}(\xi))\,\frac{c}{|\xi|} + \hat{g}(\xi)\nabla_\xi\frac{c}{|\xi|}\right\} \\ &= \frac{2\mu c}{|\xi|^3}e^{-2\mu\hat{g}(\xi)c/|\xi|}(c\frac{dh}{dt}(\frac{c}{|\xi|} - c) + \hat{g}(\xi)|\xi|)\nabla_\xi|\xi|. \end{aligned} \tag{A.10}$$

By Lemma 3.1 and (A.10), we obtain

$$\begin{aligned}
&\int_{\Omega_\infty} \nabla_x u(x)\cdot[a_{ij}(x)]\cdot(\nabla_x e^{-2\mu g(x)|x|})^T v(x)dx \\
=&\int_{\hat{\Omega}_\infty} |J(\varphi_{(n)}^{-1})|\nabla_\xi \hat{u}(\xi)\cdot J(\widehat{\varphi_{(n)}})^T\cdot[\hat{a}_{ij}(\xi)]\cdot J(\widehat{\varphi_{(n)}})\cdot\left(\nabla_\xi e^{-2\mu\hat{g}(\xi)c/|\xi|}\right)^T\hat{v}(\xi)d\xi \\
=&\int_{\hat{\Omega}_\infty}\frac{2\mu c}{|\xi|^3}e^{-2\mu\hat{g}(\xi)c/|\xi|}\left(c\frac{dh}{dt}(\frac{c}{|\xi|}-c)+\hat{g}(\xi)|\xi|\right)\nabla_\xi\hat{u}(\xi)\cdot[q_{ij}^{(n)}]\cdot(\nabla_\xi|\xi|)^T\hat{v}(\xi)d\xi \\
=&\int_{\hat{\Omega}_\infty} e^{-2\mu\hat{g}(\xi)c/|\xi|}W(\xi)\nabla_\xi\hat{u}(\xi)\cdot[q_{ij}^{(n)}]\cdot(\xi)^T\hat{v}(\xi)d\xi
\end{aligned}$$

■

A.4. EXPANSION OF $Q_{IJ}^{(N)}$

For the implementation of our method, the coefficients $q_{ij}^{(n)}$ of the transformed bilinear form are computed in a specific form.

LEMMA A.1. With the same notations as above, we obtain the following.
(i) $J(\varphi_{(n)})\circ\varphi_{(n)}^{-1} = J(\widehat{\varphi_{(n)}}) = [J(\varphi_{(n)}^{-1})]^{-1},$ for all n.
(ii) $|J(\varphi_{(2)})| = \dfrac{c^2}{|x|^4},$ for $x=(x_1,x_2)$.
(iii) The entries of the 2×2 matrix $[q_{ij}^{(2)}]$ are as follows:

$$\begin{aligned}
q_{11}^{(2)} &= [\hat{a}_{11}A^2+2\hat{a}_{12}AB+2\hat{a}_{21}AB+4\hat{a}_{22}B^2]/D,\\
q_{12}^{(2)} &= [2\hat{a}_{11}BC+\hat{a}_{12}A^2-4\hat{a}_{21}B^2+2\hat{a}_{22}AB]/D,\\
q_{21}^{(2)} &= [2\hat{a}_{11}BC+\hat{a}_{21}A^2-4\hat{a}_{12}B^2+2\hat{a}_{22}AB]/D,\\
q_{22}^{(2)} &= [4\hat{a}_{11}B^2+2\hat{a}_{12}BC+2\hat{a}_{21}BC+\hat{a}_{22}A^2]/D,
\end{aligned}$$

where $A=(-\xi_1^2+\xi_2^2), B=\xi_1\xi_2, C=(\xi_1^2-\xi_2^2), D=(\xi_1^2+\xi_2^2)^2$.
(iv) $|J(\varphi_{(3)})| = \dfrac{c^3}{|x|^6},$ for $x=(x_1,x_2,x_3)$.
(v) The entries of the 3×3 matrix $[q_{ij}^{(3)}]$ are as follows:.

$$\begin{aligned}
q_{11}^{(3)}(\xi) &= \frac{c}{|\xi|^6}(\hat{a}_{11}A^2+(\hat{a}_{12}+\hat{a}_{21})AB+\hat{a}_{22}B^2\\
&\quad +(\hat{a}_{23}+\hat{a}_{32})BC+(\hat{a}_{31}+\hat{a}_{13})AC+\hat{a}_{33}C^2),\\
q_{12}^{(3)}(\xi) &= \frac{c}{|\xi|^6}(\hat{a}_{11}AB+\hat{a}_{12}AD+\hat{a}_{21}B^2+\hat{a}_{22}BD\\
&\quad +\hat{a}_{23}BE+\hat{a}_{32}CD+\hat{a}_{31}BC+\hat{a}_{13}AE+\hat{a}_{33}CE),\\
q_{13}^{(3)}(\xi) &= \frac{c}{|\xi|^6}(-\hat{a}_{11}AC-\hat{a}_{12}AE-\hat{a}_{21}BC-\hat{a}_{22}BE\\
&\quad +\hat{a}_{23}BF-\hat{a}_{32}CE-\hat{a}_{31}C^2+\hat{a}_{13}AF+\hat{a}_{33}CF),\\
q_{21}^{(3)}(\xi) &= \frac{c}{|\xi|^6}(\hat{a}_{11}AB+\hat{a}_{12}B^2+\hat{a}_{21}AD+\hat{a}_{22}BD\\
&\quad +\hat{a}_{23}CD+\hat{a}_{32}BE+\hat{a}_{31}AE+\hat{a}_{13}BC+\hat{a}_{33}CE),
\end{aligned}$$

$$\begin{aligned}
q^{(3)}_{22}(\xi) &= \frac{c}{|\xi|^6}(\hat a_{11}B^2+(\hat a_{12}+\hat a_{21})BD+\hat a_{22}D^2\\
&\quad +(\hat a_{23}+\hat a_{32})DE+(\hat a_{13}+\hat a_{31})BE+\hat a_{33}E^2),\\
q^{(3)}_{23}(\xi) &= \frac{c}{|\xi|^6}(-\hat a_{11}BC-\hat a_{12}BE-\hat a_{21}CD-\hat a_{22}DE\\
&\quad +\hat a_{23}DF-\hat a_{32}E^2+\hat a_{13}BF-\hat a_{31}CE+\hat a_{33}EF),\\
q^{(3)}_{31}(\xi) &= \frac{c}{|\xi|^6}(-\hat a_{11}AC-\hat a_{12}BC-\hat a_{21}AE-\hat a_{22}BE\\
&\quad -\hat a_{23}CE+\hat a_{32}BF+\hat a_{31}AF-\hat a_{13}C^2+\hat a_{33}CF),\\
q^{(3)}_{32}(\xi) &= \frac{c}{|\xi|^6}(-\hat a_{11}BC-\hat a_{12}CD-\hat a_{21}BE-\hat a_{22}DE\\
&\quad -\hat a_{23}E^2+\hat a_{32}DF+\hat a_{31}BF-\hat a_{13}CE+\hat a_{33}EF),\\
q^{(3}_{33}(\xi) &= \frac{c}{|\xi|^6}(\hat a_{11}C^2+(\hat a_{12}+\hat a_{21})CE+\hat a_{22}E^2\\
&\quad -(\hat a_{23}+\hat a_{32})EF-(\hat a_{31}+\hat a_{13})CF+\hat a_{33}F^2).
\end{aligned}$$

where

$$\begin{aligned}
A &= -\xi_1^2+\xi_2^2+\xi_3^2, & B &= -2\xi_1\xi_2,\\
C &= 2\xi_1\xi_3, & D &= \xi_1^2-\xi_2^2+\xi_3^2,\\
E &= 2\xi_2\xi_3, & F &= -(\xi_1^2+\xi_2^2-\xi_3^2).
\end{aligned}$$

Proof. (a) $\varphi_{(n)}\cdot\varphi^{-1}_{(n)}=$ identity implies $[J(\varphi_{(n)})\circ\varphi_n^{-1}][J(\varphi^{-1}_{(n)})]=I_{(n)}$, the $n\times n$ unit matrix. Thus, we have $J(\varphi_{(n)})\circ\varphi^{-1}_{(n)}=[J(\varphi^{-1}_{(n)})]^{-1}$.

(b) For $n=2$, we have

$$J(\varphi_{(2)})=\left(\frac{c}{|x|^4}\right)\begin{bmatrix}-x_1^2+x_2^2 & 2x_1x_2\\ -2x_1x_2 & -(x_1^2-x_2^2)\end{bmatrix}$$

and, for $n=3$, we have

$$J(\varphi_{(3)})=\left(\frac{c}{|x|^4}\right)\begin{bmatrix}-x_1^2+x_2^2+x_3^2 & -2x_1x_2 & 2x_1x_3\\ -2x_1x_2 & x_1^2-x_2^2+x_3^2 & 2x_2x_3\\ -2x_1x_3 & -2x_2x_3 & -(x_1^2+x_2^2-x_3^2)\end{bmatrix}.$$

Thus, $|J(\varphi_{(2)})|=\dfrac{c^2}{|x|^4}$ and $|J(\varphi_{(3)})|=\dfrac{c^3}{|x|^6}$.

(c) Since $\varphi^{-1}_{(2)}=\dfrac{c}{|\xi|^2}(\xi_1,-\xi_2)$ and $\varphi^{-1}_{(3)}=\dfrac{c}{|\xi|^2}(\xi_1,\xi_2,-\xi_3)$,

$$J(\widehat{\varphi_{(2)}})=J(\varphi^{-1}_{(2)})^{-1}=\left(\frac{1}{c}\right)\begin{bmatrix}-\xi_1^2+\xi_2^2 & -2\xi_1\xi_2\\ 2\xi_1\xi_2 & -\xi_1^2+\xi_2^2\end{bmatrix}\tag{A.11}$$

and

$$\begin{aligned}
J(\widehat{\varphi_{(3)}}) &= J(\varphi^{-1}_{(3)})^{-1}\\
&= \left(\frac{1}{c}\right)\begin{bmatrix}-\xi_1^2+\xi_2^2+\xi_3^2 & -2\xi_1\xi_2 & -2\xi_1\xi_3\\ -2\xi_1\xi_2 & \xi_1^2-\xi_2^2+\xi_3^2 & -2\xi_2\xi_3\\ 2\xi_1\xi_3 & 2\xi_2\xi_3 & -(\xi_1^2+\xi_2^2-\xi_3^2)\end{bmatrix}.
\end{aligned}\tag{A.12}$$

Thus,

$$
\begin{aligned}
[q_{ij}^{(2)}] &= \frac{c^2}{|\xi|^4} J\widehat{(\varphi_{(2)})}^T \cdot [\hat{a}_{ij}(\xi)] \cdot J\widehat{(\varphi_{(2)})}, \text{ and} \\
\left[q_{ij}^{(3)}\right] &= \frac{c^3}{|\xi|^6} J\widehat{(\varphi_{(3)})}^T \cdot [\hat{a}_{ij}(\xi)] \cdot J\widehat{(\varphi_{(3)})}.
\end{aligned}
$$

lead to the formulars for $q_{ij}^{(2)}$ and $q_{ij}^{(3)}$ respectively. ■

If the elliptic operator is Δ, then the transformed bilinear $\hat{\mathcal{A}}(\cdot,\cdot)$ has a much simpler form as follows:

COROLLARY A.1. We have the following.

$$
\begin{aligned}
\int_{\Omega_\infty} e^{-2\mu g(x)|x|} \nabla_x u(x) \cdot (\nabla_x v(x))^T dx &= \int_{\hat{\Omega}_\infty} k_1(\xi) e^{-2\mu \hat{g}(\xi) c/|\xi|} \nabla_\xi \hat{u}(\xi) \cdot (\nabla_\xi \hat{v}(\xi))^T d\xi, \\
\int_{\Omega_\infty} v(x) \nabla_x u \cdot [\nabla_x e^{-2\mu g(x)|x|}]^T dx &= \int_{\hat{\Omega}_\infty} k_2(\xi) e^{-2\mu \hat{g}(\xi) c/|\xi|} (c\frac{dh}{dt}(\frac{c}{|\xi|} - c) + \hat{g}(\xi)|\xi|) \\
&\quad \nabla_\xi \hat{u}(\xi) \cdot \hat{v}(\xi)(\xi)^T d\xi, \\
\int_{\Omega_\infty} e^{-2\mu g(x)|x|} u(x) v(x) dx &= \int_{\hat{\Omega}_\infty} k_3(\xi) e^{-2\mu \hat{g}(\xi) c/|\xi|} \hat{u}(\xi) \cdot \hat{v}(\xi) d\xi, \\
\int_{\Omega_\infty} e^{-2\mu g(x)|x|} f(x) v(x) dx &= \int_{\hat{\Omega}_\infty} k_3(\xi) e^{-2\mu \hat{g}(\xi) c/|\xi|} \hat{f}(\xi) \hat{v}(\xi) d\xi,
\end{aligned}
$$

where $k_1(\xi) = 1, k_2(\xi) = \dfrac{2\mu c}{|\xi|^4}, k_3(\xi) = \dfrac{c^2}{|\xi|^4}$, when $\Omega_\infty \subset \mathbf{R}^2$ and $k_1(\xi) = \dfrac{c}{|\xi|^2}, k_2(\xi) = \dfrac{2\mu c^2}{|\xi|^6}, k_3(\xi) = \dfrac{c^3}{|\xi|^6}$, when $\Omega_\infty \subset \mathbf{R}^3$, respectively.

Proof. From (A.11) and (A.12), we have

$$
[q_{ij}^{(2)}] = |J(\varphi_{(2)}^{-1})| J\widehat{(\varphi_{(2)})}^T \cdot I_{(2)} \cdot J\widehat{(\varphi_{(2)})} = I_{(2)} \tag{A.13}
$$

and

$$
[q_{ij}^{(3)}] = |J(\varphi_{(3)}^{-1})| J\widehat{(\varphi_{(3)})}^T \cdot I_{(3)} \cdot J\widehat{(\varphi_{(3)})} = \frac{c}{|\xi|^2} I_{(3)}. \tag{A.14}
$$

The conclusion follows from (A.13), (A.14) and Lemma A.1. ■

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