# Numerical Methods for Accurate Finite Element Solutions of Elliptic Boundary Value Problems Containing Singularities

by

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**Abstract**

The Method of Auxiliary Mapping (MAM), introduced by Babuška and Oh, was proven to be very successful in dealing with monotone singularities arising in two dimensional problems. In this paper, in the framework of the $p$-version of FEM, MAM is presented for one-dimensional elliptic boundary value problems containing singularities. Moreover, in order to show the effectiveness of MAM, a detailed proof of an error estimate is also presented, which gives a sharp error bound of MAM.

## 1 Introduction

In this paper, we are concerned with special approaches (mapping techniques) to deal with singularities in the framework of the $p$-Version of the Finite El-

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ement Method. The $p$-version of Finite Element Method fixes the mesh and the accuracy is achieved by increasing the degree of the elements uniformly or selectively. For the $p$-version of FEM, we refer to ([6],[7],[24]).

The accuracy of finite element solutions of the elliptic boundary value problem,

$$-\nabla \cdot (p(x)\nabla u(x)) + q(x)u(x) = f(x) \text{ in } \Omega$$

depends on the regularity of the true solution. On the other hand, the true solution is smooth as long as the boundary $\partial\Omega$ and the data $p(x), q(x), f(x)$ are smooth. Singularities can therefore arise when $\partial\Omega$ or some part of the data are not smooth. In the theory and practice of FEM, a considerable effort has been made to design special approaches to deal with problems containing singularities. The most typical approaches are Mesh Refinement ([5],[13],[22],[26]), Use of special elements ([1],[2],[23],[27]), and Use of the Enriched(nonlocal) basis functions ([8],[15],[22]). Recently, Babuška - Oh introduced a new method, called the method of Auxiliary Mapping(MAM), to deal with the corner singularity([4]). The essence of this method involves locally transforming a region around each singularity to a new domain by use of a conformal mapping such as $\xi = z^{\beta}$. Here $\beta$ is directly determined by the known nature of the singularity in such a way as to locally transform the exact(singular) solution to a smoother function, which can be easily approximated in the new domain by the conventional use of FEM.

In ([4],[14],[17],[20]), it was demonstrated that MAM effectively handles the singularities (such as corner singularities, jump boundary data singularities, interface singularities) arising in two dimensional elliptic boundary value problems. However, estimates of error bounds of MAM were not very sharp. Hence, in this paper, MAM is presented for one-dimensional elliptic boundary value problems containing singularities and error estimates which give a sharp error bound of MAM are proved. For consistency purpose, in the presentation of MAM for one-dimensional problems, we try to keep the notations and frames that were used for the two dimensional counterpart.

# 2 Mapping Techniques for Boundary Value Problems Containing Singularities

Throughout this paper, $\alpha$ denotes a real number such that $1/2 < \alpha < 1$ and $\Omega = \{x \in \mathbf{R} : a < x < b\}$ denotes an one-dimensional domain. Let $u$ be a measurable function defined on $\Omega$ and, for a positive integer $k$,

$$\|u\|_{k,\Omega} = \left\{ \int_\Omega \left[ u^2 + \sum_{i=1}^{k} (\frac{d^i u}{dx^i})^2 \right] dx \right\}^{1/2}.$$

Then $H^k(\Omega) = \{u : \|u\|_{k,\Omega} < \infty\}, H_0^k(\Omega) = \{u \in H^k(\Omega) : u(a) = u(b) = 0\}$, denote the usual Sobolev spaces. In particular, $H^0(\Omega) = L^2(\Omega)$. For $u \in H^k(\Omega)$, $\|u\|_{k,\Omega}$ and $|u|_{k,\Omega} = \{\int_\Omega (\frac{d^k u}{dx^k})^2 dx\}^{1/2}$ are called the Sobolev norm and seminorm, respectively.

## 2.1 A Model Problem

Consider a model one-dimensional elliptic boundary value problem

$$-\frac{d}{dx}(p(x)\frac{du(x)}{dx}) + q(x)u(x) = f(x) \text{ in } \Omega \tag{2.1}$$

satisfying the homogeneous Dirichlet boundary condition $u(a) = u(b) = 0$. Here $p(x) > p_0 > 0$ and $q(x) > q_0 > 0$ are smooth functions, and $f \in H^{-1}(\Omega)$, which is the space of bounded linear functionals defined on $H_0^1(\Omega)$. We assume that the model problem contains either a **monotone singularity** of type

$$x^\alpha, x^\alpha(\log x)^k$$

or an **oscillating singularity** of type

$$x^\alpha \cos\varepsilon \ln(x).$$

Here a small real number $\varepsilon$ is called the oscillating factor.

Let $V = H_0^1(\Omega)$. Then the variational formulation of this problem is as follows: Find $u \in V$ such that

$$\mathcal{B}(u, v) = \mathcal{F}(v) \text{ for all } v \in V, \tag{2.2}$$

where

$$\mathcal{B}(u, v) = \int_\Omega (p\frac{du}{dx}\frac{dv}{dx} + quv)dx \tag{2.3}$$

$$\mathcal{F}(v) = \int_\Omega fv dx \tag{2.4}$$

## 2.2 Auxiliary Mappings

Let $\Omega_S$ be a subdomain of $\Omega$ which contains a singularity of $u$. Then an auxiliary mapping $\varphi^\beta : \hat{\Omega}_S \longrightarrow \Omega_S$ is defined by

$$\varphi^\beta(x^*) = (x^*)^\beta, \tag{2.5}$$

where $x^*$ denotes the coordinate of the points in the transformed domain $\hat{\Omega}_S$. Here $\beta$ is called the **mapping size** of the auxiliary mapping.

For example, suppose $u(x) = x^\alpha, 0.5 < \alpha < 1$ and $\beta > 1$, then $\alpha\beta > \alpha$ and hence $u \circ \varphi^\beta = (x^*)^{\alpha\beta}$ is much smoother than $x^\alpha$. In particular, if $\beta = 1/\alpha$, then $u \circ \varphi^\beta$ is smooth.

In what follows, for a function $w : \Omega_S \to \mathbf{R}, w \circ \varphi^\beta$ is denoted by $\hat{w}$. Let $[J(\varphi^\beta)] = (\beta)(x^*)^{\beta-1}$. Then the chain rule implies

$$\frac{du}{dx} = [J(\varphi^\beta)]^{-1}\frac{d\hat{u}}{dx^*}. \tag{2.6}$$

Hence, under this transformation, the variational formulation (2.2) on $\Omega_S$ becomes

$$\mathcal{B}(u, v)|_{\Omega_S} := \int_{\Omega_S} (p\frac{du}{dx}\frac{dv}{dx} + quv)dx$$

$$= \int_{\hat{\Omega}_S} [J(\varphi^\beta)]^{-1} \hat{p} \frac{d\hat{u}}{dx^*} \frac{d\hat{v}}{dx^*} + \hat{q}\hat{u}\hat{v} dx^* := \hat{\mathcal{B}}(\hat{u}, \hat{v}) \quad (2.7)$$

$$\mathcal{F}(v)|_{\Omega_S} := \int_{\Omega_S} f v dx$$

$$= \int_{\hat{\Omega}_S} [J(\varphi^\beta)] \hat{f} \hat{v} dx^* := \hat{\mathcal{F}}(\hat{v}). \quad (2.8)$$

In higher dimensional cases, $[J(\varphi^\beta)]^{-1}$ does not appear in the transformed bilinear form $\hat{\mathcal{B}}(\hat{u}, \hat{v})$ and hence no restrictions on $\beta$ are necessary. However, in one dimensional case, to make the integral in (2.7) finite, the mapping size must be $1 \le \beta < 2$. Since $0.5 < \alpha$, an optimal mapping size $\beta = 1/\alpha$ is always within this range.

## 2.3 The Construction of Finite Element Space

Without loss of generality, we assume that the model problem has only one singularity at $x = a$. Now the domain $\Omega$ is divided into parts: a singular subdomain $\Omega_S = (a, L), (L - a) < (b - a)/2$, and a regular subdomain $\Omega_R = (L, b)$. Then $u$ is smooth on $\Omega_R$ and $\Omega_S$ is called a neighborhood of the singularity.

Let $\Delta_S$ and $\Delta_R$ be quasi-uniform meshes on $\Omega_S$ and $\Omega_R$, respectively. Then a specific mesh on $\Omega = \Omega_S \cup \Omega_R$ is denoted by $\Delta = \Delta_S \cup \Delta_R$ :

$$\Delta : a = x_1 < x_2 < ... < L < ... < x_{M(\Delta)} < x_{M(\Delta)+1} = b$$

where $M(\Delta)$ is the number of elements.

Let $\Omega_{st} = \{\xi | -1 < \xi < 1\}$ be the standard (reference) element and $\Omega_k = \{x | x_k < x < x_{k+1}\}$ be the $k$th element. The conventional elemental mapping $\Phi_k$ from $\Omega_{st}$ onto the $k$th element $\Omega_k$ is defined by:

$$x = \Phi_k(\xi) = \frac{1-\xi}{2} x_k + \frac{1+\xi}{2} x_{k+1}, \qquad \xi \in \Omega_{st}. \quad (2.9)$$

Let $\hat{\Phi}_k$ be the conventional elemental mapping from $\Omega_{st}$ onto $\hat{\Omega}_k = [x_k^*, x_{k+1}^*]$, then

$$(\Phi_k^S(\xi) \equiv \varphi^\beta \circ \hat{\Phi}_k)(\xi) = \Big[\frac{1-\xi}{2} x_k^* + \frac{1+\xi}{2} x_{k+1}^*\Big]^{1/\beta}, \qquad \xi \in \Omega_{st}, \quad (2.10)$$

is called **the singular elemental mapping** from $\Omega_{st}$ onto the $k$th element $\Omega_k$.

Let $\mathcal{M}$ be the vector of elemental mappings assigned to the elements in $\Delta = \Delta_S \cup \Delta_R$ by the following rule:

- Assign the conventional elemental mappings $\Phi_k$ to the elements $\Omega_k \subset \Omega_R$;
- Assign the singular elemental mappings $\Phi_k^S$ defined by (2.10) to the elements $\Omega_k \subset \Omega_S$.

Let $\mathcal{P}_p(\Omega_{st})$ be the space of all polynomials of degree $p$ defined on $\Omega_{st}$. Then **the finite element space,** denoted by $S^p(\Omega, \Delta, \mathcal{M})$, is the set of all functions $v$ defined on $\Omega$ such that

- $v \circ \Phi_k^S \in \mathcal{P}_p(\Omega_{st})$ if $\Omega_k$ is an element in the singular zone $\Omega_S$;
- $v \circ \Phi_k \in \mathcal{P}_p(\Omega_{st})$ if $\Omega_k$ is an element in the regular zone $\Omega_R$.

The dimension of vector space $S^p(\Omega, \Delta, \mathcal{M})$ is called the **Number of Degree of Freedom.**

In **the p-version of FEM**, to obtain the desired accuracy, the mesh $\Delta$ of the domain is fixed and only the degree $p$ of the basis polynomials is increased.

**The Method of Auxiliary Mapping** (MAM) to deal with elliptic problem containing singularities is as follows: In the master element approach, the computations of local stiffness matrices and local load vectors for the elements in the singular subdomain $\Omega_S$ use singular basis functions, constructed through the singular elemental mappings (2.10), which resembles the singularity.

Obviously, it is an extra work to construct singular basis functions for computations of the stiffness matrix and the load vector. However, this problem is circumvented by computing the transformed bilinear form $\hat{\mathcal{A}}|_{\hat{\Omega}_k}(\cdot,\cdot)$ and the transformed linear functional $\hat{\mathcal{F}}|_{\hat{\Omega}_k}(\cdot)$, given by (2.7) and (2.8) respectively, on the transformed elements $\hat{\Omega}_k$ together with the elemental mapping

$$\hat{\Phi}_{\hat{\Omega}_k} : \Omega_{st} \longrightarrow \hat{\Omega}_k.$$

In other words, the local stiffness matrices and local load vectors are computed by the following rule:

- Use $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{F}(\cdot,\cdot)$ for the elements $\Omega_k$ in the regular subdomain $\Omega_R$,
- Use $\hat{\mathcal{A}}|_{\hat{\Omega}_k}(\cdot,\cdot)$ and $\hat{\mathcal{F}}|_{\hat{\Omega}_k}(\cdot)$ on the transformed elements $\hat{\Omega}_k$, for the elements $\Omega_k$ in the singular subdomain $\Omega_S$.

In summary, MAM is to apply a conventional $p$-version of FEM on the domain $\hat{\Omega}_S \cup_{\varphi^\beta} \Omega_R$ with the two phase bilinear form $\hat{\mathcal{A}}|_{\hat{\Omega}_S}(\cdot,\cdot) \cup_{\varphi_\beta} \mathcal{A}|_{\Omega_R}(\cdot,\cdot)$. and the two phase linear functional $\hat{\mathcal{F}}|_{\hat{\Omega}_S}(\cdot) \cup_{\varphi_\beta} \mathcal{F}|_{\Omega_R}(\cdot)$.

Let us note that MAM has a similar effect to using singular basis functions of type $x^{1/\beta}$. Thus, one can expect much improved results when MAM is applied for FE solution. Novelty of this method is to obtain improved results without introducing any singular basis functions. Moreover, unlike conventional singular element approaches, this method does not destroy the nice band structure of FEM.

**Remark 2.1** *Suppose $\Omega_k = [L_k, L_{k+1}]$ is an element in the singular region $\Omega_S$, then $(\varphi^\beta)^{-1}(\Omega_k) = [L_k^*, L_{k+1}^*]$, $L_k^* = L_k^{1/\beta}$, $L_{k+1}^* = L_{k+1}^{1/\beta}$. Implementing this method, the integrals of (2.7) and (2.8) are computed as follows: Let us consider the following change of variable,*

$$x^* = t^\delta, \ \textit{where}\ \delta = \frac{2}{2-\beta}. \tag{2.11}$$

*By the change of variable (2.11), the integrals of (2.7) become*

$$\int_{L_k^*}^{L_{k+1}^*} [J(\varphi^\beta)]^{-1} \frac{d\hat{\Psi}_i}{dx^*}\frac{d\hat{\Psi}_j}{dx^*} dx^* = \int_{\hat{L}_k}^{\hat{L}_{k+1}} \frac{\delta}{\beta} t \tilde{\Psi}_i(t^\delta)\tilde{\Psi}_j(t^\delta) dt, \tag{2.12}$$

$$\int_{L_k^*}^{L_{k+1}^*} [J(\varphi^\beta)] \hat{f} \hat{\Psi}_j dx^* = \int_{\hat{L}_k}^{\hat{L}_{k+1}} (\beta\delta) t^{\beta\delta-1} \hat{f}(t^\delta)\hat{\Psi}_j(t^\delta) dt, \tag{2.13}$$

*where* $\hat{L}_k = (L_k^*)^{1/\delta}$, $\hat{L}_{k+1} = (L_{k+1}^*)^{1/\delta}$, $\dfrac{d\hat{\Psi}_i}{dx^*} = \tilde{\Psi}_i$ *and* $\dfrac{d\hat{\Psi}_j}{dx^*} = \tilde{\Psi}_j$. *Here* $\hat{\Psi}_i$ *is an elemental shape function. Hence* $\hat{\Psi}_i$ *and* $\tilde{\Psi}_i$ *are polynomials in* $x^*$. *Thus, the integrals on the right hand sides contain no singular terms.*

# 3 An Error Analysis of MAM

Suppose $u_{fe}$ denotes the FE solution obtained by MAM, then from the construction of a finite element space $S^p(\Omega, \Delta, \mathcal{M})$, $u_{fe}|_{\Omega_S}$ is a singular function. In other words, the elemental shape functions $\bar{\Psi}_i \circ (\varphi^\beta \circ \hat{\Phi}_k)^{-1}$ on the elements $\Omega_k$ in the singular zone $\Omega_S$ are singular.However, $\hat{\Phi}_k^{-1}$ is linear and hence $\hat{u}_{fe}|_{\hat{\Omega}_k} = \sum \xi_i (\bar{\Psi}_i \circ \hat{\Phi}_k^{-1})$ is a polynomial. Here $\bar{\Psi}_i(\xi)$ denotes the standard basis polynomials on the reference element $\Omega_{st}$.

In this section, $e(x) = u_{ex}(x) - u_{fe}(x)$ denotes the error of MAM. We are going to estimate this error in energy norm, $\|e\|_E = \|de/dx\|_{L^2(\Omega)}$.

First of all, we prove several preliminary Lemmas for this error estimation.

Repeated applications of the integration by parts yield the following Lemma:

**Lemma 3.1** *Let* $n$ *and* $\nu$ *be integers such that* $0 \le \nu \le n$, *and* $\alpha$, $\beta$ *be real numbers such that* $\alpha > \frac{1}{2}$, $1 \le \beta < 2$. *Then*

$$\int_{-1}^{1} (s-1)^\nu (s+1)^{n-\nu+\alpha\beta-\beta} ds = \frac{(-1)^\nu \nu! 2^{n+\alpha\beta-\beta+1} \Gamma(n+\alpha\beta-\beta-\nu+1)}{\Gamma(n+\alpha\beta-\beta+2)}.$$

Using the mathematical induction, we prove the following Lemma:

**Lemma 3.2** (cf, p. 1042 of [10]) *Let* $n$ *be a positive integer and* $a$ *and* $b$ *be real numbers which are not negative integers. Then*

$$\sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(-1)^\nu \Gamma(n+a+1-\nu)\Gamma(n+b+1)}{\Gamma(n+a+1)\Gamma(n+b+1-\nu)} = \prod_{\nu=1}^{n} \frac{\nu-a+b-1}{-a-\nu}. \tag{3.1}$$

**proof 1** *We proceed the proof by induction on* $n$. *If* $n = 1$, *then the equation (3.1) holds for all real numbers* $a$ *and* $b$ *which are not negative integers. Now let us assume that for all real numbers* $a$ *and* $b$ *which are not negative integers the equation (3.1) is true when* $n = m - 1$. *Then*

$$\begin{aligned}
&\sum_{\nu=0}^{m} \binom{m}{\nu} \frac{(-1)^\nu \Gamma(m+a+1-\nu)\Gamma(m+b+1)}{\Gamma(m+a+1)\Gamma(m+b+1-\nu)} \\
&\quad = \sum_{\nu=0}^{m-1} \binom{m-1}{\nu} \frac{(-1)^\nu \Gamma(m+a+1-\nu)\Gamma(m+b+1)}{\Gamma(m+a+1)\Gamma(m+b+1-\nu)} \\
&\qquad + \sum_{\nu=1}^{m} \binom{m-1}{\nu-1} \frac{(-1)^\nu \Gamma(m+a+1-\nu)\Gamma(m+b+1)}{\Gamma(m+a+1)\Gamma(m+b+1-\nu)} \\
&\quad = \sum_{\nu=0}^{m-1} \binom{m-1}{\nu} \frac{(-1)^\nu \Gamma((m-1)+(a+1)+1-\nu)\Gamma((m-1)+(b+1)+1)}{\Gamma((m-1)+(a+1)+1)\Gamma((m-1)+(b+1)+1-\nu)}
\end{aligned}$$

$$-\frac{m+b}{m+a}\sum_{\nu=0}^{m-1}\binom{m-1}{\nu}\frac{(-1)^\nu\Gamma((m-1)+a+1-\nu)\Gamma((m-1)+b+1)}{\Gamma((m-1)+a+1)\Gamma((m-1)+b+1-\nu)}$$
$$=\prod_{\nu=1}^{m-1}\frac{\nu-a+b-1}{-a-1-\nu}-\frac{m+b}{m+a}\prod_{\nu=1}^{m-1}\frac{\nu-a+b-1}{-a-\nu}$$
$$=\prod_{\nu=1}^{m}\frac{\nu-a+b-1}{-a-\nu}.$$

*This completes the proof.*

The following Lemma is in ([11]).

**Lemma 3.3**

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)}=\frac{1}{n^{\beta-\alpha}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\qquad(n\to\infty)$$

*where* $\mathcal{O}\left(\frac{1}{n}\right)$ *depends on* $\alpha$, $\beta$.

Let us consider the error $e=u_{ex}-u_{fe}$ on the element $\Omega_1=\{x:0<x<h\}, h\leq 1$, that contains a monotone singularity of type $x^\alpha$. Then we have

$$\begin{aligned}|e|^2_{1,[0,h]} &= \int_0^h\left(\frac{d}{dx}u_{ex}(x)-\frac{d}{dx}u_{fe}(x)\right)^2dx \\ &= \int_0^{h^{1/\beta}}\left(\frac{d}{dt}\hat{u}_{ex}(t)-\frac{d}{dt}\hat{u}_{fe}(t)\right)^2\beta^{-1}t^{1-\beta}dt \\ &= \int_{-1}^1\left(\frac{d}{ds}u^*_{ex}(s)-\frac{d}{ds}u^*_{fe}(s)\right)^2\beta^{-1}(\frac{s+1}{2})^{1-\beta}2h^{-1}ds.\end{aligned}\tag{3.2}$$

As it was pointed out at the outset of this section, $u_{fe}(x)$ is not a polynomial, but $\hat{u}_{fe}$ and $u^*_{fe}$ are polynomials. Hence $\frac{d}{ds}u^*_{fe}(s)$ is a polynomial of the variable $s$. Thus, the best $L^2$-approximation of the mapping technique is given by the partial sum of Jacobi polynomial expansion(see [25]).

If the exact solution $u_{ex}|_{\Omega_1}(x)$ is $x^\alpha(\alpha>1/2)$, then

$$|e|^2_1\leq\alpha^2\beta\frac{b^{2\alpha-1}}{2^{2\alpha\beta-\beta}}\int_{-1}^1\left((1+s)^{\alpha\beta-1}-\sum_{n=0}^p a_nP_n^{(0,1-\beta)}(s)\right)^2(1+s)^{1-\beta}ds,\tag{3.3}$$

where

$$P_n^{(0,1-\beta)}(s)=\sum_{\nu=0}^n\binom{n}{n-\nu}\binom{n+1-\beta}{\nu}\left(\frac{x-1}{2}\right)^\nu\left(\frac{x+1}{2}\right)^{n-\nu}\tag{3.4}$$

and

$$a_n=\frac{\int_{-1}^1(s+1)^{\alpha\beta-1}P_n^{(0,1-\beta)}(s)(s+1)^{1-\beta}ds}{\int_{-1}^1\left(P_n^{(0,1-\beta)}(s)\right)^2(s+1)^{1-\beta}ds}\tag{3.5}$$

is the generalized Fourier coefficients in the orthogonal expansion of $(s+1)^{\alpha\beta-1}$ by Jacobi polynomials.

Since

$$(s+1)^{\alpha\beta-1} \sim \sum_{n=0}^{\infty} a_n P_n^{(0,1-\beta)}(s), \tag{3.6}$$

we obtain

$$\begin{aligned} |e|_1^2 &\leq \alpha^2\beta \frac{h^{2\alpha-1}}{2^{2\alpha\beta-\beta}} \int_{-1}^{1} \left( \sum_{n=p+1}^{\infty} a_n P_n^{(0,1-\beta)}(s) \right)^2 (s+1)^{1-\beta} ds \qquad (3.7) \\ &= \alpha^2\beta \frac{h^{2\alpha-1}}{2^{2\alpha\beta-\beta}} \sum_{n=p+1}^{\infty} a_n^2 \int_{-1}^{1} \left( P_n^{(0,1-\beta)}(s) \right)^2 (s+1)^{1-\beta} ds. \end{aligned}$$

By (p. 68 of [25]) , we have

$$\int_{-1}^{1} \left( P_n^{(0,1-\beta)}(s) \right)^2 (s+1)^{1-\beta} ds = \frac{2^{2-\beta}}{2n+2-\beta}. \tag{3.8}$$

Hence, the generalized Fourier coefficients $a_n$ can be written as follows:

$$\begin{aligned} a_n &= \frac{2n+2-\beta}{2^{2-\beta}} \int_{-1}^{1} (s+1)^{\alpha\beta-1} P_n^{(0,1-\beta)}(s)(s+1)^{1-\beta} ds \\ &= \frac{2n+2-\beta}{2^{n+2-\beta}} \int_{-1}^{1} (s+1)^{\alpha\beta-\beta} \sum_{\nu=0}^{n} \binom{n}{n-\nu} \binom{n+1-\beta}{\nu} (s-1)^{\nu}(s+1)^{n-\nu} ds \\ &= \frac{2n+2-\beta}{2^{n+2-\beta}} \sum_{\nu=0}^{n} \binom{n}{n-\nu} \binom{n+1-\beta}{\nu} \int_{-1}^{1} (s-1)^{\nu}(s+1)^{n-\nu+\alpha\beta-\beta} ds. \end{aligned}$$

Applying Lemma 3.1, we have

$$\begin{aligned} a_n &= \frac{2n+2-\beta}{2^{1-\alpha\beta}} \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(-1)^{\nu}\Gamma(n+2-\beta)\Gamma(n+\alpha\beta-\beta-\nu+1)}{\Gamma(n+2-\beta-\nu)\Gamma(n+\alpha\beta-\beta+2)} \\ &= \frac{(n+1-\beta/2)2^{\alpha\beta}}{(n+\alpha\beta-\beta+1)} \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(-1)^{\nu}\Gamma(n+2-\beta)\Gamma(n+\alpha\beta-\beta-\nu+1)}{\Gamma(n+2-\beta-\nu)\Gamma(n+\alpha\beta-\beta+1)}. \end{aligned}$$

Since $\alpha\beta-\beta > -1$ and $1-\beta > -1$, it follows from Lemma 3.2 with $a = \alpha\beta-\beta$ and $b = 1-\beta$ that

$$\begin{aligned} a_n &= \frac{(n+1-\beta/2)2^{\alpha\beta}}{(n+\alpha\beta-\beta+1)} \prod_{\nu=1}^{n} \frac{\nu-\alpha\beta}{-\alpha\beta+\beta-\nu} \\ &= \frac{(n+1-\beta/2)2^{\alpha\beta}}{(n+\alpha\beta-\beta+1)} \frac{(-1)^n\Gamma(\alpha\beta-\beta+1)}{\Gamma(n+\alpha\beta-\beta+1)} \prod_{\nu=1}^{n} (\nu-\alpha\beta). \qquad (3.9) \end{aligned}$$

Applying the identity

$$\prod_{\nu=1}^{n} (\nu-\alpha\beta) = \Gamma(n-\alpha\beta+1)\Gamma(\alpha\beta)\sin(\alpha\beta\pi)/\pi \text{ if } n > \alpha\beta-1, \tag{3.10}$$

we obtain for $n > \alpha\beta - 1$

$$a_n = \frac{2^{\alpha\beta}(-1)^n\Gamma(\alpha\beta-\beta+1)\Gamma(\alpha\beta)\sin(\alpha\beta\pi)(n+1-\beta/2)\Gamma(1-\alpha\beta+n)}{\pi\Gamma(n+\alpha\beta-\beta+2)} \quad (3.11)$$

Suppose $p > \alpha\beta - 2$. Then Lemma 3.3 implies

$$\begin{aligned}\sum_{n=p+1}^{\infty} a_n^2 \frac{2n+2-\beta}{2^{2-\beta}} &= \frac{2^{2\alpha\beta-\beta+1}\Gamma(\alpha\beta-\beta+1)^2\Gamma(\alpha\beta)^2\sin^2(\alpha\beta\pi)}{\pi^2}\sum_{n=p+1}^{\infty}\frac{1}{n^{4\alpha\beta-2\beta+1}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\\ &\le \frac{C^2 2^{2\alpha\beta-\beta+1}\Gamma(\alpha\beta-\beta+1)^2\Gamma(\alpha\beta)^2\sin^2(\alpha\beta\pi)}{\pi^2(p+1)^{4\alpha\beta-2\beta}}\end{aligned}$$

for some constant $C$.

Thus, we have proved the following theorem:

**Theorem 3.4** Suppose with an auxiliary mapping $\varphi^\beta$, MAM is applied to one dimensional elliptic problem such that $u_{ex}|_{\Omega_1}(x) = x^\alpha$, where $\Omega_1 = (0,h)$, and if $p > \alpha\beta - 2$, then we have an error estimate

$$|e|_{1,[0,h]} \le \frac{\alpha\sqrt{\beta}\sqrt{2}C\Gamma(\alpha\beta-\beta+1)|\Gamma(\alpha\beta)\sin(\alpha\beta\pi)|h^{\alpha-1/2}}{\pi(p+1)^{(2\alpha-1)\beta}}, \quad (3.12)$$

where $C$ is a constant independent of the polynomial degree $p$ and the mesh size $h$.

**Remark 3.1** $\sin(\alpha\beta\pi) = 0$ *whenever* $\alpha\beta = 1$. *Thus, Theorem 3.4 implies that the mapping size* $\beta = 1/\alpha$ *yields an optimal results. This reflects the fact that* $(u_{ex})^*(x^*) = u_{ex}\circ\varphi(x^*) = x^*$ *on* $[0, h^{1/\beta}]$, *for which the FE approximation to* $(u_{ex})^*$ *has no error.*

$\sin(\alpha\beta\pi)$ *in* (3.12) *is near* 0 *whenever* $|\alpha\beta|$ *is close to* 1. *That is, if* $0 \le 1/\alpha - \beta_1 < \beta_2 - 1/\alpha < 1$, *the mapping size* $\beta_1$ *yields better results than the mapping size* $\beta_2$ *(see Example 4.1 and Fig. 2). This is different from the effects of the two dimensional counterpart, in which MAM with respect to* $\beta_2 > 1/\alpha$ *yields a better results.*

By using Theorem 3.4 and the Gui-Babuška's error estimation (Theorem 1.1 of [12]), we complete the estimation of the error for MAM in the following:

**Theorem 3.5** *Let* $p_k$ *be the polynomial degree on the element* $\Omega_k = (x_k, x_{k+1})$, $h_k = x_{k+1} - x_k$, $r_k = (\sqrt{x_{k+1}} - \sqrt{x_k})/(\sqrt{x_{k+1}} + \sqrt{x_k})$. *Then we have, if* $k = 1$,

$$|e|_{1,\Omega_1} \le C_1\frac{|\Gamma(\alpha\beta)\sin(\alpha\beta\pi)|h_1^{\alpha-1/2}}{\pi(p_1+1)^{(2\alpha-1)\beta}}.$$

*If* $k \ge 2$ *and* $0 < r_k^2 < 1 - 1/p_k$, *then*

$$|e|_{1,\Omega_k} \le C_k\frac{h_k^{\alpha-1/2}}{\sqrt{1-r_k^2}}\frac{r_k^{p_k+1-\alpha}}{p_k^\alpha}\left(\frac{1}{p_k^{\alpha-1/2}} + (1-r_k^2)^{\alpha-1/2}\right).$$

*If* $k \geq 2$ *and* $1 - 1/p_k \leq r_k^2 \leq 1$, *then*

$$|e|_{1,\Omega_k} \leq C_k h_k^{\alpha-1/2} \frac{r_k^{p_k+1-\alpha}}{p_k^{\alpha-1/2}} \left(\frac{1}{p_k^{\alpha-1/2}} + (1 - r_k^2)^{\alpha-1/2}\right).$$

*Here* $C_k, k \geq 1$, *are constants that depend on neither* $p_k$ *nor* $h_k$. *Actually,* $C_1$ *depends on* $\alpha$ *and* $\beta$ *and* $C_k$ *depend merely on* $\alpha$.

# 4 Numerical Results

Let $\mathcal{U}(w) = \frac{1}{2}\mathcal{B}(w,w)$ and $\Pi(w) = \frac{1}{2}\mathcal{B}(w,w) - \mathcal{F}(w)$ be the strain energy and the potential energy of $w$, respectively. Here $\mathcal{B}(\cdot,\cdot)$ and $\mathcal{F}(\cdot)$ denote the bilinear form and the linear functional in the variational equations of the model problems. Then, the relative error in energy norm is defined as

$$\|e\|_{E,r} = \left[\frac{\|e\|_E^2}{\mathcal{U}(u_{ex})}\right]^{1/2}. \tag{4.13}$$

It was shown in ([24]) that $\|e\|_E^2 = \Pi(u_{fe}) - \Pi(u_{ex})$. By a similar argument, one can show that $\|e\|_E^2 = \mathcal{U}(u_{fe}) - \mathcal{U}(u_{ex})$, provided that one of the following cases applies: all boundary conditions are either homogeneous Dirichlet or arbitrary Neumann boundary conditions; some Dirichlet boundary conditions are nonhomogeneous, but all other boundary conditions are either homogeneous Neumann or homogeneous Dirichlet and the governing equations are homogeneous. Hereafter we refer to $\|e\|_{E,r}$ as the **relative error in the energy norm**. In what follows, $\|e\|_{E,r}(\%)$ denotes $\|e\|_{E,r}$ multiplied by 100 (Relative Error in Energy Norm in Percent). DOF stands for the Number of Degrees of Freedom.

In order to show the effectiveness of MAM, in the framework of the $p$-version of FEM, the method is applied to a Poisson equation on $\Omega = (0,2)$ in which the true solution contains a singularity point with intensity of singularity, $\alpha = 0.65$.

Let $\Delta_1$ be a coarse mesh of $\Omega = (0,2)$ shown as Fig. 1.

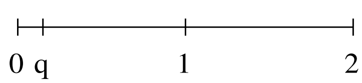


**Fig. 1.** Scheme of the Mesh $\Delta_1$, where $q = 0.15$.

If $u(x) = x^\alpha$ is the true solution of a Poisson equation MAM with $\beta = 1/\alpha$ gives rise to the FE solutions which virtually have no error. Thus, we consider the following example that contains a monotone singularity of type $x^\alpha$.

**Example 4.1** *Let* $u(x) = x^\alpha - x$, *then* $u(x)$ *solves*

$$-\frac{d^2u}{dx^2} = f \text{ in } \Omega, \tag{4.14}$$

*where* $f(x) = -\alpha(\alpha-1)x^{\alpha-2}$.

Let $\Omega_R = (1,2)$ and $\Omega_S = (0,1)$ be the regular and the singular regions, respectively. Then, $\hat{u}(x^*) = u \circ \varphi^\beta(x^*) = (x^*)^{\alpha\beta} - (x^*)^\beta$ on $\hat{\Omega}_S$. Table 1 is the relative errors in energy norm in percent ($\|e\|_{E,r}(\%)$) of FE solutions obtained by applying MAM on the mesh $\Delta_1$ with various mapping sizes $\beta$. From this example, we have the following observations:

**i.** Suppose an intensity of the singularity is $\alpha = 0.65$. Then $|\sin\alpha\beta\pi|$ in an error bound of Theorem 3.6 becomes, respectively, $0.541, 0.0$, and $0.048$ whenever the mapping sizes $\beta$ is $1/0.55, 1/0.65$, and $1/0.66$. Let us note that $|\sin\alpha\beta\pi| = 0$ does not imply $u_{ex} - u_{fe} \approx 0$ since the $(x^*)^\beta$-term of $\hat{u}$ is not highly regular.

**ii.** Table 1 shows that the optimal mapping size $\beta = 1/\alpha$ yields the best results and the mapping size $\beta = 1/0.66$ yields better results than $\beta = 1/0.55$. That is, Theorem 3.6 implies that MAM yields better results whenever the mapping size is closer to an optimal one.

**iii.** The mapped function $u \circ \varphi^{1/0.55}$ has a higher regularity than the mapped function $u \circ \varphi^{1/0.66}$. However, Table 1 shows that the results in "sub-optimal(under-sized)" is better than those in "super-optimal(over-sized)". As mentioned before, these results of MAM are different from those of MAM for two dimensional elasticity problems ([20]).

The results in Table 1 are depicted in Fig. 2.

**Remark 4.1** *For $\alpha < 1$, the right side $f(x)$ is not square integrable. However, $f(x)$ is in $H^{-1}(\Omega)$, the dual space of $H_0^1(\Omega)$, whenever $\alpha > 0.5$. In fact, it follows from Theorem 7.17 of ([9]) that if $v \in H_0^1(0,1)$, then $|v(x)| < Cx^{0.5}$. Thus, we have*

$$|\int_0^1 x^{\alpha-2} v(x)dx| \leq C \int_0^1 [x^{\alpha-2}x^{0.5}]dx = C \int_0^1 x^{\alpha-1.5}dx < \infty,$$

*whenever $(\alpha - 1.5) + 1 > 0$, that is, $\alpha > 0.5$.*

**Table 1.** Relative Error in Energy Norm (%) for various Mapping Sizes:$\beta = 1/0.65$ (optimal), $\beta = 1/0.66$ (under-sized), and $\beta = 1/0.55$ (over-sized), when the intensity of singularity is $\alpha = 0.65$.

| p-deg | DOF | No map | $\beta = 1/0.55$ | $\beta = 1/0.65$ | $\beta = 1/0.66$ |
|---|---|---|---|---|---|
| 1 | 5 | 43.837 | 28.006 | 6.161 | 8.105 |
| 2 | 10 | 30.049 | 19.417 | 0.673 | 1.995 |
| 3 | 15 | 25.931 | 15.017 | 0.112 | 1.383 |
| 4 | 20 | 23.766 | 12.610 | 0.042 | 1.153 |
| 5 | 25 | 22.297 | 11.046 | 0.027 | 1.013 |
| 6 | 30 | 21.193 | 9.929 | 0.020 | 0.915 |
| 7 | 35 | 20.319 | 9.082 | 0.016 | 0.842 |
| 8 | 40 | 19.605 | 8.412 | 0.013 | 0.784 |
| 9 | 45 | 19.007 | 7.865 | 0.011 | 0.738 |
| 10 | 50 | 18.498 | 7.408 | 0.009 | 0.699 |

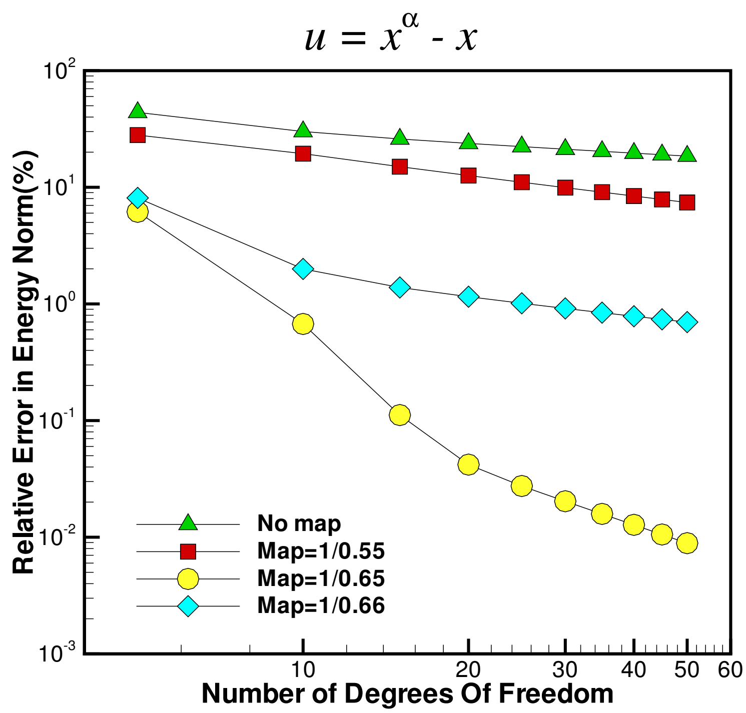


**Fig. 2.** Relative Errors in Energy Norm (%) for the results obtained by MAM with mapping sizes, $\beta = 1/0.65$ (optimal), $\beta = 1/0.66$ (sub-optimal), $\beta = 1/0.55$(super-optimal), and $\beta = 1$(No map), respectively.

By the auxiliary mapping (2.5), the pollution effect caused by the monotone singularities can be either completely removed (when $\beta$ is an optimal one) or greatly reduced (when $\beta$ is close to an optimal one). However, it fails to remove the pollution effect caused by "$\log x$". Unlike the two dimensional counterpart([21]), an exponential auxiliary mapping defined by $\varphi(\xi) = \exp(\xi)$ is of no use in one dimensional variational equation.

**Example 4.2** *Let* $u(x) = x^\alpha(\ln x) - x$. *Then* $u(x)$ *solves* $-\dfrac{d^2u}{dx^2} = f$ *in* $\Omega$, *where* $f = -x^{\alpha-2}[(\alpha^2-\alpha)\ln x + 2\alpha - 1]$.

Suppose $\beta = 1/\alpha$. Then $\hat{u} = u \circ \varphi^\beta = \beta x^* \ln x^*$ becomes smoother than $u$. However, because of the presence of $\ln x$, $\hat{u}$ is still not smooth enough to yield highly accurate FE solutions. In dimension two, the mapping size $\beta$ has no restrictions and hence, by selecting $\beta > 2$, it is possible to make $\hat{u}$ much smoother. The relative errors of FE solutions obtained by applying MAM to various mapping sizes to the second problem are shown in Table 2 and Fig. 3.

**Table 2.** Relative Error in Energy Norm (%) for various Mapping Sizes

| p-deg | DOF | No map | $\beta = 1/0.60$ | $\beta = 1/0.61$ | $\beta = 1/0.65$ |
|---|---|---|---|---|---|
| 1 | 5 | 74.770 | 20.639 | 24.224 | 36.981 |
| 2 | 10 | 66.465 | 9.094 | 12.385 | 24.942 |
| 3 | 15 | 62.250 | 5.769 | 8.598 | 20.271 |
| 4 | 20 | 59.460 | 4.213 | 6.648 | 17.584 |
| 5 | 25 | 57.383 | 3.336 | 5.432 | 15.774 |
| 6 | 30 | 55.744 | 2.799 | 4.594 | 14.446 |
| 7 | 35 | 54.402 | 2.456 | 3.979 | 13.418 |
| 8 | 40 | 53.276 | 2.231 | 3.507 | 12.590 |
| 9 | 45 | 52.313 | 2.082 | 3.133 | 11.905 |
| 10 | 50 | 51.478 | 1.982 | 2.829 | 11.326 |

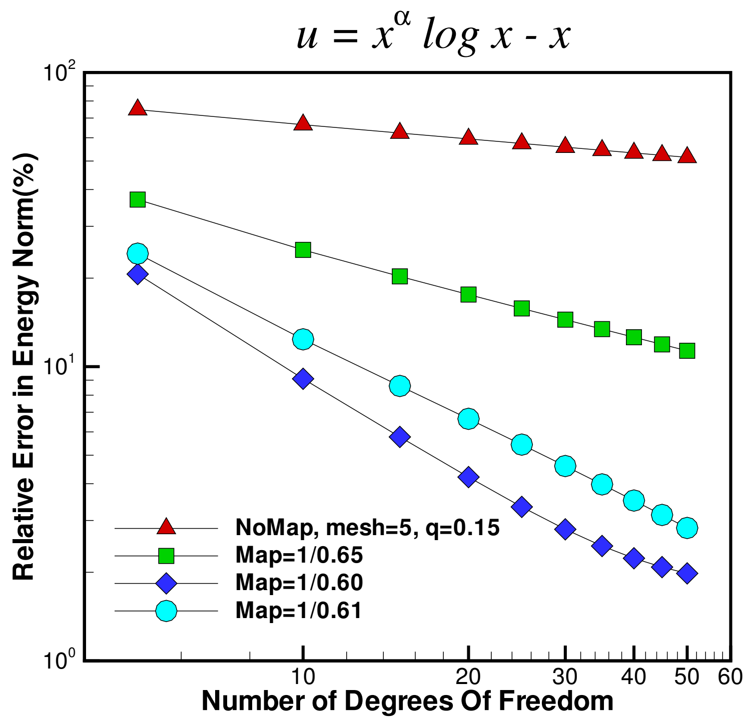


**Fig. 3.** Relative Errors in Energy Norm (%) for the results obtained by MAM with mapping sizes, $\beta = 1/0.65$ (optimal), $\beta = 1/0.66$ (sub-optimal), $\beta = 1/0.55$(super-optimal), and $\beta = 1$(No map), respectively.

Finally, in dealing with oscillating singularity, MAM is compared with Partition of Unity Finite Element Method(PUFEM), introduced by Babuška and Melenk([18],[19]), is one of the most flexible methods to deal with singularities.

**Example 4.3** *Consider Poisson equation in the domain* $\Omega = (0, 2)$ *whose solution is*

$$u(x) = x^{\alpha} \sin(\varepsilon \ln x/2),$$

*where* $\alpha = 0.65$ *and* $\epsilon = 0.1$.

Let us consider the partition

$$\Delta_2 : 0(= x_1) < 1/2(= x_2) < 1(= x_3) < 2(= x_4).$$

Let $\{Q_1^* = (x_1, x_3), Q_2^* = (x_2, x_4)\}$ be a cover of $\Omega$ and the partition of unity subordinate to this cover be defined by

$$\begin{aligned} \varphi_1 &= \begin{cases} 1 \text{ if } x \in (0, 1/2) \\ 2(1-x) \text{ if } x \in (1/2, 1) \end{cases} \\ \varphi_2 &= \begin{cases} 2x-1 \text{ if } x \in (1/2, 1) \\ 1 \text{ if } x \in (1, 2). \end{cases} \end{aligned}$$

Now, we define the PUFEM space by $V = V_1 \cup V_2$, where

$$\begin{aligned} V_1 &= \{\Psi_p^1(x)\} \cup \{\varphi_1(x)x^{0.65} \sin(0.1 \ln x/2)\}, \\ V_2 &= \{\Psi_p^2(x)\}. \end{aligned}$$

Here $\Psi_p^k$ are the usual piecewise polynomial basis functions on $Q_k^*$. Suppose prior knowledge on the oscillatory singularity is available (that is, $\alpha = 0.65, \epsilon = 0.1$). Then PUFEM yields highly accurate solution as it was shown in Fig. 4.1. Let us note that if $V_2$ were $\{\Psi_p^2(x)\} \cup \{\varphi_2(x)x^{0.65} \sin(0.1 \ln x/2)\}$ in the construction of the PUFEM space, PUFEM would have had no error.

However, suppose instead of the space $V = V_1 \cup V_2$, a perturbed space $\tilde{V} = W \cup V_2$, where $W = \{\Psi_p^1(x)\} \cup \{\varphi_1(x)x^{0.65} \sin(0.3 \ln(x/2))\}$, is used. Then PUFEM yields the results in the column "PUFEM3" of Table 4. Those results show that PUFEM fails to give a practical solution unless prior knowledge on the singularity is available. Thus, in order to have highly accurate solution, PUFEM needs to have prior knowledge on the structure of the singularity.

In order to compare the results by PUFEM with the results by MAM with respect to the same size DOF, we consider a finer mesh

$$\Delta_3 : 0(= x_1) < (0.15)^2(= x_2) < 0.15(= x_3) < 1(= x_4) < 2(= x_5).$$

The left half of Table 4 is the results obtained by applying MAM on $\Delta_3$ with $\Omega_S = (x_1, x_4)$ and the results obtained by the standard FEM on the mesh $\Delta_3$. The relative errors in energy norm (%) of the results by MAM, No Map, and PUFEM are depicted in Fig. 4.

If the exact oscillating factor is known in advance, PUFEM yields highly accurate FE solutions. Otherwise, MAM gives reasonably accurate FE solutions. Moreover, MAM is able to work in any conventional FEM codes.

**Table 4.** Relative Error in Energy Norm (%) of the FE solutions obtained by MAM ("MAM"), PUFEM with exact oscillatory basis function ($\varepsilon = 0.1$,"PUFEM1"), and PUFEM with perturbed oscillatory basis function($\varepsilon = 0.3$,"PUFEM3").

| p-deg | DOF | No map | $\beta = 1/0.65$ | PUFEM, $\varepsilon = 0.1$ | PUFEM, $\varepsilon = 0.3$ |
|---|---|---|---|---|---|
| 1 | 4 | 63.787 | 23.816 | 5.6336 | 73.842 |
| 2 | 9 | 51.999 | 9.397 | 2.1429 | 73.280 |
| 3 | 14 | 46.500 | 6.385 | 0.6616 | 71.441 |
| 4 | 19 | 43.008 | 5.400 | 0.2249 | 68.203 |
| 5 | 24 | 40.466 | 5.044 | 0.0817 | 63.913 |
| 6 | 29 | 38.487 | 4.915 | 0.0309 | 59.118 |
| 7 | 34 | 36.884 | 4.872 | 0.0120 | 54.285 |
| 8 | 39 | 35.550 | 4.860 | 0.0047 | 49.717 |
| 9 | 44 | 34.417 | 4.859 | 0.0018 | 45.567 |
| 10 | 49 | 33.441 | 4.859 | 0.0004 | 41.881 |

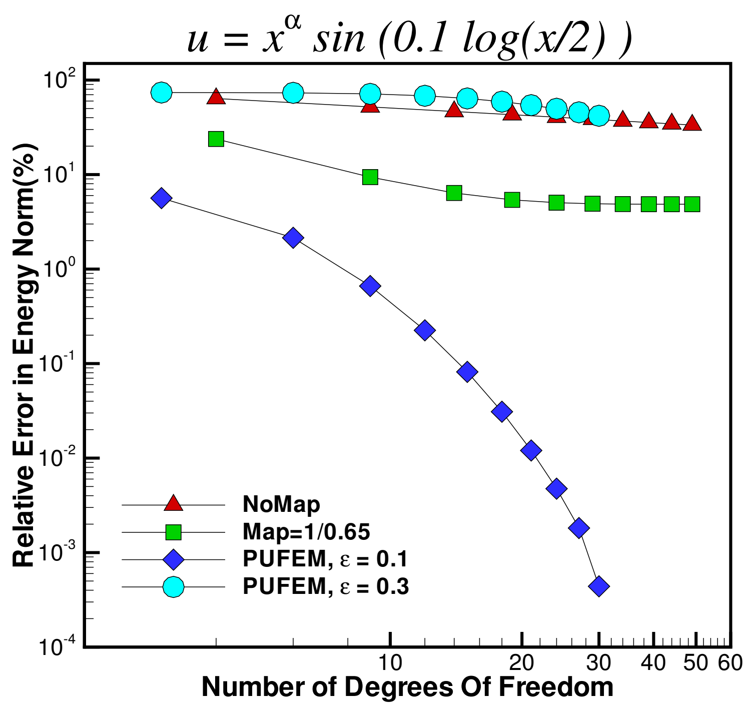


**Fig. 4.** Relative Error in Energy Norm in % for MAM, PUFEM with the exact solution $x^{0.65}\sin(\ln x/2)$ (PUFEM1), and PUFEM with a perturbed function $x^{0.65}\sin(3\ln x/2)$ (PUFEM3).

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