A Generalization of Beatty's Theorem

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In 1926 Sam Beatty made the following discovery, which he posed as a problem in [1]. If a is a positive irrational number, the sequences $m(1+a), m = 1, 2, \ldots$ and $n(1+a^{-1}), n = 1, 2, \ldots$ together contain exactly one number from each of the intervals $(k, k+1), k = 1, 2, 3, \ldots$. The problem was solved by Ostrowski and Aitken[4] and generalized to a larger class of sequences by Lambeck and Moser[3]. The authors are grateful for UNC Charlotte undergraduate James Rudzinski for suggesting the combinatorial game that lead to this paper.

The purpose of this paper is to extend this theorem from sequences to continuous functions. We first establish some notation. Let $P = (a_0 = 0, a_1, a_2, \ldots)$ be a strictly increasing unbounded sequence of real numbers. For any nonnegative number x, let $\lfloor x \rfloor_P$ be the largest member of P that does not exceed x, and for positive x,

$$\rfloor x \lfloor_P = \begin{cases} \lfloor x \rfloor_P & \text{if } x \text{ is not in } P \\ a_{i-1} & \text{if } x \in P \text{ and } \lfloor x \rfloor_P = a_i. \end{cases}$$

Also, if $t > 0$, we define $N_t = \{0, t, 2t, 3t, 4t, \cdots\}$ and $N_t^+ = N_t \backslash \{0\}$.

Main Theorem: Let F and G be real, continuous, strictly increasing, functions with domains $[0, \infty)$ satisfying $F(0) = G(0) = 0$ and $\lim_{x\to\infty}(F(x) + G(x)) = \infty$. For all $t > 0$, let $P_t = \{0, (F+G)^{-1}(t), (F+G)^{-1}(2t), (F+G)^{-1}(3t), \ldots\}$, and $P_t^+ = P_t \backslash \{0\}$. Then the two sequences A_t and B_t defined by $A_t = \{\rfloor (F^{-1})^{-1}(n) \lfloor_{P_t} : n \in N_t^+\}$ and $B_t = \{\lfloor (G^{-1})^{-1}(n) \rfloor_{P_t} : n \in N_t^+\}$ partition P_t^+ for all $t > 0$. Also, the elements of A_t are distinct and the elements of B_t are distinct.

We conjecture that the theorem remains true even when $F+G$ is bounded. Also, it is easy to see that an analogous version exists for functions $F:[a,b)\to[0,\infty)$ and $G:[a,b)\to[0,\infty)$, where F and G are continuous, strictly increasing, and satisfy $F(a)=G(a)=0$ and $\lim_{x\to b}(F(x)+G(x))=\infty$.

Note that the two functions F and G in the theorem are independent, unlike those in the Lambek and Moser paper where each one determines the other. This shows that using continuous functions provides a much richer theory. Note also that our conclusion is true for all $t>0$. Thus we have an uncountable number of partitions. This forces rigid restrictions on the sequences A_t and B_t.

The following is an example illustrating the theorem. We can choose F and G independently. However, we have chosen F and G so that we can compute $F^{-1}, G^{-1}, (F+G)^{-1}$ explicitly. Of course, if we cannot compute the inverse of one of our functions explicitly, then we just leave that inverse in implicit form just as we would for the function sin^{-1}. Let $F(x)=ax^2+bx, G(x)=cx^2+dx, a,b,c,d>0$. Then $(F+G)(x)=(a+c)x^2+(b+d)x, F^{-1}(n)=\frac{-b+\sqrt{b^2+4an}}{2a}, G^{-1}(n)=\frac{-d+\sqrt{d^2+4cn}}{2c}$ and $(F+G)^{-1}(n)=\frac{-(b+d)+\sqrt{(b+d)^2+4(a+c)n}}{2(a+c)}$. Then

$$P_t=\left\{\frac{-(b+d)+\sqrt{(b+d)^2+4(a+c)n}}{2(a+c)} : n=0,t,2t,3t,\ldots\right\},$$

$$A_t=\left\{\rfloor\frac{-b+\sqrt{b^2+4an}}{2a}\lfloor_{P_t} : n=t,2t,3t,\ldots\right\},$$

and

$$B_t = \left\{ \lfloor \tfrac{-d+\sqrt{d^2+4cn}}{2c} \rfloor_{P_t} : n = t, 2t, 3t, \ldots \right\}.$$

The theorem guarantees that for all $t > 0$, A_t and B_t partition P_t^+ and the members of A_t and B_t are distinct. The functions $F(x) = \frac{x}{1+a}$ and $G(x) = \frac{ax}{1+a}$, where a is a positive irrational number, can be used to prove Beatty's theorem. The reader might also like to use $F(x) = 3(x^2 + x), G(x) = x^3$ to give an example for himself.

We next prove the main theorem by first proving a more specialized result, for which we need more notation. Let $N = \{0, 1, 2, 3, \ldots\}$, $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, and for any $t > 0$, $Z_t = \{\cdots, -3t, -2t, -t, 0, t, 2t, 3t, \cdots\}$ and $N_t = \{0, t, 2t, 3t, \ldots\}$. Thus $N = N_1$ and $Z = Z_1$. Suppose $x = nt + \underline{\epsilon}$ where $n \in Z$ and $0 \le \underline{\epsilon} < t$. Then $\lfloor x \rfloor_t = nt$. Suppose $x = mt + \overline{\epsilon}$ where $m \in Z$ and $0 < \overline{\epsilon} \le t$. Then $\lceil x \rceil_t = mt + t$. Thus $\lfloor x \rfloor_t$ and $\lceil x \rceil_t$ are the *floor* and *ceiling* functions with respect to the grid Z_t. Of course, $\lfloor x \rfloor_1 = \lfloor x \rfloor$ and $\lceil x \rceil_1 = \lceil x \rceil$ where $\lfloor x \rfloor$ and $\lceil x \rceil$ are the usual floor and ceiling functions. If $F : A \to B$ is a function on A where A is a countable linearly ordered set, then we consider $S = \{F(x) : x \in A\}$ to be both a set and a sequence. The reason for this is that we wish to discuss the set S of functional values and we also need to discuss the fact that the members of the *sequence* are distinct.

Definition: Suppose f and g are any real functions with domain $[0, \infty)$. We say that f and g are complementary if f and g satisfy $f(0) = g(0) = 0$, f and g are strictly increasing on $[0, \infty)$, and $f(x) + g(x) = x$ for all $x \ge 0$.

It is easy to see that for any $\epsilon > 0$, $\quad f(x+\epsilon) - f(x) < \epsilon$ and $0 < g(x+\epsilon) - g(x) < \epsilon$ for all non-negative x, so complimentary functions are continuous. Also, it is easy to see that $0 < f(x) < x$ and $0 < g(x) < x$ for all positive x.

<u>Theorem 1</u>: Suppose $f : [0, \infty) \to R$, $g : [0, \infty) \to R$ are complementary. Then

a. $\forall t > 0$, the following two sets partition the set $N_t^+ = \{t, 2t, 3t, 4t, \cdots\}$: $A = \left\{\lceil f^{-1}(n) \rceil_t : n \in N_t^+\right\}$ and $B = \{\lfloor g^{-1}(n) + t \rfloor_t : n \in N_t\}$. Also, the elements of the sequence A are distinct and the elements of the sequence B are distinct.

b. Also, for every $t > 0$, the following two sets partition N_t^+ : $C = \left\{\lceil f^{-1}(n) - t \rceil_t : n \in N_t^+\right\}$ and $D = \left\{\lfloor g^{-1}(n) \rfloor_t : n \in N_t^+\right\}$. Also, the elements of the sequence C are distinct and the elements of the sequence D are distinct.

We note that if one of f, g is bounded above, then in both a. and b., one of the two sets is finite.

We will now prove theorem 1 through a series of lemmas. It is convenient for notational reasons to assume that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Although we will assume this, a slightly more clumsy version holds without the assumption. In the following lemmas, we consider $t > 0$ to be arbitrary but fixed.

<u>Lemma 1</u>: Suppose $h : N_t \to N_t$ satisfies $h(0) = 0$, $h(n + t) - h(n) \in \{0, t\}$ for all $n \in N_t$, and $\lim_{n\to\infty} h(n) = \lim_{n\to\infty}(n - h(n)) = \infty$.

Define $k : N_t \to N_t$ as follows: $\forall n \in N_t, k(n) = n - h(n)$. It is easy to see that $k(0) = 0, \lim_{n\to\infty} k(n) = \infty$ and $\forall n \in N_t, k(n+t) - k(n) \in \{0, t\}$. Also both h and k are non-decreasing on N_t.

Next, $\forall m \in N_t$, define $H(m)$ to be the smallest member of N_t such that $h(H(m)) = m$ and $K(m)$ to be the smallest member of N_t such that $k(K(m)) = m$. Then

a. $H(0) = K(0) = 0$.

b. H and K are strictly increasing.

c. the two sets $\left\{H(n) : n \in N_t^+\right\}$ and $\left\{K(n) : n \in N_t^+\right\}$ partition the set N_t^+.

Proof: Parts a and b are easy to see. To prove c, first observe that $\forall n \in N_t^+$, either $h(n) - h(n-t) = t$ and $k(n) - k(n-t) = 0$ or $h(n) - h(n-t) = 0$ and $k(n) - k(n-t) = t$. This is because $h(n) + k(n) = n$. From this it follows that $\forall x \in N_t^+, x$ is in the range set of exactly one of H and K. $\square$

Definitions: Let f, g be the two complementary functions used in the hypothesis of the Theorem 1. For convenience we again assume that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$.

Let us define for all positive numbers t, $h : N_t \to N_t$ by $h(n) = \lfloor f(n) \rfloor_t$ and define $k : N_t \to N_t$ by $k(n) = n - h(n)$. It follows that $k(n) = n - \lfloor f(n) \rfloor_t = n + \lceil -f(n) \rceil_t = \lceil n - f(n) \rceil_t = \lceil g(n) \rceil_t$. i.e., $k(n) = \lceil g(n) \rceil_t$. Note that k is being defined from h the same way as it was in lemma 1. These functions h and k are used in the remainder of our proof of the theorem.

Lemma 2: The function h defined above satisfies the hypothesis given in lemma 1, namely

1. $h(0) = 0$,
2. $\forall n \in N_t, h(n+t) - h(n) \in \{0, t\}$,
3. $\lim_{n\to\infty} h(n) = \lim_{n\to\infty}(n - h(n)) = \infty$.

Proof: Since $h(n) = \lfloor f(n) \rfloor_t$, $n - h(n) = \lceil g(n) \rceil_t$, $f(0) = 0$ and $\lim_{n\to\infty} f(x) = \lim_{n\to\infty} g(x) = \infty$, conclusions 1 and 3 are easy to see. We now prove conclusion 2.

Now $0 < f(n+t) - f(n) < t$ since $\forall \epsilon > 0, \forall x \in [0, \infty), 0 < f(x+\epsilon) - f(x) < \epsilon$.

Therefore $\lfloor f(n+t) \rfloor_t - \lfloor f(n) \rfloor_t \in \{0, t\}$. It follows that $h(n+t) - h(n) \in \{0, t\}$ since $h(n+t) = \lfloor f(n+t) \rfloor_t$ and $h(n) = \lfloor f(n) \rfloor_t$. This completes the proof of lemma 2.

Since $k(n) = n - h(n)$, as in lemma 1, it follows that $k(0) = 0, \lim_{n\to\infty} k(n) = \infty$ and $\forall n \in N_t$, $k(n+t) - k(n) \in \{0, t\}$. Now $\forall m \in N_t$, $H(m), K(m)$ are defined in lemma 1 since h and k satisfy the conditions specified in lemma 1. That is, $\forall m \in N_t$, $H(m)$ is the smallest member of N_t such that $h(H(m)) = \lfloor f(H(m)) \rfloor_t = m$ and $\forall m \in N_t$, $K(m)$ is the smallest member of N_t such that $k(K(m)) = \lceil g(K(m)) \rceil_t = m$. The following two lemmas will now add considerable power to lemma 1 when we use $h(n) = \lfloor f(n) \rfloor_t$ and $k(n) = n - h(n) = \lceil g(n) \rceil_t$ in lemma 1.

Since $f : [0, \infty) \to R$ and $g : [0, \infty) \to R$ are complementary, we recall that

$f(0) = g(0) = 0$, f, g are strictly increasing and continuous on $[0, \infty)$ and also $\forall \epsilon > 0, \forall x \in [0, \infty), 0 < f(x+\epsilon) - f(x) < \epsilon, 0 < g(x+\epsilon) - g(x) < \epsilon$. As always, for convenience, we are assuming that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. $\square$

Lemma 3: $\forall n \in N_t, H(n) = \lceil f^{-1}(n) \rceil_t$.

Proof: Let us abbreviate $f^{-1}(n) = x$. Therefore $f(x) = n$.

Define $\overline{x} = \lceil x \rceil_t$. Of course, $\overline{x} \in N_t$. Now by lemma 1, $H(0) = 0$. Also, $f^{-1}(0) = 0$. So $H(0) = \lceil f^{-1}(0) \rceil_t = \lceil 0 \rceil_t = 0$ is obvious. Therefore, we now assume that $n \in \{t, 2t, 3t, 4t, \cdots\}$. Then note that $f^{-1}(n) = x > t$ since $\forall x \in (0, \infty), 0 < f(x) < x$. Now since by lemma 2, $\forall \overline{n} \in N_t, h(\overline{n} + t) - h(\overline{n}) \in \{0, t\}$, lemma 3 is proved once we prove the following:

a. $h(\overline{x}) = \lfloor f(\overline{x}) \rfloor_t = n$

b. $h(\overline{x} - t) = \lfloor f(\overline{x} - t \rfloor_t = n - t$.

Proof of a: Now $\overline{x} - x = \overline{\epsilon}$ where $0 \le \overline{\epsilon} < t$. Now $\lfloor f(\overline{x}) \rfloor_t = \lfloor f(x + \overline{\epsilon} \rfloor_t = \lfloor f(x+\overline{\epsilon}) - f(x) + f(x) \rfloor_t = \lfloor f(x+\overline{\epsilon}) - f(x) + n \rfloor_t = \lfloor f(x+\overline{\epsilon}) - f(x) \rfloor_t + n$. Now $0 \le f(x+\overline{\epsilon}) - f(x) \le \overline{\epsilon} < t$ since $0 \le \overline{\epsilon}$. It follows that $\lfloor f(x+\overline{\epsilon}) - f(x) \rfloor_t = 0$. Therefore $\lfloor f(\overline{x}) \rfloor_t = \lfloor f(x+\overline{\epsilon}) - f(x) \rfloor_t + n = 0 + n = n$.

Proof of b: Define $\underline{x} = \overline{x} - t = \lceil x - t \rceil_t$. Let $x - \underline{x} = \underline{\epsilon}$, where $0 < \underline{\epsilon} \le t$. Now $\lfloor f(\overline{x} - t) \rfloor_t = \lfloor f(\underline{x}) \rfloor_t = \lfloor f(x - \underline{\epsilon}) \rfloor_t = \lfloor f(x-\underline{\epsilon}) - f(x) + f(x) \rfloor_t = \lfloor f(x-\underline{\epsilon}) - f(x) \rfloor_t + n$. Since $0 < \underline{\epsilon}$, we see that $0 < f(x) - f(x - \underline{\epsilon}) < \underline{\epsilon} \le t$. Therefore $-t < f(x-\underline{\epsilon}) - f(x) < 0$. It follows that $\lfloor f(x-\underline{\epsilon}) - f(x) \rfloor_t = -t$. Hence $\lfloor f(\overline{x} - t) \rfloor_t =$

$\lfloor f(x-\underline{\epsilon}) - f(x) \rfloor_t + n = n - t.$ $\square$

Lemma 4: $K(0) = 0$ and $\forall n \in N_t,\ K(n+t) = \lfloor g^{-1}(n) + t \rfloor_t$.

Proof: Recall that K was defined at the end of the proof of lemma 2. Note that $K(0) = 0$ from lemma 1. We now show that for all $n \in N_t,\ K(n+t) = \lfloor g^{-1}(n) + t \rfloor_t$. Let us first consider $n = 0$. We will show that $K(0+t) = \lfloor g^{-1}(0) + t \rfloor_t$. Now $g^{-1}(0) = 0$. Therefore, we need to show that $K(t) = \lfloor t \rfloor_t = t$. Now $0 < g(t) < t$ is true. Therefore, $k(t) = \lceil g(t) \rceil_t = t$. Since $k(0) = 0,\ k(t) = t$, we see that $K(t) = t$ must be true by the definition of K. We will now suppose $n \in \{t, 2t, 3t, \cdots\}$. We must show that $K(n+t) = \lfloor g^{-1}(n) + t \rfloor_t$. Let us abbreviate $g^{-1}(n) = x$. Therefore, $g(x) = n$. Since $n \in \{t, 2t, \cdots\}$ and for all $x \in (0, \infty),\ 0 < g(x) < x$, it follows that $x > t$. Define $\overline{x} = \lfloor x + t \rfloor_t$. Of course, $\overline{x} \in N_t$. Since $\forall \overline{n} \in N_t$, $k(\overline{n}+t) - k(\overline{n}) \in \{0, t\}$, the lemma is proved once we prove the following:

a. $k(\overline{x}) = \lceil g(\overline{x}) \rceil_t = n + t$.

b. $k(\overline{x} - t) = \lceil g(\overline{x} - t) \rceil_t = n$.

Proof of a: Define $\overline{x} - x = \overline{\epsilon}$, where $0 < \overline{\epsilon} \leq t$. Now $\lceil g(\overline{x}) \rceil_t = \lceil g(x + \overline{\epsilon}) \rceil_t = \lceil g(x+\overline{\epsilon}) - g(x) + g(x) \rceil_t = \lceil g(x+\overline{\epsilon}) - g(x) \rceil_t + n$. Now $0 < g(x+\overline{\epsilon}) - g(x) < \overline{\epsilon} \leq t$, since $0 < \overline{\epsilon}$. Therefore $\lceil g(x+\overline{\epsilon}) - g(x) \rceil_t = t$. Therefore

$\lceil g(\overline{x}) \rceil_t = \lceil g(x+\overline{\epsilon}) - g(x) \rceil_t + n = n + t.$

Proof of b: Remember $x > t$. Let $\underline{x} = \overline{x} - t = \lfloor x \rfloor_t$. We must show $\lceil g(\underline{x}) \rceil_t = n$. Define $x - \underline{x} = \underline{\epsilon}$, where $0 \leq \underline{\epsilon} < t$. Now $\lceil g(\underline{x}) \rceil_t = \lceil g(x - \underline{\epsilon}) \rceil_t = \lceil g(x-\underline{\epsilon}) - g(x) + g(x) \rceil_t =$

$\lceil g(x-\underline{\epsilon})-g(x)\rceil_t+n$. Now $0 \leq g(x)-g(x-\underline{\epsilon}) \leq \underline{\epsilon} < t$, since $0 \leq \underline{\epsilon} < t$. Therefore $-t < g(x-\underline{\epsilon})-g(x) \leq 0$. Therefore $\lceil g(x-\underline{\epsilon})-g(x)\rceil_t = 0$. Therefore $\lceil g(\underline{x})\rceil_t = \lceil g(x-\underline{\epsilon})-g(x)\rceil_t + n = 0+n = n$. $\square$

We now have enough information to prove the theorem 1.

<u>Proof of Theorem 1</u>. We know that f and g are complementary functions. Also, we know that f and g are continuous and bounded above by the identity function. We also recall that we defined $h : N_t \rightarrow N_t$ and $k : N_t \rightarrow N_t$ as follows: $\forall n \in N_t, h(n) = \lfloor f(n)\rfloor_t, k(n) = n - h(n) = \lceil g(n)\rceil_t$. In lemma 2 we showed that this h satisfies the hypothesis of lemma 1. Also this k is defined from this h in the same way as in lemma 1.

We also defined the functions $H(m)$ and $K(m)$ in lemma 1. We also know from lemma 3, 4 that $\forall n \in N_t,\ H(n) = \lceil f^{-1}(n)\rceil_t$ and $\forall n \in N_t\ K(n+t) = \lfloor g^{-1}(n)+t\rfloor_t$.

From lemma 1, we know that $\forall t > 0$, the following two sets partition the set N_t^+ :
$\left\{H(n) : n \in N_t^+\right\} = \left\{\lceil f^{-1}(n)\rceil_t : n \in N_t^+\right\}$ and $\left\{K(n) : n \in N_t^+\right\} = \{\lfloor g^{-1}(n)+t\rfloor_t : n \in N_t\}$.

Recall that H and K are strictly increasing functions whose image sets are disjoint. Therefore, the elements of these two sets, when viewed as a sequence, are all distinct.

Now since $g^{-1}(0) = 0$, we know that $\{\lfloor g^{-1}(n)+t\rfloor_t : n \in N_t\} = \{t\} \cup \left\{\lfloor g^{-1}(n)+t\rfloor_t : n \in N_t^+\right\}$. Therefore, $\left\{\lceil f^{-1}(n)\rceil_t : n \in N_t^+\right\}$ and $\left\{\lfloor g^{-1}(n)+t\rfloor_t : n \in N_t^+\right\}$ partition the set $\{2t, 3t, 4t, 5t, \cdots\}$. Therefore $\left\{\lceil f^{-1}(n)-t\rceil_t : n \in N_t^+\right\}$ and $\left\{\lfloor g^{-1}(n)\rfloor_t : n \in N_t^+\right\}$

partition N_t^+. Also, the elements of these two sets, when viewed as a sequence, are all distinct. $\square$

We now use theorem 1 to prove the Main Theorem.

Suppose $h : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing, and satisfies both $h(0) = 0$ and $\lim_{x\to\infty} h(x) = \infty$. We define for all $t > 0$, the sets P_t and P_t^+ as follows:

$$P_t = \{h(0) = 0, h(t), h(2t), \ldots\} \text{ and } P_t^+ = \{h(t), h(2t), \ldots\}.$$

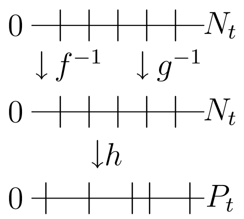
Next consider the following lemma.

<u>Lemma 5</u>: Suppose f and g are complementary functions and h satisfies the conditions listed above. Then the sets A and B defined by $A = \{\rfloor (fh^{-1})^{-1}(n) \lfloor_{P_t} : n \in N_t^+\}$ and $B = \{\lfloor (gh^{-1})^{-1}(n) \rfloor_{P_t} : n \in N_t^+\}$ partition P_t^+ for all $t > 0$. Also, the elements of A (when viewed as a sequence) are distinct and the same is true for the elements of B.

<u>Proof</u>: Note that $(fh^{-1})^{-1}(n) = hf^{-1}(n)$ and $(gh^{-1})^{-1}(n) = hg^{-1}(n)$. By part b of theorem 1, $\{\lceil f^{-1}(n) - t \rceil_t : n \in N_t^+\}$ and $\{\lfloor g^{-1}(n) \rfloor_t : n \in N_t^+\}$ partition N_t^+.

Also, the elements of each of the two sets, when viewed as sequences, are distinct. Since h is an order preserving change of variable and since $\rfloor x \lfloor_{P_t}$ is defined for the partition of P_t topologically analogous to the way $\lceil x - t \rceil_t$ is defined for the partition of N_t and since $\lfloor x \rfloor_{P_t}$ is defined for the partition P_t topologically analogous to the way $\lfloor x \rfloor_t$ is defined for the partition of N_t, the conclusion is easy to see. See figure

1. $\square$



h is an order preserving change of variable.

We now prove the Main Theorem.

Proof: Since both F and G are strictly increasing, so is $F+G$. Also, $F+G$ is unbounded and $(F+G)(0)=0$. It follows that $(F+G)^{-1}$ exists for all non-negative x. Also, $((F+G)(F+G)^{-1})(x) = (F(F+G)^{-1})(x) + (G(F+G)^{-1})(x) = x$ for all nonnegative x. For convenience of notation, denote $F(F+G)^{-1}$ by f, $G(F+G)^{-1}$ by g and $(F+G)^{-1}$ by h. Note that f, g, and h satisfy the hypothesis of the last lemma and that $fh^{-1} = F$ and $gh^{-1} = G$. The lemma completes the proof. $\square$

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