

Dynamic One-Pile Blocking Nim

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Abstract

The purpose of this paper is to solve a class of combinatorial games consisting of one-pile counter pickup games for which the number of counters that can be removed on each successive move changes during the play of the game. Both the minimum and maximum number of counters that can be removed is dependent upon the move number. Also, on each move, the opposing player can block some of the moving player's moves. The number of blocks also depends upon the move number. In a later paper, these same variables will depend upon both the move number and the pile size.

Notation: If $a \leq b$ are integers, $[a, b] = \{x : x \text{ is an integer and } a \leq x \leq b\}$.

Rules of Game: Two players alternate removing positive numbers of counters from a single pile of counters according to the following rules: A sequence of non-negative integers c_1, c_2, c_3, \dots and two sequences of positive integers m_1, m_2, m_3, \dots , M_1, M_2, M_3, \dots are given that satisfy the following conditions:

(a) $0 \leq c_1 \leq c_2 \leq c_3 \leq \dots$.

(b) $\forall i$, define $\Delta_i = M_i - m_i$. Then $0 \leq \Delta_1 - c_1 \leq \Delta_2 - c_2 \leq \Delta_3 - c_3 \leq \dots$.

Of course, (a) and (b) imply that $\forall i, 0 \leq c_i \leq \Delta_i$ and $1 \leq m_i \leq M_i$. Subject to the second rule, the player who makes the k th move, $k = 1, 2, 3, \dots$, can remove

from the pile any $x \in [m_k, M_k]$ that is in the pile. Of course, the player cannot move when the pile size is less than m_k . There is also a second rule. Before a player makes the k th move, $k = 1, 2, 3, \dots$, his opponent can block up to \underline{c}_k of his moves, and he can do this in any way that he chooses including blocking less than c_k moves.

As an example, suppose $[m_3, M_3] = [5, 10]$ and $c_3 = 2$. Also, suppose the pile size before the 3rd move is 15. Since $c_3 = 2$, the opponent, before the moving player makes the 3rd move, can block up to 2 of the moving player's moves. Suppose that the opponent blocks the removal of 6 counters and the removal of 10 counters. This means the moving player can remove from the 15 counter pile either 5,7,8 or 9 counters. Of course, if the pile size before the 3rd move is 6, then if the opponent blocks the removal of 5 and 6 counters, the moving player could not move at all since it is impossible to remove 7,8,9 or 10 counters from a 6 counter pile.

The game ends as soon as one of the two players cannot move, and the winner of the game is the one who makes the last move.

Remark: Although neither the theorem nor the proof in this paper use any graphs, graphs were used extensively in the derivation of both the theorem and the proof.

Theorem: The safe positions of this game are the following:

$$[0, m_1 + c_1 - 1],$$

$$[M_1 + m_2 - c_1 + c_2, M_2 + m_1 + m_3 + c_1 - c_2 + c_3 - 1],$$

$$[M_1 + M_3 + m_2 + m_4 - c_1 + c_2 - c_3 + c_4, M_2 + M_4 + m_1 + m_3 + m_5 + c_1 - c_2 + c_3 - c_4 + c_5 - 1], \dots$$

$$\left[\sum_{i=1}^k M_{2i-1} + \sum_{i=1}^k m_{2i} + \sum_{i=1}^{2k} (-1)^i c_i, \sum_{i=1}^k M_{2i} + \sum_{i=1}^{k+1} m_{2i-1} + \sum_{i=1}^{2k+1} (-1)^{i+1} c_i - 1 \right], \dots$$

Of course, the unsafe positions are the remaining positions, and they must be the following:

$$[m_1 + c_1, M_1 + m_2 - c_1 + c_2 - 1],$$

$$[M_2 + m_1 + m_3 + c_1 - c_2 + c_3, M_1 + M_3 + m_2 + m_4 - c_1 + c_2 - c_3 + c_4 - 1],$$

$$[M_2 + M_4 + m_1 + m_3 + m_5 + c_1 - c_2 + c_3 - c_4 + c_5, M_1 + M_3 + M_5 + m_2 + m_4 + m_6 - c_1 + c_2 - c_3 + c_4 - c_5 + c_6 - 1], \dots$$

$$\left[\sum_{i=1}^k M_{2i} + \sum_{i=1}^{k+1} m_{2i-1} + \sum_{i=1}^{2k+1} (-1)^{i+1} c_i, \sum_{i=1}^{k+1} M_{2i-1} + \sum_{i=1}^{k+1} m_{2i} + \sum_{i=1}^{2k+2} (-1)^i c_i - 1 \right], \dots$$

Proof: Let us first show that all of the above intervals of integers exist. First consider the safe intervals.

Now obviously $[0, m_1 + c_1 - 1]$ is non-empty since $0 \leq c_1$ and $1 \leq m_1$. We next show that $\forall k \in \{1, 2, 3, \dots\}$,

$$\sum_{i=1}^k M_{2i-1} + \sum_{i=1}^k m_{2i} + \sum_{i=1}^{2k} (-1)^i c_i \leq \sum_{i=1}^k M_{2i} + \sum_{i=1}^{k+1} m_{2i-1} + \sum_{i=1}^{2k+1} (-1)^{i+1} c_i - 1.$$

Recalling that $\Delta_i = M_i - m_i$, the above will be true if and only if

$$\sum_{i=1}^k (\Delta_{2i-1} - c_{2i-1}) + \sum_{i=1}^k c_{2i} \leq \sum_{i=1}^k (\Delta_{2i} - c_{2i}) + \sum_{i=1}^{k+1} c_{2i-1} + (m_{2k+1} - 1).$$

Since

$$1 \leq m_{2k+1}, 0 \leq c_1 \leq c_2 \leq c_3 \leq \dots$$

and

$$0 \leq \Delta_1 - c_1 \leq \Delta_2 - c_2 \leq \Delta_3 - c_3 \leq \dots,$$

we see that this inequality must be true.

Let us next show that all of the above unsafe intervals are non-empty. Now

$$[m_1 + c_1, M_1 + m_2 - c_1 + c_2 - 1]$$

is nonempty if and only if

$$m_1 + c_1 \leq M_1 + m_2 - c_1 + c_2 - 1$$

which is equivalent to

$$0 \leq (\Delta_1 - c_1) + (m_2 - 1) + (-c_1 + c_2).$$

Since $0 \leq \Delta_1 - c_1$, $1 \leq m_2$ and $0 \leq c_1 \leq c_2$, the inequality holds.

Next, we show that $\forall k \in \{1, 2, 3, \dots\}$,

$$\sum_{i=1}^k M_{2i} + \sum_{i=1}^{k+1} m_{2i-1} + \sum_{i=1}^{2k+1} (-1)^{i+1} c_i \leq \sum_{i=1}^{k+1} M_{2i-1} + \sum_{i=1}^{k+1} m_{2i} + \sum_{i=1}^{2k+2} (-1)^i c_i - 1.$$

This is true if and only if

$$\sum_{i=1}^k (\Delta_{2i} - c_{2i}) + \sum_{i=1}^{k+1} c_{2i-1} \leq \sum_{i=1}^{k+1} (\Delta_{2i-1} - c_{2i-1}) + \sum_{i=1}^{k+1} c_{2i} + (m_{2k+2} - 1).$$

since

$$1 \leq m_{2k+2}, 0 \leq c_1 \leq c_2 \leq c_3 \leq \dots,$$

and

$$0 \leq \Delta_1 - c_1 \leq \Delta_2 - c_2 \leq \Delta_3 - c_3 \leq \dots,$$

this inequality is obviously true.

Let us next observe that this game has different levels which we can define as starting the game on move number t , as t varies over $1, 2, 3, \dots$. Of course, the two players alternate moving and blocking, so if the players are on level 3, this means the moving player can remove any $x \in [m_3, M_3]$ from the pile that is in the pile after the blocking player has first placed a block on up to c_3 of the moving player's moves. By translational symmetry, the safe and unsafe positions (or pile sizes) on level t can be computed from the safe and unsafe pile sizes on level 1 by merely shifting the subscripts of the c_i 's, m_i 's, M_i 's. For example, corresponding to the safe level 1 pile sizes $[0, m_1 + c_1 - 1]$, we have the safe level 5 pile sizes $[0, m_5 + c_5 - 1]$.

We will use mathematical induction on the pile sizes to deal with all levels simultaneously. We will successively show that for $n = 0, 1, 2, 3, \dots$, the formulas in the theorem (that are for level 1) and the corresponding formulas for an arbitrary level t correctly state whether n is a safe or unsafe pile size on that level. By translational symmetry, we can accomplish this by focusing our attention only on

level 1 and showing that the level 1 formulas correctly state whether n is a safe or unsafe pile size on level 1. The higher level reasoning involves nothing more than a simple shift in subscripts of the level 1 c_i 's, m_i 's, M_i 's, and we can do this in our imagination.

Now $n = 0$ is obviously a safe pile size on all levels since $n = 0$ means that no counters remain on the table at all. Of course, the formulas state that $n = 0$ is a safe pile size on all levels since $[0, m_1 + c_1 - 1]$ are safe pile sizes on level 1 and $[0, m_t + c_t - 1]$ are safe pile sizes on level t . Therefore, the mathematical induction is started on all levels. Let us next suppose that the safe and unsafe positions are computed correctly on all levels by the above formulas for all pile sizes $0, 1, 2, 3, \dots, n - 1$, where $n - 1 \geq 0$.

We now show that the above formulas also correctly compute whether a pile size of n counters is safe or unsafe on all levels by merely focusing our attention on level 1.

Of course, by definition of level, when a player on level 1 makes a move, the game ends up on level 2. We will consider the safe level 1 intervals first. Then we consider the unsafe level 1 intervals. Let us first suppose $n \in [0, m_1 + c_1 - 1]$. Now $[0, m_1 + c_1 - 1] = [0, m_1 - 1] \cup [m_1, m_1 + c_1 - 1]$. Now the first player (i.e., the first moving player) cannot move when $n \in [0, m_1 - 1]$ since he can only remove $x \in [m_1, M_1]$. Next, suppose $n \in [m_1, m_1 + c_1 - 1]$. We ignore this when $c_1 = 0$,

so we are assuming $1 \leq c_1$. Let us note, by the way, that $m_1 \leq m_1 + c_1 - 1 < m_1 + (M_1 - m_1) = M_1$ since $1 \leq c_1 \leq \Delta_1 = M_1 - m_1$.

Since the opposing player can place a block on up to c_1 of the moving player's moves, we see that the opposing player can completely block all moves that the first moving player can make when $n \in [m_1, m_1 + c_1 - 1]$. Therefore, we see that n must be a safe pile size on level 1 when $n \in [0, m_1 + c_1 - 1]$, as is correctly stated in the formulas. Note that we did not have to use induction to see this.

Let us next suppose that

$$\begin{aligned} n &\in \left[\sum_{i=1}^k M_{2i-1} + \sum_{i=1}^k m_{2i} + \sum_{i=1}^{2k} (-1)^i c_i, \sum_{i=1}^k M_{2i} + \sum_{i=1}^{k+1} m_{2i-1} + \sum_{i=1}^{2k+1} (-1)^{i+1} c_i - 1 \right] \\ &= [A, B], \text{ where } k \geq 1. \end{aligned}$$

We show that n is a safe level 1 pile size, as it should be by the formulas. Remember, $0 \leq c_1 \leq c_2 \leq c_3 \leq \dots$. So $m_1 \leq M_1 < n$. Since the level 1 moving player can remove any $x \in [m_1, M_1]$ that is in the pile and is not blocked, and since $m_1 \leq M_1 < n$, we see that it is possible for the level 1 moving player to remove any $x \in [m_1, M_1]$ that the opposing player does not block. Of course, the opposing player can block up to c_1 of these x 's. We note that $[m_1, M_1]$ contains $M_1 - m_1 + 1$ consecutive integers.

Now $\forall x \in [m_1, M_1]$, we see that $n - x$ belongs to the interval

$$\begin{aligned} &\left[\sum_{i=2}^k M_{2i-1} + \sum_{i=1}^k m_{2i} + \sum_{i=2}^{2k} (-1)^i c_i - c_1, \sum_{i=1}^k M_{2i} + \sum_{i=2}^{k+1} m_{2i-1} + \sum_{i=2}^{2k+1} (-1)^{i+1} c_i - 1 + c_1 \right] \\ &= [C - c_1, D + c_1]. \text{ Note that } C, D \text{ are being defined here and that } A, B \text{ were defined} \end{aligned}$$

earlier.

We will show that the level 1 blocking player can force a level 1 moving player to choose an x in $[m_1, M_1]$ such that $n - x \in [C, D]$. Since $n - x < n$, we can use induction to see that if $n - x \in [C, D]$, then $n - x$ must be an unsafe pile size on level 2. Remember, we shift by 1 the subscripts of the level 1 unsafe intervals to find the level 2 unsafe intervals. So the theorem implies that $[C, D]$ is an unsafe interval on level 2. The reader can use a specific example, say $k = 3$, to see this move easily. This means that n must be a safe pile size on level 1. Let us write $[C - c_1, D + c_1] = [C - c_1, C - 1] \cup [C, D] \cup [D + 1, D + c_1]$.

Since this is a blocking game and the opposing player on level 1 can block up to c_1 of the moving player's moves, we see the following:

First, note that $[C - c_1, C - 1]$ contains c_1 consecutive integers and $[D + 1, D + c_1]$ contains c_1 consecutive integers. Now for all $n \in [A, B]$, $\{n - x : x \in [m_1, M_1]\}$ contains $M_1 - m_1 + 1$ consecutive integers. Also, the set $[C, D]$ contains

$$\begin{aligned} D - C + 1 &= \sum_{i=1}^k M_{2i} + \sum_{i=2}^{k+1} m_{2i-1} + \sum_{i=2}^{2k+1} (-1)^{i+1} c_i - \sum_{i=2}^k M_{2i-1} - \sum_{i=1}^k m_{2i} + \sum_{i=2}^{2k} (-1)^{i+1} c_i \\ &= \sum_{i=1}^k (\Delta_{2i} - c_{2i}) - \sum_{i=2}^k (\Delta_{2i-1} - c_{2i-1}) + \sum_{i=2}^{2k+1} (-1)^{i+1} c_i + m_{2k+1} \end{aligned}$$

consecutive integers. Each of the two sets $[C - c_1, D]$ and $[C, D + c_1]$ contains

$$D - C + 1 + c_1 = \sum_{i=1}^k (\Delta_{2i} - c_{2i}) - \sum_{i=2}^k (\Delta_{2i-1} - c_{2i-1}) + \sum_{i=1}^{2k+1} (-1)^{i+1} c_i + m_{2k+1}$$

consecutive integers. Let us now show that

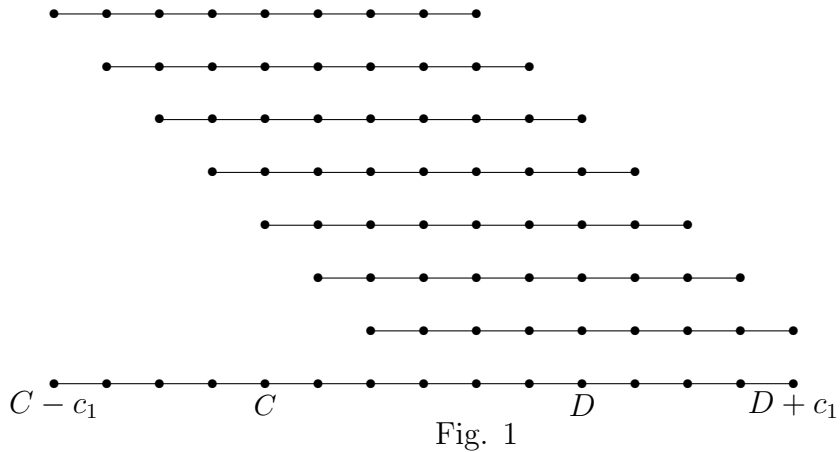
$$M_1 - m_1 + 1 = \Delta_1 + 1 \leq \sum_{i=1}^k (\Delta_{2i} - c_{2i}) - \sum_{i=2}^k (\Delta_{2i-1} - c_{2i-1}) + \sum_{i=1}^{2k+1} (-1)^{i+1} c_i + m_{2k+1}.$$

This means that for all n in $[A, B]$, the number of consecutive integers in $\{n - x : x \in [m_1, M_1]\}$ does not exceed the number of consecutive integers in $[C - c_1, D]$ and does not exceed the number of consecutive integers in $[C, D + c_1]$. This is true if and only if

$$(\Delta_1 - c_1) + 1 \leq (\Delta_2 - c_2) + \sum_{i=2}^k (\Delta_{2i} - c_{2i}) - \sum_{i=2}^k (\Delta_{2i-1} - c_{2i-1}) + \sum_{i=2}^{2k+1} (-1)^{i+1} c_i + m_{2k+1}.$$

Since $0 \leq c_1 \leq c_2 \leq c_3 \leq \dots$ and $0 \leq \Delta_1 - c_1 \leq \Delta_2 - c_2 \leq \Delta_3 - c_3 \leq \dots$ and $1 \leq m_{2k+1}$, we see that this inequality must be true.

Figure 1 shows how the sets $\{n - x : x \in [m_1, M_1]\}$ vary as n varies over $[A, B]$.



Considering the figure above, and the following list of facts we have just proved, we see that the level 1 opposing player can always block c_1 of level 1 moving player's options in such a way that the level 1 moving player is forced to choose a move $x \in [m_1, M_1]$ such that $n - x \in [C, D]$.

1. For all $n \in [A, B]$, and for all $x \in [m_1, M_1]$, $n - x \in [C - c_1, D + c_1] = [C - c_1, C - 1] \cup [C, D] \cup [D + 1, D + c_1]$.
2. For all $n \in [A, B]$, $\{n - x : x \in [m_1, M_1]\}$ contains $M_1 - m_1 + 1$ consecutive integers. From this and what we have proved it follows that
3. For all $n \in [A, B]$, the number of consecutive integers in $\{n - x : n \in [m_1, M_1]\}$ does not exceed the number of consecutive integers in $[C - c_1, D]$ and does not exceed the number of integers in $[C, D + c_1]$.
4. $[C - c_1, C - 1]$ contains c_1 consecutive integers and $[D + 1, D + c_1]$ contains c_1 consecutive integers.
5. The level 1 opposing player can block up to c_1 of the level 1 moving player's moves.

Since the level 1 opposing player can block c_1 of the level 1 moving player's moves in such a way as to force the level 1 moving player to choose an x in $[m_1, M_1]$ such that $n - x \in [C, D]$, we have already seen by induction that

$n - x$ is an unsafe pile size on level 2. Therefore, n must be a safe pilesize on level 1 as it should be by the formulas.

We now consider the unsafe level 1 intervals. Let us first suppose that $n \in [m_1 + c_1, M_1 + m_2 - c_1 + c_2 - 1]$. We show that n must be an unsafe level 1 pile size, as it should be by the formulas.

Now the first player can remove from the pile any $x \in [m_1, M_1]$ that is in the pile after the opposing player has first blocked up to c_1 of these x 's. Let us write

$$\begin{aligned} & [m_1 + c_1, M_1 + m_2 - c_1 + c_2 - 1] \\ = & [m_1 + c_1, M_1] \cup [M_1, M_1 + m_2 - c_1 + c_2 - 1]. \end{aligned}$$

Now $0 \leq c_1 \leq \Delta_1 = M_1 - m_1, 1 \leq m_2, 0 \leq c_1 \leq c_2$ implies that both of these two intervals of integers exist. We will consider two cases. First, suppose $n \in [m_1 + c_1, M_1]$. If this were not a blocking game, the moving player could remove $x = n$ from the pile and leave the pile empty since $[m_1 + c_1, M_1] \subseteq [m_1, M_1]$ and he can remove any $x \in [m_1, M_1]$. However, since his opponent can first block up to c_1 of the moving player's moves, the analysis is more complicated than this. Now $\forall n \in [m_1 + c_1, M_1]$, the moving player can remove from the pile of n counters at least one x that is in the set $\{n, n - 1, n - 2, \dots, n - c_1\}$. This is because $\{n, n - 1, n - 2, \dots, n - c_1\} \subseteq [m_1, M_1]$ and the blocking player can block only up to c_1 of the moving player's moves. If he removes such an x , this means that after

he moves, he will leave a pile size of $n - x$ counters where $n - x \in \{0, 1, 2, 3, \dots, c_1\}$. Now $\{0, 1, 2, 3, \dots, c_1\} \subseteq [0, m_2 + c_2 - 1]$ since $0 \leq c_1 \leq c_2$ and $1 \leq m_2$. Since $n - x < n$, by induction we know that $n - x$ is a safe pile size on level 2 since $[0, m_2 + c_2 - 1]$ is a safe interval on level 2. . Remember, we shift by 1 the subscripts of the level 1 safe intervals to find the level 2 safe intervals. Therefore n is an unsafe pile size on level 1, as it should be by the formulas.

Second, suppose $n \in [M_1, M_1 + m_2 - c_1 + c_2 - 1]$. Since the blocking player can block only up to c_1 of the moving player's moves and $M_1 \leq n$, we see that the moving player can remove at least one x that is in the set $\{M_1, M_1 - 1, M_1 - 2, \dots, M_1 - c_1\}$. Of course, $0 \leq c_1 \leq \Delta_1 = M_1 - m_1$ implies $m_1 \leq M_1 - c_1$. If he removes such an x , then since $n \in [M_1, M_1 + m_2 - c_1 + c_2 - 1]$ the number $n - x$ will lie in the set

$$[0, (M_1 + m_2 - c_1 + c_2 - 1) - (M_1 - c_1)] = [0, m_2 + c_2 - 1].$$

As before, by induction, since $n - x < n$, we know that $n - x$ is a safe pile size on level 2. Therefore again n is an unsafe pile size on level 1, as it should be by the formulas.

At this point, it might be appropriate to remind the reader that even though the reasoning is taking place only on level 1, by translational symmetry the corresponding reasoning is also taking place simultaneously on all levels.

Last, let us suppose that

$$\begin{aligned} n &\in \left[\sum_{i=1}^k M_{2i} + \sum_{i=1}^{k+1} m_{2i-1} + \sum_{i=1}^{2k+1} (-1)^{i+1} c_i, \sum_{i=1}^{k+1} M_{2i-1} + \sum_{i=1}^{k+1} m_{2i} + \sum_{i=1}^{2k+2} (-1)^i c_i - 1 \right] \\ &= [A, B], \text{ where } k \geq 1. \end{aligned}$$

We will show that n is an unsafe pile size on level 1, as it should be by the formulas.

The first player can remove from the pile any $x \in [m_1, M_1]$ that is in the pile after the opposing player has first blocked up to c_1 of these x 's. Let us now define

$$\begin{aligned} [C, D] &= [A - m_1 - c_1, B - M_1 + c_1] \\ &= \left[\sum_{i=1}^k M_{2i} + \sum_{i=2}^{k+1} m_{2i-1} + \sum_{i=2}^{2k+1} (-1)^{i+1} c_i, \sum_{i=2}^{k+1} M_{2i-1} + \sum_{i=1}^{k+1} m_{2i} + \sum_{i=2}^{2k+2} (-1)^i c_i - 1 \right] \end{aligned}$$

Note that $0 < C$ since $0 \leq c_1 \leq c_2 \leq c_3 \dots$. We will show that after the level 1 blocking player has blocked up to c_1 of the level 1 moving player's moves, the level 1 moving player can still find $x \in [m_1, M_1]$ such that $n - x \in [C, D]$. Again recall that we shift by 1 the subscripts of the level 1 safe intervals to find the level 2 safe intervals. By induction, since $n - x < n$, we can then see that $n - x$ is a safe pile size on level 2 since $[C, D]$ is a safe level 2 interval. The reader can use a specific example, say $k = 2$, to see this move easily. Therefore, n will be an unsafe pile size on level 1, as it should be by the formulas.

Let us first show that $c_1 \leq D - C$. This will be true if and only if

$$c_1 \leq \sum_{i=2}^{k+1} M_{2i-1} + \sum_{i=1}^{k+1} m_{2i} + \sum_{i=2}^{2k+2} (-1)^i c_i - 1 - \sum_{i=1}^k M_{2i} - \sum_{i=2}^{k+1} m_{2i-1} - \sum_{i=2}^{2k+1} (-1)^{i+1} c_i.$$

Recall that $\Delta_i = M_i - m_i$. Therefore, this is true if and only if

$$c_1 \leq \sum_{i=2}^{k+1} (\Delta_{2i-1} - c_{2i-1}) - \sum_{i=1}^k (\Delta_{2i} - c_{2i}) + \sum_{i=2}^{2k+2} (-1)^i c_i + m_{2k+2} - 1.$$

Since

$$0 \leq c_1 \leq c_2 \leq c_3 \leq \dots,$$

$$0 \leq \Delta_1 - c_1 \leq \Delta_2 - c_2 \leq \Delta_3 - c_3 \leq \dots$$

and $1 \leq m_{2k+2}$, we see that this inequality must be true.

From the definition of $[C, D]$, let us now write

$$\begin{aligned} [A, B] &= [C + m_1 + c_1, D + M_1 - c_1] \\ &= [C + m_1 + c_1, C + M_1] \cup [C + M_1, D + M_1 - c_1]. \end{aligned}$$

Since $0 \leq c_1 \leq \Delta_1 = M_1 - m_1$ and $c_1 \leq D - C$, we see that both of these intervals of integers exist. We will consider two cases. First, suppose $n \in [C + m_1 + c_1, C + M_1]$.

Now $n = C + t$, where $m_1 + c_1 \leq t \leq M_1$. Now the moving player can remove from the pile of n counters at least one x that is in the set $\{t, t-1, t-2, \dots, t-c_1\}$ since $\{t, t-1, t-2, \dots, t-c_1\} \subseteq [m_1, M_1]$ and the blocking player can block only up to c_1 of the moving player's moves. If he removes such an x , this means that after he moves, he will leave a pile size of $n-x$ counters where $n-x \in \{C, C+1, C+2, \dots, C+c_1\}$.

Now $\{C, C+1, C+2, \dots, C+c_1\} \subseteq [C, D]$ since $0 \leq c_1 \leq D - C$. Of course, this means $n-x \in [C, D]$ is true, which is what we wished to show.

Last, suppose $n \in [C + M_1, D + M_1 - c_1]$. Of course, $M_1 < n$ since $0 < C$. Now the moving player can remove from the pile of n counters at least one x that is in the set $\{M_1, M_1 - 1, M_1 - 2, \dots, M_1 - c_1\}$. That is because $\{M_1, M_1 - 1, M_1 - 2, \dots, M_1 - c_1\} \subseteq [m_1, M_1]$ and the fact that the blocking player can block only up to c_1 of the moving player's moves. Note that $0 \leq c_1 \leq \Delta_1 = M_1 - m_1$ implies $m_1 \leq M_1 - c_1$. If he removes such an x , then since $n \in [C + M_1, D + M_1 - c_1]$ and since $M_1 - c_1 \leq x \leq M_1$, we see that

$$\begin{aligned} n - x &\in [(C + M_1) - M_1, (D + M_1 - c_1) - (M_1 - c_1)] \\ &= [C, D]. \end{aligned}$$

Since this is what we wished to show, the proof of the theorem is now complete.

References

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