(Pilesize) Dynamic One-Pile Nim

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Abstract The purpose of this paper is to solve a class of combinatorial games consisting of one-pile counter pickup games for which the maximum number of counters that can be removed on each successive move changes during the play of the game. The maximum size of the move is determined by a move function f whose arguments are pile sizes. In another paper[8], we will discuss the game in which the number of counters that can be removed depends on the number removed in the previous move.

Introduction The purpose of this paper is to solve a class of combinatorial games consisting of one-pile counter pickup games for which the number of counters that can be removed on each successive move changes during the play of the game. Two players alternate removing a positive number of counters from the pile. The maximum size of the move is determined by a move function f whose arguments are pile sizes. Each player in his turn must remove from one up to the minimum of t and $f(t)$ counters, where t is the size of the pile. The game ends when either the pile is empty or $f(t) < 1$ and the winner is the last player to move. Our analysis solves a very large class of games that includes as a subclass all games whose move function satisfies both

1. $f(0) = 0$ and
2. For all $n \geq 0$, $f(n+1) - f(n) \in \{0, 1\}$.

The second condition, called the **unit jump condition** (ujc), will be used repeatedly.

The move functions $f(n) = \lfloor rn \rfloor$ and when $f(n) = \lceil rn \rceil$ for some $r, \quad 0 < r < 1$, are special cases. Our collection of games also includes the static games for which f is a constant function. The game $N(n, k)$ of static one-pile nim is well-known. A pile of n counters and a constant k are given. Two players alternately take from 1 up to k counters from the pile. The winner is the player who removes the last counter. We assume the reader is familiar with the Sprague-Grundy theory of combinatorial games. The Grundy values of positions are called Nim values in this paper.

At the end of this paper, we will show how to generate many applications of the general theory. These will include the 'primitive' move functions $f(n) = \lfloor rn \rfloor$ and $f(n) = \lceil rn \rceil, 0 < r < 1$. An extension of Beatty's theorem is also included.

In a companion paper [6], we use the basic ideas of this paper to extend Beatty's theorem to continuous functions. See [1] and [6]. Also, a further extension of this paper led us to discover a class of combinatorial games not discussed in the literature of combinatorial games [7].

Next we provide the reader with a short summary of what motivated this paper. We started by studying the Nim values for the simple move function $f(n) = \lfloor n/2 \rfloor$. Then we noticed that this function satisfies the unit jump condition, and that most of our conclusions hold for all move functions satisfying the ujc. Later we were able to extend our results for functions that do not satisfy the ujc. We admitted nonlinear functions and then added theorem 2 to make the proofs more efficient, since we were writing the same proofs repeatedly. We then proceeded to solve more related problems such as finding all possible Nim-value sequences $g(1), g(2), g(3), \ldots$ that can exist when $f(n)$ satisfies the ujc.

Finally, if the reader is not interested in the level of generalization given in theorem 1, he can just imagine that f satisfies the ujc and always substitute f for $\overline{f}$. All of our applications assume that f satisfies the ujc.

We now specify our move functions. Let N denote the set of non-negative integers and Z the set of all integers. Let $h : N \to Z$ be an arbitrary function. Define a new function $\overline{h} : N \to Z$ as follows: $\overline{h}(0) = 0$ and $\forall x \in N\backslash\{0\}, \quad \overline{h}(x) = \min(\overline{h}(x-1)+1, h(x))$.

Suppose $h(0) = 0$ and h satisfies the ujc. Then it follows by induction that $h = \overline{h}$. Also note that if $h(0) = 0$ and h is non-decreasing, then $\overline{h}$ satisfies the ujc, again by mathematical induction. Finally, note from the definition of $\overline{h}$ that $\overline{h}$ is non-decreasing if and only if $\overline{h}$ satisfies the ujc.

Theorem 1 *Let n be a non-negative integer. Consider the game (n, f) where $f : N \to N$ is a move function such that $\overline{f}$ is non-decreasing. Then the Nim values $g(n)$ of the game (n, f) satisfy the following:*

1. $g(0) = 0$

2. *If $\overline{f}(n) - \overline{f}(n-1) = 1$ and $n \geq 1$, then $g(n) = \overline{f}(n)$.*

3. If $\overline{f}(n)-\overline{f}(n-1)=0$ *and* $n\geq 1$*, then* $\overline{f}(n)=f(n)$ *and* $g(n)=g(n-1-f(n))=g(n-1-\overline{f}(n))$*.*

4. For all non-negative integers n*,* $\{g(n),g(n-1),\ldots,g(n-\overline{f}(n))\}=\{0,1,2,\ldots,\overline{f}(n)\}$*.*

5. For all non-negative integers n*,* $g(n)\leq\overline{f}(n)$*.*

Before we begin the proof, note that $\overline{\overline{f}}=\overline{f}$. Since $\overline{f}$ satisfies the hypothesis on f for the theorem, the theorem can be used with $\overline{f}$ in place of f. Thus the games using f and $\overline{f}$ have the same Nim values. Also, note that in condition 4. that $0<1<\cdots<\overline{f}(n)$, so that both sets have precisely $\overline{f}(n)$ elements.

Proof. Note that $\overline{f}(n)\leq n$ for all $n\varepsilon N$ because $\overline{f}$ satisfies the ujc. Then $\overline{f}(n)=\overline{(min(n,f(n)))}$, and there is no loss of generality in assuming that $0\leq f(n)\leq n$ for all $n\varepsilon N$ since the game is actually being played with the move function $min(n,f(n))$. The proof of the theorem is by mathematical induction on n. Starting the induction at $n=0$ is trivial because only conditions 4 and 5 apply, which hold because $g(0)=\overline{f}(0)=0$. Now assume that the theorem holds for $k\in\{0,1,2,\ldots,n-1\}$. We show that the statements also hold for n. Consider the two cases,

1. $\overline{f}(n-1)<f(n)$ and 2. $f(n)\leq\overline{f}(n-1)$.

We will tacitly assume that $f(n)>0$. If $f(n)=0$, then $\overline{f}(0)=\overline{f}(1)=\overline{f}(2)\ldots=\overline{f}(n)=0$ by the definition of $\overline{f}$.

Proof for case 1. Note that $\overline{f}(n) - \overline{f}(n-1) = 1$ since, in this case, $\overline{f}(n-1) + 1 \leq f(n)$ and $\overline{f}(n) = \min(\overline{f}(n-1)+1, f(n))$, so we need to prove condition 2. Once we prove condition 2, it is clear that condition 5 also holds. From the induction hypothesis, $\{g(n-1), g(n-2), \ldots, g(n-1-\overline{f}(n-1))\} = \{0, 1, 2, \ldots, \overline{f}(n-1)\}$, where $0 < 1 < \cdots < \overline{f}(n-1)$. Note that $\overline{f}(n) = \overline{f}(n-1) + 1 \leq f(n)$ by the definition of $\overline{f}$ or by the case 1 condition itself. Let $\varepsilon = f(n) - \overline{f}(n-1) - 1 \geq 0$. Then

$$
\begin{aligned}
g(n) &= mex(\{g(n-1), g(n-2), \ldots, g(n-f(n))\}) \\
&= mex(\{g(n-1), g(n-2), \ldots, g(n-1-\overline{f}(n-1)-\varepsilon)\}) \\
&= mex(\{g(n-1), g(n-2), \ldots, g(n-1-\overline{f}(n-1))\} \cup_{i=1}^{\varepsilon} \{g(n-1-\overline{f}(n-1)-i)\}) \\
&= mex(\{0, 1, 2 \ldots, \overline{f}(n-1)\} \cup_{i=1}^{\varepsilon} \{g(n-1-\overline{f}(n-1)-i)\})
\end{aligned}
$$

Note that $0 \leq g(n-1-\overline{f}(n-1)-i) \leq \overline{f}(n-1-\overline{f}(n-1)-i) \leq \overline{f}(n-1)$ by the induction hypothesis about condition 5 of the theorem and the fact that $\overline{f}$ is non-decreasing. Thus $g(n-1-\overline{f}(n-1)-i)$ lies in the set $\{0, 1, \ldots, \overline{f}(n-1)\}$. It follows that $g(n) = \overline{f}(n-1) + 1 = \overline{f}(n)$. To prove condition 4, note that

$$
\begin{aligned}
&\{g(n), g(n-1), g(n-2), \ldots, g(n-\overline{f}(n))\} \\
&= \{g(n)\} \cup \{g(n-1), g(n-2), \ldots, g(n-1-\overline{f}(n-1))\} \\
&= \{\overline{f}(n)\} \cup \{0, 1, 2, \ldots, \overline{f}(n-1)\}
\end{aligned}
$$

$$= \{0,1,2,\ldots,\overline{f}(n)\}, \text{ where } 0 < 1 < \cdots < \overline{f}(n).$$

Proof for case 2. We are given $f(n) \le \overline{f}(n-1)$. Now $\overline{f}(n) = \min(\overline{f}(n-1) + 1, f(n)) = f(n)$. Since $\overline{f}$ is non-decreasing, it follows that $\overline{f}(n-1) \le \overline{f}(n)$. Putting these relations together gives $\overline{f}(n) = \overline{f}(n-1) = f(n)$. The induction hypothesis gives $\{g(n-1), g(n-2), \ldots, g(n-1-\overline{f}(n-1))\} = \{0,1,2,\ldots\overline{f}(n-1)\}$ where $0 < 1 < \cdots < \overline{f}(n-1)$. From $\overline{f}(n-1) \le n-1$ and $\overline{f}(n) = \overline{f}(n-1)$, it follows that $n - \overline{f}(n) - 1 \ge 0$. That $\overline{f}(n-1) \le n-1$ follows from the ujc which $\overline{f}$ satisfies together with the fact $\overline{f}(0) = 0$. Since $\overline{f}(n) - \overline{f}(n-1) = 0$, we need to show condition 3 of the theorem, namely, $g(n) = g(n - f(n) - 1)$. Because the elements of the list $g(n-1), g(n-2), \ldots, g(n-1-\overline{f}(n-1))$ are all distinct,

$$\begin{aligned} g(n) &= mex(\{g(n-1), g(n-2), \ldots, g(n-f(n))\}) \\ &= mex(\{g(n-1), g(n-2), \ldots, g(n-\overline{f}(n-1))\}) \\ &= mex(\{g(n-1), g(n-2), \ldots, g(n-1-\overline{f}(n-1))\} \backslash \{g(n-1-\overline{f}(n-1))\}) \\ &= mex(\{0,1,2,\ldots\overline{f}(n-1)\} \backslash \{g(n-1-\overline{f}(n-1))\}), \end{aligned}$$

By induction and the fact that $\overline{f}$ is non-decreasing, we see that $0 \le g(n-1-\overline{f}(n-1)) \le \overline{f}(n-1-\overline{f}(n-1)) \le \overline{f}(n-1)$. This means that $g(n-1-\overline{f}(n-1))$ lies in the set $\{0,1,2,\ldots\overline{f}(n-1)\}$. Therefore,

$$g(n) = g(n-1-\overline{f}(n-1)) = g(n-1-\overline{f}(n)) = g(n-1-f(n)).$$

This proves condition 3. Next note that $g(n) = g(n-f(n)-1) \le \overline{f}(n-f(n)-1) \le$

$\overline{f}(n)$ by the induction hypothesis and the fact that $\overline{f}$ is non-decreasing. So condition 5 holds.

Furthermore, condition 4 is satisfied:

$$
\begin{aligned}
&\{g(n), g(n-1), g(n-2), \ldots, g(n-\overline{f}(n))\} \\
&= \{g(n)\} \cup \{g(n-1), g(n-2), \ldots, g(n-\overline{f}(n))\} \\
&= \{g(n-1-\overline{f}(n))\} \cup \{g(n-1), g(n-2), \ldots, g(n-\overline{f}(n))\} \\
&= \{g(n-1), g(n-2), \ldots, g(n-1-\overline{f}(n))\} \\
&= \{g(n-1), g(n-2), \ldots, g(n-1-\overline{f}(n-1))\} \\
&= \{0, 1, 2, \ldots \overline{f}(n-1)\} \\
&= \{0, 1, 2, \ldots \overline{f}(n)\}
\end{aligned}
$$

■

In order to use the main theorem more effectively with specific functions, we need Theorem 2.

<u>Theorem 2.</u> Let n vary over the non-negative integers. Consider the game (n, f) where f is a move function satisfying

1. $f(0) = 0$ and
2. For all $n \geq 0$, $f(n+1) - f(n) \in \{0, 1\}$.
3. $\lim_{n\to\infty} f(n) = \infty$.

4. $\lim_{n\to\infty} n - f(n) = \infty$.

Let $h(n) = n - f(n)$. Note that because of conditions 1-4, both h and f are surjections of N onto N. Thus for each integer $m \geq 0$, we may define $F(m)$ to be the smallest non-negative integer x such that $f(x) = m$, and $H(m)$ to be the smallest non-negative integer x such that $h(x) = m$. Finally, let α be any non-negative integer and $0 \leq a_1 < a_2 \cdots < a_i < \cdots$ be all the non-negative integers such that $g(a_i) = \alpha$. That is, $a_1, a_2, \ldots$ is the sequence of pile sizes whose Nim values are α. Then, the sequence $a_1, a_2, \ldots$ can be generated recursively as follows: $a_1 = F(\alpha)$ and for all $i = 2, 3, 4, \ldots,\quad a_i = H(a_{i-1} + 1)$. Notice that conditions 3 and 4 could be omitted if we wish.

Proof. First note that

1. $h(0) = 0$,

2. for all $n \geq 0$, $h(n+1) - h(n) \in \{0, 1\}$, and

3. $\lim_{n\to\infty} h(n) = \infty$.

Also note that $\overline{f} = f$ because f satisfies the ujc. We use Theorem 1 to prove Theorem 2 by induction on the index i in $a_1, a_2, \ldots$. In the conclusion of theorem 1, we use f instead of $\overline{f}$. Since $\lim_{n\to\infty} f(n) = \infty$, $\lim_{n\to\infty} n - f(n) = \lim_{n\to\infty} h(n) = \infty$, it follows from conclusion 4 of Theorem 1 that there must be infinitely many non-negative integers whose Nim value is α. Therefore it makes sense to talk about

$a_1, a_2, \ldots$ as being the infinite increasing sequence of all non-negative integers whose Nim values are α. It is easy to see that the function h satisfies $h(n+1)-h(n) \in \{0,1\}$ for all n, and together with the same property for f, it follows that $f(F(m)-1) = m-1$ for all m and $h(H(m)-1) = m-1$ for all m.

From condition 5 of theorem 1, it follows that for all $n \geq 0, \quad g(n) \leq f(n)$. Therefore, from condition 2 of Theorem 1, $a_1 = F(\alpha)$. Thus we may suppose for the induction hypothesis that $0 \leq a_1 = F(\alpha) < a_2 < a_3 < \ldots < a_i$ are all the non-negative integers up to and including a_i whose Nim values are α. We also suppose that these a_i's are generated by $a_1 = F(\alpha)$ and $a_k = H(a_{k-1}+1)$ for all $k \in \{2, 3, \ldots, i\}$.

To complete the proof it remains to show that next positive integer a_{i+1} whose Nim value is α is given by

$$a_{i+1} = H(a_i + 1).$$

To this end, let $\overline{n}$ be the smallest positive integer such that $h(\overline{n}) > a_i$. Of course, $\overline{n} = H(a_i+1)$. Also, because of how h is defined, $a_i < \overline{n}$. Since $h(n+1) - h(n) \in \{0,1\}$, we have

$$\text{i.} \quad \overline{n} - f(\overline{n}) = a_i + 1 \text{ and ii.} \quad (\overline{n}-1) - f(\overline{n}-1) = a_i.$$

Now since $(\overline{n}-1)-f(\overline{n}-1) = a_i$ and $g(a_i) = \alpha$, it follows from the definition of Nim value that $g(\overline{n}-1) \neq \alpha$. The condition $h(n+1)-h(n) \in \{0,1\}$ together with equation ii. above implies that for all $t \in \{a_i+1, a_i+2, \ldots, \overline{n}-1\}, \quad h(t) = t - f(t) \leq a_i$.

Thus for t in this range, $g(t) \neq \alpha$. We now show that $g(\overline{n}) = \alpha$. This will complete the proof since $\overline{n}$ is the next integer after a_i such that $g(\overline{n}) = \alpha$ and $\overline{n} = H(a_i + 1)$. Using equations i. and ii. above, we see that $f(\overline{n}) - f(\overline{n} - 1) = 0$. From conclusion 3 of theorem 1 and equation i. above, we see that $g(\overline{n}) = g(\overline{n} - f(\overline{n}) - 1) = g(a_i) = \alpha$.

■

In order to apply theorem 2, we will use the following material from the companion paper [6]. If the reader uses Cartesian graphs instead of formal algebra, these results are easy to prove. Thus the proofs are left to the reader. Note that [9] can also be used to generate applications of theorem 2.

Definition: Suppose u, v are real functions on the domain $[0, \infty)$. We say u and v are complementary if they satisfy

1. $u(0) = v(0) = 0$,

2. u and v are non-decreasing on $[0, \infty)$, and

3. $u(x) + v(x) = x$ for all nonnegative x.

It is easy to show that u and v must be continuous and at least one is unbounded. We do not have to assume that both are unbounded; however, we will make this assumption for convenience.

Lemma 1. Suppose u and v are complimentary functions. Define $f : N \to N$, $h : N \to N$ by $f(n) = \lfloor u(n) \rfloor, h(n) = n - f(n) = n - \lfloor u(n) \rfloor = \lceil n - u(n) \rceil = \lceil v(n) \rceil$. Since f and h satisfy the hypothesis of theorem 2, for every nonnegative integer m,

$F(m)$ and $H(m)$ are defined by theorem 2. For any nonnegative integer m, define $u^{-1}(m)$ to be the smallest nonnegative real number that satisfies $u(u^{-1}(m)) = m$ and $v^{-1}(m)$ to be the largest nonnegative real number satisfying $v(v^{-1}(m) = m$. . Then $F(m) = \lceil u^{-1}(m) \rceil$, and $H(0) = 0, H(m+1) = \lfloor v^{-1}(m) + 1 \rfloor$.

The reader can easily see that the two sequences $F(m), m = 1, 2, 3, \ldots$, and $H(m+1), m = 0, 1, 2, 3, \ldots$ together partition the set of positive integers and also that the members of each of the two sequences are distinct. In [6] we prove a much more powerful version of this result, which is our generalization of Beatty's theorem. (See appendix for a discussion of Beatty's theorem.)

Application 1 Define u, v on $[0, \infty)$ as follows:

$$u(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ \sqrt{x} & \text{if } x > 1 \end{cases}$$

and

$$v(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1 \\ x - \sqrt{x} & \text{if } x > 1 \end{cases}$$

Define the move function $f(n) = \lfloor \sqrt{n} \rfloor, n = 0, 1, 2, \ldots$. Then $h(n) = \lceil n - \sqrt{n} \rceil, n = 0, 1, 2, \ldots$. We invite the reader to go through the technicalities of lemma 1. Now $u^{-1} = n^2, n = 0, 1, 2, \ldots$, and $v^{-1}(n) = \frac{2n+1+\sqrt{4n+1}}{2}, n = 0, 1, 2, \ldots$. Let a be any nonnegative integer and $0 \le a_1 < a_2 \ldots$ be all nonnegative integers whose Nim values are a. Then $a_1, a_2, \ldots$ is generated recursively as follows: $a_1 = F(a) = \lceil u^{-1}(a) \rceil = a^2$, and for all $i = 2, 3 \ldots, a_i = H(a_{i-1} + 1) = \lfloor v^{-1}(a_{i-1}) + 1 \rfloor = \left\lfloor \frac{2a_{i-1}+3+\sqrt{4a_{i-1}+1}}{2} \right\rfloor$.

The reader can supply the proof for the next application.

Application 2 Consider the game (n, f) where $f(n) = \lfloor n - \sqrt{n} \rfloor$, $n = 0, 1, 2, \ldots$. Let a be any nonnegative integer and $0 \leq a_1 < a_2 \ldots$ be all the nonnegative integers whose Nim value is a. Then $a_1, a_2, \ldots$ is generated recursively as follows. If $a = 0$, then $a_1 = 0$, and for all $i = 2, 3, \ldots, a_i = a_{i-1}^2 + 1$. If $a \neq 0$, then $a_1 = \left\lceil \frac{2a_{i-1}+1+\sqrt{4a_{i-1}+1}}{2} \right\rceil$, and for all $i = 2, 3, \ldots, a_i = a_{i-1}^2 + 1$. The reader can supply the complete analysis for the next applications.

Applications 3,4. Let $f(n) = \lceil \sqrt{n} \rceil$ and $f(n) = \lceil n - \sqrt{n} \rceil$

Applications 5,6. Let $f(n) = \lfloor rn \rfloor$ and $f(n) = \lceil rn \rceil, 0 < r < 1$. The reader should note that the special case $r = 1/2$ is very interesting.

Applications 7,8. Let $f(n) = \lfloor 3n^{2/3} - 3n^{1/3} \rfloor$ and $f(n) = \lceil 3n^{2/3} - 3n^{1/3} \rceil$.

Appendix In 1926 Sam Beatty made the following discovery, which he posed as a problem in [1]. If a is a positive irrational number, the sequences $m(1+a), m = 1, 2, \ldots$ and $n(1+a^{-1}), n = 1, 2, \ldots$ together contain exactly one number from each of the intervals $(k, k+1), k = 1, 2, 3, \ldots$. The reader can use lemma 1 to prove this by defining $u(x) = \frac{x}{1+a}$ and $v(x) = \frac{ax}{1+a}$. The problem was solved by Ostrowski and Aitken[10] and generalized to a larger class of sequences by Lambek and Moser[9]. It appears that the Lambek and Moser approach is the inverse of our approach. Our approach arose from studying the Nim (ie, Sprague-Grundy) values of games, while it is quite possible the Lambek and Moser were not aware of connection with combinatorial games. In [4], Fraenkel used Beatty's theorem to study a generalized

Wythoff's game. He did not use Wythoff's game to derive Beatty's theorem as we have done here with our game.

References

[1] Sam Beatty, Problem 3173, *American Mathematical Monthly*, vol. 33, 1926, p.159.

[2] Elwyn Berlekamp, John H. Conway, and Richard Guy, *Winning Ways*, Academic Press, Vol. 1, A. K. Peters, 2001.

[3] John H. Conway, *On Games and Numbers*, A. K. Peters, 2001.

[4] Fraenkel, A. S. How to beat your Wythoff games' opponents on three fronts, *American Mathematical Monthly*, vol. 89, 1982, 353-361.

[5] Richard K. Guy, *Fair Game*, 2nd. ed., COMAP, New York, 1989.

[6] Holshouser, A and H. Reiter, An Extention of Beatty's Theorem, submitted.

[7] Holshouser, A and H. Reiter, Dynamic One-Pile Blocking Nim, submitted.

[8] Holshouser, A., J. Rudzinski, and H. Reiter, Dynamic One-Pile Nim, submitted.

[9] J. Lambek and L. Moser, "Inverse and Complementary Sequences of Natural Numbers", *American Mathematical Monthly*, vol. 61, 1954, p.454. MR 16,17a.

[10] Ostrowski and Aitken, Solution to Problem 3173, *American Mathematical Monthly*, vol. 34, p. 159.