THE LINEAR GROWTH IN THE LENGTHS OF A FAMILY OF THICK KNOTS

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Abstract. For any given knot K, a thick realization K_0 of K is a knot of unit thickness which is of the same knot type with K. In this paper, we show that there exist a family of prime knots $\{K_n\}$ with the property that $Cr(K_n) \to \infty$ (as $n \to \infty$) such that the arc-length of any thick realization of K_n will grow at least linearly with respect to $Cr(K_n)$.

1. Introduction

In this paper, we are interested in the relation between the crossing number of a thick knot and its arc length. Physically, this means to explore the amount of rope needed to tie certain knot. Mathematically, one needs to take care of the definition of the thickness of a knot. There are different ways to define the thickness of a knot ([**CKS2**], [**DER1**], [**LSDR**]). Alternatively, one could also use knots realized on the cubic lattice, since these knots have a naturally defined length and thickness. For any given knot K, a *thick realization* K_0 of K is a knot of unit thickness which is of the same knot type with K. In this paper, we will be using the so called *disk thickness* defined in [**LSDR**], although the results obtained here should also hold for other thickness definitions modulo a suitable scale change in the constant coefficient.

It is shown in [**B1**] and [**BS**] that the minimal crossing number of a knot of unit thickness is bounded by a constant times its length to the four-thirds power. In other word, for any knot K, a thick realization K_0 of it is at least of length $c(Cr^{3/4}(K))$ for some constant c. The constant c is estimated to be at least 1.105 by the result obtained in [**BS**] and is improved to 2.135 recently ([**RS**]). This four third power is also shown to be achievable for some knot families ([**CKS1**],[**DE**]). That is, there exist a family of knots $\{K_n\}$, such that the thick realizations of them have length at most $c_0(Cr^{3/4}(K))$ for some constant $c_0 > 0$.

On the other hand, what is the sufficient rope length to tie a particular knot? That is, given a knot K, how long does an arc have to be so that we are sure a thick realization of K can be found with that arc-length? It is a well known fact that a thick realization of K can always be found if we allow its length to reach

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the order of $Cr^2(K)$. But it is not clear up to now whether there exists a family of knots $\{K_n\}$ such that the thick realization of K_n is of order $O(Cr^2(K))$. In fact, it was unknown whether there exists a family of prime knots $\{K_n\}$ such that the thick realization of K_n is of order more than $O(Cr^{3/4}(K))$.

In this paper, we show that there exist a family of prime knots $\{K_n\}$ such that the thick realization of K_n is of order $O(Cr(K))$. This shows that a knot does not always have a thick realization with length at the order $Cr^{3/4}(K)$.

2. The case of lattice realization

The cubic lattice is a graph in R^3 whose vertices are all points with coordinates (x, y, z) where x, y, and z are integers and whose edges are the unit length line segments connecting the adjacent vertices.

THEOREM 2.1. *Let K be a knot and let $n = b(K)$ be its bridge number. If P is a polygonal knot of the same knot type as K on the cubic lattice, then the length $L(P)$ of P is at least $6n$.*

PROOF. Let P_0 be a polygonal knot of type K on the cubic lattice such that its length is the shortest among all lattice knots of type K. By the definition of the bridge number, P_0 has at least n non-removable maximal points in the z-direction. Such a maximal point cannot be as shown in Figure 1(a), since the length of P_0 can then be reduced by 2, contradicting the definition of P_0. Thus, a path joining a maximal point of P_0 to the next has at least 6 edges as shown in Figure 1(b). Hence the total length of P_0 is at least $6n$. □

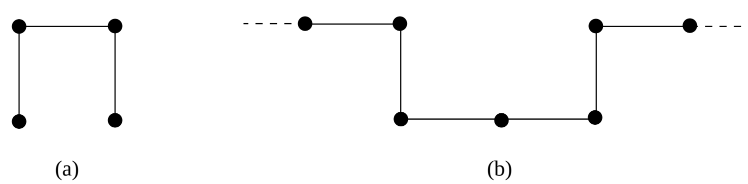

FIGURE 1. An impossible maximal point of P_0 and a shortest path on P_0 joining two maximal points

Corollary. Let K_1, K_2, ..., K_m be m prime knots and $K = K_1\#K_2\#...\#K_m$, then for any polygonal knot P on the cubic lattice that is equivalent to K, we have $L(P) \geq 6(\sum b(K_j) - m + 1)$. In particular, if the K_j's are of the same knot type, then $L(P) \geq 6m(b(K_1) - 1) + 6$.

PROOF. This follows because a result proven by Schubert [**S**] $b(K_1\#K_2) = b(K_1) + b(K_2) - 1$ where K_1 and K_2 are knots. So in our case we have $b(K) = \sum b(K_j) - m + 1$. □

THEOREM 2.2. *Let K_n denote the knot whose Conway symbol is $(\underbrace{3, 3, \ldots, 3}_{n}, 2)$ $(n \geq 2)$ (as shown in Figure 2). Then the bridge number of K_n is $n + 1$. Moreover any realization of K_n on the cubic lattice has a ratio of $\frac{length}{crossing\ number} > 2$.*

PROOF. Figure 2 shows a standard diagram of K_n together with an assignment of 2-cycles to the overpasses of the diagram. It is easily verified that this assignment respects the Wirtinger relators (see [**L**] or [**BuZ**] for a reference); therefore it determines a homomorphism of the group of K_n onto the symmetric group S_{n+2}, mapping meridians to 2-cycles. From the existence of this homomorphism and the fact that the group S_{n+2} cannot be generated by fewer than (n+1) 2-cycles, it follows that the group of K_n cannot be generated by fewer than $n+1$ meridians. Since a k-bridge presentation of a knot gives rise to a Wirtinger presentation of its group with k meridian generators, it follows that the bridge number of K_n is at least $n+1$.

On the other hand, Figure 2 exhibits an embedding of K_n in $\mathbb{R}^3$ with $n+1$ local maxima. Therefore the bridge number of K_n is at most $n+1$. Finally $Cr(K_n) = 3n+2$ since K_n is alternating and the statement about the ratio of length over crossing number follows. □

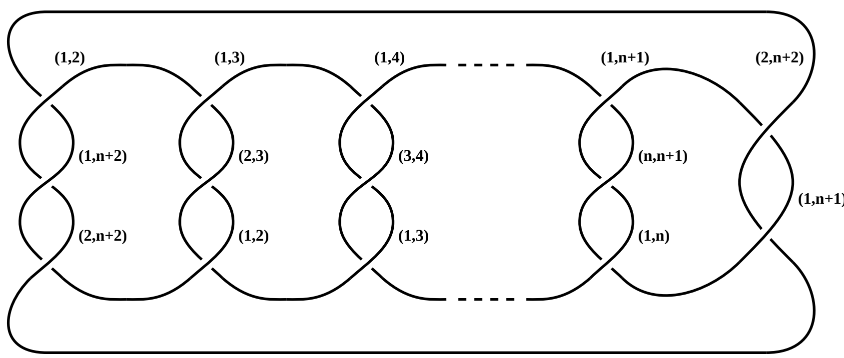

FIGURE 2. The knot K_n

Remark 2.1. If we extend the definition of K_n to include the degenerate cases K_0 = Hopf Link, $K_1 = (5,2)$-torus knot, then the conclusion of about the ratio $\frac{length}{crossing\ number} > 2$ still holds.

Remark 2.2. It is easy to see that the knot K_n can indeed be realized with $O(n)$ steps in the lattice. Thus for K_n on the cubic lattice the ratio $\frac{length}{crossing\ number}$ is bounded by some constant c.

Remark 2.3. The above example is a particular type of Montesinos knot. A Montesinos knot is a knot (or link) that has a projection as shown in Figure 3 using $r = 3$ balls. In general r can be any integer greater or equal to two. A ball with label $\frac{\beta_i}{\alpha_i}$, $1 \leq i \leq r$ stands for the rational tangle classified by that same fraction $\frac{\beta_i}{\alpha_i}$. A treatment of Montesinos knots can be found in [**BuZ**]. Assuming that $r \geq 3$ and that for all i, $\alpha_i \geq 2$, it is shown in [**BZ**] that for any such Montesinos knot K we have $b(K) = r$. The example in Figure 2 is a Montesinos knot with maximal r for a given crossing number n. While the above knot K_n is alternating, Montesinos knots are non alternating if tangles of different sign are substituted into the diagram in Figure 3. Furthermore, the minimal crossing number of such knots is known to be the number of crossings in the standard diagram ([**LT**], [**T1**], [**T2**]). Thus one could construct non-alternating prime knots whose ratio of length on the lattice over the crossing number is bounded below by a positive constant. The number of different Montesinos knots with n crossings grows exponentially [**ES**] and one can use the above technique to construct a family F_n of sets of different Montesinos knots with the following properties:

(1) $\forall K \in F_n,\ Cr(K) = n.$

(2) $\forall K \in F_n,\ \frac{L(K)}{Cr(K)} > c > 0$ for some constant c where $L(K)$ is the length of the shortest realization of K on the cubic lattice.

(3) $|F_n|$, the number of elements in F_n, grows exponentially with n.

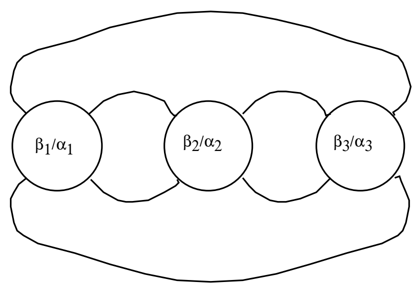

FIGURE 3. The diagram of a Montesinos knot with $r = 3$

THEOREM 2.3. *Let $\mathbf{T}_n$ be the set of all prime knots with crossing number n. For any $K \in \mathbf{T}_n$, let $P_0(K)$ be a polygonal knot on the cubic lattice which is equivalent to K and is shortest among all lattice polygons that equivalent to K. Define $\ell_n = \max\{\frac{L(P_0(K))}{n} :\ K \in \mathbf{T}_n\}$. Then for any $n \geq 3$ we have $\ell_n \geq 2 - \frac{2}{4+3n} > 1.71$.*

PROOF. This theorem follows from the Remark 2.3 above. All one has to do is to substitute a tangle with 4 or 5 crossing for any of the three crossing tangles that preserves the property that the knot is alternating. □

Remark 2.4. The above results can also be applied to prime links provided one changes the number 6 in theorem 2.1 to 4. However in the case of links it is easy to construct prime links that look like medieval chain mails which trivially must have a linear relationship between their lengths on the lattice and their crossing numbers.

Remark 2.5. Note that in the case of non prime links one can do a little better. This can be easily seen by creating a chain L_n of n unknotted circles, i.e., a Hopf link with n components. Such a link L_n with $n \geq 3$ components has $2(n-1)$ crossings and can be constructed with no fewer than $10(r-2)+16$ steps. Thus $\frac{L(L_n)}{crossing\ number} = \frac{10(r-2)+16}{2(r-1)} = 5 - \frac{3}{r-1} > 5.$

3. The case of smooth thick knots

In this section we extend the results of last section to the case of smooth knots of thickness one. We shall only use the *disk thickness* definition of C^2-curves. Similar result holds if other definitions of thicknesses are used. The disk thickness is also known as *curvature thickness* and was defined and studied in [**LSDR**]. Let $\alpha(s)$ be the arc-length parameterized equation for a C^2 knot K, and let the disk (or curvature) thickness of K be denoted by $t_c(K)$ or $t_c(\alpha)$. It is shown in [**LSDR**] that

$$t_c(\alpha) = \min\{1/\kappa(\alpha), d(\alpha)\},$$

where $\kappa(\alpha)$ is the maximum curvature of the curve α, and $d(\alpha)$ is the minimum separation between any two double critical points in α (a double critical pair $(\alpha(s), \alpha(t))$ is found when the chord between $\alpha(s)$ and $\alpha(t)$ is also normal to the tangent vectors of α at $\alpha(s)$ and $\alpha(t)$).

Let K be a smooth knot of unit disk thickness. A polygonal knot $P(K)$ on some cubic lattice is called a lattice approximation of K if there exists an $r \in (0,1)$ such that $P(K)$ is contained in the r-neighborhood of K and there is an ambient isotopy that carries K to $P(K)$ which is the identity outside the 1-neighborhood of K.

The following lemma is from [**DER2**].

LEMMA 3.1. *Let K be a smooth knot of unit disk thickness and let $L(K)$ be its length. Then K can be approximated by a lattice knot $P(K)$ on the lattice $\mathbf{Z}^3/4$ (where $\mathbf{Z}^3/m$ is the cubic lattice with vertices of coordinates of the form p/m for any $p \in \mathbf{Z}$). Moreover, the number of edges in $P(K)$ is bounded by $12L(K)$. In other word, the length of $P(K)$ is at most $3L(K)$.*

Remark 3.1. Originally, the bound on the number of edges in $P(K)$ given in [**DER2**] is $14L(K)$. It has been pointed out recently by John Sullivan that $12L(K)$ also bounds this number.

Combine the above lemma with the results of last section, we have the following theorems. The proofs are straight forward and are left to our reader.

THEOREM 3.2. *Let K be a knot and let $n = b(K)$ be its bridge number. If K_0 is a thick realization of K, then the length $L(K_0)$ of K_0 is at least $2n$. In particular, if $K = K_1\#K_2\#...\#K_m$ where K_1, K_2, ..., K_m are prime knots, then $L(K_0) \geq 2(\sum b(K_j) - m + 1)$.*

THEOREM 3.3. *Let $\mathbf{T}_n$ be the set of all prime knots with crossing number n. For any $K \in \mathbf{T}_n$, let K^t be a knot of unit thickness which is equivalent to K and is shortest among all such thick knots that are equivalent to K. Define $\ell_n^t = \max\{\frac{L(K^t)}{n} : K \in \mathbf{T}_n\}$. We have for all $n \geq 3$ that $\ell_n^t \geq \frac{1.71}{3}$.*

4. Some open questions

It is known that the length of any thick realization of a knot K is at least of the order $O(Cr^{3/4}(K))$ ([**B1**], [**BS**]). It is also known that for some knots, the length of their thick realizations is of the order $O(Cr^{3/4}(K))$. Our result in this paper shows that for some knots, the length of their thick realizations is of the order $O(Cr(K))$. Since the proof of our result is based on the bridge number of the knots, the technique here does not apply to knots with many crossings but with a small bridge number. We end this paper by raising the following two questions:

a. Can we find a family of knots $\{K_n\}$ such that the minimal length of their thick realizations is more than $O(Cr(K))$?

b. For any prime alternating knot K, is it true that the length of its thick realization is at least of the order $O(Cr(K))$?

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