The topology of the independence complex

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Abstract

We introduce a large self-dual class of simplicial complexes about which we show that each complex in it is contractible or homotopy equivalent to a sphere. Examples of complexes in this class include independence and dominance complexes of forests, pointed simplicial complexes, and their combinatorial Alexander duals.

1 Introduction

In this paper we introduce a large class of abstract simplicial complexes about which it is possible to show purely combinatorially that each of its members is contractible or homotopy equivalent to a sphere. We call the complexes of this class *constrictive* complexes. In general it is hard to say whether the geometric realizations of two abstract simplicial complexes are homotopic, since a homotopy equivalence between topological spaces may preserve little of the underlying discrete structure. There is however a notion of homotopy equivalence, called *simple-homotopy* equivalence that is close enough to the discrete world to have a combinatorial meaning. Simple-homotopy equivalence is defined as a sequence of *elementary collapses* and their inverses. These operations, studied by Gil Kalai in [13] and by J. Kahn, M. Saks, and D. Sturtevant in [12] are combinatorially defined and induce a homotopy equivalence of simplicial complexes. Another combinatorial operation that is clearly not changing the homotopy type is *contracting an edge* whenever the combinatorial structure allows it. It is essential to notice that edge contraction may be realized by a sequence of elementary collapses and inverse elementary collapses, we will prove this in the preliminary Section 2.

The basic example motivating our research was the simplicial complex of *sparse* subsets of the set $\{1, 2, \ldots, n\}$, i.e, the simplicial complex whose faces are the subsets containing no pair of consecutive integers. For all n, this complex is homeomorphic to a wedge of spheres by the results of Billera and Myers [2] on interval orders, and it is contractible or homotopy equivalent to a sphere as a consequence of Kozlov's theory of complexes of directed trees [14].

The class of constrictive simplicial complexes contains all complexes of sparse sets, and includes many other important examples. Constrictive complexes are formally defined in Section 4. They are closed under contracting an edge and their simplest examples are the empty set, and boundary complexes of simplices. The structure of constrictive complexes is best understood in terms of the structure of non-faces of simplicial complexes, this approach to edge contractions is developed in Section 3. Using the non-face approach it is almost immediate from the definition of a constrictive complex that it must be homotopy equivalent to a ball or to a sphere. What turns out to be harder to show is that many simplicial complexes arising in a combinatorial or graph theoretic setting are actually constrictive. Our examples include *branching complexes* which generalize the notion of the complex of independent sets of vertices in a forest of trees, dominance complexes of forests, and *pointed complexes*, which appear in the work of Ehrenborg and Steingrímsson [8] on playing Nim on a simplicial complex.

Branching complexes are shown to be constrictive Section 5 and an algorithm to calculate the exact homotopy type of the independence complex of a forest is described in Section 6. As a consequence we obtain the exact homotopy type of the complex of sparse subsets of $\{1, 2, \ldots, n\}$ as a function of n. The dominance complex of a forest is defined as the family of complements of its dominating sets and it is shown to be constrictive in Section 7. The proof indicates a method to calculate the actual homotopy type.

Finally, in Section 8 we revisit the notion of the *combinatorial Alexander dual*, introduced by Gil Kalai in [13]. As was already observed by Gil Kalai, an elementary collapse induces an elementary collapse at the level of dual complexes, but it is not clear in general that the combinatorial Alexander dual of a complex that is homotopy equivalent to a ball or sphere would also be homotopy equivalent to a ball or sphere. This is true, however, if we restrict our attention to constrictive complexes: it turns out that the combinatorial Alexander dual of a constrictive complex is constrictive. The only difficulty in proving this statement is establishing the fact that the combinatorial Alexander dual of the boundary complex of a simplex is constrictive, since it is pointed. Our theorem allows to state the dual of every result in the preceding sections.

It is not clear to us, what would be the "best" approach to proving homotopy equivalence to balls or spheres for simplicial complexes in general. The widely known method of shellings, even in its nonpure form as introduced by Björner and Wachs [3], seems to be more suitable for studying the homeomorphy rather than the homotopy type, and it is worth noting that for the class of constrictive simplicial complexes, a more elementary approach than discrete Morse theory [10] suffices.

2 Preliminaries

Definition 2.1 *An* abstract simplicial complex $\triangle$ *on a finite vertex set* V *is a family of sets* $\{\sigma \in \triangle : \sigma \subseteq V\}$ *satisfying the following properties.*

(i) $\{v\} \in \triangle$ *for all* $v \in V$

(ii) If $\sigma \in \triangle$ *then every subset* $\tau \subseteq \sigma$ *belongs to* $\triangle$*.*

The elements of $\triangle$ are *faces* the elements of V are called *vertices.* By property (i), giving the set of faces determines the set of vertices. Hence we may identify an abstract simplicial complex with its set of faces. The *dimension* of a face σ is $|\sigma| - 1$, maximal faces are called *facets*, 1-dimensional faces are called *edges.*

Given a subset $U \subseteq V$ of the vertex set, the *restriction* $\triangle|_U$ of the simplicial complex to U is the simplicial complex with vertex set U and face set $\triangle|_U = \{\sigma : \sigma \in \triangle, \sigma \subseteq U\}$. Another important notion is the link of a face σ in the complex $\triangle$, defined by

$$\mathrm{link}_{\triangle}(\sigma) = \{\tau \subseteq V \setminus \sigma \ : \ \tau \cup \sigma \in \triangle\}.$$

Every abstract simplicial complex has a *standard geometric realization*. We take a basis $\{e_v : v \in V\}$ in $\mathbb{R}^{|V|}$ and the union of the convex hulls of sets $\{e_v : v \in \sigma\}$ for each $\sigma \in \triangle$. Homotopic or homeomorphic properties of a finite simplicial complex are the same as those of its geometric realization.

Definition 2.2 *We call an edge $\{u, v\}$ of a simplicial complex $\triangle$* contractible *if every face $\sigma \in \triangle$ satisfying $\{u\} \cup \sigma \in \triangle$ and $\{v\} \cup \sigma \in \triangle$ also satisfies $\{u, v\} \cup \sigma \in \triangle$.*

If the edge $\{u, v\} \in \triangle$ is contractible, the contracted simplicial complex $\triangle/\{u, v\}$ is constructed as follows:

- We remove the vertices u and v from the vertex set V and add a new vertex w.
- A set $\tau \subseteq V \setminus \{u, v\} \cup \{w\}$ is a face of $\triangle/\{u, v\}$ if $w \notin \tau$ and $\tau \in \triangle$ or $w \in \tau$ and at least one of $\tau \setminus \{w\} \cup \{u\}$, $\tau \setminus \{w\} \cup \{v\}$ is a face of $\triangle$.

To simplify our notation, the "new" vertex may be identified with either u or v, hence we may talk of "contracting the edge $\{u, v\}$ to u", for example. It is visually straightforward that contracting a contractible edge induces a homotopy equivalence of the geometric realizations.

A face $\tau \in \triangle$ is *free* if it is contained in a unique facet σ. If $|\sigma \setminus \tau| = 1$ then the removal of τ and σ is called an *elementary collapse*.

Definition 2.3 *We call the simplicial complexes $\triangle$ and $\triangle'$* simple-homotopic *if there is a finite sequence $\triangle = \triangle_1, \triangle_2, \ldots, \triangle_n = \triangle'$ of simplicial complexes such that for each index i at least one of $\triangle_i$ and $\triangle_{i+1}$ may be obtained from the other by an elementary collapse.*

Elementary collapses and simple-homotopy are well-studied topological notions. A good reference is M. Cohen's book [4] where, at the beginning of Chapter II, these notions are introduced for finite *CW* complexes. It is well-known that simple-homotopy is a narrower equivalence relation than homotopy equivalence. In the literature of combinatorial papers on the homotopy type of various abstract simplicial complexes it is also customary to cite a (much later) paper of G. Kalai [13], which was one of the first ones to use this notion in a combinatorial setting. M. Cohen calls the inverse of an elementary collapse an *elementary expansion*, the combinatorial literature seems to prefer the term *anticollapse*.

It is worth noting that simple-homotopy includes the possibility of edge contraction, because of the following theorem.

Theorem 2.4 *If the edge $\{u,v\}$ is contractible in the simplicial complex $\triangle$, then the complex $\triangle$ and the contracted simplicial complex $\triangle/\{u,v\}$ are simple-homotopic.*

Proof: Consider the set of faces

$$S = \{\sigma \subseteq V \setminus \{u,v\} \ : \ \sigma \cup \{v\} \in \triangle, \sigma \cup \{u\} \notin \triangle\}.$$

Order this set $S = \{\sigma_1, \ldots, \sigma_m\}$ such that $\sigma_i \subseteq \sigma_j$ implies that $i \leq j$. (This may be achieved, for example, by writing listing the elements of S in increasing order of cardinality.) Let $\triangle_j$ be the simplicial complex

$$\triangle_j = \triangle \cup \{\sigma_i \cup \{u\}, \sigma_i \cup \{u,v\} \ : \ 1 \leq i \leq j\}.$$

Note that $\triangle_0 = \triangle$ and that $\triangle_{j+1}$ is obtained from $\triangle_j$ by an anticollapse. Moreover, if a face σ of the last complex $\triangle_m$ contains v then $\sigma \cup \{u\} \in \triangle_m$. (In other words, using the terminology of Definition 2.5 below, the link of v in $\triangle_m$, $\text{link}_{\triangle_m}(v)$, is a cone with apex u.)

Let us order now the faces in $\text{link}_{\triangle_m}(\{u,v\}) = \{\tau_1, \ldots, \tau_n\}$ such that $\tau_i \supseteq \tau_j$ implies $i \leq j$. Let

$$\Gamma_j = \triangle_m \setminus \{\tau_i \cup \{v\}, \tau_i \cup \{u,v\} \ : \ 1 \leq i \leq j\}.$$

Evidently, $\Gamma_0 = \triangle_m$ and Γ_{j+1} is obtained from Γ_j by an elementary collapse. It is straightforward to see that Γ_n is the contracted complex $\triangle/\{u,v\}$. □

Beyond edge contractions and elementary collapses there are two operations on simplicial complexes which are not homotopy equivalences, but yield homotopy spheres or balls: *coning* an arbitrary simplicial complex over a new vertex v, and *suspension* of spheres or balls.

Definition 2.5 *A simplicial complex $\triangle$ on a vertex set V is a* cone *with apex $v \in V$ if every $\sigma \in \triangle$ satisfies $\sigma \cup \{v\} \in \triangle$.*

If a simplicial complex $\triangle$ is a cone with apex v then its geometric representation may be contracted to the point e_v. Every simplicial complex $\triangle'$ may be extended to a cone by adding a new vertex u to its vertex set, and the sets $\{\sigma \cup \{u\} \ : \ \sigma \in \triangle'\}$ to the set of faces. The resulting simplicial complex is denoted by $u * \triangle'$. The extrinsic and intrinsic description of the cone may be brought together by stating that a simplicial complex $\triangle$ is a cone with apex v if and only if $\triangle = v * \triangle\Big|_{U \setminus \{v\}}$.

Remark 2.6 Not only is the geometric realization of a cone $u * \triangle$ is contractible, but contraction to a single vertex may be achieved by successively collapsing every pair of faces σ and $\sigma \cup \{u\}$. Hence a cone is also simple-homotopic to a single vertex.

Suspension is more easily described the extrinsic way.

Definition 2.7 *Let $\triangle$ be an abstract simplicial complex on the vertex set V. The* suspension $\Sigma(\triangle)$ *of $\triangle$ is defined up to isomorphism by adding two new vertices $u, v \notin V$ to the vertex set, and setting $\Sigma(\triangle) = \{\sigma, \sigma \cup \{u\}, \sigma \cup \{v\} \ : \ \sigma \in \triangle\}$.*

Alternatively, a simplicial complex $\triangle$ is a suspension of a smaller simplicial complex, if and only if $\triangle \cong \Sigma\left(\triangle\Big|_{V\setminus\{u,v\}}\right)$ for some pair of vertices $\{u, v\}$. The following lemma is well-known.

Lemma 2.8 *If $\triangle$ is contractible then $\Sigma(\triangle)$ is contractible. If $\triangle$ is homotopy equivalent to a sphere of dimension k then $\Sigma(\triangle)$ is homotopy equivalent to a sphere of dimension $k+1$.*

3 Edge contraction and non-faces

In our main results we focus on the homotopic properties of a simplicial complex $\triangle$ in terms of its *non-faces*, that is, the family $\{A \subseteq V : A \notin \triangle\}$. A minimal non-face of a simplicial complex is called a *circuit.* If there is a vertex v that is not contained any circuit then the simplicial complex is a cone with apex v and thus contractible.

We call a collection $\mathcal{B} = \{B_1, \ldots, B_n\}$ of nonempty subsets of a vertex set V a *block system.* The *independence complex* of $\mathcal{B}$ over V, denoted by $I_V(\mathcal{B})$, is the simplicial complex consisting of the faces

$$I_V(\mathcal{B}) = \{\sigma \subseteq V \ : \ B_i \not\subseteq \sigma \text{ for all } B_i \in \mathcal{B}\}.$$

The vertex set of $I_V(\mathcal{B})$ is

$$V \setminus \bigcup_{\substack{i \\ |B_i|=1}} B_i,$$

hence in general we may assume that each block B_i has cardinality at least 2. Certain operations on block systems may yield singleton blocks, at the level of the independence complex this will simply mean that we remove the corresponding vertices from the vertex set. In Section 8 we will use a generalized definition of a simplicial complex, which will make the exceptional treatment of singleton blocks unnecessary.

It is worth noting that *every simplicial complex is an independence complex*: it is the independence complex of its circuits.

Let us rephrase edge-contraction in terms of non-faces. An edge $\{u, v\} \in \triangle$ is contractible, if for any non-face $A \notin \triangle$ containing $\{u, v\}$, either $A \setminus \{u\}$ or $A \setminus \{v\}$ is a non-face. In the contracted complex, $A \subseteq V \setminus \{u, v\} \cup \{w\}$ is a non-face if either $w \notin A$ and A is a non-face in the original complex, or $w \in A$ and both $A \setminus \{w\} \cup \{u\}$ and $\sigma \setminus \{w\} \cup \{v\}$ are non-faces in the original complex.

Lemma 3.1 *An edge $\{u, v\} \in \triangle$ is contractible if and only if no circuit (minimal non-face) contains $\{u, v\}$.*

Proof: Assume that some circuit B contains $\{u, v\}$. Then neither $B \setminus \{u\}$ nor $B \setminus \{v\}$ is a non-face, and $\{u, v\}$ can not be contracted.

Assume now that no circuit contains $\{u, v\}$ and let A be an arbitrary non-face containing $\{u, v\}$. Since A can not be minimal, it properly contains a circuit B. Since B is a proper subset of A, it avoids at least one of u, v, and so either $A \setminus \{u\}$ or $A \setminus \{v\}$ is a non-face. □

Lemma 3.2 *Let $\mathcal{B}$ be a block system on the vertex set V and let $\triangle = I_V(\mathcal{B})$. Assume that the edge $\{u,v\} \in \triangle$ is contractible to the vertex w. Then the resulting simplicial complex is the independence complex of*

$$\mathcal{B}' = \{B \ : \ B \in \mathcal{B}, B \cap \{u,v\} = \emptyset\} \cup \big\{\{w\} \cup B' \cup B'' \setminus \{u,v\} \ : \ B', B'' \in \mathcal{B}, u \in B', v \in B''\big\}$$

on the vertex set $V \setminus \{u,v\} \cup \{w\}$.

Proof: We show that the non-faces of the contracted complex are exactly those subsets of $V' := V \setminus \{u,v\} \cup \{w\}$ which contain some element of $\mathcal{B}'$.

Assume first that a set $A \subseteq V'$ contains some $B \in \mathcal{B}$ that is disjoint from $\{u,v\}$. Since B is a non-face of the contracted complex, so is A. Assume next that $A \subseteq V'$ contains a union of sets $\{w\} \cup B' \cup B'' \setminus \{u,v\}$ for some $B', B'' \in \mathcal{B}$ satisfying $u \in B', v \in B''$. Then both $A \setminus \{w\} \cup \{u\}$ and $A \setminus \{w\} \cup \{v\}$ are non-faces in the original complex since the first one contains B' the second one contains B''. Hence A is a non-face in the contracted complex.

To prove the reverse inclusion, assume that A is a non-face in the contracted complex. If $w \notin A$ then A is also a non-face in the original complex, and it contains some $B \in \mathcal{B}$ which obviously satisfies $B \cap \{u,v\} = \emptyset$. Finally, if $w \in A$ then both $A \setminus \{w\} \cup \{u\}$ and $A \setminus \{w\} \cup \{v\}$ are non-faces in the original complex and the first must contain a block $B' \in \mathcal{B}$ containing u, the second must contain a block $B'' \in \mathcal{B}$ containing v. □

4 Constrictive simplicial complexes

In this section we present a class of complexes which may be shown to be contractible or homotopy equivalent to a sphere, using only edge contractions. *Constrictive* complexes are defined recursively as follows.

Definition 4.1 *A simplicial complex $\triangle$ on the vertex set V is* constrictive *if the complex $\triangle$ is the boundary of the simplex on the vertex set V or there is a vertex v in V belonging to at most one circuit with one of the following properties:*

(i) v belongs to no circuit; or

(ii) v belongs to a unique circuit $B \neq V$ and there is a vertex $u \notin B$ such that contracting the edge $\{u,v\}$ yields a constrictive complex.

Under the circumstances of condition *(ii)*, the edge $\{u,v\}$ is contractible by Lemma 3.1, since no circuit contains both u and v. Using Lemma 3.2, the contracted complex may be described as the independence complex of a block system that is easily derived from the non-faces of the original complex.

Lemma 4.2 *A constrictive simplicial complex $\triangle$ is simple-homotopic to a single vertex or to the boundary complex of a simplex.*

Proof: We proceed by induction on $|V|$. If $\triangle$ is the boundary complex of a simplex, then there is nothing to prove. In case (i) of Definition 4.1 the simplicial complex is a cone with apex v and thus by Remark 2.6 it is reducible to a single vertex by a sequence of edge contractions. In case (ii) we may apply the induction hypothesis. □

In this section we give two initial examples of constrictive complexes, further classes of constrictive complexes will be explored in Sections 5 through 7. The first one is the class of *pointed* simplicial complexes. They appeared in the work of Ehrenborg and Steingrímsson [8]. We call a simplicial complex $\triangle$ *pointed* if every circuit C of $\triangle$ contains a vertex v that does not belong to any other circuit of $\triangle$. Call the vertex v of the circuit C the *pointed* vertex of C. Using Lemma 3.2 one can prove the following.

Proposition 4.3 *Let $\triangle$ be a pointed simplicial complex on n vertices with k circuits. Then the complex $\triangle$ is constrictive. Moreover, if the vertex set V is the union of the circuits $C_1, \ldots, C_k$ then the complex $\triangle$ is simple-homotopy equivalent to an $(n-k-1)$-dimensional sphere.*

Proof: If there is a vertex v that is not contained in any circuit then the complex $\triangle$ is constrictive and also homotopy equivalent to a point. Hence we may now assume that the vertex set V is the union of the circuits. Take two circuits and contract their two pointed vertices. Observe that this falls into case (ii) of Definition 4.1. The result is a pointed simplicial complex on $n-1$ vertices and $k-1$ circuits and where every vertex belong to at least one circuit. Proceed in this manner and we obtain a simplicial complex consisting of $n-k+1$ vertices and one circuit which consists of all the vertices. The independence complex is the boundary of an $(n-k)$-dimensional simplex and hence is an $(n-k-1)$-dimensional sphere. We may also conclude by induction on the number of circuits that a pointed simplicial complex is constrictive. □

Our next example is *the independence complex of a family of intervals on* $[1, n] = \{1, 2, \ldots, n\}$. We assume that our vertex set is $[1, n]$. An *interval* $I = [i, j] \subseteq [1, n]$ is a set $\{i, i+1, \ldots, j\}$. Here we allow $i = j$ yielding a singleton as an interval.

Theorem 4.4 *The independence complex of a family of intervals on $[1, n]$ is constrictive.*

Proof: We proceed by induction on n. The case $n = 1$ is trivial. Since nested blocks may be removed without changing the independence complex, we may assume that our family of intervals is an antichain, that is, no interval contains another. Then our family of intervals may be written as $\{[a_1, b_1], \ldots, [a_k, b_k]\}$ for some $1 \leq a_1 < a_2 < \cdots < a_k \leq n$ and $1 \leq b_1 < b_2 < \cdots < b_k \leq n$ satisfying $a_i \leq b_i$ for $i = 1, 2, \ldots, k$. If $b_k < n$ then the independence complex is a cone with apex $v = n$. Otherwise the vertex $v = b_k = n$ belongs to the unique circuit $[a_k, b_k]$. If $a_k = 1$ then the entire vertex set is a circuit and we have the boundary of a simplex. If $a_k > 1$ then consider vertex $u = a_k - 1$. The edge $\{u, v\}$ is contractible to u and the resulting simplicial complex is the independence complex of the following blocks:

- intervals $[a_i, b_i]$ for $i \leq k-1$ satisfying $b_i < u$, and
- intervals $[a_i, b_i] \cup [a_k, b_k] \setminus \{n\} = [a_i, n-1]$ for $i \leq k-1$ satisfying $b_i \geq u$.

Note that $a_i \leq u$ always holds for $i < k$, since $a_i < a_k$. Therefore we obtain the independence complex of a family of intervals on $[1, n-1]$, and we may invoke the induction hypothesis. □

We call a subset of $\{1, \ldots, n\}$ *sparse* if it does not contain two consecutive integers.

Corollary 4.5 *The simplicial complex consisting of all sparse sets on $\{1, \ldots, n\}$ is constrictive.*

In fact, this is just the independence complex of the family of intervals $\{[1,2], [2,3], \ldots, [n-1, n]\}$. For a more detailed discussion of this simplicial complex and its homotopy type see Corollary 6.3 and the paragraphs thereafter.

5 Branching block systems

Definition 5.1 *A* branching block system $\mathcal{B} = \{B_1, \ldots, B_n\}$ *is a set of blocks such that for every $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$ at least one of $B_{i_1} \cap B_{i_2}, B_{i_2} \cap B_{i_3}, \ldots, B_{i_k} \cap B_{i_1}$ is contained in (and hence equal to) $B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_k}$.*

This definition may be rephrased as follows. Consider the graph whose vertices are $\{i_1, \ldots, i_k\}$, and $\{i, j\} \subseteq \{i_1, \ldots, i_k\}$ is an edge if and only if $B_i \cap B_j$ properly contains $B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_k}$. Then this graph contains no k-cycle.

Any subfamily of a branching block system is evidently a branching block system. In particular, if $B_i \subseteq B_j$ for some $i, j \in \{1, 2, \ldots, n\}$ then B_j may be removed from our family, without changing the independence complex. We say that B_j was a *nested block* of $\mathcal{B}$.

Proposition 5.2 *A branching system $\mathcal{B} = \{B_1, \ldots, B_n\}$ of at least two blocks either contains a nested block or at least two blocks B_i, B_j such that*

$$B_i \not\subseteq \bigcup_{t \neq i} B_t \quad \text{and} \quad B_j \not\subseteq \bigcup_{t \neq j} B_t.$$

In the proof of Theorem 5.3 we need only the existence of one such block, but technically it is easier to prove the existence of two such blocks.

Proof of Proposition 5.2: Assume that $\mathcal{B}$ contains no nested blocks. We prove by induction on $\sum_{i=1}^{n} |B_i|$ the existence of two blocks none of which is contained in the union of the other blocks.

As a consequence of Definition 5.1, $B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_k}$ is not empty whenever none of $B_{i_1} \cap B_{i_2}, B_{i_2} \cap B_{i_3}, \ldots, B_{i_k} \cap B_{i_1}$ is the empty set. Consider the following graph G. Its vertex set is $\{1, 2, \ldots, n\}$ and $\{i, j\} \subseteq \{1, 2, \ldots, n\}$ is an edge if and only if $B_i \cap B_j \neq \emptyset$. By our observation G is a "forest of cliques", in other words, every 2-connected component of G is a clique. In fact, if there are two vertex-disjoint paths between i and j, then there is also a cycle $(i_1, \ldots, i_k)$ containing both vertices, and $B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_k} \neq \emptyset$ implies that any unordered pair $\{i_s, i_t\}$ is an edge.

Case 1: G is not 2-connected.
In this case after contracting each 2-connected component to a single vertex, we obtain a forest with at least 2 vertices. Such a forest has at least 2 leaves or isolated vertices. Assume that $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_l\}$ are two different cliques that contract to a leaf or isolated vertex. It is sufficient to show that at least one of $B_{i_1}, \ldots, B_{i_k}$ is not contained in the union of the remaining blocks, and then the same argument may be repeated for the j_t's. If $k = 1$ then B_{i_1} has nonzero intersection only with at most one other block, and that block can not contain it unless it is nested. If k is at least 2, then by our induction hypothesis there are at least two blocks B_{i_r} and B_{i_s} such that

$$B_{i_r} \not\subseteq \bigcup_{t \neq r} B_{i_t} \quad \text{and} \quad B_{i_s} \not\subseteq \bigcup_{t \neq s} B_{i_t}.$$

Since the 2-connected component containing i_r and i_s contract to a leaf or isolated vertex, only at most one of B_{i_r} and B_{i_s} may have a nonempty intersection with any B_j satisfying $j \not\in \{i_1, \ldots, i_k\}$. The other one is not contained in the union of all the other blocks.

Case 2: G is 2-connected (hence a clique).
In this case $B_1 \cap \cdots \cap B_n \neq \emptyset$. Consider the block system $\mathcal{B}' = \{B'_1, \ldots, B'_n\}$ where $B'_i = B_i \setminus (B_1 \cap \cdots \cap B_n) \neq \emptyset$. The system $\mathcal{B}'$ is also branching and non-nested. Moreover $\sum_{i=1}^{n} |B'_i|$ is strictly less than $\sum_{i=1}^{n} |B_i|$. Hence we may apply our induction hypothesis. If, say, B'_i is not contained in the union of the other B'_j's then B_i is not contained in the union of the other B_j's. □

Theorem 5.3 *The independence complex of a branching block system $\mathcal{B} = \{B_1, \ldots, B_n\}$ is constrictive. As a consequence, the independence complex of a branching block system is simple-homotopic to a single vertex or to a sphere.*

Proof: We proceed by induction on n. The basis of the induction is $n = 1$. It is straightforward to observe that the independence complex $I_V(\{B_1\})$ is constrictive.

If $\mathcal{B}$ contains a nested block, we may remove it without changing the independence complex. Otherwise, as a consequence of Proposition 5.2, there is at least one block not contained in the union of the others. Without loss of generality we may assume $B_n \not\subseteq \bigcup_{i=1}^{n-1} B_i$. Let v be an element of $B_n \setminus \left(\bigcup_{i=1}^{n-1} B_i \right)$. Choose an $m < n$ such that $B_m \cap B_n$ is a maximal element of the family of sets $\{B_i \cap B_n \ : \ i < n\}$ ordered by inclusion. (In particular, if B_n is disjoint from all the other B_i's, m may be any index less than n.) Since $\mathcal{B}$ has no nested blocks, there is a vertex $u \in B_m \setminus B_n$. The vertices u and v are not contained in any minimal non-face of the independence complex, hence they are contractible to a single vertex w. By abuse of notation let us denote the new vertex w also by u. Using Lemma 3.2, this identification allows us to describe the contracted simplicial complex as the independence complex of $\mathcal{B}' = \{B_1, \dots, B'_{n-1}\}$, where

$$B'_i = \begin{cases} B_i & \text{if } u \notin B_i \\ B_i \cup (B_n \setminus \{v\}) & \text{if } u \in B_i \end{cases}$$

It is sufficient to show that $\mathcal{B}'$ is a branching block system, and we are done by induction.

Consider a subset $\{i_1, \dots, i_k\}$ of $\{1, 2, \dots, n-1\}$ and assume first that $m \notin \{i_1, \dots, i_k\}$. Since $\mathcal{B}$ is branching, two cyclically consecutive elements of the list $(B_{i_1}, \dots, B_{i_k})$ intersect in $B_{i_1} \cap \cdots \cap B_{i_k}$. Without loss of generality we may assume that

$$B_{i_1} \cap B_{i_2} = B_{i_1} \cap \cdots \cap B_{i_k}. \tag{5.1}$$

It is sufficient to show that

$$B'_{i_1} \cap B'_{i_2} = B'_{i_1} \cap \cdots \cap B'_{i_k} \tag{5.2}$$

also holds. If $u \in B_{i_1} \cap B_{i_2}$ then u belongs to all B_{i_t}'s and equation (5.2) may be obtained from (5.1) by joining the same $B_n \setminus \{v\}$ to both sides. If u belongs to neither B_{i_1} nor B_{i_2} then we have

$$B'_{i_1} \cap B'_{i_2} = B_{i_1} \cap B_{i_2} = B_{i_1} \cap \cdots \cap B_{i_k} \subseteq B'_{i_1} \cap \cdots \cap B'_{i_k}$$

while the reverse inclusion obviously holds. Hence we may assume that u belongs to exactly one of B_{i_1}, B_{i_2}, by cyclic symmetry we may assume that $u \in B_{i_1} \setminus B_{i_2}$.

Consider the following cyclic list of blocks:

$$(B_{i_1}, B_m, B_n, B_{i_2}, B_{i_3}, \dots, B_{i_k}). \tag{5.3}$$

By the branching property for $\mathcal{B}$, at least two cyclically consecutive blocks on this list intersect in the intersection of all blocks on the list. If $B_{i_j} \cap B_{i_{j+1}}$ is such an intersection for some $j \in \{2, 3, \dots, k-1\}$ then we may remove B_{i_j} from our list without changing the intersection of all blocks since in that case we have

$$B_{i_j} \cap B_{i_{j+1}} \subseteq B_{i_1} \cap B_{i_2} = B_{i_1} \cap \cdots \cap B_{i_k}$$

and by the obvious reverse inclusion $B_{i_j} \cap B_{i_{j+1}}$ contributes the same set to the meet of all blocks on the list as $B_{i_1} \cap B_{i_2}$. Similarly if $B_{i_k} \cap B_{i_1}$ is equal to the intersection of all blocks then we may remove B_{i_k} from our cyclic list (5.3). Repeated application of this observation yields a cyclic list of blocks containing $B_{i_1}, B_m, B_n, B_{i_2}$ consecutively, with the same intersection of all blocks on the list, and such that the only consecutive pair of blocks intersecting in the intersection of all blocks on the list is either $B_{i_1} \cap B_m$, or $B_m \cap B_n$, or $B_n \cap B_{i_2}$. The intersection $B_{i_1} \cap B_m$ contains u which does not belong to B_{i_2} hence we are left with the other two possibilities. By the choice of B_m, the intersection $B_m \cap B_n$ can not be a proper subset of $B_n \cap B_{i_2}$, hence we get

$$B_n \cap B_{i_2} \subseteq B_{i_1} \cap \cdots \cap B_{i_k} \cap B_m \cap B_n \subseteq B_{i_1} \cap B_{i_2}.$$

This implies

$$B'_{i_1} \cap B'_{i_2} = (B_{i_1} \cup (B_n \setminus \{v\})) \cap B_{i_2} \subseteq B_{i_1} \cap B_{i_2} = B_{i_1} \cap \cdots \cap B_{i_k},$$

therefore

$$B'_{i_1} \cap B'_{i_2} \subseteq B'_{i_1} \cap \cdots \cap B'_{i_k}$$

and the reverse inclusion obviously holds.

We conclude our proof by describing the adjustments that have to be made for the above argument if m belongs to $\{i_1, \ldots, i_k\}$. If the pair $\{i_1, i_2\}$ found at the beginning of our argument does not contain m then the only adjustment to the above argument is at the introduction of the cyclic list (5.3). There we will skip $i_j = m$ from the list (and keep the item B_m occurring after i_1 and before B_n.) Finally, if $m \in \{i_1, i_2\}$ then upon reaching the assumption $u \in B_{i_1} \setminus B_{i_2}$ we must conclude $m = i_1$. Instead of the cyclic list (5.3) we start out considering the list

$$(B_m, B_n, B_{i_2}, B_{i_3}, \ldots, B_{i_k})$$

and keep removing B_{i_j}'s for $j > 2$ until we get the shortest possible list with the same intersection of all blocks, still containing the items B_m, B_n, B_{i_2} consecutively. Again the consecutive pair intersecting in the intersection of all blocks is either $B_m \cap B_n$ or $B_n \cap B_{i_2}$, and from here the argument is the same. □

6 The independence complex of a forest

A simple undirected graph G with no loops or parallel edges may be considered as a block system $\mathcal{B}$ where each block of $\mathcal{B}$ is of the form $\{u, v\}$ for some edge uv in the graph. Moreover, the independence complex of $\mathcal{B}$ consists of all independent sets of the graph G.

When the graph is a forest then the associated block system is a branching block system. Thus the following is a direct corollary of Theorem 5.3.

Corollary 6.1 *Let F be a forest on a vertex set V, that is, a graph without cycles. Then the independence complex of F is constrictive and thus simple-homotopy equivalent to a single vertex or to a sphere.*

Proposition 6.2 lets us recursively calculate the homotopy type of the independence complex of a forest. Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a block system on the vertex set V and let x be a vertex in V. Let $\mathcal{B}_{x,k}$ denote the block system $\mathcal{B}$ with a path of length k attached to the vertex x. That is, $\mathcal{B}_{x,k}$ is a block system on the disjoint union of the set V and $\{x_1, \ldots, x_k\}$ with the added blocks $\{x, x_1\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}$. Similarly, let $\mathcal{B}_{x,k,h}$ denote the block system $\mathcal{B}$ with two paths attached to the vertex x, one of length k and one of length h. In our notation, $\mathcal{B}_{x,k,h} = (\mathcal{B}_{x,k})_{x,h}$.

Proposition 6.2 *For a block system $\mathcal{B}$ we have the following simple-homotopy equivalences:*

(i) $I(\mathcal{B}_{x,1,1}) \cong I(\mathcal{B}_{x,1})$,

(ii) $I(\mathcal{B}_{x,3}) \cong \Sigma(I(\mathcal{B}))$,

(iii) $I(\mathcal{B}_{x,2,2}) \cong \Sigma(I(\mathcal{B}_{x,2}))$ *and*

(iv) $I(\mathcal{B}_{x,2,1})$ *is simple-homotopy equivalent to a point, that is, contractible.*

Proof: In the block system $\mathcal{B}_{x,k,h}$ let $x_1, \ldots, x_k$ denote the vertices of the first path added and let $y_1, \ldots, y_h$ denote the vertices of the second path added. To prove (i) contract x_1 and y_1 and denote the contracted vertex also by x_1. By Lemma 3.2 the resulting complex is the independence complex of $\mathcal{B}_{x,1}$.

To prove (ii) contract x_1 and x_3 and denote the contracted vertex also by x_3. Using Lemma 3.2 again yields that the resulting complex is the independence complex of the following block system on $V \cup \{x_2, x_3\}$. The blocks are the blocks of $\mathcal{B}$ and the two blocks $\{x, x_2, x_3\}$ and $\{x_2, x_3\}$. The block $\{x, x_2, x_3\}$ contains $\{x_2, x_3\}$ hence it may be discarded without changing the independence complex. The independence complex of the resulting block system is isomorphic to $\Sigma(I(\mathcal{B}))$.

To prove (iii) contract x and x_2, and denote the contracted vertex by x_2. An argument similar to the proof of (ii) shows that the resulting complex $\triangle_2$ is identifiable with the independence complex of $\{B \in \mathcal{B} : x \notin B\} \cup \{\{x_1, x_2\}, \{y_1, y_2\}\}$ on $V \setminus \{x\}$. Observe now that the same contraction applied to $I(\mathcal{B}_{x,2})$ yields the independence complex $\triangle_1$ of $\{B \in \mathcal{B} : x \notin B\} \cup \{\{x_1, x_2\}\}$ on $V \setminus \{x\}$. The statement now follows from the straightforward observation $\triangle_2 \cong \Sigma(\triangle_1)$.

Finally, to prove (iv) contract again x and x_2, and denote the contracted vertex by x_2. The resulting complex is the independence complex of a block system on $V \setminus \{x\}$ in which no block contains y_1. Thus we obtain a cone with apex y_1. □

As indicated at the end of Section 6, we are now able to determine the homotopy type of the simplicial complex of sparse sets on the set $\{1, \ldots, n\}$ precisely.

Corollary 6.3 *The simplicial complex consisting of all sparse sets on* $\{1, \ldots, n\}$ *is contractible if* $n \equiv 1 \bmod 3$. *Otherwise the complex is homotopy equivalent to a* $\lfloor (n-1)/3 \rfloor$*-dimensional sphere.*

Proof: The simplicial complex in the statement is the independence complex of a path on n vertices. By Proposition 6.2, part (ii), it is enough to verify the statement for $n = 1, 2, 3$. □

The simplicial complex of sparse sets was previously studied by Billera and Myers [2], and Kozlov [14]. Billera and Myers consider sparse sets as a special case of *interval orders* and they prove that such an order in general is non-pure shellable in the sense of Björner and Wachs [3] and hence homeomorphic to a wedge of spheres. Kozlov proved Corollary 6.3 as a special case of results on complexes of directed trees [14, Proposition 4.5]. Kozlov studies complexes whose *vertices* are edges of some directed graph, and faces are directed forests. The circuits (minimal non-faces) in such complexes are particularly nice: a set $\{e_1, \ldots, e_k\}$ is a circuit if and only if $\{e_1, \ldots, e_k\}$ forms a directed cycle in some order or $k = 2$ and e_1 and e_2 have the same target vertex. Therefore the study of such complexes from the non-face perspective might yield interesting results.

Note that the simplicial complex of sparse sets is not a pure simplicial complex in general. It is easy to show that the dimensions of facets range between $\lfloor (n+2)/3 \rfloor - 1$ and $\lceil n/2 \rceil - 1$. Thus this simplicial complex is pure only when $n \leq 2$ or when $n = 4$.

7 The dominance complex of a forest

Let G be a graph on the vertex set V. A *dominance set* of the graph G is a subset S of vertices such that each vertex in the graph is either in the set S or adjacent to a vertex in the set S. Observe that if S is a dominance set and the set T contains S then T is also a dominance set. Thus the complements of dominance sets are closed under inclusion. Hence we define the *dominance complex* of a graph G to be the simplicial complex consisting of the faces

$$D_V(G) = \{\sigma \subseteq V \ : \ V \setminus \sigma \text{ is a dominating set of } G\}.$$

Theorem 7.1 *The dominance complex of a forest F is simple-homotopy equivalent to a sphere. In fact, the dominance complex $D_V(F)$ is constrictive.*

For each vertex of the graph G let $N[v]$ denote the set of all neighbors of v together with the vertex v. The dominance complex $D_V(G)$ may be described as the independence set of the block system $\{N[v] \ : \ v \in V\}$. In fact, the set σ contains $N[v]$ for some vertex v if and only if the complement $V \setminus \sigma$ is not dominating the vertex v. In general, the block system $\{N[v] : v \in V\}$ is not branching. This can be seen using a path consisting of six vertices.

We will prove a more general statement than Theorem 7.1; see Theorem 7.2. In order to proceed, we need to introduce the notion of a forest on a partition. Let π be a partition of the vertex set V, that is, $\pi = \{S_1, \ldots, S_k\}$ is a collection of non-empty disjoint subsets of V whose union is V. The usual terminology is to call the subsets of the partition π blocks. We will follow this terminology in this section and call the blocks in a block system blocking sets. Let F be a forest on the set of blocks of the partition π. We write $S \sim T$ if S and T are two adjacent blocks in the forest. Define the neighborhood of a block S in π as the set

$$N[S] = S \cup \bigcup_{S \sim T} T.$$

Define the dominance complex $D_V(F)$ as the independence complex

$$D_V(F) = I_V\left(\{N[S] \ : \ S \in \pi\}\right).$$

Now we can introduce the stronger statement:

Theorem 7.2 *Let F be a forest on a partition π. Then the dominance complex $D_V(F)$ is constrictive and it is simple-homotopy equivalent to a sphere.*

In order to work with forests on partitions we need to introduce some notation. Let π be a partition of the set V and let F a forest on π. Let B and C be two non-empty disjoint sets that are also disjoint from the set V. Let $F \cup \{B\}$ denote the forest where we add the set B as a new block to the partition π and let this block be an isolated node in the forest. Similarly, let $F \cup \{B, C\}$ be forest where we add two singleton blocks to the forest F. Let $F \cup \{B \sim C\}$ be forest where we add the two nodes B and C, and we attach them with an edge together. Let A be a block of π. Let $B_1, \ldots, B_k$ be disjoint non-empty sets that are also disjoint from the vertex set V. Let $F_{A;B_1,\ldots,B_k}$ denote the forest on the partition $\pi \cup \{B_1, \ldots, A_k\}$ where we add the adjacency relations $A \sim B_1, B_1 \sim B_2, \ldots, B_{k-1} \sim B_k$. Similarly, let $F_{A;B_1,\ldots,B_k;C_1,\ldots,C_m}$ denote the forest $(F_{A;B_1,\ldots,B_k})_{A;C_1,\ldots,C_m}$, that is, we attach two paths to the forest F at the node A.

Similar to Proposition 6.2 is the following one for dominance complexes of forests on partitions:

Proposition 7.3 *We have the following list of one equality and five simple-homotopy equivalences.*

(i) $D(F \cup \{A \sim B\}) = D(F \cup \{A \cup B\})$,

(ii) $D(F \cup \{A \cup \{u\}, B \cup \{v\}\}) \cong D(F \cup \{A \cup B \cup \{w\}\})$,

(iii) $D(F_{A;B\cup\{u\};C\cup\{v\}}) \cong D(F_{A;B\cup C\cup\{w\}})$,

(iv) $D(F_{A;B\cup\{u\};C,D\cup\{v\}}) \cong D(F_{A;B\cup C\cup D\cup\{w\}})$,

(v) $D(F_{A;B,C\cup\{u\};D\cup\{v\},E}) \cong D(F_{A;B\cup C\cup D\cup\{w\},E})$ *and*

(vi) $D(F_{A;B\cup\{u\},C,D\cup\{v\}}) \cong D(F_{A;B\cup C\cup D\cup\{w\}})$.

Proof: To prove statement (i), observe that on the left hand side forest the neighborhoods of the blocks A and B are the same, that is $N[A] = N[B] = A \cup B$. But this is the neighborhood of the block $A \cup B$ in the right hand side forest. Thus the two dominance complexes are the same.

In each statement (ii) trough (vi) observe that $\{u, v\}$ do not belong to any minimal non-face. Hence we may contract the vertices u and v to obtain the new vertex w. This contraction alone yields the right hand side in each of these five statements.

For instance, let us consider statement (v). Observe that the neighborhoods essential to us are $N[C \cup \{u\}] = B \cup C \cup \{u\}$, $N[E] = D \cup E \cup \{v\}$, and $N[A] = A \cup B \cup D \cup \{v\} \cup S$, where S is the neighborhood of A in the original forest F. We do need to consider the neighborhoods $N[B]$ and $N[D \cup \{v\}]$ since they contain $N[B]$ respectively $N[D \cup \{v\}]$. Contracting u and v we obtain the following two blocking sets in the contracted complex: $B \cup C \cup D \cup E \cup \{w\}$ and $A \cup B \cup C \cup D \cup \{w\} \cup S$. In the forest $F_{A;B\cup C\cup D\cup\{w\},E}$ these two sets are the neighborhoods $N[E]$ and $N[A]$ proving statement (v). □

Proof of Theorem 7.2: We prove the statement by induction on the number of blocks in the underlying partition. The induction basis is when there is only one block A in the partition. Then we have that the dominating complex is a sphere of dimension $|A| - 2$.

If there are more than one block in the partition one of the rules (i) through (vi) applies and we obtain a smaller forest. Observe that when we are contracting, one of the contracted vertices are in a unique circuit. Hence the dominance complex is constrictive. □

Lemma 7.4 *The dominance complex of a path on k vertices is simple-homotopy equivalent to a sphere of dimension $\lfloor k/2 \rfloor - 1$. More generally, if π is a partition of an n-element set into k blocks and F is a path on these k blocks the dominance complex $D(F)$ is simple-homotopy equivalent to a sphere of dimension $n - \lceil k/2 \rceil - 1$.*

Proof: We prove the more general statement by induction on k. The induction basis being $k \leq 2$ and in this case the dominance complex is the boundary of $(n-1)$-dimensional simplex, that is, it is a $(n-2)$-dimensional sphere. When $k = 3$ apply rules (iii) and (i) to obtain a path of one node and one underlying vertex less since we contracted two vertices. When $k \geq 4$ apply rule (vi) to obtain a path with two nodes less one underlying vertex less. Observe that the quantity $n - \lceil k/2 \rceil - 1$ remains invariant under these transformations and hence it is the dimension of the sphere. The first statement of the lemma follows by considering the case when $n = k$. □

Observe that the dominance complex in this lemma can also also be viewed as the independence complex of a family of intervals on $[1, n]$. When $n = k$, the intervals are $[1, 2]$, $[2, 4]$, $[3, 5]$, ..., $[n-3, n-1]$ and $[n-1, n]$.

8 The Alexander dual of a constrictive complex

We now consider the *Alexander dual* or *blocker* of a simplicial complex. In order to make its definition work properly, we prefer to drop the requirement that a singleton has to be a face from the definition of a simplicial complex, as it is done in [7, Section 2]. A *generalized (abstract) simplicial complex* $\triangle$ on a vertex set V is simply a family of subsets of V, closed under inclusion. If we think of the subsets of V as a Boolean algebra, then a simplicial complex is a *lower ideal* of this partially ordered set. The notions of edge contraction, elementary collapse, coning and suspension may be generalized to generalized abstract simplicial complexes in a straightforward manner. In this section only, by the term "simplicial complex" we will always mean "generalized abstract simplicial complex".

For a generalized abstract simplicial complex $\triangle$ define the set of *genuine vertices* as $\text{vert}(\triangle) = \{v \in V : \{v\} \in \triangle\}$. Observe that there are two simplicial complexes on the empty vertex set. First there is $\triangle = \{\emptyset\}$. This simplicial complex should be considered as a (-1)-dimensional sphere. Second, there is the complex $\triangle = \emptyset$. This complex is contractible since it is obtained from the point $\{\emptyset, \{v\}\}$ by a collapse and should be considered as a (-1)-dimensional simplex.

Definition 8.1 *Let $\triangle$ be a simplicial complex on the vertex set V. We define the* Alexander dual *of $\triangle$ as $\mathcal{D}(\triangle) = \{\sigma \subseteq V : V \setminus \sigma \notin \triangle\}$.*

A simplicial complex $\triangle$ is a lower ideal in the Boolean algebra B_V generated by the set V. The complement $B_V \setminus \triangle$ in the Boolean algebra B_V is an upper ideal. Finally, the complements of the sets in $B_V \setminus V$ form again a lower ideal, namely the Alexander dual. Thus a facet σ in the complex $\triangle$ correspond to the circuit $V \setminus \sigma$ in the Alexander dual $\mathcal{D}(\triangle)$. Similarly, a circuit B in the complex $\triangle$ correspond to the facet $V \setminus B$ in the Alexander dual. A free face $\tau \in \triangle$ is an element of the lower ideal contained in a unique maximal element σ of $\triangle$. If $|\tau| = |\sigma| - 1$, then the collection $\mathcal{D}(\triangle) \cup \{V \setminus \sigma, V \setminus \tau\}$ is a lower ideal. This reasoning provides a combinatorial proof of the following statement.

Proposition 8.2 *Let $\triangle$ and $\triangle'$ be simplicial complexes on the same vertex set V. Then $\triangle'$ may be obtained from $\triangle$ via an elementary collapse if and only if $\mathcal{D}(\triangle)$ may be obtained from $\mathcal{D}(\triangle')$ via an elementary collapse.*

This is property 7 of the Alexander dual in Kalai's paper [13]. He also notes that $\triangle$ is isomorphic to $\triangle'$ if and only if $\mathcal{D}(\triangle)$ is isomorphic to $\mathcal{D}(\triangle')$ and that

$$\mathcal{D}(\mathcal{D}(\triangle)) = \triangle \tag{8.4}$$

for every simplicial complex. The same fact is also noted by J. Kahn, M. Saks, and D. Sturtevant on p. 301 in [12], and cited in a setting of PL-manifolds by X. Dong in [6, Lemma 10]. Repeated application of Proposition 8.2 yields the following theorem.

Theorem 8.3 *Let $\triangle$ and $\triangle'$ be simplicial complexes on the same vertex set V. Then $\triangle$ is simple-homotopic to $\triangle'$ if and only if $\mathcal{D}(\triangle)$ is simple-homotopic to $\mathcal{D}(\triangle')$.*

From a topological view point, the geometric realization of $\mathcal{D}(\triangle)$ is homotopy equivalent to the the set difference between the geometric realization of the boundary of the simplex with vertex set V and the geometric realization of the complex $\triangle$. From this interpretation and using the well-known Alexander Duality Theorem one can prove that $\triangle$ is a homology sphere if and only if its Alexander dual is. See the papers [7, 13] for details.

Remark 8.4 Since we allow the vertex set V of the simplicial complex $\triangle$ to be a larger set than the set of genuine vertices vert($\triangle$), the natural question arises how does the Alexander dual change when we enlarge the vertex set with additional non-genuine vertices. This also seems to be an issue that has not been addressed explicitly in the literature. It is relatively easy to prove the following. Let $\triangle'$ be the simplicial complex obtained from $\triangle$ by adding a new (non-genuine) vertex V. Then the combinatorial Alexander dual of $\triangle'$ is homotopy equivalent to the suspension of the combinatorial Alexander dual of $\triangle$. Hence either both Alexander duals are homotopy equivalent to a single vertex or a sphere, or none of them are.

As a consequence of Lemma 4.2 and Theorem 8.3 we obtain:

Corollary 8.5 *The Alexander dual of a constrictive simplicial complex is simple homotopic to a single vertex or the boundary complex of a sphere.*

We may use this result to obtain more classes of simplicial complexes that are contractible or homotopy equivalent to spheres. In particular, as the Alexander duals of Theorems 4.4, 5.3 and 7.1 we obtain the following four corollaries.

Corollary 8.6 *Let $\mathcal{I}$ be a family of intervals on the set $[1, n]$. Then the simplicial complex*

$$\triangle_{\mathcal{I}} = \{\sigma \,:\, \sigma \subseteq [1, n] \setminus I \text{ for some } I \in \mathcal{I}\}$$

is simple-homotopic to a single vertex or to a sphere.

This is [1, Theorem 3] and it is equivalent to a result of Kahn [11] on interval generated lattices which was rediscovered independently by Linusson [15, Theorem 15.1].

Corollary 8.7 *Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be branching block system on a vertex set V. Then the simplicial complex*

$$\triangle = \{\sigma \,:\, \sigma \subseteq V \setminus B_i \text{ for some } B_i \in \mathcal{B}\}$$

is simple-homotopic to a single vertex or to a sphere.

As a corollary to the previous corollary or to Corollary 6.1 we have the next dual result.

Corollary 8.8 *Let F be a forest on the vertex set V. Then the simplicial complex $\triangle$ consisting of all subsets σ of V that do not contain all the edges, that is,*

$$\triangle = \{\sigma \,:\, \sigma \subseteq V \setminus \{u, v\} \text{ for some } uv \in E(F)\}$$

is simple-homotopic to a single vertex or to a sphere.

Corollary 8.9 *Let F be a forest on the vertex set V. Then the simplicial complex $\triangle_F$ consisting of all subsets σ of V that are not dominating, that is,*

$$\triangle_F = \{\sigma \,:\, \sigma \subseteq V \setminus N[v] \text{ for some } v \in V\}$$

is simple-homotopy equivalent to a sphere.

Note that $\triangle_F$ is the independence complex of the collection of dominating sets of the forest.

9 Concluding questions

Given a graph G what can be said about the topology of the independence complex $I(G)$? As it was pointed out to us by a referee, the first barycentric subdivision of a simplicial complex is the independence complex of the complement of the comparability graph of the underlying face poset.

Therefore every simplicial complex arising as the barycentric subdivision of a CW complex may be represented as the independence complex of a graph. As a consequence, the independence complex of a graph may have any homotopy type. This makes the question which *graph theoretic* properties imply homotopy equivalence to a single vertex or a sphere, even more interesting. The same question may be raised about the topology of the dominance complex $D(G)$.

Given a forest F we know that its dominance complex is homotopy equivalent to a sphere. Thus the dimension of this sphere is an invariant of the forest. Is there a simple way to compute this invariant? Similarly, is there a simple way to determine if independence complex of a forest is contractible and if not determine the dimension of associated sphere? One suggestion is to consider the algorithms occurring in work of Contenza [5], Farber [9] and Mynhardt [16]. Moreover, can our homotopy results be extended to other classes of graphs, for instance, strongly chordal graphs?

Other questions that occur naturally are: Can the class of constrictive simplicial polytopes be classified? When is a constrictive simplicial complex non-pure shellable; for this extension of the notion of shelling see the paper by Björner and Wachs [3].

The *Stirling complex* is the simplicial complex $\triangle_n$ on the vertex set $V_n = \{(i,j) \ : \ 1 \leq i < j \leq n\}$ with the minimal non-faces (circuits) are the pairs $\{(i,j),(i,k)\}$ and $\{(i,k),(j,k)\}$ where i, j and k range over $1 \leq i < j < k \leq n$. Observe that the Stirling complex is the independence complex of a graph, since all its circuits have cardinality 2. Another way to describe this complex is that the collection of all faces is the set of all rook placements on the board V_n. The number of k-dimensional faces is given by the Stirling number of the second kind $S(n, n-k-1)$; see [17, Proposition 2.4.2]. What can be said about the homotopy type of the Stirling complex $\triangle_n$?

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