Global Uniqueness of a Multidimensional Inverse Problem for a Nonlinear Parabolic Equation by a Carleman Estimate

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Abstract

An inverse problem with the single measurement data for a general nonlinear parabolic equation $u_t = F(u_{ij}, \nabla_x u, x, t, q(u,y))$ in n-D is considered. The unknown coefficient $q(u,y)$ depends on the solution u and $(n-1)$ spatial variables $y = (x_2, \cdots, x_n)$. Such problems were not studied in the past for $n \geq 2$. A global uniqueness result is proven by the method of Carleman estimates.

1 Introduction

For $x \in R^n$, denote $x = (x_1, y)$, where $y = (x_2, \cdots, x_n)$. For a function $f \in C^1(R^n)$, denote $f_i = \dfrac{\partial f}{\partial x_i}$. Let the domain $\Omega \subset R^n$ be $\Omega = (0,1) \times \Omega_1$, where the domain $\Omega_1 \subseteq R^{n-1}$. For a $T > 0$, denote $Q_T = \Omega \times (0,T)$. Let a function $q \in C(R^n)$ and a function $F \in C^3(R^m)$, where $m = n^2 + 2n + 2$. We assume that $F(u_{ij}, \nabla_x u, x, t, q(u,y))$ is a nonlinear elliptic operator and

$$\frac{\partial F}{\partial q}(s) = \frac{\partial F}{\partial s_m}(s) \neq 0, \quad \forall s \in R^m. \tag{1.1}$$

The ellipticity of the operator F means that

$$\mu_1 |\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial F}{\partial (u_{ij})}(s)\, \xi_i \xi_j \leq \mu_2 |\xi|^2, \tag{1.2}$$

$$\forall \xi \in R^n, \quad \forall s \in R^m; \quad \mu_1, \mu_2 = const > 0.$$

Consider the nonlinear parabolic equation in Q_T,

$$u_t = F(u_{ij}, \nabla_x u, x, t, q(u,y)), \text{ in } Q_T. \tag{1.3}$$

Let

$$u|_{x_1=0} = \varphi_0(t), \quad u_{x_1}|_{x_1=0} = \psi_0(y,t), \tag{1.4}$$

$$u|_{x_1=1} = \varphi_1(y,t), \quad u_{x_1}|_{x_1=1} = \psi_1(y,t). \tag{1.5}$$

We study the following

Inverse Problem. Given functions $F, \varphi_0, \psi_0, \varphi_1, \psi_1$, determine the vector valued function $(u, q(u,y))$.

Two examples of (1.3) are quasilinear parabolic equations,

$$u_t = \sum_{i,j=1}^{n} a^{ij} \left(\nabla_x u, u, x, t \right) u_{ij} + b \left(\nabla_x u, u, x, t \right) + q \left(u, y \right)$$

and

$$q \left(u, y \right) u_t = \sum_{i,j=1}^{n} a^{ij} \left(\nabla_x u, u, x, t \right) u_{ij} + b \left(\nabla_x u, u, x, t \right),$$

$$\mu_1 \left| \xi \right|^2 \leq \sum_{i,j=1}^{n} a^{ij} \left(p \right) \xi_i \xi_j \leq \mu_2 \left| \xi \right|^2,$$

$$\forall s \in R^n, \quad \forall p \in R^{n+3}.$$

In the second case the condition (1.1) should be replaced with: $u_t > 0$ and $q \geq const > 0$. Uniqueness, stability and existence results for initial boundary value problems for such equations can be found in the book of Ladyzhenskaya, Solonnikov and Uraltceva [17].

We assume that

$$\varphi_0 \in C^1 \left[0, T \right], \quad \varphi_0' \left(t \right) \geq \gamma_1 = const > 0, \tag{1.6}$$

$$\frac{\partial \varphi_1}{\partial t} \equiv 0, \tag{1.7}$$

$$\frac{\partial u}{\partial x_1} \left(x, t \right) \leq -\gamma_2 = const < 0, \tag{1.8}$$

where γ_1 and γ_2 are certain positive constants. By (1.7) and (1.8), $\varphi_1 \left(y \right) \leq u \left(x, t \right) \leq \varphi_0 \left(t \right)$. Hence, one should expect to recover the function $q \left(z, y \right)$ on the set D, where

$$D = \left\{ \left(z, y \right) : \quad \varphi_1 \left(y \right) < z < \varphi_0 \left(T \right), \quad y \in \Omega_1 \right\}. \tag{1.9}$$

Introduce the set $D_0 \subset D$,

$$D_0 = \left\{ \left(z, y \right) : \quad \varphi_1 \left(y \right) < z < \varphi_0 \left(0 \right), \quad y \in \Omega_1 \right\}. \tag{1.10}$$

The main result of this paper is

Theorem 1 *Suppose, there exist two pairs of functions* $\left(u_i, q_i \right), i = 1, 2$ *satisfying (1.3)-(1.8) and such that*

$$\left(u_i \right)_{tt}, \quad D_x^{\alpha} D_t^{\beta} u_i \in C \left(\overline{Q}_T \right), \quad \left| \alpha \right| \leq 3; \quad \beta = 0, 1; \quad i = 1, 2; \quad q_i \in C \left(\overline{D} \right), \quad i = 1, 2$$

and the function $q \left(z, y \right)$ *is given on the set* D_0*, thus*

$$q_1 \left(z, y \right) = q_2 \left(z, y \right) \text{ in } D_0. \tag{1.11}$$

Then $u_1 = u_2$ *in* Q_T *and* $q_1 = q_2$ *in* D.

With the aim of simplifying the presentation, we are not concerned here with minimal smoothness conditions. Note that Theorem 1 does not require knowledge of neither the initial condition $u(x,0)$ nor the boundary condition at $\{(z,y,t) : z \in (0,1), y \in \partial G, t \in (0,T)\}$. This is mainly due to the knowledge of the function $q(z,y)$ on the set D_0 and the well known uniqueness theorem for the Cauchy problem with the lateral data for the parabolic equation, see, e.g., books of Lavrent'ev, Romanov and Shishatskii [19] (Chapter 4, Section 1) and Isakov [8] (Chapter 3, section 3.3). Using the maximum principle, one can draw some specific examples of boundary value problems for quasilinear parabolic equations, for which conditions (1.6)-(1.8) are satisfied. However, such examples are outside of the scope of this paper.

Problems like the one considered here arise in the processes of the heat transfer with significant dependencies of material properties from the temperature. such processes have a broad range of applications in engineering, see, e.g., the book of Alifanov [1]. In such a process the assumption $q := q(u,y)$ means that the dependence of a material property from the temperature is substantially more important than its dependence on the spatial variable x_1, in the direction of which the temperature changes most rapidly (condition (1.8)). In conditions of Theorem 1, the temperature on the left boundary $\{x_1 = 0\}$ is controlled in such a way that it does not change in space and grows whit time. On the right boundary $\{x_1 = 1\}$, the temperature is controlled in such a way that it does not change with time. Suppose that the temperature on the left boundary is substantially higher than on the right boundary. Then the inequality (1.8) means that the temperature inside the medium is decreasing when moving from the left to the right, which is natural for a medium without cavities. The fact that the function $q(z,y)$ is given on the set D_0 can be interpreted as the knowledge of that material property at the initial moment of time $\{t = 0\}$, while the temperature is still low.

In this paper we study an inverse problem with the single measurement data for a multidimensional nonlinear PDE with the unknown coefficient depending on both the solution and some spatial variables. Such problems were not considered in the past. However, some uniqueness results were published for the one dimensional case with $q := q(u)$. The author [14] has proven a global uniqueness theorem for the 1-D parabolic case, using the Bukhgeim-Klibanov method of Carleman estimates [2], [3], [10]. The data in [14] are measurements of the function $u(x,t), (x,t) \in (0,1) \times (0,T)$ at $k+1$ interior points $\{x_i\}_{i=1}^{k+1} \subset (0,1)$, where $k \geq 1$ is the dimension of the unknown vector valued function $q(u) = (q_1(u), \cdots, q_k(u))$. Therefore, Theorem 1 is a new result even in the 1-D case, since only boundary measurements are considered here. Kügler [16] has proven uniqueness for a 1-D inverse problem for a quasilinear elliptic equation. Muzylev has published an uniqueness theorem for a piecewise analytic unknown coefficient $q(u)$ in a parabolic operator [21]. Pilant and Rundell have established uniqueness under a smallness condition for an (n-D) / (1-D) problem [22]. That is, in [22] the unknown source function $q(u)$ is a part of an n-D parabolic operator, and the data are given at a single point of the boundary.

A more complete set of results is available for multidimensional inverse problems for nonlinear parabolic and elliptic equations with the multiple measurement data, i.e., the Dirichlet-to-Neumann map. This series of publications has started from the paper of Isakov [7], in which the so-called linearization method was introduced; also see, e.g., [8], [9] and references cited there.

The main idea of Theorem 1 consists in an extension of the method of Carleman estimates to the problem considered here. As to nonlinear equations: In addition to [14], this method was also applied by the author to prove a global uniqueness theorem for a multidimensional inverse problem for a nonlinear elliptic equation in R^{n+1} [12], [15]. However, the unknown coefficient in [12] and [15] depends on n spatial variables, rather than on the solution of that equation. In the rest of publications about this method, it has been used so far for proofs of uniqueness and stability results for multidimensional inverse problems for linear PDEs only (including systems of PDEs), see, e.g., Imanuvilov and Yamamoto [5], Imanuvilov, Isakov and Yamamoto [6], Klibanov [11,13,15], Lin and Wang [20], as well as references cited there.

A crucial part of the proof of Theorem 1 is Theorem 2 (section 5), which is a new pointwise Carleman estimate for the parabolic operator. This theorem might be interesting in its own right. Some features of Theorem 2, as well as a part of its proof are naturally similar with the Carleman estimate of Lemma 3 in Section 1 of Chapter 4 of the book [19]. There are significant differences, however. First, the Carleman Weight Function (CWF) of Theorem 2 is different from one in [19]. However, the most important new element in Theorem 2 is the estimate (5.2) from the below of a certain boundary integral over the curvilinear boundary $\partial_3 E \subset R^{n+1}$. The latter, in turn is important for the proof of Theorem 1, as it can be seen from a comparison of (6.2) with (6.3) and (6.4) (section 6). The peculiarity here is that the n-D manifold $\partial_3 E$ is not a level surface of the CWF. Therefore, in a traditional setting both Dirichlet and Neumann zero boundary data should be assigned on such a part of the boundary. In our case, however only the zero Dirichlet boundary condition is given on $\partial_3 E$. This makes it necessary to carefully analyze all boundary terms in the corresponding pointwise Carleman estimate, including even the 1-D case. Such an analysis, in turn requires a complete proof of that estimate, which is inevitably space consuming, as it is always the case when these estimates are derived.

The third important difference with [19] is the presence of terms with the derivatives u_t^2 and u_{ij}^2 in Theorem 2. These derivatives are involved in the principal part of the parabolic operator. Whereas only lower order derivatives are present in the Carleman estimate of [19]. In principle, it is well known that such terms can be included in the elliptic case (see, e.g., Theorem 8.3.1 in the book of Hörmander [4]), and this can be done analogously in the parabolic case as well. Still, however we need to provide a detailed proof in our specific setting (Lemma 5.4), because we need to estimate that boundary integral.

The second auxiliary result, which might be interesting in its own right is a new estimate from the above of a certain integral in Lemma 2.1 (section 2). This estimate is stronger than one explored in previous works, because of the presence of the multiplier $1/\lambda^2$, which is important as $\lambda \to \infty$ (compare, e.g., with Lemma 3.7 and the inequality (4.23) in [15])

The paper is organized as follows. In Section 2 we prove Lemma 2.1. In section 3 we begin the proof of Theorem 1. The main result of this section is a certain integro-differential inequality for a function w. In Section 4 we introduce the CWF and show that it is sufficient to prove that the function $w = 0$ in a certain small domain E. In section 5 we prove the Carleman estimate. We complete the proof of Theorem 1 in section 6.

2 Estimating an Integral

Lemma 2.1. *Let the function $f(t) \in C^1[0,a]$ and $f'(t) \leq -b$, where $b = const > 0$. Then for all $\lambda > 0$ and for all real valued functions $g \in L_2(0,a)$ the following estimate holds*

$$\int_0^a \exp[2\lambda f(t)] \cdot \left(\int_0^t g(\tau)\, d\tau \right)^2 dt \leq \frac{1}{\lambda^2 b^2} \int_0^a g^2(t) \exp[2\lambda f(t)]\, dt. \tag{2.1}$$

Proof. Obviously,

$$\begin{aligned} \exp(2\lambda f) &= \frac{2\lambda f'}{2\lambda f'} \exp(2\lambda f) \\ &= \left(-\frac{1}{2\lambda f'}\right) \frac{d}{dt}[-\exp(2\lambda f)] \leq \frac{1}{2\lambda b} \frac{d}{dt}[-\exp(2\lambda f)]. \end{aligned}$$

Hence,

$$I := \int_0^a \exp(2\lambda f(t)) \left(\int_0^t g(\tau)\, d\tau \right)^2 dt \leq \frac{1}{2\lambda b} \int_0^a \frac{d}{dt}[-\exp(2\lambda f(t))] \cdot \left(\int_0^t g(\tau)\, d\tau \right)^2 dt$$

$$= -\frac{1}{2\lambda b} \cdot \exp(2\lambda f(a)) \left(\int_0^a g(\tau)\, d\tau \right)^2$$

$$+\frac{1}{\lambda b} \int_0^a \left\{ \left[\exp(\lambda f(t)) \cdot \int_0^t g(\tau)\, d\tau \right] \cdot [g(t) \exp(\lambda f(t))] \right\} dt.$$

Since,

$$-\frac{1}{2\lambda b} \cdot \exp(2\lambda f(a)) \left(\int_0^a g(\tau)\, d\tau \right)^2 \leq 0,$$

then

$$\begin{aligned} I &\leq \frac{1}{\lambda b} \int_0^a \left\{ \left[\exp(\lambda f) \cdot \int_0^t g(\tau)\, d\tau \right] \cdot [g(t) \exp(\lambda f(t))] \right\} dt \\ &\leq \frac{1}{\lambda b} \sqrt{I} \cdot \left[\int_0^a g^2(t) \exp(2\lambda f(t))\, dt \right]^{\frac{1}{2}}. \end{aligned}$$

Dividing this inequality by $\sqrt{I}$ and squaring both sides then, we obtain (2.1). ■

3 Integro-Differential Inequality

The proof of Theorem 1 begins in this section. First, we introduce a 'pseudo spatial' variable z by

$$u\left(v\left(z,y,t\right),y,t\right)=z,$$

which is possible because of (1.8). The equation (1.3) becomes

$$v_{t}=\widetilde{F}\left(v_{11},v_{ij},\nabla_{z,y}v,z,y,t,q\left(z,y\right)\right),(i,j)\neq(1,1), \tag{3.1}$$

where $v_1 = v_z$ and $\widetilde{F}$ is the nonlinear elliptic operator generated by the operator F and this change of variables. The equation (3.1) is satisfied in the domain D_T with curvilinear boundaries $\{z=\varphi_1(y)\}$ and $\{z=\varphi_0(t)\}$,

$$D_T=\left\{(z,y,t):\varphi_1(y)<z<\varphi_0(t),\quad y\in\Omega_1,\quad t\in(0,T)\right\}.$$

Let

$$D_{0T}=D_0\times(0,T)=\left\{(x,y,t):\varphi_1(y)<z<\varphi_0(0),\quad y\in\Omega_1,\quad t\in(0,T)\right\}.$$

By (1.6), $\varphi_0(t)>\varphi_0(0)$ for $t>0$. Hence, the domain $D_{0T}\subset D_T$. Let Γ_l and Γ_r be the left and right curvilinear boundaries of the domain D_T respectively,

$$\Gamma_l=\left\{(z,y,t):z=\varphi_1(y),\quad y\in\Omega_1,\quad t\in(0,T)\right\},$$

$$\Gamma_r=\left\{(z,y,t):z=\varphi_0(t),\quad y\in\Omega_1,\quad t\in(0,T)\right\}.$$

Relations (1.4) and (1.5) imply that

$$v|_{\Gamma_l}=1,\quad v_z|_{\Gamma_l}=\frac{1}{\psi_1(y,t)} \tag{3.2}$$

$$v|_{\Gamma_r}=0,\quad v_z|_{\Gamma_r}=\frac{1}{\psi_0(y,t)}. \tag{3.3}$$

Suppose that there exist two pairs of functions (u_1,q_1) and (u_2,q_2) satisfying (1.3)-(1.8). Then there also exist two pairs of functions $(v_1(z,y,t),q_1(z,y))$ and $(v_2(z,y,t),q_2(z,y))$ satisfying (3.1)-(3.3). Denote $\widetilde{v}(z,y,t)=v_1(z,y,t)-v_2(z,y,t)$, $\widetilde{q}(z,y)=q_1(z,y)-q_2(z,y)$. Then

$$\widetilde{q}(z,y)=0\text{ in }D_0. \tag{3.4}$$

Relations (3.1)-(3.3) lead to

$$a^0\cdot\widetilde{v}_t-L\widetilde{v}:=a^0(z,y,t)\cdot\widetilde{v}_t-\widetilde{v}_{zz}-\sum_{\substack{i,j=1\\(i,j)\neq(1,1)}}^{n}a^{ij}(z,y,t)\,\widetilde{v}_{ij}$$

$$-\sum_{i=1}^{n}b^{j}(z,y,t)\,\widetilde{v}_j-b^0(z,y,t)\,\tilde{v}=c(z,y,t)\,\widetilde{q}(z,y),\text{ in }D_T, \tag{3.5}$$

$$\widetilde{v}|_{\Gamma_l} = \frac{\partial \widetilde{v}}{\partial \widehat{n}}|_{\Gamma_l} = 0, \tag{3.6}$$

$$\widetilde{v}|_{\Gamma_r} = \frac{\partial \widetilde{v}}{\partial \widehat{n}}|_{\Gamma_r} = 0, \tag{3.7}$$

where $\widehat{n}$ is the unit outward normal vector on either of boundaries.

Let the function $d\left(z,y,t\right)$ be either a^0 or an arbitrary coefficient of the operator L. Then $d \in C^2\left(\overline{D}_T\right)$. Also, (1.2) implies that the operator L is elliptic in D_T, i.e.,

$$\widetilde{\mu}_1 \left|\xi\right|^2 \leq \xi_1^2 + \sum_{\substack{i,j=1 \\ (i,j)\neq(1,1)}}^{n} a^{ij}\xi_i\xi_j \leq \widetilde{\mu}_2 \left|\xi\right|^2, \tag{3.8}$$

$$\forall \left(z,y,t\right) \in \bar{D}_\Gamma, \quad \forall \xi \in R^n, \quad \widetilde{\mu}_1, \widetilde{\mu}_2 = const > 0.$$

Let $G \subseteq D_T$ be an arbitrary bounded subdomain. Then there exists a positive constant $C\left(G\right)$ such that

$$\min_{\overline{G}} \left[a^0\left(z,y,t\right)\right] \geq C\left(G\right) \tag{3.9}$$

and by (1.1)

$$\min_{\overline{G}} \left|c\left(z,y,t\right)\right| \geq C\left(G\right). \tag{3.10}$$

Also, $\widetilde{v}_{tt}, D_{z,y}^{\alpha} D^{\beta} \widetilde{v} \in C\left(\overline{D}_T\right)$ for $|\alpha| \leq 3, \beta = 0,1$. Denote

$$M\left(G\right) = \max\left\{ \left\|a^0\right\|_{C^1\left(\overline{G}\right)}, \ \max_{i,j} \left\|a^{ij}\right\|_{C^1\left(\overline{G}\right)}, \ \max_{j} \left\|b^j\right\|_{C^1\left(\overline{G}\right)}, \left\|\frac{1}{c}\right\|_{C^2\left(\overline{G}\right)} \right\}.$$

From now on, given a bounded subdomain $G \subseteq D_T$, M denotes different positive constants depending on the constant $M\left(G\right)$.

Consider the equation (3.5) in the domain D_{0T}. Note that Γ_l is also the left boundary of the domain D_{0T}. And the right boundary of this domain is $\{(z,y,t) : z = \varphi_0(0), y \in \Omega_1, \ t \in (0,T)\}$. The right hand side of the equation (3.5) $c\left(z,y,t\right)\widetilde{q}\left(z,y\right) = 0$ in D_{0T}. Hence, the zero Dirichlet and Neumann boundary conditions (3.6) on Γ_l and the uniqueness of the Cauchy problem for the parabolic equation with the lateral data imply that

$$\widetilde{v}\left(z,y,t\right) = 0 \text{ in } D_{0T}. \tag{3.11}$$

Thus, from now on we shall consider the equation (3.5) only in the domain $H_T = D_T \cap \left\{z > \varphi_0\left(0\right)\right\}$. Hence,

$$H_T = \left\{(z,y,t) : \varphi_0\left(0\right) < z < \varphi_0\left(t\right), \ \left(y,t\right) \in \ \Omega_1 \times (0,T)\right\}. \tag{3.12}$$

By (3.11) the condition (3.6) can be replaced with

$$\widetilde{v}|_{z=\varphi_0(0)} = \widetilde{v}_z|_{z=\varphi_0(0)} = 0. \tag{3.13}$$

To apply the Carleman estimate, we obtain an integro-differential inequality with Volterra-like integrals first. An important feature of this inequality is that it does not depend explicitly on the unknown coefficient $\widetilde{q}$. Since $c \neq 0$ in $\overline{H}_T$, then (3.5) leads to

$$\widetilde{q}\,(z,y) = \frac{a^0 \cdot \widetilde{v}_t - L\widetilde{v}}{c}, \text{ in } H_T.$$

Differentiating with respect to t, we obtain

$$\frac{\partial}{\partial t}\left[\frac{a^0 \cdot \widetilde{v}_t - L\widetilde{v}}{c}\right] = 0, \text{ in } H_T.$$

This equation can be rewritten as

$$a^0 \cdot \widetilde{v}_{tt} - L\widetilde{v}_t + a^0_t \cdot \widetilde{v}_t - L_t\widetilde{v} = \frac{c_t}{c}\left[a^0 \cdot \widetilde{v}_t - L\widetilde{v}\right], \text{ in } H_T, \tag{3.14}$$

where L_t is the linear operator whose coefficients are t-derivatives of the coefficients of the operator L. Denote

$$h\,(z,y,t) = \frac{c_t}{c} = \left[\ln |c|\right]_t, \tag{3.15}$$

$$w\,(z,y,t) = (\widetilde{v}_t - h\widetilde{v})\,(z,y,t) \tag{3.16}$$

and use $\widetilde{v}_{tt} - h\widetilde{v}_t = (\widetilde{v}_t - h\widetilde{v})_t + h_t\widetilde{v} = w_t + h_t\,\widetilde{v}$ and similar formulas for other derivatives. Then (3.14)-(3.16) lead to

$$a^0 w_t - Lw = \sum_{\substack{i,j=1 \\ (i,j)\neq(1,1)}}^{n} k^{ij}\widetilde{v}_{ij} + \sum_{i=1}^{n} l^i\widetilde{v}_i + l^0\widetilde{v}, \text{ in } H_T, \tag{3.17}$$

where functions

$$k^{ij}, l^i \in C\left(\overline{H}_T\right). \tag{3.18}$$

To express the function $\widetilde{v}$ through the function w, introduce a new function $t = g\,(z)$, which is the inverse for the function $\varphi_0\,(t)$,

$$\varphi_0\,(g\,(z)) = z. \tag{3.19a}$$

Hence,

$$t > g\,(z)\,, \text{ in } H_T. \tag{3.19b}$$

Since by (3.7) $\widetilde{v}|_{z=\varphi_0(t)} = 0$, then we obtain the following Cauchy problem for the linear ordinary differential equation (3.16) in $\{t > g\,(z)\}$

$$\widetilde{v}_t - h\widetilde{v} = w, \; \widetilde{v}|_{t=g(z)} = 0,$$

Hence, (3.15) implies that

$$\widetilde{v}\,(z,y,t) = \int_{g(z)}^{t} K\,(z,y,t,\tau)\,w\,(z,y,\tau)\,d\tau, \tag{3.20}$$

$$K(z,y,t,\tau)=\frac{c(z,y,t)}{c(z,y,\tau)}.$$

Consider now values of the function $w(z,y,t)$ and its first derivatives on the right boundary Γ_r of the domain H_T. Differentiating $\widetilde{v}(\varphi_0(t),y,t)=0$ with respect to t and using the fact that $\widetilde{v}_z(\varphi_0(t),y,t)=0$, we obtain $\widetilde{v}_t(\varphi_0(t),y,t)=0$. Hence, (3.16) implies that

$$w(\varphi_0(t),y,t)=w|_{z=\varphi_0(t)}=0, \text{for } (y,t)\in\ \Omega_1\times(0,T). \tag{3.21}$$

Hence, (3.20)-(3.23) lead to

$$w_j|_{z=\varphi_0(t)}=0,\quad j=2,\cdots,n,\quad \text{in }\ \Omega_1\times(0,T), \tag{3.22}$$

$$w_t|_{z=\varphi_0(t)}=-\varphi_0'(t)\cdot w_z|_{z=\varphi_0(t)},\quad \text{in }\ \Omega_1\times(0,T). \tag{3.23}$$

Thus, differentiating (3.20) and taking into account (3.21) and (3.22), we obtain

$$\widetilde{v}_j(z,y,t)=\int\limits_{g(z)}^{t}(Kw)_j(z,y,t,\tau)\,d\tau,\quad j=1,\cdots,n, \tag{3.24}$$

$$\widetilde{v}_{ij}(z,y,t)=\int\limits_{g(z)}^{t}(Kw)_{ij}(z,y,t,\tau),\quad \text{for }(i,j)\neq(1,1);\quad i,j=1,\cdots,n, \tag{3.25}$$

$$\widetilde{v}_t(z,y,t)=w(z,y,t)+\int\limits_{g(z)}^{t}(K_tw)(z,y,t,\tau)\,d\tau. \tag{3.26}$$

Fix an arbitrary bounded subdomain $\widetilde{G}\subseteq\Omega_1$. Define the domain $G\subseteq H_T$ as

$$G=\left\{(z,y,t):\varphi_0(0)<z<\varphi_0(t),y\in\widetilde{G},t\in(0,T)\right\},\ \text{for } n\geq 2 \tag{3.27}$$

and $G=H_T$ in the 1-D case. Let L_0 be the principal part of the elliptic operator L,

$$L_0w=w_{zz}+\sum_{\substack{i,,j=1\\(i,j)\neq(1,1)}}^{n}a^{ij}(z,t,y)\,w_{ij}. \tag{3.28}$$

Then (3.17)-(3.28) imply that the function $w(z,y,t)$ satisfies the following integro-differential inequality and boundary conditions

$$\left|a^0w_t-L_0w\right|\leq M\left(|\nabla w|+|w|\right)$$

$$+M\int\limits_{g(z)}^{t}\left[\left(|\nabla w|+|w|\right)(z,y,\tau)\right]d\tau+M\int\limits_{g(z)}^{t}\left[\sum_{\substack{i,,j=1\\(i,j)\neq(1,1)}}^{n}|w_{ij}|(z,y,\tau)\right]d\tau,\ \text{in } G, \tag{3.29}$$

$$w|_{z=\varphi_0(0)} = w_z|_{z=\varphi_0(0)} = 0, \text{ for } (y,t) \in \widetilde{G} \times (0,T), \tag{3.30}$$

$$w|_{z=\varphi_0(t)} = 0, \text{ for } (y,t) \in \widetilde{G} \times (0,T), \tag{3.31}$$

where function $D^{\alpha}_{z,y}w, w_t \in C\left(\bar{G}\right), |\alpha| \leq 3$ and $M = M(G)$.

The main effort of the rest of the paper is focused on the proof that relations (3.29)-(3.31) imply that the function $w(z,y,t) = 0$ in G and, therefore $w(z,y,t) = 0$ in H_T. This and (3.20) would imply that $\widetilde{v}(z,y,t) = 0$ in H_T. Finally, substituting $\widetilde{v} := 0$ in the equation (3.5) and using the fact that by (3.10) the function $c(z,y,t) \neq 0$ in $\overline{H}_T$, we would obtain that the function $\widetilde{q}(z,y) = 0$ in the domain D, which was defined in (1.9). The latter would prove Theorem 1. Below the dependence from $y \in R^{n-1}$ should be ignored if $n = 1$; all formulations and proofs remain almost the same for this case.

4 Domains and some Notations

4.1 Domains

Let δ be sufficiently small positive number, which we will choose later (see (4.24)). Let the function ψ be

$$\psi(z,y,t) = z - \varphi_0(0) + \sqrt{\delta}\,|y|^2 + t + \frac{1}{2},$$
$$\text{for } (z,y,t) \in \left\{\varphi_0(0) < z < \varphi_0(t), t > 0, y \in R^{n-1}\right\}.$$

For sufficiently large parameters $\lambda, \nu > 1$ define the CWF as

$$\mathcal{C}(z,y,t) = \exp\left[\lambda\psi^{-\nu}\right]. \tag{4.1}$$

For brevity, we omit to mark the dependence of the CWF from parameters λ, ν and δ.

Let $\eta \in [0,T), y_0 \in \widetilde{G}$ and $dist\left(y_0, \partial\widetilde{G}\right) < \delta^{1/4}$, i.e., the distance between y_0 and $\partial\widetilde{G}$ is less than $\delta^{1/4}$. Define the domain $E(\delta,\eta,y_0)$ as

$$E(\delta,\eta,y_0) \tag{4.2}$$
$$= \left\{(z,y,t) : \psi(z, y-y_0, t-\eta) + \varphi_0(0) - \varphi_0(\eta) < \delta + \frac{1}{2}, \varphi_0(\eta) < z < \varphi_0(t), t > \eta\right\}.$$

Also, (4.2) can be rewritten as

$$E(\delta,\eta,y_0) = \left\{(z,y,t) : z - \varphi_0(\eta) + \sqrt{\delta}\,|y-y_0|^2 + (t-\eta) < \delta, \varphi_0(\eta) < z < \varphi_0(t), t > \eta\right\}.$$

Clearly, there exists a $\delta = \delta(\eta) > 0$ such that $E(\delta,\eta,y_0) \subset H_T$. The goal of this subsection is to demonstrate that to prove Theorem 1, it is sufficient to prove that the function $w(z,y,t) = 0$ in $E(\delta,0,y_0)$ for an $y_0 \in \Omega_1$ and a sufficiently small δ. This assertion follows from

Lemma 4.1. *Suppose that for an arbitrary $\eta \in [0,T)$ there exists a method of proving that relations*

$$w|_{z=\varphi_0(\eta)} = w_z|_{z=\varphi_0(\eta)} = 0, \forall\ t \in (\eta,T), \forall y_0 \in \widetilde{G} \cap \left\{dist\left(y_0, \partial\widetilde{G}\right) < \delta^{1/4}\right\}, \tag{4.3}$$

being satisfied for an arbitrary bounded subdomain $\widetilde{G} \subseteq \Omega_1$*, imply that*

$$w(z,y,t) = 0 \text{ in } E(\delta,\eta,y_0), \forall y_0 \in \widetilde{G} \cap \left\{ dist\left(y_0, \partial\widetilde{G}\right) < \delta^{1/4} \right\}$$

for all sufficiently small $\delta \in (0, \delta(\eta)]$*. Then the function* $w(z,y,t) = 0$ *in* H_T *and the function* $\widetilde{q}(z,y) = 0$ *in* D*.*

Proof. Choose an arbitrary $\eta \in [0,T)$ and suppose that (4.3) is true for this value of η. Then $w(z,y_0,t) = 0$ in $\{(z,t) : \varphi_0(\eta) < z < \varphi_0(t), z+t-\eta < \delta + \varphi_0(\eta), t > \eta\}$. Since $\delta \in (0, \delta(\eta))$ and $\widetilde{G} \subseteq \Omega_1$ is an arbitrary subdomain, the latter leads to

$$w(z,y,t) = 0 \text{ in } \widetilde{E}(\eta), \tag{4.4}$$

where

$$\widetilde{E}(\eta) = \{(z,y,t) : \varphi_0(\eta) < z < \varphi_0(t), z+t-\eta < \delta(\eta) + \varphi_0(\eta), t > \eta, y \in \Omega_1\}. \tag{4.5}$$

Suppose now that $\eta = 0$ and $\delta \in (0,T)$. In the (z,t) space, consider the point $\left(z, \widetilde{t}_1\right) = \left(\varphi_0\left(\widetilde{t}_1\right), \widetilde{t}_1\right)$ of the intersection of the straight line $\{z+t = \delta + \varphi_0(0)\}$ with the curve $\{z = \varphi_0(t), t > 0\}$. Hence, $\varphi_0\left(\widetilde{t}_1\right) + \widetilde{t}_1 = \delta + \varphi_0(0)$. Consider the function $\alpha(t) = \varphi_0(t) + t$. Since $\alpha(0) = \varphi_0(0) < \delta + \varphi_0(0)$, by (1.6) $\alpha'(t) \geq \gamma_1 + 1$ and $\delta \in (0,T)$, then the point $\widetilde{t}_1 \in (0,T)$ exists and is unique. Choose an integer β such that $\beta + 1 > \alpha'(t)$ in $[0,T]$. Denote $t_1 = \delta/(\beta+1)$. Then $t_1 \in \left(0, \widetilde{t}_1\right)$. Indeed, there exists a $\xi \in \left(0, \widetilde{t}_1\right)$ such that

$$\frac{\alpha\left(\widetilde{t}_1\right) - \alpha(0)}{\widetilde{t}_1 - 0} = \alpha'(\xi).$$

Hence,

$$\widetilde{t}_1 = \frac{\alpha\left(\widetilde{t}_1\right) - \alpha(0)}{\alpha'(\xi)} = \frac{\delta}{\alpha'(\xi)} > \frac{\delta}{\beta+1} = t_1. \tag{4.6}$$

Furthermore, since $\widetilde{t}_1 \in (0,T)$, then $t_1 \in (0,T)$ also. Denote

$$P(t_1) = \{(z,y,t) : \varphi_0(0) < z < \varphi_0(t), t \in (0,t_1), y \in \Omega_1\}.$$

Since by (4.6) $\widetilde{t}_1 > t_1$, then $z + t < \delta + \varphi_0(0)$ in $P(t_1)$. This and (4.5) imply that $P(t_1) \subseteq \widetilde{E}(0)$. Hence, (3.30), (4.4) and the assumption of this lemma lead to

$$w(z,y,t) = 0, \text{ in } P(t_1). \tag{4.7}$$

Further, (3.19a,b) imply that the definition of the domain $P(t_1)$ can be rewritten as

$$P(t_1) = \{(z,y,t) : \varphi_0(0) < z < \varphi_0(t_1), t \in (g(z), t_1), y \in \Omega_1\}.$$

Hence, using (3.20) and (4.7), we obtain $\widetilde{v}(z,y,t) = 0$ in $P(t_1)$. Substituting $\widetilde{v}(z,y,t) := 0$ in the equation (3.5) for $(z,y,t) \in P(t_1)$ and using (3.10), we obtain that

$$\widetilde{q}(z,y) = 0, \text{ in } \{(z,y) : \varphi_0(0) < z < \varphi_0(t_1), y \in \Omega_1\}.$$

Hence, (3.5) implies that

$$a^0 \cdot \widetilde{v}_t - L\widetilde{v} = 0, \text{ in } S(t_1), \tag{4.8}$$

where the domain $S(t_1)$ is defined as

$$S(t_1) = \{(z, y, t) : z \in (\varphi_0(0), \varphi_0(t_1)), (y, t) \in \Omega_1 \times (t_1, T)\}. \tag{4.9}$$

Also, by (3.13)

$$\widetilde{v}|_{z=\varphi_0(0)} = \widetilde{v}_z|_{z=\varphi_0(0)} = 0, \text{ for } (y, t) \in \Omega_1 \times (t_1, T). \tag{4.10}$$

Thus, (4.8)-(4.10) and the uniqueness theorem for the Cauchy problem for the parabolic equation with the lateral data imply that the function $\widetilde{v}(z, y, t) = 0$ in $S(t_1)$. This and (3.16) lead to

$$w|_{z=\varphi_0(t_1)} = w_z|_{z=\varphi_0(t_1)} = 0, \text{ for } (y, t) \in \Omega_1 \times (t_1, T).$$

hence, we now obtain (4.4) with $\eta := t_1$.

Therefore, the following iterative process can be arranged. Let $t_0 := 0$. For $s \geq 1$ let

$$t_s = s \cdot \frac{\delta}{\beta + 1}. \tag{4.11}$$

On the step $s \geq 1$ of this process one starts from the set $E(\delta, t_{s-1}, y_0)$ and proceeds similarly with the above. After s steps we obtain that

$$\widetilde{q}(z, y) = 0, \text{ in } \{(z, y) : \varphi_0(0) < z < \varphi_0(t_s), \quad y \in \Omega_1\}. \tag{4.12}$$

Let

$$P(t_s) = \{(z, y, t) : \varphi_0(t_{s-1}) < z < \varphi_0(t_s), t \in (g(z), t_s), y \in \Omega_1\}.$$

This process can be continued as long as $P(t_s) \subset H_T$, i.e., $P(t_s) \subset \{t \in (0, T)\}$. So, in the rest of the proof of Lemma 4.1 we show that (4.12) implies that

$$\widetilde{q}(z, y) = 0 \text{ in } \{(z, y) : \varphi_0(0) < z < \varphi_0(T), \quad y \in \Omega_1\}, \tag{4.13}$$

which is sufficient for establishing the validity of this lemma.

One can choose sufficiently small number δ_0 such that

$$T = k \cdot \frac{\delta_0}{\beta + 1}, \text{ where } k = \frac{\beta + 1}{\delta_0} \cdot T \tag{4.14}$$

is an integer. Let $\delta_0 := \delta$. Consider all integers $s \geq 1$ such that

$$t_s + \delta < T. \tag{4.15}$$

By (4.11), (4.14) and (4.15) $s < k - (\beta + 1)$. Given the integer β, one can always choose a sufficiently small $\delta_0 = \delta_0(\beta, T) := \delta$ such that (4.14) holds and $k - (\beta + 1) \geq 2$. Thus, the set of integers s satisfying (4.15) is not empty. Now, take $s := s_0 = k - (\beta + 2)$. By (4.5) $\widetilde{E}(\eta) \subset \{\eta < t < \eta + \delta(\eta)\}$. Hence $\widetilde{E}(t_{s_0}) \subset \{t_{s_0} < t < \delta + t_{s_0}\}$. The latter and (4.15) imply that $\widetilde{E}(t_s) \subset H_T$ for $s = 1, \cdots, s_0$. Thus, one can take $s := s_0$ in (4.12). On the other hand, the number

$$T - t_{s_0} = \left(1 + \frac{2}{\beta + 1}\right)\delta$$

can be made arbitrary small by decreasing δ, which implies (4.13). ■

4.2 Notations for Section 5

We assume from now on that $0 \in \Omega_1$ and $\varphi_0(0) = 0$. The function ψ takes the form

$$\psi(z, y, t) = z + \sqrt{\delta}\,|y|^2 + t + \frac{1}{2}.$$

Denote $\varphi(t) := \varphi_0(t)$ and let

$$E = \left\{(z, y, t) : z + \sqrt{\delta}\,|y|^2 + t < \delta, \quad t > 0, \quad 0 < z < \varphi(t)\right\}.$$

Hence, the definition of the domain E can also be written as

$$E = \left\{(z, y, t) : \psi(z, y, t) < \delta + \frac{1}{2},\ t > 0,\ 0 < z < \varphi(t)\right\}.$$

Because of Lemma 4.1, it is sufficient to prove that $w(z, y, t) = 0$ in the domain E for a sufficiently small $\delta > 0$. The boundary ∂E of E consists of three parts,

$$\partial E = \bigcup_{i=1}^{3} \partial_i E,$$

where

$$\partial_1 E = \{z = 0\} \cap \overline{E},$$

$$\partial_2 E = \left\{\psi = \frac{1}{2} + \delta\right\} \cap \overline{E},$$

$$\partial_3 E = \{z = \varphi(t)\} \cap \overline{E}.$$

Hence,

$$\psi(z, y, t)\,|_{\partial_3 E} = \psi(\varphi(t), y, t) = \varphi(t) + \sqrt{\delta}\,|y|^2 + t + \frac{1}{2} \leq \delta + \frac{1}{2}.$$

By (3.30) and (3.31),

$$w = \nabla w = w_t = 0 \ \text{ on } \partial_1 E, \tag{4.16}$$

$$w = 0 \text{ on } \partial_3 E. \tag{4.17}$$

By (4.1) $\partial_2 E$ is a level surface of the CWF $\mathcal{C}(z, y, t)$ and this function attains its minimal value (over $\overline{E}$) on $\partial_2 E$. This and zero Dirichlet and Neumann boundary conditions (4.16) on $\partial_1 E$ imply that one should not be concerned with boundary integrals over $\partial_1 E$ and $\partial_2 E$ in the Carleman estimate. However, one should be concerned with such an integral over $\partial_3 E$, because the Neumann boundary condition is not given on $\partial_3 E$ and $\partial_3 E$ is not a level surface of the function $\mathcal{C}(z, y, t)$.

Let $\widetilde{t}_1$ be the number, which was introduced in the proof of Lemma 4.1, i.e., $\widetilde{t}_1 > 0$ is the unique solution of the equation $\varphi\left(\widetilde{t}_1\right) + \widetilde{t}_1 = \delta$. Denote

$$r(t) = \delta^{-1/4} \cdot \sqrt{\delta - \varphi(t) - t},\ \text{ for } t \in \left(0, \widetilde{t}_1\right).$$

Then $r(t) \in \left(0, \delta^{1/4}\right)$ and

$$\partial_3 E = \left\{(z, y, t) : z = \varphi(t), \ |y| \leq r(t), t \in \left[0, \widetilde{t}_1\right]\right\}, \quad \text{for} \quad n \geq 2.$$

Also,

$$\partial_3 E = \left\{(z, t) = (\varphi(t), t) : t \in \left[0, \widetilde{t}_1\right]\right\}, \text{ for } \ n = 1.$$

Hence,

$$\int\limits_{\partial_3 E} f(y, t)\, dS = \int\limits_0^{\widetilde{t}_1} \sqrt{1 + [\varphi'(t)]^2} \cdot \left(\int\limits_{|y| < r(t)} f(y, t)\, dy \right) dt, \text{ for } \ n \geq 2, \quad \forall f \in C(\partial_3 E).$$

Thus, the Gauss' formula implies that for all functions $f \in C^1(\partial_3 E)$

$$\int\limits_{\partial_3 E} f_j(y, t)\, dS = \int\limits_0^{\widetilde{t}_1} \sqrt{1 + [\varphi'(t)]^2} \cdot \left(\int\limits_{|y| = r(t)} f(y, t) \cos(\overline{n}, y_j)\, d\sigma \right) dt, \tag{4.18}$$

$$\text{for } j = 2, \cdots, n,$$

where $\overline{n}$ is the outward unit normal vector to the sphere $\{|y| = r(t)\} \subset R^{n-1}$. It is convenient for us to write the following inequality instead of (4.18)

$$\int\limits_{\partial_3 E} f_j(y, t)\, dS \geq -C \int\limits_0^{\widetilde{t}_1} \int\limits_{|y| = r(t)} |f(y, t)|\, d\sigma dt, \text{ for } j = 2, \cdots, n, \ \forall f \in C^1(\partial_3 E), \tag{4.19}$$

where $C = C\left(\|\varphi'\|_{C[0,T]}\right)$ is a positive constant. If $n \geq 3$, then it is clear, of course what the interior integral in (4.19) means. If $n = 2$, i.e., $R^{n-1} = R^1$, then (4.18) implies that the inequality (4.19) will still hold if defining the integral over $\{|y| = r(t)\}$ as

$$\int\limits_{|y| = r(t)} |f(y, t)|\, d\sigma \triangleq |f(r(t), t)| + |f(-r(t), t)|, \text{ if } R^{n-1} = R^1.$$

Let B be the set of functions defined as

$$B = \left\{u : u_t, D^\alpha_{z,y} u \in C\left(\overline{E}\right), |\alpha| \leq 3, \quad u|_{\partial_1 E} = \nabla u|_{\partial_1 E} = \ u|_{\partial_3 E} = 0\right\}.$$

Obviously, $u_t = 0$ on $\partial_1 E$, $\forall u \in B$. By (4.16) and (4.17), the above function $w \in B$. Also, (3.22) and (3.23) lead to

$$u_j = 0, \text{ on } \partial_3 E, \text{ for } j = 2, \cdots, n, \quad \forall u \in B. \tag{4.20}$$

$$u_t = -\varphi'(t) \cdot u_z, \quad \text{on } \partial_3 E, \ \forall u \in B. \tag{4.21}$$

Let $G \subseteq H_T$ be the bounded subdomain defined in (3.27) and $E \subseteq G$ for all $\delta \in (0, \delta_0)$, where δ_0 is sufficiently small. In addition to the constant $M = M(G)$ (section 3), introduce the constant A,

$$A = \max \left\{ \frac{1}{\|a^0\|_{C(\overline{G})}}, \ \ \|\varphi'\|_{C[0,T]} \right\}.$$

In section 5 $O(1/\lambda)$ and $O\left(\delta^{3/4}\right)$ denote different $C^1\left(\overline{E}\right)$-functions such that

$$\left| O\left(\frac{1}{\lambda}\right) \right| \leq \frac{K_0}{\lambda}, \quad \left|O(\delta^{3/4})\right| \leq K_0 \cdot \delta^{3/4} \text{ in } E, \quad \forall \lambda, \nu > 1, \quad \forall \delta \in (0,1), \tag{4.22}$$

together with their first derivatives. Here and below $K_0 = K_0(A, \widetilde{\mu}_1, M(G), G)$ denotes different positive constants dependent on $A, \widetilde{\mu}_1, M(G), G$ and independent on parameters $\lambda, \nu, \lambda_0, \nu_0$ and $\delta \in (0,1)$. Recall that the positive constant $\widetilde{\mu}_1$ was defined in (3.8). Also, $\lambda_0 = \lambda_0(A, \widetilde{\mu}_1, M(G), G)$ and $\nu_0 = \nu_0(A, \widetilde{\mu}_1, M(G), G)$ denote different sufficiently large positive parameters depending only on $A, \widetilde{\mu}_1, M(G)$ and G. We choose $\delta \in (0,1)$ so small and λ_0 so large that $E = E(\delta) \subset G$ and

$$\frac{K_0}{\lambda_0} < \frac{1}{8}, \tag{4.23}$$

$$K_0 \cdot \delta^{3/4} < \frac{1}{8}, \tag{4.24}$$

for all constants K_0 occurring in the proof of Theorem 2. From now on, we fix the parameter δ, while the parameter λ_0 can still be increased in the course of the proof of Theorem 2. Choices (4.23) and (4.24) are possible, because only a finite number of functions $O(1/\lambda), O\left(\delta^{3/4}\right)$ and constants K_0 occur in that proof.

5 The Carleman Estimate

Theorem 2 *There exist sufficiently large positive constants $\nu_0 = \nu_0(A, \tilde{\mu}_1, M(G), G), \lambda_0 = \lambda_0(A, \tilde{\mu}_1, M(G), G)$ and a positive constant $K = K(A, \widetilde{\mu}_1, M(G), G)$ such that if $\nu = \nu_0$, then the following pointwise Carleman estimate is valid in the domain E for all functions $u \in B$ and for all $\lambda > \lambda_0$*

$$\left(a^0 \cdot u_t - L_0 u\right)^2 \cdot \mathcal{C}^2$$
$$\geq \frac{K}{\lambda}\left(u_t^2 + \sum_{i,,j=1}^{n} u_{ij}^2\right) \cdot \mathcal{C}^2 + K\left[\lambda\left|\nabla u\right|^2 + \lambda^3 u^2\right] \cdot \mathcal{C}^2 + \nabla \cdot U + V_t, \tag{5.1}$$

where the vector valued function (U, V) is such that

$$\int\limits_{\partial_3 E} \left[(U, V), \tilde{n}\right] dS \geq K\lambda \int\limits_{\partial_3 E} u_z^2 \cdot \mathcal{C}^2 dS - \frac{K}{\lambda} \int\limits_0^{\widetilde{t}_1} \int\limits_{|y|=r(t)} u_z^2 \cdot \mathcal{C}^2 d\sigma dt, \tag{5.2}$$

$$|(U,V)| \leq K\lambda^3 \left[u_t^2 + \sum_{i,,j=1}^{n} u_{ij}^2 + |\nabla u|^2 + u^2 \right] \mathcal{C}^2, \text{ in } \overline{E}, \tag{5.3}$$

$$(U,V) = 0 \text{ on } \partial_1 E, \tag{5.4}$$

where $\tilde{n}$ is the unit outward normal vector on $\partial_3 E$ and $[,]$ denotes the dot product in R^{n+1}. If $n = 1$, then the second integral in the right hand side of (5.2) should be ignored.

In sections 5 and 6 $K = K(A, \widetilde{\mu}_1, \nu_0, M(G), G)$ denotes different positive constants depending only on $A, \widetilde{\mu}_1, \nu_0, M(G), G$. We assume in section 5 that parameters λ_0 and ν_0 are sufficiently large. It can be seen in the course of the proof of Theorem 2 that their choice depends only on numbers $A, \widetilde{\mu}_1, M(G)$ and the domain G. We break the proof of Theorem 2 in proofs of four lemmas. In all estimates of this section $(z, y, t) \in E$. Also in this section $u \in B$ is an arbitrary function and $\nabla u = \nabla_{z,y} u$.

Lemma 5.1. *The following inequality is valid for all $\lambda, \nu > 2$*

$$\left(a^0 u_t - L_0 u\right) u \cdot \mathcal{C}^2 \geq \widetilde{\mu}_1 |\nabla u|^2 \cdot \mathcal{C}^2 - K_0 \lambda^2 \nu^2 \cdot \psi^{-2\nu-2} \left[1 + O\left(\frac{1}{\lambda}\right)\right] u^2 \cdot \mathcal{C}^2 + \nabla \cdot U_1 + (V_1)_t ,$$

where the vector valued function (U_1, V_1) is such that

$$(U_1, V_1) = (0,0), \text{ on } \partial_1 E \cup \partial_3 E,$$

$$|(U_1, V_1)| \leq K_0 \lambda \nu \cdot \psi^{-\nu-1} \left(|\nabla u|^2 + u^2\right) \cdot \mathcal{C}^2.$$

We omit the proof of this result, because it is almost identical to the proof of Lemma 1 in section 1 of Chapter 4 of the book [19].

Lemma 5.2. *The following estimate is valid for all $\nu > 2$ and for all $\lambda > \lambda_0$*

$$\left(a^0 \cdot u_t - L_0 u\right)^2 \psi^{\nu+2} \cdot \mathcal{C}^2$$

$$\geq -K_0 \lambda \nu |\nabla u|^2 \cdot \mathcal{C}^2 + K_0 \lambda^3 \nu^4 \psi^{-2\nu-2} \cdot u^2 \cdot \mathcal{C}^2 + \nabla \cdot U_2 + (V_2)_t , \tag{5.5}$$

where the vector valued function (U_2, V_2) is such that

$$[(U_2, V_2), \tilde{n}] \geq K_0 \lambda \nu u_z^2 \cdot \mathcal{C}^2, \text{ on } \partial_3 E, \tag{5.6a}$$

$$|(U_2, V_2)| \leq K_0 \lambda^3 \nu^3 \psi^{-2\nu-1} \left(u_t^2 + |\nabla u|^2 + u^2\right) \cdot \mathcal{C}^2, \text{ in } \overline{E}, \tag{5.6b}$$

$$(U_2, V_2) = 0, \text{ on } \partial_1 E. \tag{5.6c}$$

Proof. Not that (5.6c) follows from (5.6b). Let $v = u \cdot \exp(\lambda \psi^{-\nu}) = u \cdot \mathcal{C}$. Express derivatives of the function $u = v \cdot \mathcal{C}^{-1}$ through those of the function v,

$$u_z = \left(v_z + \lambda \nu \cdot \psi^{-\nu-1} \cdot v\right) \cdot \mathcal{C}^{-1},$$

$$u_{zz} = \left[v_{zz} + 2\lambda \nu \psi^{-\nu-1} \cdot v_z + \lambda^2 \nu^2 \cdot \psi^{-2\nu-2} \left(1 + O\left(\frac{1}{\lambda}\right)\right) \cdot v\right] \cdot \mathcal{C}^{-1}, \tag{5.7}$$

$$u_{zj} = \left[v_{zj} + \lambda \nu \psi^{-\nu-1} v_j + \lambda \nu \psi_j \psi^{-\nu-1} \cdot v_z + \lambda^2 \nu^2 \psi_j \psi^{-2\nu-2} \left(1 + O\left(\frac{1}{\lambda}\right)\right) v\right] \cdot \mathcal{C}^{-1}. \tag{5.8}$$

Also, for $i,j=2,\cdots,n$

$$u_i=\left(v_i+\lambda\nu\psi_i\psi^{-\nu-1}\cdot v\right)\cdot\mathcal{C}^{-1},$$

$$u_{ij}=\left[v_{ij}+\lambda\nu\cdot\psi^{-\nu-1}\left(\psi_iv_j+\psi_jv_i\right)+\lambda^2\nu^2\cdot\psi^{-2\nu-2}\left(\psi_i\psi_j+O\left(\frac{1}{\lambda}\right)\right)\cdot v\right]\cdot\mathcal{C}^{-1}. \tag{5.9}$$

Thus, denoting $a^{11}:=1$, and noting that $\psi_1=1$, we obtain

$$\left(a^0\cdot u_t-L_0u\right)\cdot\mathcal{C} \tag{5.10}$$

$$=\left\{a^0v_t-L_0v-2\lambda\nu\psi^{-\nu-1}\sum_{i,,j=1}^{n}a^{ij}\psi_jv_i-\lambda^2\nu^2\psi^{-2\nu-2}\left[\sum_{i,,j=1}^{n}a^{ij}\left(\psi_i\psi_j+O\left(\frac{1}{\lambda}\right)\right)\right]v\right\}.$$

Denote

$$z_1=a^0\cdot v_t,$$

$$z_2=-L_0v=-\sum_{i,,j=1}^{n}a^{ij}v_{ij},$$

$$z_3=-2\lambda\nu\cdot\psi^{-\nu-1}\sum_{i,,j=1}^{n}a^{ij}\psi_jv_i,$$

$$z_4=-\lambda^2\nu^2\cdot\psi^{-2\nu-2}\left[\sum_{i,,j=1}^{n}a^{ij}\left(\psi_i\psi_j+O\left(\frac{1}{\lambda}\right)\right)\right]\cdot v.$$

By (5.10), the left hand side of the inequality (5.5) can be estimated as

$$\left(a^0u_t-L_0u\right)^2\cdot\psi^{\nu+2}\cdot\mathcal{C}^2=\left[(z_1+z_3)+(z_2+z_4)\right]^2\cdot\psi^{\nu+2}$$

$$\geq\left[z_1^2+z_3^2+2z_1z_3+2z_1z_2+2z_2z_3+2z_1z_4+2z_3z_4\right]\cdot\psi^{\nu+2}. \tag{5.11}$$

In the rest of the proof of this lemma, we estimate from the below either terms or groups of terms in the right hand side of (5.11). We do this in several steps.

Step 1. Estimate $2z_1z_2\cdot\psi^{\nu+2}$. Denote $b^{ij}=a^0\cdot a^{ij}$. Then

$$2z_1z_2\cdot\psi^{\nu+2}=-\sum_{i,,j=1}^{n}b^{ij}\left(v_{ij}+v_{ij}\right)v_t\cdot\psi^{\nu+2}$$

$$=\sum_{i,,j=1}^{n}\left[\left(-b^{ij}\cdot\psi^{\nu+2}\cdot v_iv_t\right)_j+\left(-b^{ij}\cdot\psi^{\nu+2}\cdot v_jv_t\right)_i\right]$$

$$+v_t\cdot\sum_{i,,j=1}^{n}\left[\left(b^{ij}\cdot\psi^{\nu+2}\right)_j\cdot v_i+\left(b^{ij}\cdot\psi^{\nu+2}\right)_i\cdot v_j\right]+\sum_{i,,j=1}^{n}b^{ij}\cdot\psi^{\nu+2}\left(v_iv_{jt}+v_jv_{it}\right)$$

$$=2v_t\sum_{i,,j=1}^{n}\left(b^{ij}\cdot\psi^{\nu+2}\right)_j\cdot v_i+\sum_{i,,j=1}^{n}b_{ij}\cdot\psi^{\nu+2}\cdot\left(v_iv_j\right)_t+\nabla\cdot U_{21},$$

where

$$\nabla \cdot U_{21} = \sum_{j=1}^{n} \left[-2 \sum_{i=1}^{n} b^{ij} \cdot \psi^{\nu+2} \cdot v_i v_t \right]_j \quad . \tag{5.12}$$

Further,

$$\sum_{i,,j=1}^{n} b^{ij} \cdot \psi^{\nu+2} \cdot (v_i v_j)_t = \sum_{i,,j=1}^{n} \left(b^{ij} \cdot \psi^{\nu+2} \cdot v_i v_j \right)_t - \sum_{i,,j=1}^{n} \left(b^{ij} \cdot \psi^{\nu+2} \right)_t v_i v_j.$$

Hence,

$$2z_1 z_2 \cdot \psi^{\nu+2} = 2v_t \sum_{i,,j=1}^{n} \left(b^{ij} \cdot \psi^{\nu+2} \right)_j v_i - \sum_{i,,j=1}^{n} \left(b^{ij} \cdot \psi^{\nu+2} \right)_t v_i v_j + \nabla \cdot U_{21} + (V_{21})_t \,, \tag{5.13}$$

where $\nabla \cdot U_{21}$ was defined in (5.12) and

$$V_{21} = \sum_{i,,j=1}^{n} b^{ij} \psi^{\nu+2} \cdot v_i v_j. \tag{5.14}$$

Note that

$$\left(b^{ij} \cdot \psi^{\nu+2} \right)_j = \left(b^{ij} \right)_j \cdot \psi^{\nu+2} + (\nu+2)\, b^{ij} \psi_j \cdot \psi^{\nu+1}$$

$$\left(b^{ij} \cdot \psi^{\nu+2} \right)_t = \left(b^{ij} \right)_t \cdot \psi^{\nu+2} + (\nu+2)\, b^{ij} \cdot \psi^{\nu+1},$$

$$|\psi_j| \leq 1 \text{ in } \overline{E}, j = 1, \cdots, n, \tag{5.15}$$

$$\frac{1}{2} \leq |\psi| \leq \frac{1}{2} + \delta < 1 \text{ in } \overline{E}. \tag{5.16}$$

Recall that $b^{ij} = a^0 \cdot a^{ij}$. Thus, (5.13)-(5.16) imply that for all $\nu > 2$

$$2z_1 z_2 \cdot \psi^{\nu+2} \geq -K_0 \nu \cdot \psi^{\nu+1} \cdot |\nabla v|^2 + 2(\nu+2) \cdot \psi^{\nu+1} a^0 \cdot v_t \sum_{i,,j=1}^{n} a^{ij} \psi_j v_i$$

$$+ 2\psi^{\nu+2} \cdot v_t \cdot \sum_{i,,j=1}^{n} \left(b^{ij} \right)_j v_i + \nabla \cdot U_{21} + (V_{21})_t \,.$$

Since $z_1 = a^0 \cdot v_t$, then

$$2z_1 z_2 \cdot \psi^{\nu+2} \geq -K_0 \nu \cdot \psi^{\nu+1} \cdot |\nabla v|^2 \tag{5.17}$$

$$+ 2z_1 \cdot \psi^{\nu+2} \left[(\nu+2)\, \psi^{-1} \sum_{i,,j=1}^{n} a^{ij} \psi_j v_i + \frac{1}{a^0} \cdot \sum_{i,j=1}^{n} \left(b^{ij} \right)_j v_i \right] + \nabla \cdot U_{21} + (V_{21})_t \,.$$

Now we estimate the magnitude of the vector valued function (U_{21}, V_{21}) on the surface $\partial_3 E$. Since

$$v_t = u_t \cdot \mathcal{C} + u \cdot \mathcal{C}_t, \quad v_j = u_j \cdot \mathcal{C} + u \cdot \mathcal{C}_j,$$

and $u = 0$ on $\partial_3 E$, then (5.12), (5.14) and (5.16) imply that

$$|(U_{21}, V_{21})| \leq K_0 u_z^2 \cdot \mathcal{C}^2, \text{ on } \partial_3 E. \tag{5.18}$$

Step 2. Estimate $\left(z_1^2+z_3^2+2z_1z_3+2z_1z_2\right)\psi^{\nu+2}$. Using (5.17), we obtain

$$\begin{aligned}
&\left(z_1^2+z_3^2+2z_1z_3+2z_1z_2\right)\cdot\psi^{\nu+2}\\
\geq\ &\left\{z_1^2+z_3^2+2z_1\left[z_3+(\nu+2)\,\psi^{-1}\sum_{i,,j=1}^{n}a^{ij}\psi_jv_i+\frac{1}{a^0}\cdot\sum_{i,j=1}^{n}\left(b^{ij}\right)_jv_i\right]\right\}\cdot\psi^{\nu+2}\\
&-K_0\nu\cdot\psi^{\nu+1}\cdot|\nabla v|^2+\nabla\cdot U_{21}+\left(V_{21}\right)_t.
\end{aligned}$$

By the Cauchy-Schwarz inequality

$$2z_1\left[z_3+(\nu+2)\,\psi^{-1}\sum_{i,,j=1}^{n}a^{ij}\psi_jv_i+\frac{1}{a^0}\sum_{i,,j=1}^{n}\left(b^{ij}\right)_jv_i\right]$$

$$\geq -z_1^2-z_3^2-2z_3\,(\nu+2)\,\psi^{-1}\sum_{i,,j=1}^{n}a^{ij}\psi_jv_i-2z_3\cdot\frac{1}{a^0}\cdot\sum_{i,,j=1}^{n}\left(b^{ij}\right)_jv_i$$

$$-(\nu+2)^2\,\psi^{-2}\left(\sum_{i,,j=1}^{n}a^{ij}\psi_jv_i\right)^2-K_0\,|\nabla v|^2.$$

Hence, the formula for z_3 and (5.16) lead to

$$\left(z_1^2+z_3^2+2z_1z_3+2z_1z_2\right)\cdot\psi^{\nu+2}$$

$$\geq -K_0\lambda\nu\,|\nabla v|^2+4\lambda\nu\,(\nu+2)\left(\sum_{i,,j=1}^{n}a^{ij}\psi_jv_i\right)^2-(\nu+2)^2\,\psi^{\nu}\left(\sum_{i,,j=1}^{n}a^{ij}\psi_jv_i\right)^2$$

$$+4\lambda\nu\psi\cdot\frac{1}{a^0}\left(\sum_{i,j=1}^{n}a^{ij}\psi_jv_i\right)\cdot\left(\sum_{i,j=1}^{n}\left(b^{ij}\right)_j\psi_jv_i\right)+\nabla\cdot U_{21}+\left(V_{21}\right)_t. \tag{5.19}$$

By (5.16), $\psi^{\nu}<1$ in E. Hence, for all $\lambda>\lambda_0$ and for all $\nu>2$ we obtain $\lambda\nu\,(\nu+2)>(\nu+2)^2\,\psi^{\nu}$ in $\overline{E}$. Hence,

$$\begin{aligned}
&4\lambda\nu\,(\nu+2)\left(\sum_{i,,j=1}^{n}a^{ij}\psi_jv_i\right)^2-(\nu+2)^2\,\psi^{\nu}\left(\sum_{i,,j=1}^{n}a^{ij}\psi_jv_i\right)^2\\
\geq\ &3\lambda\nu\,(\nu+2)\left(\sum_{i,,j=1}^{n}a^{ij}\psi_jv_i\right)^2\geq 0.
\end{aligned}$$

In addition, since $a^0\geq A$ in E, then

$$4\lambda\nu\psi\cdot\frac{1}{a^0}\left(\sum_{i,,j=1}^{n}b_j^{ij}v_i\right)\cdot\left(\sum_{i,,j=1}^{n}a^{ij}\psi_jv_i\right)\geq -K_0\lambda\nu\cdot|\nabla v|^2.$$

Therefore, (5.19) leads to

$$\left(z_1^2+z_3^2+2z_1z_3+2z_1z_2\right)\cdot\psi^{\nu+2}\geq -K_0\lambda\nu\cdot|\nabla v|^2+\nabla\cdot U_{21}+\left(V_{21}\right)_t, \tag{5.20}$$

where the vector valued function (U_{21}, V_{21}) satisfies (5.18).

Step 3. Estimate $2z_2z_3 \cdot \psi^{\nu+2}$. Observe that $\psi_1 = 1$ and $\psi_j = 2\sqrt{\delta}y_j = O\left(\delta^{3/4}\right)$ in E, for $j = 2, \cdots, n$. Hence, (4.24) implies that

$$|\psi_j| < \frac{1}{8} \text{ , in } E, \quad j = 2, \cdots, n. \tag{5.21}$$

Recalling that $a^{11} = 1$, we single out the derivative v_z in z_3 and rewrite z_3 as

$$z_3 = z_{31} + z_{32},$$

where

$$z_{31} = -2\lambda\nu \cdot \psi^{-\nu-1} \cdot v_z,$$
$$z_{32} = -2\lambda\nu \cdot \psi^{-\nu-1}\left[\sum_{j=2}^{n} a^{1j}v_j + \sum_{i,j=2}^{n} a^{ij}\psi_j v_i.\right]$$

Likewise, rewrite z_2 as

$$z_2 = -v_{zz} - \sum_{j=2}^{n} a^{1j}v_{zj} - \sum_{i,j=2}^{n} a^{ij}v_{ij}.$$

Step 3.1. Estimate $2z_2z_{31} \cdot \psi^{\nu+2}$.

$$2z_2z_{31} \cdot \psi^{\nu+2} = 4\lambda\nu\psi v_z \cdot \left[v_{zz} + \sum_{j=2}^{n} a^{1j}v_{zj}\right] + 2\lambda\nu\psi v_z \cdot \sum_{i,j=2}^{n} a^{ij}\left(v_{ij} + v_{ji}\right)$$

$$= \left(2\lambda\nu\psi \cdot v_z^2\right)_z - 2\lambda\nu v_z^2 + \sum_{j=2}^{n}\left[2\lambda\nu\psi a^{1j} \cdot v_z^2\right]_j - 2\lambda\nu\sum_{j=2}^{n}\left[\psi a^{1j}\right]_j v_z^2 \tag{5.22}$$

$$+\sum_{i,j=2}^{n}\left(2\lambda\nu\psi a^{ij} \cdot v_z v_i\right)_j + \sum_{i,j=2}^{n}\left(2\lambda\nu\psi a^{ij} \cdot v_z v_j\right)_i - 2\lambda\nu\psi\sum_{i,j=2}^{n} a^{ij}\left(v_{zj}v_i + v_j v_{zi}\right)$$

$$-4\lambda\nu\sum_{i,j=2}^{n}\left(\psi a^{ij}\right)_j v_i v_z.$$

Finally, since

$$-2\lambda\nu\psi\sum_{i,j=2}^{n} a^{ij}\left(v_{zj}v_i + v_j v_{zi}\right) = -2\lambda\nu\psi\sum_{i,j=2}^{n} a^{ij}\left(v_i v_j\right)_z$$

$$= \sum_{i,j=2}^{n}\left(-2\lambda\nu\psi a^{ij} \cdot v_i v_j\right)_z + 2\lambda\nu\sum_{i,j=2}^{n}\left(\psi a^{ij}\right)_z v_i v_j,$$

then (5.22) leads to

$$2z_2z_{31} \cdot \psi^{\nu+2} \geq -K_0\lambda\nu\left|\nabla v\right|^2 + \nabla \cdot U_{22}^{(1)}, \tag{5.23}$$

where

$$\nabla \cdot U_{22}^{(1)} = \left[2\lambda\nu\psi \cdot v_z^2 - 2\lambda\nu\psi \sum_{i,j=2}^{n} a^{ij} \cdot v_i v_j \right]_z$$

$$+ \sum_{j=2}^{n} \left[2\lambda\nu\psi a^{1j} \cdot v_z^2 \right]_j + \sum_{i,j=2}^{n} \left[4\lambda\nu\psi a^{ij} \cdot v_z v_i \right]_j . \tag{5.24}$$

Note that

$$\cos(\tilde{n}, x_j) = 0, \text{ on } \partial_3 E, \quad j = 2, \cdots, n. \tag{5.25}$$

In addition, since $u|_{\partial_3 E} = 0$, then $v_z = u_z \cdot \mathcal{C}$ on $\partial_3 E$ and by (4.20) $v_j = u_j \cdot \mathcal{C} + u \cdot \mathcal{C}_j = 0$ on $\partial_3 E$ for $j = 2, \cdots, n$. Thus, (5.16), (5.24) and (5.25) imply that

$$\left[\left(U_{22}^{(1)}, 0 \right) \cdot \tilde{n} \right] \geq K_0 \lambda\nu u_z^2 \cdot \mathcal{C}^2 \text{ on } \partial_3 E. \tag{5.26}$$

Step 3.2. Estimate $2z_2 z_{32} \cdot \psi^{\nu+2}$.

$$2z_2 z_{32} \cdot \psi^{\nu+2} = 4\lambda\nu\psi \left[\sum_{j=2}^{n} a^{j1} v_j + \sum_{i,j=2}^{n} a_{ij} \psi_j v_i \right] \cdot \left(v_{zz} + \sum_{i,j=2}^{n} a^{ij} v_{ij} \right).$$

We carry out this estimate in the same manner as one in Step 3.1. Similarly with Step 3.1, we obtain

$$2z_2 z_{32} \cdot \psi^{\nu+2} \geq -K_0 \lambda\nu \cdot |\nabla v|^2 + \nabla \cdot U_{22}^{(1)}. \tag{5.27}$$

Since functions $v_j = 0$ on $\partial_3 E$ for $j = 2, \cdots, n$, then (5.25) implies that

$$\left[\left(U_{22}^{(2)}, 0 \right), \tilde{n} \right] = 0. \tag{5.28}$$

To finalize Step 3, we sum up inequalities (5.23) and (5.27). Denote $U_{22} = U_{22}^{(1)} + U_{22}^{(2)}$ and take into account estimates (5.26) and (5.28). We obtain

$$2z_2 (z_{31} + z_{32}) \cdot \psi^{\nu+2} = 2z_2 z_3 \cdot \psi^{\nu+2} \geq -K_0 \lambda\nu |\nabla v|^2 + \nabla \cdot U_{22}, \tag{5.29}$$

$$[(U_{22}, 0), \tilde{n}] \geq K_0 \lambda\nu u_z^2 \cdot \mathcal{C}^2, \text{ on } \partial_3 E. \tag{5.30}$$

Step 4. Estimate $2z_1 z_4 \cdot \psi^{\nu+2}$. Since $a^{11} = 1, \psi_1 = 1$ and $\psi_j = O\left(\delta^{3/4}\right)$, for $j = 2, ..., n$, then one can rewrite z_4 as

$$z_4 = -\lambda^2 \nu^2 \psi^{-2\nu-2} \left[1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right) \right] \cdot v.$$

Hence,

$$\begin{aligned} 2z_1 z_4 \cdot \psi^{\nu+2} &= -2\lambda^2 \nu^2 \cdot \psi^{-\nu} a^0 \left[1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right) \right] \cdot v_t v \\ &= \left[-\lambda^2 \nu^2 \cdot \psi^{-\nu} a^0 \left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right) \right) v^2 \right]_t \\ &\quad + \lambda^2 \nu^2 \left[a^0 \cdot \psi^{-\nu} \left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right) \right) \right]_t \cdot v^2. \end{aligned}$$

Hence, using (4.22)-(4.24), we obtain

$$2z_1 z_4 \cdot \psi^{\nu+2} \geq -K_0 \lambda^2 \nu^3 \cdot \psi^{-\nu-1} \cdot v^2 + (V_{22})_t\,, \tag{5.31}$$

where

$$V_{22} = -\lambda^2 \nu^2 \cdot \psi^{-\nu} a^0 \left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right)\right) v^2,$$

which implies that

$$V_{22} = 0, \text{ on } \partial_3 E. \tag{5.32}$$

Step 5. Estimate $2z_3 z_4 \cdot \psi^{\nu+2}$.

$$\begin{aligned}
2z_3 z_4 \cdot \psi^{\nu+2} &= 2\left(z_{31} + z_{32}\right) z_4 \cdot \psi^{\nu+2} \\
&= 4\lambda^3 \nu^3 \cdot \psi^{-2\nu-1} \left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right)\right) v_z v \\
&\quad + 4\lambda^3 \nu^3 \cdot \psi^{-2\nu-1} \left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right)\right) \cdot \sum_{i=2}^{n} \sum_{j=1}^{n} a^{ij} \psi_j v_i v \\
&= \left[2\lambda^3 \nu^3 \cdot \psi^{-2\nu-1} \left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right)\right) \cdot v^2\right]_z \\
&\quad + 2\lambda^3 \nu^3 \left(2\nu + 1\right) \cdot \psi^{-2\nu-2} \left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right)\right) \cdot v^2 \\
&\quad + \sum_{i=2}^{n} \left[2\lambda^3 \nu^3 \cdot \psi^{-2\nu-1} \left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right)\right) \cdot \sum_{j=1}^{n} a^{ij} \psi_j \cdot v^2\right]_i
\end{aligned}$$

$$+2\lambda^3 \nu^3 \left(2\nu + 1\right) \cdot \psi^{-2\nu-2} \left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right)\right) \left(\sum_{i=2}^{n} \psi_i \sum_{j=1}^{n} a^{ij} \psi_j\right) \cdot v^2$$

$$-2\lambda^3 \nu^3 \cdot \psi^{-2\nu-1} \sum_{i=2}^{n} \left[\left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right)\right) \sum_{j=1}^{n} a^{ij} \psi_j\right]_i \cdot v^2.$$

Since $\psi_i = O\left(\delta^{3/4}\right)$, for $i = 2, ..., n$, then (4.23) and (4.24) imply that

$$\left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right)\right) \left|\sum_{i=2}^{n} \psi_i \sum_{j=1}^{n} a^{ij} \psi_j\right| \leq \frac{K_0}{\lambda_0} + K_0 \delta^{3/4} < \frac{1}{4}$$

Thus,

$$2z_3 z_4 \cdot \psi^{\nu+2} \geq K_0 \lambda^3 \nu^4 \cdot \psi^{-2\nu-2} \left(1 - \frac{\psi}{\nu}\right) \cdot v^2 + \nabla \cdot U_{23}, \tag{5.33}$$

where

$$\nabla \cdot U_{23} = \left[2\lambda^3 \nu^3 \cdot \psi^{-2\nu-1} \left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right)\right) \cdot v^2\right]_z$$

$$+ \sum_{i=2}^{n} \left\{2\lambda^3 \nu^3 \cdot \psi^{-2\nu-1} \left(1 + O\left(\frac{1}{\lambda}\right) + O\left(\delta^{3/4}\right)\right) \cdot \sum_{j=1}^{n} a^{ij} \psi_j \cdot v^2\right\}_i$$

Hence,

$$U_{23} = 0 \text{ on } \partial_3 E. \tag{5.34}$$

Since $\nu > 2$, then (5.16) implies that

$$\left|\frac{\psi}{\nu}\right| < \frac{1}{2}.$$

This and (5.33) lead to

$$2z_3 z_4 \cdot \psi^{\nu+2} \geq K_0 \lambda^3 \nu^4 \cdot \psi^{-2\nu-2} \cdot v^2 + \nabla \cdot U_{23}. \tag{5.35}$$

Finally, summing up inequalities (5.20), (5.29), (5.31) and (5.35), which result from the above steps, noticing that $\lambda^3\nu^4\psi^{-2\nu-2} > \lambda^2\nu^2\psi^{-\nu-1}$ (compare (5.31) with (5.35)), denoting $U_2 = U_{21} + U_{22} + U_{23}, V_2 = V_{21} + V_{22}$, taking into account estimates (5.18), (5.30), (5.32) and (5.34) of boundary terms, as well as the above explicit formulas for $U_{21}, U_{22}, U_{23}, V_{21}$ and V_{22} and returning to the function $u = v \cdot \mathcal{C}^{-1}$, we obtain estimates (5.5) and (5.6a,b) of this lemma. ■

In the following lemma we set $\nu := \nu_0$.

Lemma 5.3. *Let $\nu := \nu_0$. Then the following estimate is valid for all $\lambda > \lambda_0$*

$$\left(a^0 u_t - L_0 u\right)^2 \cdot \mathcal{C}^2 \geq K\left(\lambda\left|\nabla u\right|^2 + \lambda^3 u^2\right) \cdot \mathcal{C}^2 + \nabla \cdot U_3 + \left(V_3\right)_t, \tag{5.36}$$

where the vector valued function (U_3, V_3) is such that

$$\left[\left(U_3, V_3\right), \tilde{n}\right] \geq K\lambda u_z^2 \cdot \mathcal{C}^2, \text{ on } \partial_3 E, \tag{5.37}$$

$$\left|\left(U_3, V_3\right)\right| \leq K\lambda^3\left[u_z^2 + \left|\nabla u\right|^2 + u^2\right] \cdot \mathcal{C}^2, \text{ in } \overline{E}, \tag{5.38a}$$

$$\left(U_3, V_3\right) = 0, \text{ on } \partial_1 E. \tag{5.38b}$$

Proof. Multiply the estimate of Lemma 5.1 by $2K_0\lambda\nu\left(\widetilde{\mu}_1\right)^{-1}$ and add to the estimate (5.5) of Lemma 5.2. We obtain

$$\begin{aligned} &2K_0\left(\widetilde{\mu}_1\right)^{-1} \cdot \lambda\nu\left(a^0 \cdot u_t - L_0 u\right) \cdot u \cdot \mathcal{C}^2 + \left(a^0 u_t - L_0 u\right)^2 \cdot \psi^{\nu+2} \cdot \mathcal{C}^2 \\ &\geq K_0\lambda\nu\left|\nabla u\right|^2 \cdot \mathcal{C}^2 + K_0\lambda^3\nu^4 \cdot \psi^{-2\nu-2}\left(1 - \frac{2K_0}{\widetilde{\mu}_1} \cdot \frac{1}{\nu}\right) \cdot u^2 \cdot \mathcal{C}^2 + \nabla \cdot U_3 + \left(V_3\right)_t, \end{aligned}$$

where $U_3 = 2K_0\lambda\nu\left(\widetilde{\mu}_1\right)^{-1} U_1 + U_2$ and $V_3 = 2K_0\lambda\nu\left(\widetilde{\mu}_1\right)^{-1} V_1 + V_2$. By (5.6a,b,c), the vector valued function (U_3, V_3) satisfies (5.37) and (5.38a,b). Choose a $\nu_0\left(A, \widetilde{\mu}_1, G\right) > 1$ such that

$$1 - \frac{2K_0}{\widetilde{\mu}_1} \cdot \frac{1}{\nu_0} > \frac{1}{2}$$

and set $\nu := \nu_0$. Since by (5.16) $\psi^{\nu+2} < 1$ in E, then

$$\begin{aligned} &2K_0\left(\widetilde{\mu}_1\right)^{-1}\lambda\nu\left(a^0 u_t - L_0 u\right) \cdot u \cdot \mathcal{C}^2 + \left(a^0 u_t - L_0 u\right)^2 \cdot \psi^{\nu+2} \cdot \mathcal{C}^2 \\ \leq\ &2\left(a^0 \cdot u_t - L_0 u\right)^2 \cdot \mathcal{C}^2 + K\lambda^2 u^2 \cdot \mathcal{C}^2. \end{aligned}$$

These lead to (5.36). ■

The Carleman estimate (5.36) has terms only with the low order derivatives in its right hand side. To incorporate terms with the derivatives u_t^2 and u_{ij}^2, we prove the following lemma first.

Lemma 5.4. *Let $\nu := \nu_0$. Then the following estimate is valid for all $\lambda > \lambda_0$*

$$\begin{gathered}(a^0u_t - L_0u)^2 \cdot \mathcal{C}^2 \geq K\left(u_t^2 + \sum_{i,j=1}^n u_{ij}^2\right)\cdot \mathcal{C}^2 \\ -K\lambda^2 \left|\nabla u\right|^2 \cdot \mathcal{C}^2 + \nabla \cdot U_4 + (V_4)_t\,,\end{gathered} \tag{5.39}$$

where the vector valued function (U_4, V_4) satisfies

$$\int_{\partial_3 E} \left[(U_4, V_4)\,, \tilde{n}\right] dS \geq -K\lambda \int_{\partial_3 E} u_z^2 \cdot \mathcal{C}^2 dS - K \int_0^{\tilde{t}_1} \int_{|y|=r(t)} u_z^2 \cdot \mathcal{C}^2 d\sigma dt, \tag{5.40}$$

$$|(U_4, V_4)| \leq K\left(u_t^2 + \sum_{i,j=1}^n u_{ij}^2 + |\nabla u|^2\right)\cdot \mathcal{C}^2, \text{ in } \overline{E}, \tag{5.41}$$

$$(U_4, V_4) = 0, \text{ on } \partial_1 E. \tag{5.42}$$

Proof. Again, let $b^{ij} = a^0 \cdot a^{ij}$. We have

$$\begin{aligned}\left(a^0u_t - L_0u\right)^2 \cdot \mathcal{C}^2 &= \left(a^0\right)^2 u^2 \cdot \mathcal{C}^2 + \sum_{i,j=1}^n \left[-b^{ij}\left(u_{ij} + u_{ij}\right)u_t\right]\cdot \mathcal{C}^2 \\ +\left(L_0u\right)^2 \cdot \mathcal{C}^2 &= y_1 + y_2 + y_3.\end{aligned}$$

Obviously,

$$y_1 = \left(a^0\right)^2 u_t^2 \geq K u_t^2 \cdot \mathcal{C}^2. \tag{5.43}$$

We estimate y_2 and y_3 from the below in two steps.

Step 1. Estimate y_2.

$$\begin{aligned} y_2 &= \sum_{i,j=1}^n \left[\left(-b^{ij}u_iu_t \cdot \mathcal{C}^2\right)_j + \left(-b^{ij}u_ju_t \cdot \mathcal{C}^2\right)_i\right] \\ &\quad + \sum_{i,j=1}^n b^{ij}\left(u_iu_{jt} + u_ju_{it}\right)\cdot \mathcal{C}^2 + u_t \sum_{i,j=1}^n \left[\left(b^{ij}\right)_j u_i + \left(b^{ij}\right)_i u_j\right]\cdot \mathcal{C}^2 \\ &\quad -2\lambda\nu \cdot \psi^{-\nu-1} \cdot u_t \sum_{i,j=1}^n b^{ij}\left(u_i\psi_j + u_j\psi_i\right)\cdot \mathcal{C}^2. \end{aligned}$$

Note that

$$\begin{aligned}&\sum_{i,j=1}^n b^{ij}\left(u_iu_{jt} + u_ju_{it}\right)\cdot \mathcal{C}^2 \\ &= \sum_{i,j=1}^n \left[b^{ij}u_iu_j \cdot \mathcal{C}^2\right]_t - \sum_{i,j=1}^n \left(b^{ij}\right)_t u_iu_j \cdot \mathcal{C}^2 + 2\lambda\nu\psi^{-\nu-1}\sum_{i,j=1}^n b^{ij}u_iu_j \cdot \mathcal{C}^2 \\ &\geq K\lambda\left|\nabla u\right|^2 \cdot \mathcal{C}^2 + \sum_{i,j=1}^n \left[b^{ij}u_iu_j \cdot \mathcal{C}^2\right]_t \geq \sum_{i,j=1}^n \left[b^{ij}u_iu_j \cdot \mathcal{C}^2\right]_t.\end{aligned}$$

Hence applying the Cauchy-Schwarz inequality 'with $\varepsilon > 0$', $2ab \geq -\varepsilon a^2 - b^2/\varepsilon$, we obtain

$$y_2 \geq -\varepsilon u_t^2 \cdot \mathcal{C}^2 - K\frac{\lambda^2}{\varepsilon} \cdot |\nabla u|^2 \cdot \mathcal{C}^2 + \nabla \cdot U_{41} + (V_4)_t \,, \tag{5.44}$$

where

$$\nabla \cdot U_{41} = \sum_{i,j=1}^{n} \left(-2b^{ij} u_i u_t \cdot \mathcal{C}^2\right)_j , \tag{5.45}$$

$$V_4 = \sum_{i,j=1}^{n} b^{ij} u_i u_j \cdot \mathcal{C}^2. \tag{5.46}$$

Using (4.20) and (4.21), we obtain

$$|(U_{41}, V_4)| \leq K u_z^2 \cdot \mathcal{C}^2, \text{ on } \partial_3 E. \tag{5.47}$$

Summing up (5.43) and (5.44) and choosing $\varepsilon = K/2$, we obtain with a different constant K

$$y_1 + y_2 \geq K u_t^2 \cdot \mathcal{C}^2 - K\lambda^2 |\nabla u|^2 \cdot \mathcal{C}^2 + \nabla \cdot U_{41} + (V_4)_t \,. \tag{5.48}$$

Step 2. Estimate y_3.

$$y_3 = (L_0 u)^2 \cdot \mathcal{C}^2 = \sum_{i,j,k,s=1}^{n} a^{ij} a^{ks} u_{ij} u_{ks} \cdot \mathcal{C}^2$$

$$= \sum_{\substack{i,j,k,s=1 \\ (i,j)=(k,s)}}^{n} a^{ij} a^{ks} (u_{ij})^2 \cdot \mathcal{C}^2 + \sum_{\substack{i,j,k,s=1 \\ (i,j)\neq(k,s)}}^{n} a^{ij} a^{ks} u_{ij} u_{ks} \cdot \mathcal{C}^2$$

$$= \sum_{i,j,k,s=1}^{n} a^{ij} a^{ks} (u_{ij})^2 \cdot \mathcal{C}^2 + \sum_{\substack{i,j,k,s=1 \\ (i,j)\neq(k,s)}}^{n} \left(a^{ij} a^{ks} u_i u_{ks} \cdot \mathcal{C}^2\right)_j - \sum_{\substack{i,j,k,s=1 \\ (i,j)\neq(k,s)}}^{n} a^{ij} a^{ks} u_i u_{ksj} \cdot \mathcal{C}^2$$

$$- \sum_{\substack{i,j,k,s=1 \\ (i,j)\neq(k,s)}}^{n} \left(a^{ij} a^{ks} \cdot \mathcal{C}^2\right)_j u_i u_{ks}.$$

Hence,

$$y_3 = \sum_{\substack{i,j,k,s=1 \\ (i,j)\neq(k,s)}}^{n} \left[\left(a^{ij} a^{ks} u_i u_{ks} \cdot \mathcal{C}^2\right)_j - \left(a^{ij} a^{ks} u_i u_{kj} \cdot \mathcal{C}^2\right)_s\right] - \sum_{\substack{i,j,k,s=1 \\ (i,j)\neq(k,s)}}^{n} \left(a^{ij} a^{ks} \cdot \mathcal{C}^2\right)_j \cdot u_i u_{ks}$$

$$+ \sum_{\substack{i,j,k,s=1 \\ (i,j)\neq(k,s)}}^{n} \left(a^{ij} a^{ks} \cdot \mathcal{C}^2\right)_s \cdot u_i u_{kj} + \sum_{i,j,k,s=1}^{n} a^{ij} a^{ks} u_{is} u_{kj} \cdot \mathcal{C}^2.$$

It was proven in the book of Ladyzhenskaya and Uraltceva [18] (Chapter 3, formula (7.6)) that

$$\sum_{i,j,k,s=1}^{n} a^{ij}a^{ks}u_{is}u_{kj} \geq \widetilde{\mu}_1^2 \sum_{i,j=1}^{n} u_{ij}^2.$$

Hence, applying the Cauchy Schwarz inequality with $\varepsilon = \widetilde{\mu}_1^2/2$, we obtain

$$y_3 \geq \frac{1}{2}\widetilde{\mu}_1^2 \sum_{i,j=1}^{n} u_{ij}^2 \cdot \mathcal{C}^2 - K\lambda^2 \left|\nabla u\right|^2 \cdot \mathcal{C}^2 + \nabla \cdot U_{42}, \tag{5.49}$$

where

$$\nabla \cdot U_{42} = \sum_{\substack{i,j,k,s=1 \\ (i,j)\neq(k,s)}}^{n} \left[\left(a^{ij}a^{ks}u_i u_{ks} \cdot \mathcal{C}^2\right)_j - \left(a^{ij}a^{ks}u_i u_{kj} \cdot \mathcal{C}^2\right)_s\right]. \tag{5.50}$$

To establish (5.40), we estimate from the below the boundary integral

$$\int\limits_{\partial_3 E} \left[\left(U_{42}, 0\right), \tilde{n}\right] dS, \text{ for } n \geq 2.$$

To do this, we estimate from the below integrals I_{ijks},

$$I_{ijks} = \int\limits_{\partial_3 E} a^{ij}a^{ks}u_i u_{ks} \cdot \mathcal{C}^2 \cos\left(\tilde{n}, x_j\right) dS, \text{ for } \ (i,j) \neq (k,s).$$

By (4.20) and (5.25), $I_{ijks} = 0$ for $i \geq 2$ and for $j \geq 2$. Hence, we have to evaluate only integrals I_{11ks},

$$I_{11ks} = \int\limits_{\partial_3 E} a^{ks}u_z u_{ks} \cdot \mathcal{C}^2 \cos\left(\tilde{n}, z\right) dS, \text{ for } \ (k,s) \neq (1,1).$$

Introduce the set E_{23} as

$$\begin{aligned} E_{23} &= \left\{(z,y,t) : |y| = r(t), t \in \left(0, \widetilde{t}_1\right)\right\} \\ &= \left\{(z,y,t) : z = \varphi(t), \varphi(t) + \sqrt{\delta}\left|y\right|^2 + t = \delta, \ t \in \left(0, \widetilde{t}_1\right)\right\}. \end{aligned}$$

Then $E_{23} = \partial_3 E \cap \partial_2 E$.

First, let both $k \geq 2$ and $s \geq 2$. Then (4.20) implies that $u_k = 0$ on $\partial_3 E$. Hence, on $\partial_3 E$

$$a^{ks}u_z u_{ks} \cdot \mathcal{C}^2 \cos\left(\tilde{n}, z\right)$$

$$= \frac{\partial}{\partial x_s}\left[\frac{a^{ks}u_z u_k \cdot \mathcal{C}^2}{\sqrt{1 + \left[\varphi'\left(t\right)\right]^2}}\right] - u_k \cdot \frac{\partial}{\partial x_s}\left(\frac{a^{ks}u_z \cdot \mathcal{C}^2}{\sqrt{1 + \left[\varphi'\left(t\right)\right]^2}}\right) = \frac{\partial}{\partial x_s}\left[\frac{a^{ks}u_z u_k \cdot \mathcal{C}^2}{\sqrt{1 + \left[\varphi'\left(t\right)\right]^2}}\right].$$

Since $E_{23} = \partial_3 E \cap \partial_2 E$, then $u_k = 0$ on E_{23}. Therefore (4.19) leads to $I_{11ks} = 0$, if both $k \geq 2$ and $s \geq 2$. Suppose now that $k = 1$ and $s \geq 2$. Then $u_{ks} = u_{zs}$. Hence, on $\partial_3 E$

$$
\begin{aligned}
a^{1s} u_z u_{zs} \cdot \mathcal{C}^2 \cos(\tilde{n}, z) &= \frac{\partial}{\partial x_s}\left[\frac{a^{1s} u_z^2 \cdot \mathcal{C}^2}{2\sqrt{1+[\varphi'(t)]^2}}\right] - \frac{\partial}{\partial x_s}\left(\frac{a^{1s} \cdot \mathcal{C}^2}{2\sqrt{1+[\varphi'(t)]^2}}\right) \cdot u_z^2 \\
&\geq -K\lambda u_z^2 \cdot \mathcal{C}^2 + \frac{\partial}{\partial x_s}\left[\frac{a^{1s} u_z^2 \cdot \mathcal{C}^2}{2\sqrt{1+[\varphi'(t)]^2}}\right].
\end{aligned}
$$

Thus, (4.19) leads to

$$
I_{111s} \geq -K\lambda \int\limits_{\partial_3 E} u_z^2 \cdot \mathcal{C}^2 dS - K \int\limits_0^{\tilde{t}_1} \int\limits_{|y|=r(t)} u_z^2 \cdot \mathcal{C}^2 d\sigma dt, \text{ for } s \geq 2.
$$

Summarizing, we see that

$$
I_{ijks} \geq -K\lambda \int\limits_{\partial_3 E} u_z^2 \cdot \mathcal{C}^2 dS - K \int\limits_0^{\tilde{t}_1} \int\limits_{|y|=r(t)} u_z^2 \cdot \mathcal{C}^2 d\sigma dt, \quad \text{for } (i,j) \neq (k,s).
$$

Therefore, (5.49) and (5.50) imply that

$$
y_3 \geq K \sum_{i,j=1}^{n} u_{ij}^2 \cdot \mathcal{C}^2 - K\lambda^2 |\nabla u|^2 \cdot \mathcal{C}^2 + \nabla \cdot U_{42}, \tag{5.51}
$$

where

$$
\int\limits_{\partial_3 E} [(U_{42}, 0), \tilde{n}] \, dS \geq -K\lambda \int\limits_{\partial_3 E} u_z^2 \cdot \mathcal{C}^2 dS - K \int\limits_0^{\tilde{t}_1} \int\limits_{|y|=r(t)} u_z^2 \cdot \mathcal{C}^2 d\sigma dt. \tag{5.52}
$$

Summing up estimates (5.48) and (5.51), we obtain the estimate (5.39) of this lemma. Also, estimates (5.47) and (5.52) imply the estimate (5.40) of the boundary integral with $U_4 := U_{41} + U_{42}$. Recalling that $u = \nabla u = u_t = 0$ on $\partial_1 E$, we conclude that (5.41) and (5.42) follow from (5.45), (5.46) and (5.50). ■

Completion of the Proof of Theorem 2.

Divide the estimate (5.39) of Lemma 5.4 by 2λ and add to the estimate (5.36) of Lemma 5.3. Then divide both sides of the resulting inequality by $(1 + 1/2\lambda)$, denote

$$
U = \frac{1}{(1+1/2\lambda)}\left(U_3 + \frac{1}{2\lambda} U_4\right), \quad V = \frac{1}{(1+1/2\lambda)}\left(V_3 + \frac{1}{2\lambda} V_4\right)
$$

and take into account estimates (5.37) and (5.38a,b) of Lemma 5.3, as well as estimates (5.40)-(5.42) of Lemma 5.4. Then we obtain estimates (5.1)-(5.4) of Theorem 2. ■

6 Completion of the Proof of Theorem 1

In the CWF $\mathcal{C}(z,y,t)=\exp(\lambda\psi^{-\nu})$ set $\nu:=\nu_0$ and $\lambda>\lambda_0$, where λ_0 and ν_0 are parameters of Theorem 2. Multiply both sides of the integro-differential inequality (3.29) by $\mathcal{C}(z,y,t)$. Then square both of them, apply the Cauchy-Schwarz inequality and integrate over the domain E with the parameter δ being chosen in (4.24). We obtain

$$\int\limits_E \left(a^0 w_t - L_0 w\right)^2 \cdot \mathcal{C}^2 dr$$

$$\leq M\int\limits_E \left(\left|\nabla w\right|^2+w^2\right)\cdot\mathcal{C}^2 dr + M\int\limits_E\left[\int\limits_{g(z)}^{t}\left[\left|\nabla w\right|+\left|w\right|\right](z,y,\tau)\,d\tau\right]^2\cdot\mathcal{C}^2 dr$$

$$+M\sum_{\substack{i,j=1\\(i,j)\neq(1,1)}}^{n}\int\limits_E\left[\int\limits_{g(z)}^{t}\left|w_{ij}\right|(z,y,\tau)\,d\tau\right]^2\cdot\mathcal{C}^2 dr,\quad dr:=dzdydt.$$

Applying Lemma 2.1, we obtain with a different constant M,

$$\int\limits_E \left(a^0 w_t - L_0 w\right)^2 \cdot \mathcal{C}^2 dr$$

$$\leq M\int\limits_E \left(\left|\nabla w\right|^2+w^2\right)\cdot\mathcal{C}^2 dr + \frac{M}{\lambda^2}\sum_{\substack{i,j=1\\(i,j)\neq(1,1)}}^{n}\int\limits_E \left(w_{ij}\right)^2\cdot\mathcal{C}^2 dr. \tag{6.1}$$

Since the function $w\in B$, then Theorem 2 enables us to estimate the left hand side of the inequality (6.1) from the below. Using Gauss' formula and taking into account that by (5.4) the boundary integral over $\partial_1 E$ equals zero, we obtain

$$\int\limits_E \left(a^0 w_t - L_0 w\right)^2 \cdot \mathcal{C}^2 dr \geq \frac{K}{\lambda}\int\limits_E\left[w_t^2+\sum_{i,j=1}^{n}\left(w_{ij}\right)^2\right]\cdot\mathcal{C}^2 dr + K\int\limits_E\left[\lambda\left|\nabla w\right|^2+\lambda^3 w^2\right]\cdot\mathcal{C}^2 dr$$

$$+K\lambda\int\limits_{\partial_3 E} w_z^2\cdot\mathcal{C}^2 dS - K\lambda^3\int\limits_{\partial_2 E}\left[w_t^2+\sum_{i,j=1}^{n}\left(w_{ij}\right)^2+\left|\nabla w\right|^2+\left|w\right|^2\right]\cdot\mathcal{C}^2 dS \tag{6.2}$$

$$-\frac{K}{\lambda}\int\limits_0^{\widetilde{t}_1}\int\limits_{|y|=r(t)} w_z^2\cdot\mathcal{C}^2 d\sigma dt.$$

Since $\partial_2 E=\left\{\psi=\delta+\frac{1}{2}\right\}$ is a level surface of the CWF $\mathcal{C}(z,y,t)$, then

$$\mathcal{C}^2(z,y,t)=\exp\left[2\lambda\left(\frac{1}{2}+\delta\right)^{-\nu}\right],\text{ on }\ \partial_2 E.$$

Since $\{(z,y,t):|y|=r(t),t\in(0,\widetilde{t}_1)\}=E_{23}=\partial_3 E\cap\partial_2 E_2$, then

$$-\frac{K}{\lambda}\int\limits_0^{\widetilde{t}_1}\int\limits_{|y|=r(t)} w_z^2\cdot\mathcal{C}^2 d\sigma dt=-\frac{K}{\lambda}\cdot\exp\left[2\lambda\left(\frac{1}{2}+\delta\right)^{-\nu}\right]\int\limits_0^{\widetilde{t}_1}\int\limits_{|y|=r(t)} w_z^2\cdot\mathcal{C}^2 d\sigma dt.$$

Because $\partial_3 E$ is not a level surface of the function $\mathcal{C}^2(z,y,t)$, it is important that the integral over $\partial_3 E$ in (6.2) is non-negative, which emphasizes the importance of the estimate (5.2) in Theorem 2. Dropping the integral over $\partial_3 E$, we make the estimate (6.2) stronger,

$$\int\limits_E \left(a^0\cdot w_t-L_0 w\right)^2\cdot\mathcal{C}^2 dr$$

$$\geq\frac{K}{\lambda}\int\limits_E\left[w_t^2+\sum_{i,j=1}^n (w_{ij})^2\right]\cdot\mathcal{C}^2 dr+K\int\limits_E\left[\lambda|\nabla w|^2+\lambda^3 w^2\right]\cdot\mathcal{C}^2 dr \tag{6.3}$$

$$-K\lambda^3\exp\left[2\lambda\left(\frac{1}{2}+\delta\right)^{-\nu}\right]\int\limits_{\partial_2 E}\left[w_t^2+\sum_{i,j=1}^n (w_{ij})^2+|\nabla w|^2+w^2\right]dS$$

$$-\frac{K}{\lambda}\cdot\exp\left[2\lambda\left(\frac{1}{2}+\delta\right)^{-\nu}\right]\cdot\int\limits_0^{\widetilde{t}_1}\int\limits_{|y|=r(t)} w_z^2\cdot\mathcal{C}^2 d\sigma dt.$$

Choose a sufficiently large $\lambda_1>\lambda_0$ such that

$$\frac{M}{\lambda_1}<\frac{K}{2}.$$

Comparing (6.3) with (6.1), we obtain for $\lambda>\lambda_1$

$$\frac{K}{\lambda}\int\limits_E\left[w_t^2+\sum_{i,j=1}^n w_{ij}^2\right]\cdot\mathcal{C}^2 dr+K\int\limits_E\left[\lambda|\nabla w|^2+\lambda^3 w^2\right]\cdot\mathcal{C}^2 dr$$

$$\leq 2K\lambda^3\exp\left[2\lambda\left(\frac{1}{2}+\delta\right)^{-\nu}\right]\cdot\int\limits_{\partial_2 E}\left[w_t^2+\sum_{i,j=1}^n w_{ij}^2+|\nabla w|^2+w^2\right]dS \tag{6.4}$$

$$+\frac{2K}{\lambda}\cdot\exp\left[2\lambda\left(\frac{1}{2}+\delta\right)^{-\nu}\right]\cdot\int\limits_0^{\widetilde{t}_1}\int\limits_{|y|=r(t)} w_z^2\cdot\mathcal{C}^2 d\sigma dt.$$

Choose an arbitrary $\varepsilon\in(0,\delta)$ and denote

$$E(\varepsilon)=\left\{(z,y,t)\in E:\psi(z,y,t)<\frac{1}{2}+\varepsilon\right\}.$$

Hence, $E_\varepsilon \subset E$ and

$$\mathcal{C}^2(z,y,t) \geq \exp\left[2\lambda\left(\frac{1}{2}+\varepsilon\right)^{-\nu}\right], \text{ in } E_\varepsilon.$$

Replacing in (6.4) integrals over E with integrals over E_ε and dropping integrals with derivatives, we obtain a stronger estimate,

$$\begin{aligned}
&\lambda^3 \exp\left[2\lambda\left(\frac{1}{2}+\varepsilon\right)^{-\nu}\right]\int\limits_{E_\varepsilon} w^2 dr \\
\leq\ & 2\lambda^3 \exp\left[2\lambda\left(\frac{1}{2}+\delta\right)^{-\nu}\right]\int\limits_{\partial_2 E}\left[w_t^2+\sum_{i,j=1}^n w_{ij}^2+|\nabla w|^2+w^2\right]dS \\
&+\frac{2}{\lambda}\cdot\exp\left[2\lambda\left(\frac{1}{2}+\delta\right)^{-\nu}\right]\cdot\int\limits_0^{\widetilde{t}_1}\int\limits_{|y|=r(t)} w_z^2\cdot\mathcal{C}^2 d\sigma dt.
\end{aligned}$$

Divide this inequality by $\exp\left[-2\lambda\left(\frac{1}{2}+\varepsilon\right)^{-\nu}\right]\cdot\lambda^3$ and note that

$$\lim_{\lambda\to\infty}\exp\left\{-2\lambda\left[\left(\frac{1}{2}+\varepsilon\right)^{-\nu}-\left(\frac{1}{2}+\delta\right)^{-\nu}\right]\right\}=0.$$

Thus, letting $\lambda\to\infty$, we obtain

$$\int\limits_{E_\varepsilon} w^2 dr = 0.$$

Since $\varepsilon\in(0,\delta)$ is an arbitrary number, then the function $w(z,y,t)=0$ in E.■

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