LOCAL EXISTENCE AND GEVREY REGULARITY OF 3-D PERIODIC NSE WITH ℓ_p INITIAL DATA.

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We obtain local existence and Gevrey regularity of 3-D periodic Navier-Stokes equations with ℓ_p $(p < 3/2)$ initial data.

1. Introduction

C. Foias and R. Temam introduced in [FT], a method for estimating the space-analyticity radius of solutions of Navier-Stokes equations (henceforth abbreviated as NSE) with space-periodic boundary condition. The basic idea of interpolating between a suitably defined analyticity (Gevrey) norm and a Sobolev norm leads to a very simple energy method which eliminates the need of traditional estimates on higher order derivatives (for example, as in [M]). This also provides an explicit estimate of the radius of analyticity in terms of the size of the initial data and the forcing term. Subsequently, in [GK], estimate of the analyticity radius was obtained for L^p initial data. Unlike in [FT], instead of estimating the Gevrey norm directly, in [GK] the authors achieve the relevent estimates on the analyticity radius by interpolating between the L^p norm of the initial data and the L^p norm of the complexified solution.

Here, we consider the 3-D Navier-Stokes equations with space periodic boundary condition. In this set-up, the NSE can be reformulated in terms of its Fourier coefficients. The resulting system can be regarded as a nonlinear evolution equation in appropriate sequence space. This is the so-called wave-vectors formulation of NSE (see [F] for a detailed exposition). In [F], by employing only elementary functional analytic techniques, which completely bypasses Sobolev inequalities, it is proven that if the initial data has finite ℓ_1 norm, then there exists a local in time solution of the 3D-NSE with bounded Gevrey norm. A treatment of the 2-D case, in wave-vectors formulation and involving the ℓ_p norms, can be found in [MS]. However, their assumption on the initial data is much more restrictive.

Motivated by the approach in [FK] and [GM], in Section 2, we first obtain a local in time solution which is bounded in ℓ_p $(p < 3/2)$ norm on the Fourier coefficients, in case the intial data also belongs to ℓ_p $(p < 3/2)$. We also show that this solution is regular. Subsequently, in Section 3, we obtain local in time solution, the Gevrey norm of which is bounded. This provides an alternative approach to [GK] and is more in the spirit of the results in [FT] and [F]. The treatment here is essentially self-contained and elementary, completely avoiding the use of Sobolev inequalities.

2. Notation and Preliminaries

We consider the Navier-Stokes equations of viscous incompressible fluids in $\Omega = [0, L]^3$ with space periodic boundary condition:

$$\frac{\partial u}{\partial t}\mathbf{u}(\mathbf{x}, \mathbf{t}) - \nu \boldsymbol{\Delta} \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \mathbf{p} = \mathbf{f} \quad (2.1)$$

$$\nabla \mathbf{u} = 0 \quad (2.2)$$

The unknown (real-valued) functions are the vector-valued velocity function $\mathbf{u}(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ and the scalar-valued pressure $p = p(x, t),\ x \in \mathbb{R}^3, t \geq 0$. The volume force $\mathbf{f}(x, t) = (f_1(x, t), f_2(x, t), f_3(x, t))$ is given and $\nu > 0$ is the kinematic viscosity. For notational simplicity, henceforth, we will set $\nu \equiv 1$. We assume that $\mathbf{f}, \mathbf{u}, p$ are periodic in space variables with period L. For a $L-$periodic complex-valued scalar or vector function ϕ which is integrable over Ω, we define its Fourier coefficients by

$$\phi(k) = \frac{1}{L^3}\int_\Omega e^{-\frac{2\pi\imath}{L}k\cdot x}\phi(x)dx, \qquad (k \in \mathbb{Z}^3).$$

and its corresponding Fourier series is defined by

$$\sum_{k\in\mathbb{Z}^3} \phi(k)e^{-\frac{2\pi\imath}{L}k\cdot x}.$$

If ϕ, ψ are two complex vector functions, square-integrable on Ω, Parseval's identity says that

$$\frac{1}{L^3}\int_\Omega \phi(x)\cdot\psi(x)^*dx = \sum_{k\in\mathbb{Z}^3}\phi(k)\cdot\psi(k)^*,$$

where for $\mathbf{a}, \mathbf{b} \in \mathbb{C}^3$

$$b^* = (\bar{b}_1, \bar{b}_2, \bar{b}_3),\ \mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

For the remainder of the paper, for notational simplicity, we set the kinematic viscosity $\nu \equiv 1$. Also, for scalar or vector valued function $\phi = \phi(x, t) : \mathbb{R}^3 \times [0, T] \to \mathbb{C}^n,\ T \leq \infty, n \in \mathbb{N}$, $L-$periodic in the space variable, we denote by $(\phi(k, t))_{k\in\mathbb{Z}^3}$ the sequence of Fourier coefficients of the function $\phi(\cdot, t)$.

Rewriting (2.1) and (2.2) in terms of its Fourier coefficient as is done in [F], one obtains the so-called wave-vectors formulation of Navier-Stokes equation as follows:

$$\frac{d}{dt}\mathbf{u}(k, t) = \mathbf{f}(k, t) - \frac{2\pi\imath}{L}kp(k, t) - \left(\frac{2\pi}{L}\right)^2|k|^2\vec{u}(k, t) - Q[\mathbf{u}, \mathbf{v}](k, t) \quad (2.3)$$

$$k\cdot\mathbf{u}(k, t) = 0 \quad (k \in \mathbb{Z}^3), \quad (2.4)$$

where, for two $\mathbb{C}^3$ or $\mathbb{R}^3$-valued sequences $(\mathbf{u}(k))_{k\in\mathbb{Z}^3}, (\mathbf{v}(k))_{k\in\mathbb{Z}^3}$,

$$Q[\mathbf{u},\mathbf{v}](k) = \sum_{h\in\mathbb{Z}^3}(k\cdot\mathbf{u}(k))\mathbf{v}(k-h).$$

Note that if the sequences $(\mathbf{u}(k))_k \in \mathbb{Z}^3$ and $(\mathbf{v}(k))_k \in \mathbb{Z}^3$ are square- summable, $Q[\mathbf{u},\mathbf{v}](k)$ is well defined for each $k\in\mathbb{Z}^3$. Since the functions $\mathbf{u},\mathbf{f},p$ are all real, we also have

$$\mathbf{u}(-k,t) = \mathbf{u}(k,t)^*,\ p(-k,t) = \bar{p}(k,t),\ \mathbf{f}(-k,t) = \mathbf{f}(k,t)^*,\quad (k\in\mathbb{Z}^3, t\geq 0).$$

Moreover, without loss of generality (see [T]), we may also assume

$$\mathbf{u}(0,t) = \mathbf{f}(0,t) = \mathbf{0},\ p(0,t) = 0,\quad (k\in\mathbb{Z}^3).$$

Using (2.4) and taking dot-product with k on both sides of (2.3), one readily obtains

$$\frac{2\pi\imath}{L}p(k,t) = [k\cdot\mathbf{f}(k,t) - k\cdot Q[\mathbf{u},\mathbf{u}](k,t)]/|k|^2.$$

Reintroducing this in (2.3) we obtain

$$\begin{aligned}\frac{d}{dt}\mathbf{u}(k,t) &= \mathbf{g}(k,t) - \left(\frac{2\pi}{L}\right)^2|k|^2\mathbf{u}(k,t) - B[\mathbf{u},\mathbf{v}](k,t)\\ k\cdot\mathbf{u}(k,t) &= 0 \quad (k\in\mathbb{Z}^3)\end{aligned}$$

where

$$B[\mathbf{u},\mathbf{v}](k,t) = Q[\mathbf{u},\mathbf{u}](k,t) - \frac{k(k\cdot Q[\mathbf{u},\mathbf{u}](k,t))}{|k|^2},\ \mathbf{g}(k,t) = \mathbf{f}(k,t) - \frac{k(k\cdot\mathbf{f}(k,t))}{|k|^2}.$$

In view of the above discussion, following the treatment in [F], one may thus obtain infinite dimensional ODE formulation of Navier-Stokes equations in sequence space. We describe the set-up below in detail.

Let

$$\mathcal{K} = \{\vec{v} = \{(\vec{v}(k))_{k\in\mathbb{Z}^3} : \vec{v}(k)\in\mathbb{C}^3,\ \vec{v}(0) = 0, \vec{v}(-k) = \vec{v}(k)^*,\ k\cdot\vec{v}(k) = 0\}.$$

The vector space space K, when endowed with the distance

$$d(\vec{v}_1,\vec{v}_2) = \sum_{k\in\mathbb{Z}^3}\frac{|\vec{v}_1(k)-\vec{v}_2(k)|}{1+|\vec{v}_1(k)-\vec{v}_2(k)|}2^{-|k|^2}$$

is a Frechet space. For $\alpha \geq 0$ and $p \geq 1$ define

$$V_{\alpha,p} = \{\vec{u} \in K : \|\vec{u}\|_{\alpha,p} := \left(\sum_{k\in\mathbb{Z}^3} [|k|^\alpha |u(k)|]^p\right)^{1/p} < \infty\}.$$

Clearly, $V_{\alpha_1,p} \subset V_{\alpha_2,p}$ if $\alpha_1 \geq \alpha_2$ and in this case,

$$\|\vec{u}\|_{\alpha_2,p} \leq \|\vec{u}\|_{\alpha_1,p} \quad (\vec{u} \in V_{\alpha_1,p}). \tag{2.5}$$

In case $\alpha = 0$, for notational simplicity, we will refer to $V_{0,p}$ as V_p and the corresponding norm $\|\cdot\|_{0,p}$ is denoted by $\|\cdot\|_p$. For $\vec{u}, \vec{v} \in V_p$ define $Q[\vec{u}, \vec{v}]$ and $B[\vec{u}, \vec{v}]$ by

$$Q[\vec{u}, \vec{v}](k) = \sum_{h\in\mathbb{Z}^3} (k \cdot u(k))v(k-h), \quad B[\vec{u}, \vec{v}](k) = Q[\vec{u}, \vec{v}](k) - \frac{k \cdot Q[\vec{u}, \vec{v}](k)}{|k|^2}k. \tag{2.6}$$

We will first state here Young's inequality for convolution. For $\vec{u}, \vec{v} \in \ell_p(\mathbb{Z}^3)$ and $p \leq 2$, Young's inequality implies that the convolution $\vec{w} = \vec{u} * \vec{v}$

$$w(k) = \sum_{h\in\mathbb{Z}^3} u(h)v(k-h), \;\; \vec{w} \in \ell_r(\mathbb{Z}^3), r = \frac{p}{2-p}, \;\; \|\vec{w}\|_r \leq \|\vec{u}\|_p \|\vec{v}\|_p. \tag{2.7}$$

Note that by Young's inequality, for all $\vec{u}, \vec{v} \in V_p$, $Q[\vec{u}, \vec{v}](k)$ is well-defined for each $k \in \mathbb{Z}^3$. Moreover, it can be easily checked that

$$|B[\vec{u}, \vec{v}](k)| \leq |Q[\vec{u}, \vec{v}](k)| \qquad (k \in \mathbb{Z}^3) \tag{2.8}$$

and $B[\vec{u}, \vec{v}]$ is in K.

Let A be the positive, unbounded, densely defined operator on V_p given by

$$A\vec{u} = \left((\frac{2\pi}{L})^2 |k|^2 u(k)\right)_{k\in\mathbb{Z}^3}, \quad (\vec{u} \in V_p).$$

We note here that for any $\vec{v} \in V_{\alpha+2\delta,p}$ for some $\alpha \geq 0$ and $\delta \geq 0$

$$\|A^\delta \vec{v}\|_{\alpha,p} = (\frac{2\pi}{L})^{2\delta} \|\vec{v}\|_{\alpha+2\delta,p}. \tag{2.9}$$

For $T \leq \infty$, denote

$$L^q([0,T]; V_{\alpha,p}) = \{\vec{v} : [0,T] \to V_{\alpha,p} : \int_0^T \|\vec{v}(s)\|_{\alpha,p} ds < \infty\}$$

and

$$L^\infty([0,T]; V_{\alpha,p}) = \{\vec{v} : [0,T] \to V_{\alpha,p} : \sup_{0\leq s\leq T} \|\vec{v}(s)\|_{\alpha,p} < \infty.\}$$

We will denote the $k-th$ "co-ordinate" of $\vec{v} \in L^q([0,T];V_{\alpha,p}$ by $\vec{v}(k,\cdot)$. Moreover, $C([0,T];X),\ X = V_{\alpha,p}$ or K denotes the set of all $X-$valued continuous function on $[0,T]$ where the continuity is with respect to the norm topology if $X = V_{\alpha,p}$ and with respect to the metric defined before if $X = K$.

Definition Given $\vec{u}_0 \in V_p, p \leq 2$ and $\vec{g}(\cdot) \in L^1([0,\infty);V_p)$, a function $\vec{u}(\cdot)$ is said to be a weak solution of the Navier-Stokes initial value problem if it is in $C([0,T],K) \cap L^\infty([0,T];V_p)$ and satisfies

$$\begin{aligned} \frac{d}{dt}\vec{u}(k,t) &= \vec{g}(k,t) - \left(\frac{2\pi}{L}\right)^2 |k|^2\vec{u}(k,t) - B[\vec{u},\vec{u}](k,t), \\ \vec{u}(k,0) &= \vec{u}_0(k), \qquad (t>0, k \in \mathbb{Z}^3). \end{aligned} \tag{2.10}$$

Definition We say that $\vec{u}$ is a Leray-strong solution of the Navier-Stokes initial value problem if it is a weak solution which moreover satisfies

$$\vec{u}(\cdot) \in L^\infty([0,T];V_{1,p}). \tag{2.11}$$

3. Local Existence and Regularity

Let $\vec{g} \in L^1([0,\infty);V_p)$. For $\vec{u}_0 \in V_p$ define

$$\vec{G}(t) = e^{-tA}\vec{u}_0 + \int_0^t e^{-(t-s)A}\vec{g}(s)ds. \tag{3.1}$$

Remark 1 Since e^{-tA} is a contraction semigroup on V_p we have

$$\|\vec{G}(t)\| \leq \|\vec{u}_0\|_p + \int_0^\infty \|\vec{g}(s)\|_p := M \quad (t \geq 0). \tag{3.2}$$

We denote by $C([0,T];V_p)$ the set of all V_p- valued continuous function on $[0,T]$. It is a Banach space when equipped with the sup norm. In other words, for $\vec{u} \in C([0,T];V_p)$, we denote

$$\|\vec{u}\| = \sup_{0\leq t\leq T} \|\vec{u}(t)\|_p.$$

The assumption $\vec{g} \in L^1([0,\infty);V_p)$ and the fact that e^{-tA} is a contractive semigroup on V_p implies that G belongs to $C([0,T];V_p)$. Define

$$E = \{\vec{v} \in C([0,T];V_p) : \sup_{[0,T]} \|v(t) - \vec{G}(t)\|_p \leq M\}. \tag{3.3}$$

Clearly,

$$\sup_{0\leq t\leq T} \|v(t)\| \leq 2M \text{ for all } \vec{v} \in E. \tag{3.4}$$

Let $S : E \to C([0,T]; V_p)$ be the map defined by the formula

$$(S\vec{v})(t) = \vec{G}(t) - \int_0^t e^{-(t-s)A} B[\vec{v}(s), \vec{v}(s)]ds, \qquad (\vec{v} \in E). \tag{3.5}$$

We will show below that S maps E to E. We first state an elementary inequality which would be used repeatedly. For $a > 0, b > 0$, we have

$$f(\lambda) = \lambda^a e^{-b\lambda} \le \left(\frac{a}{e}\right)^a \frac{1}{b^a} \text{ for all } \lambda > 0. \tag{3.6}$$

Lemma 3.1. *Assume that $\vec{u}, \vec{v} \in V_{\alpha,p}$ for some $\alpha \ge 0$ and let $\eta > 0$. Then, for $p \le 2$ and $\beta > 0$ satisfying $\beta p > 3(p-1)$, there exists a constant $C = C(p, \alpha, \beta)$ such that*

$$\|e^{-\eta A} B[\vec{u}, \vec{v}]\|_{\alpha,p} \le C \frac{1}{\eta^{\frac{(\beta+1)}{2}}} \|\vec{u}\|_{\alpha,p} \|\vec{v}\|_{\alpha,p} \tag{3.7}$$

Proof. Recalling (2.8), we have

$$\begin{aligned}
&\|e^{-\eta A} B[\vec{u}, \vec{v}]\|_{\alpha,p}^p = \\
&\quad \sum_k |k|^{\alpha p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} |B[\vec{u}, \vec{v}](k)|^p \le \sum_k |k|^{\alpha p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} |Q[\vec{u}, \vec{v}](k)|^p = \\
&\quad \sum_k |k|^{\alpha p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} |\sum_h (k \cdot u(h)) v(k-h)|^p \\
&\quad \le \sum_k |k|^{\alpha p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} |\sum_h |k||u(h)||v(k-h)||^p \\
&\quad = \sum_k |k|^{p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} ||k|^\alpha \sum_h |u(h)||v(k-h)||^p \\
&\quad \le C \sum_k |k|^{p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} |\sum_h (|h|^\alpha + |k-h|^\alpha)|u(h)||v(k-h)||^p \qquad (3.8) \\
&\quad = C(\sum_k |k|^{p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} |\sum_h |h|^\alpha |u(h)||v(k-h)||^p \\
&\qquad\qquad + \sum_k |k|^{p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} |\sum_h |u(h)||k-h|^\alpha |v(k-h)||^p). \qquad (3.9)
\end{aligned}$$

To obtain (3.8)above, we used the inequality $(a+b)^\alpha \le C(a^\alpha + b^\alpha)$ where the constant C may depend only on α.

Let $w_1(k) = \sum_h |h|^\alpha |u(h)||v(k-h)|$ and $\vec{w_1} = (w_1(k))_{k\in\mathbb{Z}^3}$. By Young's inequality for convolution, we have

$$\|\vec{w_1}\|_r \le \|\vec{u}\|_{\alpha,p}\|\vec{v}\|_p \le \|\vec{u}\|_{\alpha,p}\|\vec{v}\|_{\alpha,p}, \qquad r = \frac{p}{2-p}. \tag{3.10}$$

The first term in the inequality in (3.9) is

$$\begin{aligned}\sum_k |k|^p e^{-p\eta(\frac{2\pi}{L})^2|k|^2}|\sum_h |h|^\alpha |u(h)||v(k-h)||^p &= \sum_k |k|^p|k|^{\beta p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2}\frac{|w_1(k)|^p}{|k|^{\beta p}} \\ \le C\frac{1}{\eta^{\frac{(\beta+1)p}{2}}}\sum_k \frac{|w_1(k)|^p}{|k|^{\beta p}} &\le C\frac{1}{\eta^{\frac{(\beta+1)p}{2}}}\|\vec{w_1}\|_r^p\left(\sum_k \frac{1}{|k|^{\frac{\beta pr}{r-p}}}\right)^{\frac{r-p}{r}} \\ &\le C\frac{1}{\eta^{\frac{(\beta+1)p}{2}}}\|\vec{u}\|_{\alpha,p}^p\|\vec{v}\|_{\alpha,p}^p \end{aligned} \tag{3.11}$$

where, to obtain the inequalities in the second last line, we first used (3.6) and then applied Holder's inequality. To obtain the last inequality, we used (3.10) and the fact that $\sum_k \frac{1}{|k|^{\frac{\beta pr}{r-p}}} < \infty$ if $\frac{\beta pr}{r-p} > 3$, which is indeed the case since $r = \frac{p}{2-p}$ and $\beta p > 3(p-1)$. A similar computation shows that for the second term in (3.9) we have

$$\sum_k |k|^p e^{-p\eta(\frac{2\pi}{L})^2|k|^2}|\sum_h |u(h)||k-h|^\alpha|v(k-h)||^p \le C\frac{1}{\eta^{\frac{(\beta+1)p}{2}}}\|\vec{u}\|_{\alpha,p}^p\|\vec{v}\|_{\alpha,p}^p \tag{3.12}$$

Putting together the inequalities (3.9), (3.11)and (3.12), the proof of the lemma is now complete.

□

In order to bootstrap higher order regularity, we will also need the following estimate on the bilinear term.

Lemma 3.2. *Assume that $\vec{u},\vec{v} \in V_{\alpha,p}$ for some $\alpha \ge 0$ and let $\eta > 0$. Then, for $p \le 2$ and $\beta > 0$ satisfying $\beta p > 3(p-1)$, there exists a constant $C = C(p,\alpha,\beta)$ such that*

$$\|e^{-\eta A}B[\vec{u},\vec{v}]\|_{\alpha,p} \le C\frac{1}{\eta^{\frac{(\alpha+\beta+1)}{2}}}\|\vec{u}\|_p\|\vec{v}\|_p \tag{3.13}$$

Proof. Let $w_2(k) = \sum_h |u(h)||v(k-h)|$ and $\vec{w_2} = (w_2(k))_{k\in\mathbb{Z}^3}$. By Young's inequality for convolution, we have

$$\|\vec{w_2}\|_r \le \|\vec{u}\|_p\|\vec{v}\|_p, \qquad r = \frac{p}{2-p}. \tag{3.14}$$

Proceeding as in the proof of the previous lemma, we have

$$\|e^{-\eta A}B[\vec{u},\vec{v}]\|_{\alpha,p}^p =$$
$$\sum_k |k|^{\alpha p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2}|B[\vec{u},\vec{v}](k)|^p \le \sum_k |k|^{\alpha p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2}|Q[\vec{u},\vec{v}](k)|^p =$$
$$\sum_k |k|^{\alpha p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2}|\sum_h (k\cdot u(h))v(k-h)|^p$$
$$\le \sum_k |k|^{\alpha p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2}|\sum_h |k||u(h)||v(k-h)||^p$$
$$= \sum_k |k|^{(\alpha+1)p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2}|\sum_h |u(h)||v(k-h)||^p$$
$$= \sum_k |k|^{(\alpha+1+\beta)p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2}\frac{|w_2(k)|^p}{|k|^{\beta p}} \le C\frac{1}{\eta^{\frac{(\alpha+1+\beta)}{2}p}}\sum_k \frac{|w_2(k)|^p}{|k|^{\beta p}}$$
$$\le C\frac{1}{\eta^{\frac{(\alpha+1+\beta)p}{2}}}\|\vec{w}_2\|_r^p\left(\sum_k \frac{1}{|k|^{\frac{\beta pr}{r-p}}}\right)^{\frac{r-p}{r}} \le C\frac{1}{\eta^{\frac{(\alpha+1+\beta)p}{2}}}\|\vec{u}\|_p^p\|\vec{v}\|_p^p,$$

where, to obtain the inequality in the second last line, we used (3.6) and to obtain the inequalities in the last line, we first used Holder's inequality and then (3.14). □

We will now prove that the map S defined in (3.5) takes E into $C([0,T];V_p)$. Note first that if $p < 3/2$ then there exists $\frac{3(p-1)}{p} < \beta < 1$.

Lemma 3.3. *Assume that* $\frac{3(p-1)}{p} < \beta < 1$*. For* $\vec{v} \in E$*, we have* $S\vec{v}$ *is in* $C([0,T]:V_p)$ *and*

$$\|(S\vec{v}-G)(t)\|_p \le \frac{4C}{1-(\beta+1)/2}MT^{1-(\beta+1)/2}M. \tag{3.15}$$

Proof. For $\vec{v} \in E$ and $t < T$ using Lemma 3.1 we have

$$\|(S\vec{v}-G)(t)\|_p =$$
$$\|\int_0^t e^{-(t-s)A}B[\vec{v}(s),\vec{v}(s)]ds\|_p \le \int_0^t \|e^{-(t-s)A}B[\vec{v}(s),\vec{v}(s)]\|_p ds \le$$
$$C\int_0^t \frac{1}{(t-s)^{(\beta+1)/2}}\|\vec{v}(s)\|_p^2 \le \frac{4C}{1-(\beta+1)/2}MT^{1-(\beta+1)/2}M$$

where in the last line we used (3.4). This establishes that $S\vec{v}$ is in $L^\infty([0,T];V_p)$. In fact, the above calculations also show that $S\vec{v}$ belongs to $C([0,T];V_p)$. □

Lemma 3.4. *Assume that* $\dfrac{3(p-1)}{p} < \beta < 1$. *For* $\vec{v}, \vec{w} \in E$, *we have*

$$\|(S\vec{v} - S\vec{w})(t)\|_p \leq \frac{2C}{1-(\beta+1)/2} M T^{1-(\beta+1)/2} \sup_{0\leq t\leq T} \|(\vec{v}-\vec{w})(s)\|_p. \tag{3.16}$$

Proof. Note that by linearity,

$$B[\vec{v},\vec{v}] - B[\vec{w},\vec{w}] = B[\vec{v},\vec{v}-\vec{w}] + B[\vec{v}-\vec{w},\vec{w}]. \tag{3.17}$$

For $0 \leq t \leq T$, using (3.17) and Lemma 3.1, we have

$$\begin{aligned}
&\|(S\vec{v} - S\vec{w})(t)\|_p \leq \\
&\quad \int_0^t \|e^{-(t-s)A} B[\vec{v}(s), (\vec{v}-\vec{w})(s)] ds\|_p + \int_0^t \|e^{-(t-s)A} B[(\vec{v}-\vec{w})(s), \vec{v}(s)]\|_p ds \\
&\quad \leq 4C \int_0^t \frac{1}{(t-s)^{(\beta+1)/2}} \|(\vec{v}-\vec{w})(s)\|_p \|\vec{v}(s)\|_p \\
&\quad \leq \frac{4C}{1-(\beta+1)/2} M T^{1-(\beta+1)/2} \sup_{0\leq t\leq T} \|(\vec{v}-\vec{w})(s)\|_p \ ,
\end{aligned}$$

□

where in the last inequality, we used (3.4).

We are now ready to state the main theorem. For a function $\vec{v} : [0,T] \to V_{\alpha,p}$, for each $k \in \mathbb{Z}^3$, we denote by $\vec{v}(k,t)$ the $k-$th component of $\vec{v}(t)$.

Theorem 3.1. *Let* $1 \leq p < 3/2$, $T < \dfrac{C}{M^{4p/(3-2p)}}$ *and* M *as in (3.2). Then, there exists* $\vec{u}$ *in* $C([0,T]; V_p)$ *with* $\sup\limits_{0\leq t\leq T} \|\vec{u}(s)\| < 2M$ *satisfies (2.10). The constant* C *may depend on* p *only as* $p \to 3/2$.

Proof. Recall that $C([0,T]; V_p)$ is a Banach space with respect to the norm $\|\vec{u}(\cdot)\| = \sup\limits_{0\leq t\leq T} \|\vec{u}(t)\|_p$. Set

$$\beta_0 = (\frac{3(p-1)}{p} + 1)/2 = \frac{4p-3}{2p}. \tag{3.18}$$

Then, β_0 satisfies the condition in (3.15). Moreover, if $T < \frac{C}{M^{4p/(3-2p)}}$ for appropriate C, then by Lemma 3.3 and Lemma 3.4, the map S defined in (3.5) is a contractive map from E into E. Thus, by Banach fixed point theorem, there exists a $\vec{u}$ in $E \subset C([0,T]; V_p)$ such that

$$\vec{u}(t) = e^{-tA}\vec{u}_0 + \int_0^t e^{-(t-s)A}\vec{g}(s)ds - \int_0^t e^{-(t-s)A} B[\vec{u}(s), \vec{u}(s)]ds. \tag{3.19}$$

From (3.19) and the definition of the operator A, it follows that for all $k \in \mathbb{Z}^3$, $\vec{u}(\cdot)$ satisfies

$$\begin{aligned}\vec{u}(k,t) &= e^{-t(\frac{2\pi}{L})^2|k|^2}\vec{u}_0(k) + \int_0^t e^{-(t-s)(\frac{2\pi}{L})^2|k|^2}\vec{g}(k,s)\,ds \\ &\quad - \int_0^t e^{-(t-s)(\frac{2\pi}{L})^2|k|^2}B[\vec{u}(s),\vec{u}(s)](k)ds. \qquad (3.20)\end{aligned}$$

Note that for any vectors $\vec{w},\vec{v} \in V_p,\ p \le 2$ we have

$$\begin{aligned}|B[\vec{w},\vec{v}](k)| \le |Q[\vec{w},\vec{v}](k)| &\le |k|\sum_{h\in\mathbb{Z}^3}|\vec{w}(h)||\vec{v}(k-h)| \\ \le |k|(\sum_{h\in\mathbb{Z}^3}|\vec{w}(h)|^2)^{1/2}(\sum_{k\in\mathbb{Z}^3}|\vec{v}(k)|^2)^{1/2} &\le |k|\|\vec{w}\|_p\|\vec{v}\|_p \qquad (3.21)\end{aligned}$$

where to obtain the first inequality in (3.21), we used Cauchy-Schwartz and the last inequality in (3.21) follows from the fact that for $1 \le p \le q$ and $\vec{v} \in V_p,\ \|\vec{v}\|_q \le \|\vec{v}\|_p$. Since $\vec{u}(\cdot)$ is in $C([0,T];V_p)$, from (3.21) it immediately follows that for every $k \in \mathbb{Z}^3$, the map $s \to B[\vec{u}(s),\vec{u}(s)](k)$ is continuous on $[0,T]$. Thus we may differentiate under the integral sign in (3.20) to conclude that $\vec{u}$ satisfies (2.10).

□

Remark If $p = 1$, then β can be taken to be zero in Lemma 3.1 and hence in Lemma 3.3 and Lemma 3.4. In this case, there exists a solution $\vec{u}$ as in (3.19) for $T < \dfrac{C}{M^2}$. This is precisely the result in [F].

We now proceed to show that the mild solution $\vec{u}$ obtained in Theorem 3.1 is indeed a strong solution. We first state a lemma.

Lemma 3.5. *Let $\vec{u} \in V_{\alpha,p}$. Then, there exists a constant $C = C(\alpha,\delta,p)$ depending only on α,δ and p such that*

$$\|A^\delta e^{-\eta A}\vec{u}\|_{\alpha,p} \le C\frac{1}{\eta^\delta}\|\vec{u}\|_{\alpha,p} \qquad (3.22)$$

Proof. The proof is a straight forward calculation as shown below.

$$\begin{aligned}&\|A^\delta e^{-\eta A}\vec{u}\|^p_{\alpha,p} = \\ \sum_k |k|^{\alpha p}(\frac{2\pi}{L}|k|)^{2\delta p}e^{-\eta p(\frac{2\pi}{L})^2|k|^2}|u(k)|^p &\le C\frac{1}{\eta^{\delta p}}\sum_k |k|^{\alpha p}|u(k)|^p \\ &= C\frac{1}{\eta^{\delta p}}\|\vec{u}\|^p_{\alpha,p},\end{aligned}$$

where to obtain the inequality above we used (3.6). $\square$

Theorem 3.2. *Assume that the force $\vec{g}$ satisfies*

$$\sup_{0\le t\le T} \|\vec{g}(t)\|_{\alpha_0,p} < \infty \tag{3.23}$$

and the initial data u_0 is in V_p, $1 \le p < 3/2$. The mild solution obtained in (3.19) $\vec{u} = \vec{u}(t)$ is in fact a strong solution satisfying

$$\sup_{0\le t\le T} t^{\alpha/2}\|\vec{u}(t)\|_{\alpha,p} < \infty \tag{3.24}$$

for all $0 \le \alpha < \alpha_0 + 2\delta$ where $0 \le \delta < 1$.

Proof. Using (2.9) and (3.22), for all $\alpha \ge 0$ we have

$$\sup_{0\le t\le T} t^{\alpha/2}\|e^{-tA}\vec{u}_0\|_{\alpha,p} = (\frac{2\pi}{L})^{2\delta} \sup_{0\le t\le T} t^{\alpha/2}\|A^{\alpha/2}e^{-tA}\vec{u}_0\|_p \le C\|\vec{u}_0\|_p. \tag{3.25}$$

Furthermore, once again using (2.9) and (3.22), for $0 \le \delta < 1$ and $0 \le \alpha \le \alpha_0$, we have

$$\begin{aligned}&\|\int_0^t e^{-(t-s)A}\vec{g}(s)ds\|_{\alpha+2\delta,p} = \\ &(\frac{2\pi}{L})^{2\delta}\|A^\delta\int_0^t e^{-(t-s)A}\vec{g}(s)ds\|_{\alpha,p} \le (\frac{2\pi}{L})^{2\delta}\int_0^T \|A^\delta e^{-(t-s)A}\vec{g}(s)\|_{\alpha,p}ds \le \\ &C\int_0^T \frac{1}{(t-s)^\delta}\|\vec{g}(s)\|_{\alpha,p}ds \le C\sup_{0\le t\le T}\|\vec{g}(t)\|_{\alpha,p}\int_0^T\frac{1}{(t-s)^\delta}ds < \infty\end{aligned}$$

In the above inequalities, we implicitly used the fact that $\sup_{0\le t\le T}\|\vec{g}(t)\|_{\alpha,p} \le \sup_{0\le t\le T}\|\vec{g}(t)\|_{\alpha_0,p} < \infty$ for all $\alpha \le \alpha_0$. In view of the above estimate and (3.25), for all $0 \le \alpha \le \alpha_0$ and $0 \le \delta < 1$ we have

$$\sup_{0\le t\le T} t^{\frac{\alpha}{2}+\delta}\|\vec{G}(t)\|_{\alpha+2\delta,p} < \infty. \tag{3.26}$$

We will now bootstrap regularity from a lower order bound. Assume (3.24) holds for some $\alpha \ge 0$. Set $\delta_0 = \frac{1-(\beta_0+1)/2}{2}$ where β_0 is as in (3.18). Clearly, with this choice of δ_0, we have $(\beta_0+1)/2+\delta_0<1$. Note first that

$$\|A^{\delta_0}(S\vec{u}-G)(t)\|_{\alpha,p} =$$

$$\|A^{\delta_0}\int_0^t e^{-(t-s)A}B[\vec{u}(s),\vec{v}(s)]ds\|_{\alpha,p} \le \int_0^t \|A^{\delta_0}e^{-(t-s)A}B[\vec{u}(s),\vec{u}(s)]\|_{\alpha,p}ds$$

$$= \int_0^t \|A^{\delta_0}e^{-\frac{1}{2}(t-s)A}e^{-\frac{1}{2}(t-s)A}B[\vec{u}(s),\vec{u}(s)]\|_{\alpha,p}ds$$

$$\le C\int_0^t \frac{1}{(t-s)^{\delta_0}}\|e^{-\frac{1}{2}(t-s)A}B[\vec{u}(s),\vec{u}(s)]\|_{\alpha,p}ds \tag{3.27}$$

$$= C(\int_0^\epsilon \frac{1}{(t-s)^{\delta_0}}\|e^{-\frac{1}{2}(t-s)A}B[\vec{u}(s),\vec{u}(s)]\|_{\alpha,p}ds + \int_\epsilon^t \frac{1}{(t-s)^{\delta_0}}\|e^{-\frac{1}{2}(t-s)A}B[\vec{u}(s),\vec{u}(s)]\|_{\alpha,p}ds)$$

$$\le C(\int_0^\epsilon \frac{1}{(t-s)^{\delta_0+(\beta_0+\alpha+1)/2}}\|\vec{u}(s)\|_p^2ds + \int_\epsilon^t \frac{1}{(t-s)^{\delta_0+(\beta_0+1)/2}}\|\vec{u}(s)\|_{\alpha,p}^2ds) \tag{3.28}$$

where to obtain (3.27) we used (3.22) and to obtain the two inequalities in (3.28), we used (3.13) and (3.7) respectively. We know from Theorem 3.1 that $\sup_{0\le t\le T}\|\vec{u}(t)\|_p \le 2M$. Thus, the first integral in (3.28) is finite. Concerning the second integral in (3.28), we have

$$\begin{aligned}\int_\epsilon^t \frac{1}{(t-s)^{\delta_0+\frac{(\beta_0+1)}{2}}}\|\vec{u}(s)\|_{\alpha,p}^2ds &= \int_\epsilon^t \frac{1}{(t-s)^{\delta_0+\frac{(\beta_0+1)}{2}}s^\alpha}s^\alpha\|\vec{u}(s)\|_{\alpha,p}^2ds \\ &\le (\sup_{0\le t\le T} t^{\alpha/2}\|\vec{u}\|_{\alpha,p})^2\int_\epsilon^t \frac{1}{(t-s)^{\delta_0+\frac{(\beta_0+1)}{2}}s^\alpha}ds < \infty.\end{aligned}$$

where the last integral is finite since $\delta_0+(\beta_0+1)/2<1$. We thus obtained the estimate estimate obtained above, we have

$$\sup_{0\le t\le T}\|A^{\delta_0}(S\vec{u}-G)(t)\|_{\alpha,p} = \left(\frac{2\pi}{L}\right)^{2\delta}\|(S\vec{u}-G)(t)\|_{\alpha+2\delta,p} < \infty.$$

The above estimate together with (3.26) yields

$$\sup_{0\le t\le T} t^{\frac{\alpha}{2}+\delta_0}\|\vec{u}(\cdot)\|_{\alpha+2\delta_0} < \infty.$$

This finishes the proof. □

4. Gevrey Regularity

We will now proceed to obtain solution of NSE which is in Gevrey class. For $p\ge 1$, the Gevrey class $X_{Gv(\gamma),p}$ is defined as

$$X_{Gv(\gamma),p} = \{\vec{u}\in K : \|\vec{u}\|_{Gv(\gamma),p} := \|e^{\gamma A^{1/2}}\vec{u}\|_p = \left(\sum_{k\in\mathbb{Z}^3} e^{p\gamma\frac{2\pi}{L}|k|}|u(k)|^p\right)^{1/p} < \infty\}.$$

Obviously, $X_{Gv(\gamma),p} \subset V_p$, and in fact, $\|\vec{u}\|_p \le \|\vec{u}\|_{Gv(\gamma),p}$, $\gamma > 0$. Throughout this section, we assume

$$\mu \le \frac{2\pi}{L}.$$

Define the Banach space

$$C_G = \left\{ \vec{u}(\cdot) \in C([0,T]; V_p) : \|\vec{u}(\cdot)\|_{Gv} = \sup_{0 \le t \le T} \|\vec{u}(t)\|_{Gv(\mu t),p} < \infty \right\}.$$

Note that

$$\vec{u} \in C_G, \ \|\vec{v}(\cdot)\|_{Gv} = \sup_{0 \le t \le T} \|e^{\mu t A^{1/2}} \vec{v}(\cdot)\|_p. \tag{4.1}$$

We will need a Young-type inequality for Gevrey norms.

Lemma 4.1. *Let* $\vec{u}$, $\vec{v} \in X_{Gv(\gamma),p}$ *and let* $\vec{w} = (w(k))_{k \in \mathbb{Z}^3}$ *be defined by* $w(k) = \sum_h |\vec{u}(h)||\vec{v}(k-h)|$. *Then,*

$$\|\vec{w}\|_{Gv(\gamma),r} = \left(\sum_k e^{\gamma r \left(\frac{2\pi}{L}\right)|k|} w(k) \right)^{1/r} \le \|\vec{u}\|_{Gv(\gamma),p} \|\vec{v}\|_{Gv(\gamma),p}, \ r = \frac{p}{2-p}. \tag{4.2}$$

Proof. Since $\vec{u}$ and $\vec{v}$ are in $X_{Gv(\gamma),p}$, the vectors $\vec{u}_1, \vec{v}_1$ defined by

$$\vec{u}_1(k) = e^{\gamma \frac{2\pi}{L}|k|} |\vec{u}(k)|, \quad \vec{v}_1(k) = e^{\gamma \frac{2\pi}{L}|k|} |\vec{v}(k)|, \qquad (k \in \mathbb{Z}^3)$$

are in $\ell_p(\mathbb{Z}^3)$ and clearly,

$$\|\vec{u}_1\|_p = \|\vec{u}\|_{Gv(\gamma),p} \text{ and } \|\vec{v}_1\|_p = \|\vec{v}\|_{Gv(\gamma),p}.$$

Consequently, by Young's inequality, the vector $\vec{w}_1 = \vec{u}_1 * \vec{v}_1$ is in $\ell_r(\mathbb{Z}^3)$, $r = \frac{p}{2-p}$ and

$$\|\vec{w}_1\|_r \le \|\vec{u}_1\|_p \|\vec{v}_1\|_p \le \|\vec{u}\|_{Gv(\gamma),p} \|\vec{v}\|_{Gv(\gamma),p}. \tag{4.3}$$

Using the inequality $|k| \le |h| + |k-h|$ and noting that $\vec{w}_1(k) = (\vec{u}_1 * \vec{v}_1)(k) = \sum_h e^{\gamma \frac{2\pi}{L}|h|} |\vec{u}(h)| e^{\gamma \frac{2\pi}{L}|k-h|} |\vec{v}(k-h)|$ we have

$$\begin{aligned} \sum_k e^{\gamma r \frac{2\pi}{L}|k|} w(k)^r &= \sum_k \left[e^{\gamma \frac{2\pi}{L}|k|} \left(\sum_h |\vec{u}(h)||\vec{v}(k-h)|\right) \right]^r \le \\ \sum_k \left[\sum_h e^{\gamma \frac{2\pi}{L}|h|} |u(h)| e^{\gamma \frac{2\pi}{L}|k-h|} |v(k-h)| \right]^r &= \sum_k \vec{w}_1(k)^r = \\ \|\vec{w}_1\|_r^r &\le \|\vec{u}_1\|_p^r \|\vec{v}_1\|_p^r \le \|\vec{u}\|_{Gv(\gamma),p}^r \|\vec{v}\|_{Gv(\gamma),p}^r. \end{aligned}$$

Thus we obtain (4.2). ∎

Lemma 4.2. *Let $\vec{u}, \vec{v} \in X_{\gamma,p}$ and $\eta > 0$. Let $p < 2$ and $\beta > 0$ is such that $\beta p > 3(p-1)$. Then, there exists a constant C (which may depend on p, β) such that*

$$\|e^{-\eta A} B[\vec{u}, \vec{v}]\|_{\gamma,p} \le C \frac{1}{\eta^{\frac{(\beta+1)}{2}}} \|\vec{u}\|_{\gamma,p} \|\vec{v}\|_{\gamma,p}. \tag{4.4}$$

Proof. For $\vec{u}, \vec{v}$ as in lemma, and $\vec{w}$ defined by

$$\vec{w}(k) = \sum_h |u(h)||v(k-h)|$$

we have

$$\begin{aligned} \|e^{-\eta A} B[\vec{u}, \vec{v}]\|^p_{Gv(\gamma),p} &\le \sum_k e^{-\eta p (\frac{2\pi}{L})^2 |k|^2} e^{\gamma p \frac{2\pi}{L}|k|} |\sum_h (k \cdot u(h)) v(k-h)|^p \le \\ \sum_k e^{-\eta p (\frac{2\pi}{L})^2 |k|^2} e^{\gamma p \frac{2\pi}{L}|k|} |k|^p |w(k)|^p &= \sum_k |k|^{(\beta+1)p} e^{-\eta p (\frac{2\pi}{L})^2 |k|^2} \frac{1}{|k|^{\beta p}} e^{\gamma p \frac{2\pi}{L}|k|} |k|^p |w(k)|^p \le \\ C \frac{1}{\eta^{\frac{(\beta+1)p}{2}}} \sum_k \frac{1}{|k|^{\beta p}} e^{\gamma p \frac{2\pi}{L}|k|} |w(k)|^p &\le C \frac{1}{\eta^{\frac{(\beta+1)p}{2}}} \left(\sum_k \frac{1}{|k|^{\frac{\beta p r}{r-p}}} \right)^{\frac{r-p}{r}} \left(\sum_k e^{\gamma r \frac{2\pi}{L}|k|} |w(k)|^r \right)^{\frac{p}{r}} \\ = C \frac{1}{\eta^{\frac{(\beta+1)p}{2}}} \|\vec{w}\|^p_{Gv(\gamma),r} &\le C \frac{1}{\eta^{\frac{(\beta+1)p}{2}}} \|\vec{u}\|^p_{\gamma,p} \|\vec{v}\|^p_{\gamma,p} \end{aligned}$$

where, to obtain the inequalities in the second last line, we used (3.6) followed by Holder's inequality, and for the inequality in the last line we used (4.2). ☐

Let G be as in (3.1) and

$$M := \|\vec{G}(\cdot)\|_{Gv} = \sup_{0 \le t \le T} \|e^{\mu t A^{1/2}} \vec{G}(t)\|_p < \infty \tag{4.5}$$

Remark Since $\mu < \frac{2\pi}{L}$, the condition (4.5) is satisfied if

$$\vec{u}_0 \in V_p, \text{ and } \sup_{0 \le t \le T} \|e^{\mu A^{1/2}} \vec{g}(t)\|_p < \infty.$$

Let $E_1 \subset C_{Gv}$ be defined as

$$E_1 = \{\vec{v} \in C_{Gv} : \|\vec{v}(\cdot) - \vec{G}(\cdot)\|_{Gv} \le M\}. \tag{4.6}$$

Lemma 4.3. *For $\vec{v} \in E_1$, we have $S\vec{v}$ is in C_{Gv} and*

$$\|(S\vec{v} - G)(t)\|_{t,p} \le \frac{4C}{1-(\beta+1)/2} M T^{1-(\beta+1)/2} M, \quad \frac{3(p-1)}{p} < \beta < 1 \tag{4.7}$$

Proof. Recall that for any $\gamma > 0$ we can write $\|\vec{u}\|_{\gamma,p} = \|e^{\gamma A^{1/2}}\vec{u}\|_p$. Moreover, since $\mu \leq \frac{\pi}{L}$

$$e^{-\beta[\frac{1}{2}A-\mu A^{1/2}]} \quad (\beta > 0) \text{ is a contraction on } V_p. \tag{4.8}$$

$$\begin{aligned}
&\|(S\vec{v} - G)(t)\|_{\mu t,p} = \\
&\quad \|e^{\mu t A^{1/2}}(S\vec{v} - G)(t)\|_p \leq \int_0^t \|e^{\mu t A^{1/2}} e^{-(t-s)A} B[\vec{v}(s), \vec{v}(s)]\|_p ds \\
&\quad \leq \int_0^t \|e^{\mu(t-s)A^{1/2}} e^{-\frac{(t-s)}{2}A} e^{-\frac{(t-s)}{2}A} e^{\mu s A^{1/2}} B[\vec{v}(s), \vec{v}(s)]\|_p ds \\
&\quad \leq \int_0^t \|e^{-\frac{(t-s)}{2}A} e^{\mu s A^{1/2}} B[\vec{v}(s), \vec{v}(s)]\|_p ds = \int_0^t \|e^{-\frac{(t-s)}{2}} B[\vec{v}(s), \vec{v}(s)]\|_{Gv(\mu s),p}\, ds \\
&\quad \leq C \int_0^t \frac{1}{(t-s)^{(\beta+1)/2}} \|\vec{v}(s)\|^2_{Gv(\mu s),p}\, ds \leq \frac{4C}{1-(\beta+1)/2} M T^{1-(\beta+1)/2} M
\end{aligned}$$

where in the last inequality, we used Lemma 4.2 with $\eta = (t-s)/2$ and $\gamma = \mu s$ as well as the fact that $\sup_{0\leq s\leq T} \|e^{\mu s A^{1/2}}\vec{v}(s)\|_p \leq 2M$. □

We will next need a lemma which shows that S is a contraction on E with respect to the gevrey norm on C_{Gv}.

Lemma 4.4. *For $\vec{v}, \vec{w} \in E_1$, and for $\frac{3(p-1)}{p} < \beta < 1$ we have*

$$\|(S\vec{v} - S\vec{w})(t)\|_p \leq \frac{2C}{1-(\beta+1)/2} M T^{1-(\beta+1)/2} \sup_{0\leq t\leq T} \|e^{\mu t A^{1/2}}(\vec{v} - \vec{w})(t)\|_p. \tag{4.9}$$

The proof of this lemma is analogous to the previous one and is omitted.

Theorem 4.1. *Let $1 \leq p < 3/2, \mu < \frac{\pi}{L},\ T < \dfrac{C}{M^{4p/(3-2p)}}$ and M as in (4.5). Then, there exists $\vec{u}$ in C_{Gv} such that $\sup\limits_{0\leq t\leq T} \|e^{\mu s A^{1/2}}\vec{u}(s)\| < 2M$ satisfying*

$$\frac{d\vec{u}(t)}{dt} = -A\vec{u}(t) + \vec{g}(t) - B[\vec{u}(t), \vec{u}(t)],\ \vec{u}(0) = \vec{u}_0 \quad a.e.\ 0 \leq t \leq T, \tag{4.10}$$

where the constant C depends on p only as $p \to 3/2$.

Proof. Take $\beta = (\frac{3(p-1)}{p} + 1)/2 = \frac{4p-3}{2p}$. Then, β satisfies the condition in (3.15). Moreover, if $T < \frac{C}{M^{4p/(3-2p)}}$ for appropriate C, then by Lemma 3.3 and Lemma 3.4, the map S defined in (3.5) is a contractive map from E_1 into E_1. Thus, by Banach fixed point theorem, there exists a $\vec{u}$ in $E_1 \subset C_{Gv}$ such that

$$\vec{u}(t) = e^{-tA}\vec{u}_0 + \int_0^t e^{-(t-s)A}\vec{g}(s)ds - \int_0^t e^{-(t-s)A}B[\vec{u}(s), \vec{u}(s)]ds.$$

The conclusion of the theorem now follows quite easily as in Theorem 3.1.

□

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