Global uniqueness for a non-overdetermined inverse conductivity problem in unbounded domains

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Abstract

In this paper we show that sufficiently smooth coefficients of the elliptic operator $\nabla_x \cdot \sigma(x')\nabla_x - c(x'),\ x \in R^3,\ x' \in R^2$ can be uniquely determined from the Cauchy data given on a strip in the plane $\{x_3 = 0\}$. This is an extension of Tikhonov's formulation of the one-dimensional inverse problem of electric prospecting to two dimensions. In this formulation, the number of variables in the Cauchy data equals the number of variables in unknown coefficients. This is referred to the concept of non-overdeterminacy. Unlike the Dirichlet-to-Neumann map defined on the entire boundary of a bounded domain, the position of a pointlike electrode injecting electric currents into a domain is assumed to be fixed, and such a domain is assumed to be unbounded. The method of Carleman estimates combined with both the direct Fourier and inverse Laplace transforms is adopted to establish the global uniqueness theorem. Also, we establish the global uniqueness result for a corresponding inverse source problem arising in gravimetry prospecting.

1 Introduction

In this paper we concern a global uniqueness for the longstanding non-overdetermined inverse conductivity problem in unbounded domains. Motivated by electric prospecting, this problem was first formulated in [1] for a half-plane filled with the isotropic analytical conductivity σ depending only on one spatial variable. In that formulation, knowledge of the Dirichlet-to-Neumann (D-to-N) map Λ_σ, i.e., knowledge of all possible current density-voltage potential measurements at any point on the entire boundary, was required to determine the conductivity. Because of this, the inverse problem was overdetermined. This means

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that the number of independent variables in the D-to-N data is greater than the number of independent variables in the conductivity function, which is one. In the mathematics literature, this is referred sometimes as to an inverse problem with multiple measurements. If the numbers of independent variables indicated above are equal, an inverse problem is said to be non-overdetermined or it is referred as to an inverse problem with a single measurement (see [2], Chapter 1). Since Calderon's paper [3], the uniqueness of overdetermined inverse conductivity problems in high dimensions ($n \geq 3$) has been extensively studied for bounded domains (see, e.g., [4] and an overview [5]). Since an overdetermined formulation requires vasty measurements, it is not attractive for applied sciences.

A non-overdetermined inverse conductivity problem of electric prospecting in a half-space was first formulated by Tikhonov in his seminal paper [6] (see also the textbook [7], pp. 426–433). Specifically, a steady-state electric field resulted from a pointlike source on the plane $x_3 = 0$ was considered in the half-space $R^3_+ = \{x = (x_1, x_2, x_3) \in R^3 : x_3 > 0\}$ filled with the conductivity depending on the x_3 variable. The half-space $R^3_- = \{x = (x_1, x_2, x_3) \in R^3 : x_3 < 0\}$ was assumed to be filled with the dielectric. The steady-state electric field was modelled by the problem

$$\Delta_{x_1 x_2} u + \sigma^{-1}(x_3) \frac{\partial}{\partial x_3} \left(\sigma(x_3) \frac{\partial u}{\partial x_3} \right) = 0 \quad \text{in} \quad R^3_+, \tag{1.1}$$

$$\frac{\partial u}{\partial z} \mid_{x_3=0} = 0, \tag{1.2}$$

$$\lim_{|x| \to \infty} u = 0, \tag{1.3}$$

where the voltage potential was represented in the form

$$u(x) = \frac{I}{2\pi\sigma(0)} \cdot \frac{1}{|x|} + \overline{u}(x). \tag{1.4}$$

Here, $\overline{u}(x)$ is a regular component of the voltage potential bounded at the origin and I is the current injected into the half-space R^3_+ by a pointlike ground situated at the origin. Due to the axial symmetry, $u = u(\rho, z)$, where (ρ, z) are cylindrical coordinates. In Tikhonov's formulation, given the function $u(\rho, 0)$, find the conductivity distribution $\sigma(z)$ of a layered medium. Since the numbers of independent variables in both the unknown function and data are equal, this formulation is non-overdetermined. In [6], the uniqueness was established by transforming the original problem to the Sturm-Liouville inverse problem.

Since the D-to-N data contain $(2n - 1)$ independent variables, the problem of reconstructing the conductivity distribution σ from the D-to-N data is non-overdetermined in two dimensions. The global uniqueness question in two dimensions was first resolved in 1996 in [8] for a bounded domain Ω with the Lipschitz boundary $\partial\Omega$ and $\sigma \in W^{2,p}(\Omega),\ p > 1$. In this paper, both the voltage potential and conductivity distribution were assumed to be two-dimensional. One may, therefore, refer to the 2-D/2-D inverse conductivity problem. However, it is difficult to use such a model in practice. Indeed, the Nachman's formulation requires the D-to-N data on the entire boundary of a bounded domain. Furthermore, the assumption of the two-dimensional voltage potential is meaningful only if the conductivity distribution, the source function and boundary conditions do not depend on one of spatial variables and the domain Ω has a specific form. As an example, we indicate an infinite cylinder whose directrix is elongated in the third variable, and the input data, i.e., the conductivity, the source function and boundary conditions, remain unchanged along the directrix. In this case, the steady-state electric field is the same at any cross-section of the cylinder by the plane perpendicular to its axis. Clearly, such conditions cannot be satisfied for any pointlike ground or for a superposition of such grounds. The boundary measurements always represent a trace of the three-dimensional voltage potential resulted from injecting the electric currents into a conductive 3-D object. In addition, the global uniqueness theorem established in [8] has not been extended to the inverse conductivity problem for the equation (2.7), in which $c(x') \neq 0$.

The Tikhonov's formulation takes into account this fact considering the 3-D/1-D inverse conductivity problem. In this paper, we extend both the Tikhonov's formulation and global uniqueness result to the 3-D/2-D inverse conductivity problem in unbounded domains. As far as the authors are aware, no such results have been published in the mathematics literature. Meanwhile, many real objects, such as geological structures, pipes and slabs, organs of a human body, can be modelled by cylinders. It should also be pointed out that we consider the inverse conductivity problem as a generic case representing the broad class of applied problems in steady-state physical processes. Such processes are often represented mathematically by the second order elliptic differential equation

$$\nabla \cdot (a(x)\nabla u) = -f(x) \quad \text{in} \quad \Omega \subset R^3$$

with corresponding boundary conditions. For instance, this equation may describe not only the distribution of direct currents in a conductive medium, but also the stationary heat transfer or the diffusion of matter, or the electrostatic field from electric charges in a dielectric medium, etc. A common part of these modelling processes involves the reconstruction of the coefficient $a(x)$ from boundary measurements within the framework of the 3-D/2-D inverse model.

To establish the global uniqueness result for the 3-D/2-D inverse conductivity problem with $\sigma = \sigma(x_1, x_3)$, we first apply to the problem the Fourier transform with respect to the x_2 variable. Doing so, we follow the paper [6], where the Hankel transform was used in the 3-D/1-D inverse conductivity problem with respect to the radial variable $\rho = \sqrt{x_1^2 + x_2^2}$. Next, the inverse Laplace transform is applied to the resulting problem to derive an auxiliary initial boundary value problem for a hyperbolic equation. Since the Laplace transform is a one-to-one operator, the global uniqueness for the original inverse problem follows from the global uniqueness for the corresponding inverse problem for the auxiliary hyperbolic equation. To establish the latter, we exploit the method of Carleman estimates (see [2], Chapter 3 for details). We use Carleman estimates for both the Laplacian and d'Alembertian. Such a combination is necessary, because the unknown conductivity distribution is contained in the equation together with its derivatives. This differs significantly from the framework indicated in [2]. Also, we stress that the inverse Laplace transform is used only for establishing the global uniqueness result. It was shown in [2] that the convexification approach can be applied to solving a coefficient inverse problem for the equation (3.1) resulted from applying the Fourier transform to the equation (2.7). Therefore, in the forthcoming numerical study, we plan to adopt the convexification approach (see [2], Chapter 5 for details) avoiding the use of the inverse Laplace transform, which is unstable. This is advantageous because of its global convergence property.

In our formulation, the three-dimensional source function is represented in the form $f(x_2)S_2(x_1, x_3)$, where $f(x_2) \in C_0^\infty(-\infty, \infty), \|f\|_{L_2(-\infty,\infty)} \neq 0$, $S_2(x_1, x_3) \in C_0^\infty(R^2)$, $S_2(x_1, x_3) = S_2(x_1, -x_3)$, and $S_2(x_1, x_3) \neq 0$ in a certain bounded closed domain of the half-plane R^2_+. The latter assumption is essential for the proof of the global uniqueness theorem. It may, therefore, seem that imposing such conditions precludes the δ-like source generated by a pointlike ground. However, one may consider the form indicated above as a sort of regularization ('spreading') of the 3-D δ-function, such that it is negligibly small outside of the vicinity of a pointlike ground. In this case, the perturbations of the electric field caused by such a regularization are small as well. Furthermore, the conditions imposed on the function $f(x_2)$ are needed only for establishing the existence of a unique solution of the forward 3-D/2-D conductivity problem. They may be omitted in the 3-D/2-D inverse conductivity problem, i.e., the function $f(x_2)$ can be replaced with $\delta(x_2)$. To simplify the presentation, we do not concern imposing the minimal smoothness conditions on both the coefficients and right-hand side in the equation (2.7).

The paper is organized as follows. In section 2 we formulate both the forward and inverse conductivity problem for the two-dimensional conductivity distribution. In section 3 we formulate the global uniqueness theorem. Section 3 establishes the proof of this theorem. For convenience, we conduct the proof in several stages. In addition, in section 4 we formulate a problem of determining the source function $S_2(x')$ in the equation (2.7) and prove the uniqueness theorem. Such a problem is closely related to an inverse problem of the potential theory and used in gravimetry prospecting (see, e.g., [14], Chapter 1). Finally, the section 5 concludes the paper.

2 Formulations

2.1 A mathematical model of the steady-state electric field

We first indicate a mathematical model describing the steady-state electric field generated by direct currents in a conductive medium. We start with Maxwell's equations governing the time-harmonic electric $\mathbf{E}$ and magnetic $\mathbf{H}$ components of the electromagnetic field (e^{iwt} is a time factor) in a conductive medium

$$\nabla \times \mathbf{H} = \sigma(x)\mathbf{E} + \mathbf{j}_{ex}, \tag{2.1}$$

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}, \tag{2.2}$$

where $\mu > 0$ is the magnetic permeability, which is assumed to be constant, ω is an angular frequency, $\mathbf{j}_{ex}$ is the current density of extrinsic sources. The displacement currents are neglected. In the case of direct currents, the angular frequency equals zero. Then we obtain from (2.2)

$$\nabla \times \mathbf{E} = 0. \tag{2.3}$$

This expresses the second Kirchhoff's law in the differential form. Acting by the operator $\nabla\cdot$ on both parts of the equation (2.1) and using the vector identity $\nabla \cdot (\nabla \times \mathbf{a}) = 0$, we obtain

$$\nabla \cdot \mathbf{j} = 0, \tag{2.4}$$

where $\mathbf{j} = \sigma(x)\mathbf{E} + \mathbf{j}_{ex}$. Because of the curl-free electric field (2.3), there exists the voltage potential $u(x)$, such that $\mathbf{E} = -\nabla u$. We then obtain from (2.4)

$$\nabla \cdot (\sigma(x)\nabla u) = \nabla \cdot \mathbf{j}_{ex}. \tag{2.5}$$

Adding to this equation the corresponding boundary and radiation conditions, one may derive a mathematical model governing the steady-state electric field produced by direct currents with the density $\mathbf{j}_{ex}$. For instance, the continuity boundary conditions for both the voltage potential u and current density $-\sigma\frac{\partial u}{\partial n}$ are implied by the continuity conditions for tangential components of both the electric and magnetic fields at any interface between conductive media. If the dielectric bounds a conductive medium, the electric current cannot flow through the boundary into the dielectric. In this case, the normal component of the current density $-\sigma\frac{\partial u}{\partial n}$ equals zero at the interface.

Also, we outline briefly the issue of modelling the extrinsic sources $\nabla \cdot \mathbf{j}_{ex}$. In electric prospecting, the sources are formed by grounds, which are metallic electrodes embedded into the conductive medium to inject the electric currents. The theory of grounds is available in the geophysics literature (see, e.g., [9], [10]). In accordance with this theory, a pointlike ground plays the key role, because the steady-state electric fields from any other grounds can be represented as superpositions of such fields. Therefore, we consider a pointlike ground and show that it generates a δ-like source. Assume that the space R^3 is divided by the plane $x_3 = 0$ into two half-spaces. The lower half-space $R^3_- = \{x_3 < 0\}$ is filled with the dielectric medium, whereas the upper half-space $R^3_+ = \{x_3 > 0\}$ is filled with the inhomogeneous conductive medium. Imagine a metallic ball with a negligibly small radius placed at the origin of Cartesian coordinate system. Let the electric current with the magnitude I be flowing uniformly through the surface of the ball into the conductive medium, which is assumed to be homogeneous in a sufficiently small vicinity of the origin. Clearly, in this vicinity, the voltage potential is given by

$$u(x) = \frac{I}{2\pi\sigma_0} \cdot \frac{1}{|x|}, \tag{2.6}$$

where σ_0 is the conductivity at the origin. Then the relation $\Delta(|x|^{-1}) = -2\pi\delta(x)$ in the half-space implies that $\nabla \cdot \mathbf{j}_{ex} = -I\delta(x)$. Thus, a pointlike ground can be modelled by a δ -like source. According to this, the voltage potential was represented in [6] in the form (1.4). Such a representation allows for considering a homogeneous elliptic equation instead of (2.5).

2.2 The forward problem

The mathematical model (2.5) describing the steady-state electric field in the half-space R^3_+ motivates the following generic model. Denote $x' = (x_1, x_3) \in R^2$ and consider the boundary value problem for the elliptic differential equation

$$\nabla \cdot (\sigma(x')\nabla u) - c(x')u = -f(x_2)S_2(x') \text{ in } R^3_+, \tag{2.7}$$

$$\partial_{x_3} u \mid_{x_3=0} = 0, \tag{2.8}$$

where $\lim_{|x|\to\infty} u(x) = 0$ and $f(x_2)S_2(x')$ is a source function. The forward problem consists of finding the function $u(x)$ in R^3_+ given the coefficients $\sigma(x')$ and $c(x')$ and source function.

In the equation (2.7), the source function $f(x_2)S_2(x')$ is assumed not to be equal zero in a certain bounded domain in R^3_+. Such an assumption is needed for establishing the global uniqueness theorem for the inverse conductivity problem. On the other hand, this assumption prohibits the use of the δ-like source motivated by the model (2.5). In this case, one may consider the function $f(x_2)S_2(x')$ as a regularization of the δ-like source function (see, e.g., [11]). Such an approximation may be given a physical interpretation using the current function. Along with the voltage potential $u(x)$, the current function

$$\psi(x) = 2\pi \int_{R^3_+} \sigma(\xi) j_{\xi_3}(\xi) d\xi$$

is usually used as the second scalar quantity characterising the steady-state electric field (see, e.g., [9]). Here, j_{x_3} is the vertical component of the current density vector. The physical meaning of the current function $\psi(x)$ is that it equals the current flowing between isosurfaces of the function $j_{x_3}(x)$. If it is possible to introduce a system of curvilinear coordinates, so that the coordinate surfaces coincide with such isosurfaces, then one may speak about the so-called current tubes. The form of isosurfaces depends on the specific conductivity distribution $\sigma(x)$. For instance, if $\sigma = const > 0$, the isosurfaces are cones with the common vertex at the origin. If the function $\sigma(x)$ is sufficiently smooth, the isosurfaces are cones whose generatrixes are smooth curves. If the function $\sigma(x)$ has jump discontinuities, the isosurfaces are cones whose generatrixes are polygon lines. For brevity, consider the homogeneous half-space R^3_+ and introduce the cylindrical coordinates (ρ, ϑ, z) whose axis coincides with the z-axis. Because of the axial symmetry, the voltage potential u_0 generated by the source function $I\delta(x_1, x_2, x_3)$ is given by (2.6). Since $j_z = -\sigma \frac{\partial u}{\partial z}$, we obtain

$$\psi(\rho, z) = 2\pi\sigma_0 \int_0^\rho j_z(\rho', z)\rho' d\rho' = I \cdot \left(1 - \frac{z}{\sqrt{\rho^2 + z^2}}\right).$$

Without loss of generality, assume that the regularization of the δ-source function results in a voltage potential $u_\varepsilon(\rho, z)$ approximating $u_0(\rho, z)$ everywhere except the ε-vicinity of the origin, in which $u_\varepsilon(\rho, z) = (\varepsilon^2 + z^2)^{-1/2}$. Then, the corresponding current function $\psi_\varepsilon(\rho, z)$ is given by

$$\psi_\varepsilon(\rho, z) = I \cdot \left[\left(1 + O\left(\frac{\varepsilon}{z}\right) \right) - \frac{z}{\sqrt{\rho^2 + z^2}} \right],$$

where $2\varepsilon > 0$ is the diameter of a ε-vicinity of the pointlike ground. Clearly, the current function ψ_ε approximates the current function ψ for all $z \gg \varepsilon$. If $\varepsilon/z = O(1)$, then the asymptotic expansion is meaningless. Thus, it is meaningful to expect that the regularization of the $\delta(x)$- source function implies only small perturbations of the current function in any domain situated sufficiently far from a pointlike ground. It should also be stressed that it is not required to regularize the function $\delta(x_2)$ when establishing the global uniqueness theorem. We impose the regularity conditions on the function $f_(x_2)$ in order to establish the existence of a unique solution of the forward problem.

Let $\varepsilon \in (0, L)$ be a positive number and $\Omega \subset R^2_+$ be a domain with a piecewise smooth boundary $\partial\Omega$, such that

$$\Omega = \{x' = (x_1, x_3) : -A < x_1 < A,\ x_3 \in (0, L)\} \bigcap \{x_3 > \varepsilon\}. \tag{2.9}$$

To demonstrate that the existence and uniqueness of the solution $u \in H^2(R^3_+)$ of the forward problem (2.7), (2.8) takes place under relatively non-restrictive conditions, we prove the following lemma.

Lemma 2.1 *Let the following conditions be fulfilled*

$$\sigma(x') \in C^{3+\alpha}\left(\overline{R^2_+}\right), c(x') \in C^{1+\alpha}\left(\overline{R^2_+}\right), \tag{2.10}$$

$$\sigma(x') \geq const > 0, c(x') \geq 0 \ \ in\ R^2_+,$$

$$\sigma(x') = 1,\ c(x') = 0 \ for\ x' \in R^2_+ \setminus \Omega, \tag{2.11}$$

$$f(x_2) \in C_0^\infty(-\infty, \infty),\ \|f\|_{L_2(-\infty,\infty)} \neq 0, \tag{2.12}$$

$$S_2(x_1, x_3) \in C_0^\infty(R^2),\ S_2(x_1, x_3) = S_2(x_1, -x_3),\ S_2(x_1, x_3) \neq 0\ in\ \overline{\Omega}, \tag{2.13}$$

where $\alpha \in (0,1)$, $f(x_2) = -g''(x_2)$, *and* $g(x_2) \in C_0^\infty(-\infty, \infty)$. *Then there exists a unique solution of the problem (2.7)-(2.8) belonging to the function space* $H^2(R^3_+)$.

Remark 2.1. The relation $f(x_2) = -g''(x_2)$ means that we first approximate the distribution $\delta(x_2)$ by the function $g(x_2) \in C_0^\infty(-\infty, \infty)$. After this, we differentiate the approximating function in terms of distributions.

Proof. Denote $\tilde{f}(s)$ the Fourier transform of the function $f(y)$

$$\tilde{f}(s) = \int_{-\infty}^{\infty} f(y) e^{-isy} dy.$$

Then we have $\tilde{f}(s) = s^2\tilde{g}(s)$ and $s^n\tilde{g}(s) \in L_2(-\infty, \infty)$ for all integer numbers $n \geq 0$. Consider the even extensions with respect to x_3 of coefficients $\sigma(x_1, x_3)$ and $c(x_1, x_3)$ in the half-plane R^2_-. Assuming the existence and uniqueness of the solution $u \in H^2(R^3_+)$ of the problem (2.7)-(2.8), consider its even extension to the half-space R^3_- with respect to the x_3 variable. For brevity, we preserve the same notations for the extension. Then the relations (2.7)-(2.11) imply that $u \in H^2(R^3)$. The reverse implication is true as well. Specifically, if we assume that the existence and uniqueness of the solution of the elliptic equation (2.7) with extended coefficients is establsihed, then the conditions (2.9) - (2.13) imply that $u(x_1, x_2, x_3) = u(x_1, x_2, -x_3)$, which leads to the condition (2.8). Hence, it is sufficient to prove the existence and uniqueness of the problem

$$\nabla \cdot (\sigma(x') \nabla u) - c(x') u = -f(x_2) S_2(x_1, x_3) \text{ in } R^3, \tag{2.14}$$

$$u \in H^2(R^3). \tag{2.15}$$

Assume that there eixists the Fourier transform of the function $u(x_1, x_2, x_3)$ with respect to the x_2 variable and denote this $w(x', s)$. Applying the Fourier transform to the equation (2.14), we obtain

$$\nabla_{x_1,x_3} \cdot (\sigma(x') \nabla w) - \left(s^2\sigma(x') + c(x')\right) w = -s^2\tilde{g}(s) S_2(x_1, x_3) \text{ in } R^2. \tag{2.16}$$

Let $p(x', s) \in H^2(R^2)$ be an arbitrary function depending on the parameter s. For $s \neq 0$, we multiply both sides of the equation (2.16) by its complex conjugate $\overline{p}(x', s)$ and integrate over the whole space R^2 using the integration by parts. We then obtain

$$\int_{R^2} \sigma\nabla w \cdot \nabla\overline{p} dxdz + \int_{R^2} \left(s^2\sigma + c\right) w\overline{p} dx_1 dx_3 = s^2\tilde{g}(s) \int_{R^2} S_2\overline{p} dx_1 dx_3. \tag{2.17}$$

The Riesz theorem and (2.17) imply that for every $s \neq 0$ there eixists a unique solution $w(x', s) \in H^1(R^2)$ of the equation (2.16).

To justify the inverse Fourier transform, we need to estimate from above the integral

$$\int_{R^3} \left[|\nabla_{x_1,x_3} w|^2 + \left(1 + s^2 + s^4\right) |w|^2 \right] dx_1 dx_3 ds.$$

Denoting $p = w$ in (2.17), recalling that $\sigma \geq 1$ and using the Cauchy-Bunyakovsky inequality, we obtain

$$\int_{R^2} \sigma |\nabla_{x_1,x_3} w|^2 dx_1 dx_3 + \frac{s^2}{2} \int_{R^2} |w|^2 dx_1 dx_3 \leq \frac{s^2}{2} |\tilde{g}(s)|^2 \|S_2\|^2_{L_2(R^2)}.$$

Hence, we have

$$\left(1 + s^2 + s^4\right) \int_{R^2} |\nabla_{x_1,x_3} w|^2 dx_1 dx_3 + \left(1 + s^2 + s^4 + s^6\right) \int_{R^2} |w|^2 dx_1 dx_3 \leq \left(1 + s^2 + s^4 + s^6\right) |\tilde{g}(s)|^2 \|S_2\|^2_{L_2(R^2)}.$$

This inequality implies that

$$\int_{R^3} [(1+s^2+s^4)|\nabla_{x_1,x_3} w|^2 \quad + \quad (1+s^2+s^4)|w|^2] dx_1 dx_3 ds \leq \tag{2.18}$$

$$\|S_2\|^2_{L_2(R^2)} \int_{-\infty}^{\infty} \left(1+s^2+s^4+s^6\right) |\tilde{g}(s)|^2 \, ds.$$

Therefore, we obtain

$$\left(1+s^2+s^3\right) w \in L_2\left(R^3\right) \tag{2.19}$$

and

$$\left(1+s^2\right) \partial_{x_1} w, \;\; \left(1+s^2\right) \partial_{x_3} w \in L_2\left(R^3\right). \tag{2.20}$$

Let the function $\tilde{w}(\xi_1, \xi_3, s)$ be the Fourier transform of the function $w(x_1, x_3, s)$ with respect to the variables x_1 and x_3. Then (2.20) implies that

$$\left(1+s^2+\xi_1^2+\xi_3^2\right)^{1/2} \tilde{w} \in L_2\left(R^3\right). \tag{2.21}$$

Suppose that

$$\left(1+s^2\right) \partial^2_{x_1} w, \;\; \left(1+s^2\right) \partial^2_{x_3} w, \;\; \left(1+s^2\right) \partial^2_{x_1 x_3} w \in L_2\left(R^3\right). \tag{2.22}$$

By virtue of (2.19)-(2.22), we then have

$$\left(1+s^2+\xi_1^2+\xi_3^2\right) \tilde{w} \in L_2\left(R^3\right). \tag{2.23}$$

Let $u(x_1, x_2, x_3)$ be the inverse Fourier transform of the function $w(x_1, x_2, s)$ with respect to the parameter s. Then the relations (2.16) and (2.23) imply that the function satisfies the conditions (2.14) and (2.15). The uniqueness of the solution of the problem (2.14), (2.15) follows from (2.17).

To prove (2.22), we consider a certain integral equation. Changing the variables $v = \sqrt{\sigma(x')} \cdot w$, we transform the equation (2.16) to the equation

$$\Delta_{x_1,x_3} v - s^2 v - q(x') v = -s^2 \tilde{g}(s) \cdot \frac{S_2(x_1, x_3)}{\sqrt{\sigma(x')}} \quad \text{in} \quad R^2, \tag{2.24}$$

where

$$q(x_1, x_3) = \left[\frac{\Delta\left(\sqrt{\sigma}\right)}{\sqrt{\sigma}} + \frac{c}{\sigma}\right](x_1, x_3). \tag{2.25}$$

By virtue of (2.10) and (2.11), the function $q(x_1, x_3) \in C^{1+\alpha}\left(R^2\right)$ is compactly supported. Denote SP its support. Note that the fundamental solution of the equation $\Delta Z - s^2 Z = -\delta(x_1, x_3)$ in R^2 is $K_0\left(|s| \sqrt{x_1^2 + x_3^2}\right)$, where K_0 is the MacDonald's function (see [7] for details). It is well known that the function $K_0(|s|)$ decays exponentially together with its derivatives as $|s| \to \infty$, and its asymptotic behaviour is given by

$$K_0(|s|) = -\ln|s|(1+o(1)) \quad \text{as} \quad |s| \to 0.$$

Since the function $w \in H^1\left(R^2\right)$ for every $s \neq 0$, then the condition (2.10) and well known results on the smoothness of solutions of elliptic equations imply that $w \in C^{3+\alpha}\left(\overline{P}\right)$ for each bounded domain $P \in R^2, \forall s \neq 0$ (see, e.g., [12], Chapter 8 for details). Hence, the relations (2.24) and (2.25) imply that for all $s \neq 0$

$$v(x', s) = s^2 \tilde{g}(s)\,(s) \qquad \int_{R^2} K_0\left(|s|\left|x' - \xi'\right|\right) \cdot \frac{\tilde{S}_2\left(\xi'\right)}{\sqrt{\sigma\left(\xi'\right)}} d\xi' + \qquad (2.26)$$

$$\int_{R^2} K_0\left(|s|\left|x' - \xi'\right|\right) q\left(\xi'\right) v\left(\xi', s\right) d\xi'.$$

For instance, consider the derivative $\partial_{x_1}^2 v$. One can differentiate the equation (2.26) at least twice, because the function $q\left(x'\right) v\left(x', s\right) = 0$ outside of the bounded domain SP and $q\left(x'\right) v\left(x', s\right) \in C^{1+\alpha}\left(R^2\right), \forall s \neq 0$. Differentiating twice the equation (2.26), we obtain

$$\partial_{x_1}^2 v(x', s) = \int_{R^2} \frac{\partial}{\partial x_1}\left[K_0\left(|s|\left|x' - \xi'\right|\right)\right] \frac{\partial}{\partial \xi_1}\left[s^2 \tilde{g}\left(s\right) \frac{S_2\left(\xi'\right)}{\sqrt{\sigma\left(\xi'\right)}} + q\left(\xi'\right) v\left(\xi', s\right)\right] d\xi'.$$

Hence, $\partial_{x_1}^2 v \in L_2\left(R^2\right)$. This implies that $\partial_{x_1}^2 w \in L_2\left(R^2\right)$. Similarly, one can show that $\partial_{x_1 x_3}^2 w,\ \partial_{x_3}^2 w \in L_2\left(R^2\right)$. Therefore, the functions $\partial_{x_1} w,\ \partial_{x_3} w \in H^1\left(R^2\right), \forall s \neq 0$. On the other hand, differentiating the equation (2.16) with respect to x_1 and denoting $w_1 = w_{x_1}$, we obtain

$$\nabla_{x_1, x_3} \cdot \left(\sigma \nabla w_1\right) - \left(s^2 \sigma + c\right) w_1 = s^2 \tilde{g}\left(s\right) \partial_{x_1} \tilde{S}_2(x_1, x_3) -$$

$$\nabla \cdot \left(\partial_{x_1} \sigma \nabla w\right) - (s^2 \partial_{x_1} \sigma + \partial_{x_1} c) w \quad \text{in} \quad R^2.$$

Since this equation is uniquely solvable in $H^1\left(R^2\right)$, then this solution is $\partial_{x_1} w = w_1$. Thus, for the function $\partial_{x_1} w$ we obtain by analogy with the relation (2.21) that

$$\left(1 + s^2 + \xi_1^2 + \xi_3^2\right)^{1/2} \tilde{w}_1 \in L_2\left(R^3\right). \qquad (2.27)$$

The similar result can be established for the funtion $\partial_{x_3} w = w_3$. Since the Fourier transforms of these functions are $i\xi_1 \tilde{w}$ and $i\xi_3 \tilde{w}$, the relation (2.23) follows from (2.19), (2.21), and (2.27).

■

The following lemma establishes the uniqueness result for the forward problem.

Lemma 2.2 *Suppose the conditions (2.10)-(2.13) hold. Then the forward problem (2.7)- (2.8) has at most one solution* $u \in H^2(R_+^3)$. *Also, the equation (2.16) has at most one weak solution belonging to the space* $H^1\left(R^2\right)$ *for every non-zero value of the parameter* s.

The proof follows directly from (2.17).

2.3 The 3-D/2-D inverse conductivity problem

Our inverse problem is formulated as follows.

Let the conditions (2.10)-(2.13) be fulfilled, and the function $u(x)$ be a solution of the boundary value problem (2.7), (2.8). Let $B = const > 0$. Given the function

$$u(x_1, x_2, 0) = \varphi(x_1, x_2), \quad \forall (x_1, x_2) \in (-B, B) \times (-\infty, \infty), \tag{2.28}$$

determine either the coefficient $\sigma(x')$ or the coefficient $c(x')$ in the domain Ω assuming that another one is known in this domain.

In this formulation, the function $u(x)$ depends on three spatial variables, whereas the unknown coefficients $\sigma(x')$ or $c(x')$ depend only on two variables. This motivates the term the 3-D/2-D inverse conductivity problem. Such a formulation reflects the fact that being locally injected into a 3-D conductive medium, the direct currents generate the voltage potenial distributed all over this medium. Some typical examples of such media include various geological structures elongated in one direction, pipes and slabs and certain organs of a human body, such as the heart and lungs.

Since the number of variables in the data $\varphi(x_1, x_2)$ equals the number of variables in the unknown coefficients, the 3-D/2-D inverse conductivity problem is non-overdetermined. Unlike the Nachman's formulation [8], which is non - overdetermined as well, our formulation takes into account the spatial dependence of the voltage potential $u(x)$. Furthermore, it is not required to measure the voltage potential on the entire boundary of the domain Ω for all possible source positions at the same boundary. Instead, it is sufficient to measure the voltage potential on a strip $(-B, B) \times (-\infty, \infty)$ for a fixed position of the source. In other words, in our formulation, the data are assumed to be incomplete. This is consistent with realistic operational conditions. In [8], the inverse scattering theory for a first order elliptic system was used to establish the global uniqueness result. However, such a theory is not applicable within the framework of our formulation. Instead, the method of Carleman estimates combined with the direct Fourier and inverse Laplace transforms is exploited.

3 The global uniqueness theorem

Denote $k(x')$ any unknown coefficient in the equation (2.7) and formulate the global uniqueness theorem as follows.

Theorem 3.1 *Let the conditions (2.10)-(2.13) be fulfilled. Suppose that $\sigma(x') \in H^5(R^2_+)$ and $c(x') \in H^4(R^2_+)$. Then there exists at most one pair of functions $(k(x'), u(x))$ satisfying the relations (2.7), (2.8), and (2.28), and such that the function $u \in H^2(R^3_+)$.*

Proof. The proof of this theorem is based on the uniqueness result for the auxiliary inverse problem for a hyperbolic equation. Conversely, to establish the latter result, we need to derive both the new Carleman weight function and estimate for the Laplacian $\Delta_{x_1x_3}$ and d'Alembertian $\partial_t^2 - \Delta_{x_1x_3}$. For convenience, we accomplish the proof in four stages.

3.1 The auxiliary inverse problem for a hyperbolic equation

In the first stage, we pose an auxiliary hyperbolic problem and formulate the corresponding inverse problem. Denote $w(x',s)$ the Fourier transform of this solution with respect to the x_2 variable. Consider the even extension of this function to R_-^2 with respect to the x_3 variable preserving the same notation for this extension. By virtue of the condition (2.12), the function $f(x_2)$ is compactly supported and its Fourier transform $\tilde{f}(s)$ is an analytic function. Hence, the function $\tilde{f}(s)$ may have only a finite number of zeros in each finite interval $(s_1, s_2) \subset (-\infty, \infty)$. Denote

$$Y(x',s) = w(x',s) \cdot \frac{1}{\tilde{f}(s)}, \quad \Phi(x_1,s) = \tilde{\varphi}(x_1,s) \cdot \frac{1}{\tilde{f}(s)},$$

where $\tilde{\varphi}(x,s)$ is the Fourier transform of the function $\varphi(x_1,x_2)$ with respect to the x_2 variable. Hence, $Y(x_1,x_3,s) = Y(x_1,-x_3,s)$. By virtue of (2.16), we have

$$\nabla \cdot (\sigma \nabla Y) - s^2 \sigma Y - cY = -S_2(x') \quad \text{in} \quad R^2, \ \forall s \neq 0, \tag{3.1}$$

$$\partial_{x_3} Y(x',s) \mid_{x_3=0} = 0, \tag{3.2}$$

$$Y(x',s) \in H^2\left(R^2\right), \ \forall s \neq 0, \tag{3.3}$$

Then we obtain

$$Y(x',s) \mid_{x_3=0} = \Phi(x_1,s) \quad \text{for} \quad x_1 \in (-B,B), \ \forall s \neq 0. \tag{3.4}$$

Let $\Omega' = \{(x_1,x_3) \in R_-^2 : (x_1,-x_3) \in \Omega\}$ and $\Omega'' = \Omega \bigcup \Omega'$. Since the functions $\sigma(x')$ and $c(x')$ are known in $R^2 \setminus \Omega''$, it follows from the uniqueness theorem for the Cauchy problem for elliptic equations and (3.1) - (3.2) that the function $Y(x',s)$ is known for all $(x',s) \in \left(R^2 \setminus \Omega''\right) \times (0,\infty)$. Hence, the following function $\psi_1(x',s)$ can be determined as

$$Y(x',s) = \psi_1(x',s), \quad \frac{\partial Y(x',s)}{\partial \nu} = \psi_2(x',s), \ \forall (x',s) \in (R^2 \backslash \Omega'') \times (0,\infty). \tag{3.5}$$

Consider an auxiliary Cauchy problem for the hyperbolic equation

$$\sigma \partial_t^2 v = \nabla \cdot (\sigma \nabla v) - cv, \quad (x',t) \in R^2 \times (0,\infty), \tag{3.6}$$

$$v \mid_{t=0} = 0, \quad \partial_t v \mid_{t=0} = S_2(x') \sigma^{-1}(x'). \tag{3.7}$$

By the embedding theorem, $\sigma \in C^3(R^2)$ and $c \in C^2(R^2)$. Using the standard energy estimates and embedding theorems (see, e.g., [13], Chapter 4), one can prove that for each $T > 0$ there exists a unique solution $v \in H^7\left(R^2 \times (0,T)\right)$

of this problem and $v(x',t) = 0$ for all $(x',t) \in \{|x| > R, t \in (0,T)\}$, where $R = R(T) > 0$ is a certain number. Also, $v \in C^5(R^2 \times [0,T])$ and there exists the positive constants C and s_0 depending only on functions $S_2(x'), \sigma(x')$ and $c(x')$, such that

$$|D^\alpha v(x',t)| \leq Ce^{s_0 t}, \quad (x',t) \in R^2 \times (0,\infty),$$

$$\int_{R^2} |D^\alpha v(x',t)|^2 dx' \leq Ce^{s_0 t}, \quad t \in (0,\infty)$$

for $|\alpha| \leq 5$. Therefore, the Laplace transform

$$\tilde{v}(x',s) = \int_0^\infty v(x',t)e^{-st}dt, s > s_0$$

exists for all $s > s_0$, and the function $\tilde{v}(x',s) \in H^5\left(R^2\right)$ for all $s > s_0$. Hence, by virtue of the lemma 2.2 and (3.1), (3.3), (3.6), and (3.7), we have $\tilde{v}(x',s) = Y(x',s)$ for all $s > s_0$. Since the Laplace transform is an one-to-one operator, then the conditions (3.5) imply that the function $\psi_3(x',t)$ and $\psi_4(x',t)$, such that

$$v(x',t) = \psi_3(x',t)$$

can be uniquely determined for all $(x',t) \in (R^2 \setminus \Omega'') \times (0,\infty)$. Hence, to prove the theorem 3.1, it is sufficient to establish the uniqueness result for an auxiliary inverse problem.

Let $G \subset R^2$ be a disk. Assume that $\sigma(x') = 1$ and $c(x') = 0$ for $x' \in R^2 \setminus G$, $\sigma(x') \geq 1$ for $x' \in R^2$, and the functions $\sigma \in C^6(R^2)$, $c \in C^4(R^2)$, $S_2 \in C_0^\infty(R^2)$, and $S_2(x_1,x_3) \neq 0$ in $\overline{G}$. Let $v(x',t) \in C^5(R^2 \times (0,\infty))$ be the solution of the Cauchy problem (3.6), (3.7). Given the functions $\psi_4(x',t)$ and $\psi_5(x',t)$, such that

$$v(x',t) \mid_{\partial G} = \psi_4(x',t), \quad \frac{\partial v(x',t)}{\partial \nu} \mid_{\partial G} = \psi_5(x',t) \tag{3.8}$$

for all $(x',t) \in \partial G \times (0,\infty)$, find the coefficient $\sigma(x')$ or the coefficient $c(x')$ in G assuming that another coefficient is known.

3.2 A Carleman weight function for the Laplacian

In the second stage, we derive a new Carleman Weight Function (CWF) for the Laplacian. Unlike [2] (Chapter 2) and [14] (Chapter 4), the level sets of this CWF are spheres. Furthermore, the corresponding Carleman estimate is constructed for the Cauchy data given on the entire boundary of a bounded domain, which actually is a ball. In the standard CWF, the level sets are paraboloids, and it is constructed for the Cauchy data given on a part of the boundary. For the sake of explicitness, the notations in sections 3.2, 3.3 and 3.4 may differ from the corresponding notations indicated above.

Lemma 3.2 *Let $R = const > 0$, $r \in (0, R)$, and $P(r, R) = \{x \in R^n : r < |x| < R\}$, where $(n = 1, ..., 13)$. Then the function $\tilde{\mathcal{C}}(x)$*

$$\tilde{\mathcal{C}}(x) = \exp\left(\lambda |x|^2\right), \lambda > 0$$

is a CWF for the Laplacian in the domain $P(r, R)$. That is, there exists a positive constant $C = C(r, R)$ and a sufficiently large positive constant $\lambda_0 = \lambda_0(r, R)$, such that the following pointwise Carleman estimate holds in $P(r, R)$ for all functions $u \in C^2\left(\overline{P}(r, R)\right)$ and for all $\lambda \geq \lambda_0$

$$(\Delta u)^2 \tilde{\mathcal{C}}^2 \geq C\lambda |\nabla u|^2 \tilde{\mathcal{C}}^2 + C\lambda^3 u^2 \tilde{\mathcal{C}}^2 + \nabla \cdot U, \tag{3.9}$$

where U is a certain vector function satisfying the inequality

$$|U| \leq C\left(\lambda |\nabla u|^2 + \lambda^3 u^2\right)\tilde{\mathcal{C}}^2.$$

Proof.

In this proof, the notation C denotes different positive constants depending on r, R, and n. Denote $v(x) = u(x)\widetilde{\mathcal{C}}(x)$ and express Δu in terms of the function v. We have

$$u = v \exp\left(-\lambda |x|^2\right),$$

$$u_i = (v_i - 2\lambda x_i v) \exp\left(-\lambda |x|^2\right),$$

$$u_{ii} = \left[v_{ii} - 4\lambda x_i v_i + 4\lambda^2 x_i^2 \left(1 + O\left(\frac{1}{\lambda}\right)\right) v\right] \exp\left(-\lambda |x|^2\right).$$

Here and below, the notation $O\left(\frac{1}{\lambda}\right)$ denotes different $C^1\left(\overline{P}(r, R)\right)$ functions satisfying the inequality

$$\left|O\left(\frac{1}{\lambda}\right)\right| \leq \frac{C}{\lambda}, \quad \forall \lambda > 1.$$

Thus, we obtain

$$(\Delta u)^2 \tilde{\mathcal{C}}^2(x) = \left[\left(\Delta v + 4\lambda^2 |x|^2 \left(1 + O\left(\frac{1}{\lambda}\right)\right) v\right) - 4\lambda \sum_{i=1}^{n} x_i v_i\right],$$

$$\geq 2z_3 (z_1 + z_2) + (z_1 + z_2)^2, \tag{3.10}$$

where

$$z_1 = \Delta v, \quad z_2 = 4\lambda^2 |x|^2 \left(1 + O\left(\frac{1}{\lambda}\right)\right) v,$$

$$z_3 = -4\lambda \sum_{i=1}^{n} x_i v_i.$$

Step 1. We estimate $2z_3z_1$. Let $z_{3i}=-4\lambda x_iv_i$. We have

$$2z_{3i}z_1=-8\lambda\sum_{j=1}^{n}x_iv_iv_{jj}=\sum_{j=1}^{n}\left(-8\lambda x_iv_iv_j\right)_j+\sum_{j=1}^{n}8\lambda x_iv_{ij}v_j+8\lambda v_i^2=$$

$$\sum_{j=1}^{n}\left(-8\lambda x_iv_iv_j\right)_j+\sum_{j=1}^{n}\left(4\lambda x_iv_j^2\right)_i-4\lambda\left|\nabla v\right|^2+8\lambda v_i^2.$$

Hence, we obtain

$$2z_3z_1=-4\lambda\left(n-2\right)\left|\nabla v\right|^2+\nabla\cdot U_1, \tag{3.11}$$

where

$$\left|U_1\right|\leq C\left(\lambda\left|\nabla u\right|^2+\lambda^3u^2\right)\tilde{\mathcal{C}}^2. \tag{3.12}$$

Step 2. We estimate $2z_2z_3$. We have

$$\begin{aligned}2z_2z_3 &= -32\lambda^3\sum_{i=1}^{n}x_i\left|x\right|^2\left(1+O\left(\frac{1}{\lambda}\right)\right)v_iv=\\ &\sum_{i=1}^{n}\left(-16\lambda^3x_i\left|x\right|^2\left(1+O\left(\frac{1}{\lambda}\right)\right)v^2\right)_i+\\ &\sum_{i=1}^{n}\left(16\lambda^3\left|x\right|^2v^2+32\lambda^3x_i^2v^2\right)\left(1+O\left(\frac{1}{\lambda}\right)\right)\geq\\ &15\lambda^3\left|x\right|^2\left(n+2\right)v^2+\nabla\cdot U_2.\end{aligned}$$

Thus, we obtain

$$2z_2z_3\geq 15\lambda^3\left|x\right|^2\left(n+2\right)v^2+\nabla\cdot U_2, \tag{3.13}$$

where

$$\left|U_2\right|\leq C\lambda^3u^2\tilde{\mathcal{C}}^2. \tag{3.14}$$

Summing up the inequalities (3.11)-(3.14), we obtain

$$2z_3\left(z_1+z_2\right)\geq -4\lambda\left(n-2\right)\left|\nabla v\right|^2+15\lambda^3\left|x\right|^2\left(n+2\right)v^2+\nabla\cdot U_3, \tag{3.15}$$

where

$$\left|U_3\right|\leq C\left(\lambda\left|\nabla u\right|^2+\lambda^3u^2\right)\tilde{\mathcal{C}}^2. \tag{3.16}$$

One can see that in the estimate (3.15), the term with $\left|\nabla v\right|^2$ contains a non-positive multiplier $-4\lambda\left(n-2\right)$ (if $n\geq 2$), which is inconvenient. Therefore, we proceed further with

Step 3. We estimate $\left(z_1+z_2\right)^2$. Let α be a positive number. It is important that α is independent of the parameter λ. Then we have

$$\left(z_1+z_2\right)^2=\left(z_1+z_2+\lambda\alpha v-\lambda\alpha v\right)^2\geq -2\lambda\alpha vz_1-2\lambda\alpha v\left(z_2+\lambda\alpha v\right). \tag{3.17}$$

We estimate separately each of two terms in the right hand side of (3.17).

$$-2\lambda\alpha v(z_2+\lambda\alpha v) = -2\lambda\alpha v$$
$$[4\lambda^2|x|^2(1+O(\frac{1}{\lambda}))v+\lambda\alpha v] \geq -10\lambda^3\alpha|x|^2v^2 \tag{3.18}$$

and

$$-2\lambda\alpha vz_1 = \sum_{i=1}^{n} -2\lambda\alpha vv_{ii} = \sum_{i=1}^{n}(-2\lambda\alpha vv_i)_i + \sum_{i=1}^{n} 2\lambda\alpha v_i^2 =$$
$$2\lambda\alpha\,|\nabla v|^2 + \nabla\cdot U_4. \tag{3.19}$$

Summing up the inequalities (3.15), (3.18), and (3.19) and taking into account the inequalities (3.10) and (3.16), we obtain

$$(\Delta u)^2\,\tilde{\mathcal{C}}^2(x) \geq 2\lambda\,(\alpha-2n+4)\,|\nabla v|^2 + 15\lambda^3\,|x|^2\left(n+2-\frac{2}{3}\alpha\right)v^2 + \nabla\cdot U, \tag{3.20}$$

$$|U| \leq C\left(\lambda\,|\nabla u|^2+\lambda^3u^2\right)\tilde{\mathcal{C}}^2.$$

Choose the number

$$\alpha = \frac{7}{4}n - \frac{1}{2}.$$

Then we obtain

$$\alpha-2n+4 = \frac{14-n}{4} > 0,$$
$$n+2-\frac{2}{3}\alpha = \frac{14-n}{6} > 0.$$

Hence, the estimate (3.20) implies that

$$(\Delta u)^2\,\tilde{\mathcal{C}}^2(x) \geq C\lambda\,|\nabla v|^2 + C\lambda^3v^2 + \nabla\cdot U.$$

Finally, one should replace v with $u = v\cdot\tilde{\mathcal{C}}^{-1}$ and $C\lambda\,|\nabla v|^2$ with $\varepsilon C\lambda\,|\nabla v|^2$, where $\varepsilon \in (0,1)$ is an appropriate number. Then we obtain the estimate (3.9). ■

3.3 A pointwise Carleman estimate for the d'Alembertian

In the third stage, we derive the Carleman estimate for the operator $\partial_t^2 - \Delta$. To simplify the presentation, we assume that the domain Ω is a ball, i.e., $\Omega = \{|x| < R\} \subset R^n$. Let $T = const > R$. Denote $Q'_T = \Omega\times(-T,T)$, $S'_T = \partial\Omega\times(-T,T)$, and

$$\psi(x,t) = |x|^2 - \eta t^2, \quad \mathcal{C}(x,t) = \exp\left[\lambda\psi(x,t)\right],$$

where the parameter $\eta \in (0,1)$. It is well known that the function $\mathcal{C}(x,t)$ is the CWF for the operator $\partial^2/\partial t^2 - \Delta$ (see, [2], Chapter 2 and [14], Chapter 4). Choose a constant $c \in (0,R)$, such that

$$\frac{R^2 - c^2}{\eta} < T^2. \tag{3.21}$$

Consider the domain G_{c^2}, which is an intersection of a hyperboloid with the cylinder Q_T', i.e.,

$$G_{c^2} = \left\{ (x,t) : |x|^2 - \eta t^2 > c^2, |x| \le R \right\}.$$

Then we have that $G_{c^2} \cap \{|x| = R\} \subset S_T'$. The boundary

$$\partial G_{c^2} = \partial_1 G_{c^2} \cup \partial_2 G_{c^2},$$

of the domain G_{c^2} consists of two parts

$$\partial_1 G_{c^2} = \left\{ \psi(x,t) = c^2 \right\} \cap \{|x| \le R\}$$

and

$$\partial_2 G_{c^2} = \overline{G}_{c^2} \cap S_T'.$$

The following lemma establishes the pointwise Carleman estimate (see [2], Chapter 2 for the proof).

Lemma 3.3 *Let $G_{c^2}^+ = G_{c^2} \cap \{t > 0\}$, $G_{c^2}^- = G_{c^2} \cap \{t < 0\}$. The following pointwise Carleman estimate holds for all functions $U \in H^2(G_{c^2}) \bigcap \left[C^2\left(\overline{G}_{c^2}^+\right) \bigcup C^2\left(\overline{G}_{c^2}^-\right)\right]$ and for all $\lambda \ge \lambda_0 > 1$*

$$\left(\partial_t^2 U - \Delta U\right)^2 \mathcal{C}^2(x,t) \ge C\lambda \left[|\nabla U|^2 + (\partial_t U)^2 \right] \mathcal{C}^2(x,t) + C\lambda^3 |U|^2 \mathcal{C}^2(x,t) + \nabla \cdot W + \partial_t V,$$

where the positive constants C, λ_0 depend only on the domain G_{c^2} and they are independent of the function U. The vector function $\left(\tilde{U}, \tilde{V}\right)$ satisfies the inequality

$$|(W,V)| \le C \left[\lambda \left(|\nabla U|^2 + U_t^2 \right) + \lambda^3 |U|^2 \right] \mathcal{C}^2(x,t).$$

Therefore,

$$(W,V) \mid_{S_T'} = 0 \quad \text{if} \quad U \mid_{S_T'} = \frac{\partial U}{\partial \nu} \mid_{S_T'} = 0.$$

Also, there exists the limit $\lim_{t \to 0} V(x,t)$. This implies that

$$\int_{G_{c^2}} V_t dxdt = \int_{\partial_1 G_{c^2}} V \cos(\nu, t) dS.$$

Lemma 3.4 *For all real functions $f \in L_2(G_{c^2})$ and for all $\lambda \ge 1$ the following estimate holds*

$$\int_{G_{c^2}} \left[\int_0^t f(x,\tau) d\tau \right]^2 \mathcal{C}^2(x,t) dxdt \le \frac{1}{\lambda \eta} \int_{G_{c^2}} \left(f^2 \mathcal{C}^2 \right)(x,t) dxdt,$$

Proof. Let $G_{c^2}^0 = G_{c^2} \cap \{t = 0\}$. For each point $(x, 0) \in G_{c^2}^0$ denote $(x, t^+(x))$ the intersection of the straight line passing through this point and parallel to the $t-$ axis with the positive part of the surface of the hyperboloid $\left\{(x,t) : |x|^2 - \eta t^2 = c^2, t > 0\right\}$. Hence,

$$t^+(x) = \frac{\sqrt{|x|^2 - c^2}}{\sqrt{\eta}}.$$

Thus, we have consequtively

$$\int_{G_{c^2}^+} \left[\int_0^t f(x,\tau)d\tau\right]^2 \mathcal{C}^2(x,t)dxdt = \int_{G_{c^2}^0} dx \int_0^{t^+(x)} \left[\int_0^t f(x,\tau)d\tau\right]^2 \mathcal{C}^2(x,t)dt \leq$$

$$\int_{G_{c^2}^0} dx \int_0^{t^+(x)} \mathcal{C}^2(x,t)t\left[\int_0^t f^2(x,\tau)d\tau\right]dt =$$

$$\int_{G_{c^2}^0} \exp\left(2\lambda |x|^2\right) dx \int_0^{t^+(x)} \exp\left(-2\lambda\eta t^2\right) t \left[\int_0^t f^2(x,\tau)d\tau\right] dt =$$

$$\int_{G_{c^2}^0} \exp\left(2\lambda |x|^2\right) dx \int_0^{t^+(x)} f^2(x,\tau)d\tau \int_\tau^{t^+(x)} \exp\left(-2\lambda\eta t^2\right) tdt =$$

$$-\frac{1}{\lambda\eta}\int_{G_{c^2}^0} \exp\left(2\lambda |x|^2\right) dx \int_0^{t^+(x)} f^2(x,\tau)d\tau \int_\tau^{t^+(x)} \frac{d}{dt}\left[\exp\left(-2\lambda\eta t^2\right)\right] dt =$$

$$\frac{1}{\lambda\eta}\int_{G_{c^2}^0} \exp\left(2\lambda |x|^2\right) dx \int_0^{t^+(x)} f^2(x,\tau)\left[\exp\left(-2\lambda\eta\tau^2\right) - \exp\left(-2\lambda\eta^2 t^+(x)\right)\right] d\tau \leq$$

$$\frac{1}{\lambda\eta}\int_{G_{c^2}^0} \exp\left(2\lambda |x|^2\right) dx \int_0^{t^+(x)} f^2(x,\tau)\exp\left(-2\lambda\eta\tau^2\right) d\tau =$$

$$\frac{1}{\lambda\eta}\int_{G_{c^2}^+} \left(f^2\mathcal{C}^2\right)(x,t)dxdt.$$

The integral over $G_{c^2}^-$ can be estimated by analogy. ■

3.4 The main lemma

In the fourth stage, we complete the proof by establishing the global uniqueness of an auxiliary inverse problem for a hyperbolic equation.

Denote $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$. Let $a(x), f(x) \in C^2\left(\overline{\Omega}\right), F \in C^4\left(R^{2n+3}\right)$. Let the function $u(x,t)$ be a solution of the following hyperbolic initial boundary value problem

$$\partial_t^2 u = \Delta u + F\left(\Delta a, \nabla a, a, \nabla u, u\right) \text{ in } Q_T, \tag{3.22}$$

$$u\mid_{t=0}= f(x), \quad \partial_t u\mid_{t=0}= 0, \tag{3.23}$$

$$u\mid_{S_T}= \varphi(x,t). \tag{3.24}$$

Without loss of generality, we assume that the function $F(\Delta a, \nabla a, a, \nabla u, u)$ is linear with respect to ∇u and u.

The inverse problem for the auxiliary initial boundary value problem (3.22)-(3.24) is formulated as follows.

Let the function $u(x,t)$ satisfies the problem (3.22)-(3.24). Given the functions F, f, φ, and

$$\frac{\partial u}{\partial \nu}\mid_{S_T}= \psi(x,t), \tag{3.25}$$

where ν is the outward normal vector on S_T, determine the function $a(x)$.

Note that this problem differs essentially from analogous problems indicated in [2]. The main distinction is that in our formulation, the unknown coefficient $a(x)$ is contained in the equation together with ∇a and Δa. This requires the essential changes in the method of Carleman estimates indicated in [2] (see Chapter 3). The following lemma establishes the uniqueness result for this inverse problem.

Lemma 3.5 *Suppose that the function F is linear with respect to $\partial_{x_i} u$ and u, $F \in C^4(R^{2n+3})$, $T > R = diam(\Omega)/2$, the functions*

$$a(x)\mid_{\partial\Omega}, \quad \frac{\partial a(x)}{\partial \nu}\mid_{\partial\Omega}$$

are known and the following condition holds

$$\frac{\partial F(\Delta a(x), \nabla a(x), a(x), \nabla f(x), f(x))}{\partial(\Delta a)} \neq 0 \quad in \quad \overline{\Omega}. \tag{3.26}$$

Let $n \in [1,13]$. Then there exists at most one pair of functions $(a,u) \in C^2\left(\overline{\Omega}\right) \times C^4\left(\overline{Q}_T\right)$ satisfying the conditions (3.22)-(3.25).

Proof. We conduct the proof by contradiction. Suppose that there exist two such pairs (a_1,u_1) and (a_2,u_2). Denote $b(x) = a_1(x) - a_2(x), \tilde{u}(x,t) = u_1(x,t) - u_2(x,t)$, Then the relations (3.22)-(3.26) imply that

$$\tilde{u}_{tt} = \Delta\tilde{u} + \sum_{j=1}^{n} c^j(x,t)\tilde{u}_j + c^0(x,t)\tilde{u} + g(x,t)\Delta b(x) + \tag{3.27}$$

$$(h(x,t), \nabla b(x)) + h^0(x,t)b(x),$$

$$\tilde{u}\mid_{t=0}= \tilde{u}_t\mid_{t=0}= 0, \tag{3.28}$$

$$\tilde{u}\mid_{S_T}= \frac{\partial\tilde{u}}{\partial\nu}\mid_{S_T}= 0, \tag{3.29}$$

where

$$c^j,\ c^0,\ g,\ h,\ h^0 \in C^2\left(\overline{Q}_T\right), \tag{3.30}$$

$$h_t(x,0) = h_t^0(x,0) = g_t(x,0) = 0, \tag{3.31}$$

and

$$g(x,t) \neq 0 \text{ in } \overline{\Omega}. \tag{3.32}$$

In $\Omega \times (-T, 0)$, consider the even extensions of all coefficients in the equation (3.27) including the functions $g(x,t), h^0(x,t)$ and the vector function $h(x,t)$ with respect to the t variable. For convenience, we preserve the same notations for such extensions. Let $d(x,t)$ be one of such functions. Then, the relations (3.30) and (3.31) imply that

$$d \in C^2\left(\overline{Q}_T\right) \cup C^2\left(\overline{\Omega} \times [-T, 0]\right).$$

Let $\hat{u}(x,t)$ be the even extension of the function $\tilde{u}(x,t)$ in $\Omega \times (-T, 0)$

$$\hat{u}\left(x,t\right) = \begin{cases} \tilde{u}(x,t) & \text{if} \quad t > 0 \\ \tilde{u}(x,-t) & \text{if} \quad t < 0. \end{cases}$$

Consider the functions $v(x,t) = \hat{u}_t\left(x,t\right), w(x,t) = \partial_t v(x,t) = \partial_t^2 \hat{u}\left(x,t\right)$. Then the relations (3.27), (3.28), and (3.31) imply that

$$v(x,0) = 0,$$

$$\lim_{t \to 0^+} v_t(x,t) = \lim_{t \to 0^-} v_t(x,t) = \partial_t^2 \hat{u}(x,0) = w(x,0) =$$

$$g(x,0)\Delta b(x) + (h(x,0), \nabla b(x)) + h^0(x,0)b(x). \tag{3.33}$$

Also, since by virtue of (3.27) and (3.31)

$$\lim_{t \to 0^+} D_t^3 \hat{u}_t(x,t) = \lim_{t \to 0^-} D_t^3 \hat{u}(x,t) = 0$$

and $D_t^3 \hat{u}_t(x,t) = w_t(x,t)$, then we obtain

$$\lim_{t \to 0^+} w_t(x,t) = \lim_{t \to 0^-} w_t(x,t) = 0.$$

Hence, the functions v, w belong to the space $H^2\left(Q_T'\right)$. This implies that

$$v, w \in H^2\left(G_{c^2}\right) \bigcap \left[C^2\left(\overline{G}_{c^2}^+\right) \bigcup C^2\left(\overline{G}_{c^2}^-\right)\right].$$

Thus, we may apply the lemma 3.3 to $\partial_t^2 v - \Delta v$ and $\partial_t^2 w - \Delta w$.

Orserve that

$$\hat{u}(x,t) = \int_0^t v(x,\tau)d\tau, \quad v(x,t) = \int_0^t w(x,\tau)d\tau \text{ in } Q_T'.$$

Hence, differentiating (3.27), we obtain that the functions $v(x,t)$ and $w(x,t)$ satisfy the folowing relations in Q_T'

$$\left|\partial_t^2 v - \nabla^2 v\right| \leq M\left(|\nabla v| + |v| + sgn(t)\int_0^t \left(|\nabla v| + |v|\right)(x,\tau)d\tau\right) + M\left(|\Delta b| + |\nabla b| + |b|\right), \tag{3.34}$$

$$\partial_t v(x,0) = g(x,0)\Delta b(x) + (h(x,0), \nabla b(x)) + h^0(x,0)b(x), \tag{3.35}$$

$$v\mid_{S'_T} = \frac{\partial v}{\partial \nu}\mid_{S'_T} = 0, \tag{3.36}$$

$$\left|\partial_t^2 w - \Delta w\right| \le M\left(|\nabla w| + |w| + sgn(t)\int_0^t \left(|\nabla w| + |w| + |\nabla v| + |v|\right)(x,\tau)d\tau\right) +$$
$$M\left(|\Delta b| + |\nabla b| + |b|\right), \tag{3.37}$$

$$w\mid_{S'_T} = \frac{\partial w}{\partial \nu}\mid_{S'_T} = 0. \tag{3.38}$$

Throughout this proof, the notation M denotes different positive constants depending on $C^2\left(\overline{Q}_T\right)$- norms of functions c^j, c^0, g, h, h^0.

Now we use the lemma 3.4. Multiplying both sides of (3.34) by the CWF $\mathcal{C}(x,t)$, squaring them and integrating over G_{c^2}, we obtain

$$\int_{G_{c^2}} \left|\partial_t^2 v - \Delta v\right|^2 \mathcal{C}^2 dxdt \le M\int_{G_{c^2}} \left(|\nabla v|^2 + v^2\right)\mathcal{C}^2 dxdt + \int_{G_{c^2}} \left[(\Delta b)^2 + |\nabla b|^2 + b^2\right]\mathcal{C}^2 dxdt.$$

Using the Carleman estimate from the lemma 3.3, for suffciently large $\lambda \ge \lambda_1(M)$ we obtain

$$\int_{G_{c^2}} \left[(\Delta b)^2 + |\nabla b|^2 + b^2\right]\mathcal{C}^2 dxdt \ge$$

$$C\lambda\int_{G_{c^2}} (\partial_t v)^2 \mathcal{C}^2 dxdt + C\lambda\int_{G_{c^2}} |\nabla v|^2 \mathcal{C}^2 dxdt + C\lambda^3\int_{G_{c^2}} v^2\mathcal{C}^2 dxdt - \tag{3.39}$$
$$C\exp\left(2\lambda c^2\right)\int_{\partial_1 G_{c^2}} \left(\lambda(\partial_t v)^2 + \lambda|\nabla v|^2 + \lambda^3 v^2\right)dS.$$

On the other hand, by virtue of (3.35), we have

$$\partial_t v(x,t) = \partial_t v(x,0) + \int_0^t \partial_t^2 v(x,\tau)d\tau = g(x,0)\Delta b(x) +$$

$$(h(x,0), \nabla b(x)) + h^0(x,0)b(x) + \int_0^t \partial_t w(x,\tau)d\tau.$$

Because of (3.32), one may assume that $|g(x,0)| \ge 1$ in $\overline{\Omega}$. Hence, we have

$$(\partial_t v)^2(x,t) \ge |\Delta b(x)|^2 - M\left(|\nabla b(x)|^2 + |b(x)|^2\right) - M\left[\int_0^t w_t(x,\tau)d\tau\right]^2. \tag{3.40}$$

Hence, the relations (3.39), (3.40) and the lemma 3.4 imply that for $\lambda \ge \lambda_1(M)$

$$C\exp\left(2\lambda c^2\right)\int_{\partial_1 G_{c^2}} \left(\lambda(\partial_t v)^2 + \lambda|\nabla v|^2 + \lambda^3 v^2\right)dS \ge$$

$$C\lambda \int_{G_{c^2}} |\nabla v|^2 \mathcal{C}^2 dxdt + C\lambda^3 \int_{G_{c^2}} v^2 \mathcal{C}^2 dxdt - M \int_{G_{c^2}} (\partial_t w)^2 \mathcal{C}^2 dxdt + \tag{3.41}$$

$$C\lambda \int_{G_{c^2}} (\Delta b)^2 \mathcal{C}^2 dxdt - M\lambda \int_{G_{c^2}} \left(|\nabla b|^2 + b^2\right) \mathcal{C}^2 dxdt.$$

Analogously, using the relations (3.37), (3.38) and lemmata 3.3, 3.4, we obtain

$$C \exp\left(2\lambda c^2\right) \int_{\partial_1 G_{c^2}} \left(\lambda (\partial_t w)^2 + \lambda |\nabla w|^2 + \lambda^3 w^2\right) dS +$$

$$M \int_{G_{c^2}} \left(|\nabla v|^2 + v^2\right) \mathcal{C}^2 dxdt + M \int_{G_{c^2}} \left((\Delta b)^2 + |\nabla b|^2 + b^2\right) \mathcal{C}^2 dxdt \geq$$

$$C\lambda \int_{G_{c^2}} \left(|\nabla w|^2 + (\partial_t w)^2 + \lambda^2 w^2\right) \mathcal{C}^2 dxdt.$$

Summing up this inequality with (3.41), we obtain for sufficiently large $\lambda \geq \lambda_1(M)$

$$C \exp\left(2\lambda c^2\right) \lambda \int_{\partial_1 G_{c^2}} \left((\partial_t w)^2 + |\nabla w|^2 + |\nabla v|^2 + (\partial_t v)^2 + \lambda^2 w^2 + \lambda^2 v^2\right) dS \geq$$

$$\lambda \int_{G_{c^2}} \left(|\nabla w|^2 + (\partial_t w)^2 + |\nabla v|^2 + \lambda^2 w^2 + \lambda^2 v^2\right) \mathcal{C}^2 dxdt + \tag{3.42}$$

$$\lambda \int_{G_{c^2}} (\Delta b)^2 \mathcal{C}^2 dxdt - M\lambda \int_{G_{c^2}} \left(|\nabla b|^2 + b^2\right) \mathcal{C}^2 dxdt.$$

Denote

$$t_+ = \sqrt{\frac{R^2 - c^2}{\eta}}, \quad t_- = -\sqrt{\frac{R^2 - c^2}{\eta}},$$

$$K(t) = \left\{x : \sqrt{c^2 + \eta t^2} < |x| < R\right\}, \ \partial_1 K(t) = \left\{x : |x| = \sqrt{c^2 + \eta t^2}\right\}.$$

Then, for any function $f \in C\left(\overline{G}_{c^2}\right)$, we have

$$\int_{G_{c^2}} f(x,t) dxdt = \int_{t_-}^{t_+} dt \int_{K(t)} f(x,t) dx.$$

Recall that

$$b \mid_{\partial\Omega} = \frac{\partial b}{\partial \nu} \mid_{\partial\Omega} = 0$$

These conditions, the lemma 3.2 and the Gauss-Ostrogradsky's formula imply that

$$C\lambda \int_{G_{c^2}} (\Delta b)^2 \mathcal{C}^2 dxdt = C\lambda \int_{t_-}^{t_+} \exp\left(-2\lambda\eta t^2\right) dt \int_{K(t)} (\Delta b)^2 \exp\left(2\lambda |x|^2\right) dx \geq$$

$$C\lambda^2 \int_{t_-}^{t_+} \exp\left(-2\lambda\eta t^2\right) dt \int_{K(t)} |\nabla b|^2 \exp(2\lambda|x|^2) dx +$$

$$C\lambda^4 \int_{t_-}^{t_+} \exp\left(-2\lambda\eta t^2\right) dt \int_{K(t)} b^2 \exp\left(2\lambda\left|x\right|^2\right) dx -$$

$$C\lambda^2 \int_{t_-}^{t_+} [\exp\left(-2\lambda\eta t^2\right) \exp\left(2\lambda(c^2+\eta t^2)\right)] dt \int_{\partial_1 K} \left(\left|\nabla b\right|^2 + \lambda^2 b^2\right) ds =$$

$$C\lambda^2 \int_{G_{c^2}} \left(\left|\nabla b\right|^2 + \lambda^2 b^2\right) \mathcal{C}^2 dx dt - C\lambda^2 \exp(2\lambda c^2) \int_{\partial_1 G_{c^2}} \left(\left|\nabla b\right|^2 + \lambda^2 b^2\right) dS.$$

Hence,

$$C\lambda \int_{G_{c^2}} (\Delta b)^2 \mathcal{C}^2 dx dt \geq C\lambda^2 \int_{G_{c^2}} \left(\left|\nabla b\right|^2 + \lambda^2 b^2\right) \mathcal{C}^2 dx dt -$$

$$C\lambda^2 \exp(2\lambda c^2) \int_{\partial_1 G_{c^2}} \left(\left|\nabla b\right|^2 + \lambda^2 b^2\right) dS.$$

Combining this estimate with (3.42), we obtain

$$C \exp\left(2\lambda c^2\right) \lambda \int_{\partial_1 G_{c^2}} \left((\partial_t w)^2 + \left|\nabla w\right|^2 + \left|\nabla v\right|^2 + (\partial_t v)^2 + \lambda^2 w^2 + \lambda^2 v^2 + \lambda \left|\nabla b\right|^2 + \lambda^3 b^2\right) dS \geq$$

$$\lambda \int_{G_{c^2}} \left(\left|\nabla w\right|^2 + (\partial_t w)^2 + \left|\nabla v\right|^2 + \lambda^2 w^2 + \lambda^2 v^2\right) \mathcal{C}^2 dx dt +$$

$$\lambda^2 \int_{G_{c^2}} \left(\left|\nabla b\right|^2 + \lambda^2 b^2\right) \mathcal{C}^2 dx dt.$$

Choose a sufficiently small number $\varepsilon > 0$ and consider the domain $G_{c^2+\varepsilon} \subset G_{c^2}$. Since $\mathcal{C}^2(x,t) \geq \exp\left[2\lambda\left(c^2+\varepsilon\right)\right]$ in this domain, then the latter estimate implies that

$$C \exp\left(2\lambda c^2\right) \lambda \int_{\partial_1 G_{c^2}} \left((\partial_t w)^2 + \left|\nabla w\right|^2 + \left|\nabla v\right|^2 + \lambda^2 w^2 + \lambda^2 v^2 + \lambda \left|\nabla b\right|^2 + \lambda^3 b^2\right) dS \geq$$

$$C\lambda^4 \exp\left[2\lambda\left(c^2+\varepsilon\right)\right] \int_{G_{c^2+\varepsilon}} b^2(x) dx dt.$$

Dividing this inequality by $\exp\left[2\lambda\left(c^2+\varepsilon\right)\right]$, letting $\lambda \to \infty$ and noticing that ε is an arbitrary positive number, we obtain

$$b(x) = 0 \quad \text{in} \quad \{x : |x| \in (c, R)\}. \tag{3.43}$$

Since $T > R$, $\eta \in (0,1)$ is an arbitrary number, and the number $c = c(\eta, T) \in (0, R)$ is chosen to satisfy the condition (3.21), we obtain from (3.43) that $b(x) = 0$ in Ω. This implies that $\tilde{u}(x,t) = 0$ in Q_T. ■

Now we complete the proof of the theorem 3.1 by proving the uniqueness of the auxiliary inverse problem formulated in the section 3.1. Let $w(x',t) = \sigma(x')\partial_t v(x',t)$. Then we have that $w \in C^4\left(R^2 \times [0,\infty)\right)$. This is the classical solution of the following hyperbolic Cauchy problem

$$\partial_t^2 w = \nabla^2 w - \frac{\nabla\sigma}{\sigma} \cdot \nabla w - q\left(\Delta\sigma, \nabla\sigma, \sigma, c\right) w, \quad (x',t) \in R^2 \times (0,\infty)\,, \quad (3.44)$$

$$w\mid_{t=0} = S_2\left(x'\right), \quad \partial_t w\mid_{t=0} = 0,$$

where

$$q\left(\Delta\sigma, \nabla\sigma, \sigma, c\right) = \left[\frac{\Delta\sigma}{\sigma} - \frac{|\nabla\sigma|^2}{\sigma^2} + \frac{c}{\sigma}\right](x')\,. \quad (3.45)$$

Since the function $\sigma(x') = 1$ for $x' \in \partial G$, then it follows from (3.8) that

$$w(x',t)\mid_{\partial G} = \psi_4(x',t), \quad \frac{\partial w(x',t)}{\partial\nu}\mid_{\partial\Omega} = \psi_5(x',t),$$

$$\forall (x',t) \in \partial G \times (0,\infty)\,.$$

Hence, if the coefficient $c(x')$ is unknown, then the uniqueness follows directly from Theorem 3.2.1 in [2]. If the coefficient $\sigma(x')$ is unknown, then the proof becomes more sophisticated. This is because this coefficient is contained in the equation (3.44) together with its derivatives. In this case, the proof follows from the lemma 3.5.

4 The global uniqueness of an inverse source problem

This section discusses the problem of determining the function $S_2(x_1,x_3)$ in the right-hand side of the equation (2.7). Based on the results indicated above, we show that the global uniqueness for such a problem can easily be established. We formulate an inverse source problem as follows.

Let the function $u(x)$ be a solution of the problem (2.7), (2.8). Given the functions $\sigma(x'), c(x'), f(x_2)$ and the data $\varphi(x_1,x_2)$ in (2.28), find the function $S_2(x')$.

Theorem 4.1 *Suppose that $\sigma(x') \in C^{1+\alpha}(\overline{R}_+^2)$, $c(x') \in C^{\alpha}(\overline{R}_+^2)$, $\sigma(x') = 1$, $c(x') = 0$ for $x' \in R_+^2 \setminus \Omega$, where $\alpha \in (0,1)$, and the domain Ω is defined by (2.9), and the condition (2.12) is fulfilled. Also, let the function $S_2 \in C^{2+\alpha}(R^2)\bigcap H^2(R^2)$ and $S_2(x_1,x_3) = S_2(x_1,-x_3)$. Then there exists at most one pair of functions (u,S_2), such that the function $u \in H^2(R_+^3)$ satisfies the equations (2.7), (2.8), and (2.28).*

Proof. We start with the problem (3.1)-(3.4). Consider the Cauchy problem for the parabolic equation

$$\sigma\partial_t v = \nabla \cdot (\sigma\nabla v) - cv, \quad (x',t) \in R^2 \times (0,\infty) \quad (4.1)$$

$$v\mid_{t=0} = S_2(x')\sigma^{-1}(x'). \quad (4.2)$$

It is well known that this problem has a unique solution $v(x',t) \in C^{2+\alpha,1+\alpha/2}(R^2 \times [0,T])$ for any $T > 0$ (see [15], Chapter 5). Furthermore, there exist the positive constants C and s_0, such that

$$|D_{x'}^{\gamma} D_t^{\beta} v(x',t)| \leq C \exp(s_0^2 t), \quad |\gamma| + 2\beta \leq 2, \quad (x',t) \in R^2 \times (0,\infty).$$

Therefore, there exists the Laplace transform

$$V(x',s) = \int_0^{\infty} v(x',t) \exp(-s^2 t) dt, \quad s > s_0.$$

By analogy with the previous proof, we conclude that $V(x',s) = Y(x',s)$. Since the Laplace transform is a one-to-one operator, then we have

$$\partial_{x_3} v \mid_{x_3=0} = 0, \quad v \mid_{x_3=0} = p(x_1,t), \tag{4.3}$$

where the function $p(x_1,t)$ is known for $(x_1,t) \in (-B,B) \times (0,\infty)$. Thus, we have obtained the Cauchy problem for the parabolic equation (4.1) with the lateral data (4.3) at the plane $\{x_3 = 0\}$. It is well known (see, e.g., [2], Chapter 2 and [14], Chapter 4) that such a problem has at most one solution $v \in C^{2,1}(R^2 \times [0,T])$.

■

5 Conclusions

The new formulation of a non-overdetermined inverse conductivity problem in unbounded domains and technique for the proof of the global uniqueness theorem have been presented. Specifically, we have extended the Tikhonov's non-overdetermined formulation of the 3-D/1-D inverse conductivity problem to two dimensions. Using the modified method of Carleman estimates combined with both the direct Fourier and inverse Laplace transforms, the global uniqueness theorem has been proven. The weakening of smoothness conditions on both coefficients of the equation (2.7) is the subject for further work. We shall also extend the global uniqueness result to tube domains. Based on both the global uniqueness result and convexification approach to coefficient inverse problems, we have started a comprehensive numerical study aimed on developments of practical reconstruction globally convergent algorithms.

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