Lipschitz Stability of an Inverse Problem for an Acoustic Equation

Michael V. Klibanov[†] and Masahiro Yamamoto[‡]
[†]Department of Mathematics, The University of North Carolina at Charlotte, Charlotte, NC 28223, U.S.A.
[‡]Graduate School of Mathematical Sciences,The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan
E-mail: mklibanv@email.uncc.edu and myama@ms.u-tokyo.ac.jp

Abstract

An inverse problem of the determination of the coefficient $p(x)$ in the equation $u_{tt} = \nabla \cdot (p(x)\nabla u), x \in \Omega \subset R^n, t \in (0,T)$ is considered. The main difficulty here as compared with the previous results is that the function $p(x)$ is involved together with its derivatives. Lipschitz stability estimate is obtained using the method of Carleman estimates.

1 Introduction

Let $\Omega \subset R^n$ be a bounded domain with a piecewise C^6 boundary $\partial\Omega$. For a $T = const > 0$, denote $Q_T = \Omega \times (0,T), S_T = \partial\Omega \times (0,T)$. Let the coefficient $p(x) \in C^1(\overline{\Omega})$ and $p(x) \geq const > 0$ in Ω. Consider the initial boundary value problem in the cylinder Q_T

$$u_{tt} = \nabla \cdot (p(x)\nabla u) \text{ in } Q_T, \tag{1.1}$$

$$u(x,0) = f(x), u_t(x,0) = 0 \text{ in } \Omega, \tag{1.2}$$

$$u \mid_{S_T} = g(x,t). \tag{1.3}$$

In acoustics, $\Omega \subset R^3$, $\sqrt{p(x)}$ is the speed of sound and $u(x,t)$ is the amplitude of a wave field, see, e.g., Tikhonov and Samarskii [1], Chapter 2.

Inverse Problem. Determine the coefficient $p(x)$ in Ω assuming that the normal derivative $h(x,t)$ is given:

$$\frac{\partial u}{\partial \nu} \mid_{S_T} = h(x,t), \tag{1.4}$$

where ν is the outward normal vector at the cylindrical surface S_T.

Remark 1.1. In (1.4), it is possinble to adopt $\frac{\partial u}{\partial \nu}$ on a suitable subboundary to establish a Lipschitz stability estimate similar to (1.12). To do this, one should modify the key Carleman estimate Lemma 2.1 by evaluating more carefully the vector function $(\mathcal{U}, \mathcal{V})$. Ideas of such modifications can be found in [12 - 16]. However, to simplify the presentation, we do not follow this root.

Choose an $x_0 \notin \overline{\Omega}$. Without loss of generality, assume that the function $p^{-1}(x)$ is such that $p^{-1}(x) \geq 1$ in Ω. Let a and M be two numbers and $a, M = const > 1$. Consider a set of functions $W(a, M, x_0)$ defined as

$$W(a,M,x_0) = \left\{ \begin{array}{c} p^{-1}(x) \in C^3(\overline{\Omega}) : 1 \leq p^{-1}(x) \leq a, \|p\|_{C^3(\overline{\Omega})} \leq M, \\ \frac{1}{2} + (x - x_0, \nabla(p^{-1})) \geq 0 \end{array} \right\}, \tag{1.5}$$

where $(,)$ denotes the scalar product in R^n. Denote

$$\mathcal{P} = \mathcal{P}(x_0, \Omega) = \left[\max_{x \in \overline{\Omega}} |x - x_0|^2 - \min_{x \in \overline{\Omega}} |x - x_0|^2 \right]^{1/2}.$$

The main result of this paper is

Theorem 1. *Let the function $f \in C^6(\overline{\Omega})$ and $n \in [1, 13]$. Suppose that there exist two pairs of functions $(p_1, u_1), (p_2, u_2)$ satisfying (1.1), (1.2) and*

$$p_1 |_{\partial\Omega} = p_2 |_{\partial\Omega}, \quad \frac{\partial^k p_1}{\partial^k \nu} |_{\partial\Omega} = \frac{\partial^k p_2}{\partial^k \nu} |_{\partial\Omega}, \quad k = 1, 2, \tag{1.6}$$

$$u_i |_{S_T} = g_i(x, t), \quad \frac{\partial u_i}{\partial \nu} |_{S_T} = h_i(x, t), \quad i = 1, 2, \tag{1.7}$$

for some functions g_i, h_i. Also, assume that there exist a point $x_0 \notin \overline{\Omega}$ and numbers $a, M > 1$ such that the initial condition $f(x)$ satisfies

$$\min_{x \in \overline{\Omega}} (x - x_0, \nabla f(x)) = \mu(x_0) > 0 \tag{1.8}$$

and

$$p_1, p_2 \in W(a, M, x_0). \tag{1.9}$$

Denote

$$\sqrt{\eta_0} = \min \left[\frac{1}{4a^2(1 + \mathcal{P}M)}, \frac{\sqrt{3}}{\sqrt{(n+3)a}} \right].$$

Assume that

$$T > \frac{\mathcal{P}}{\sqrt{\eta_0}}, \tag{1.10}$$

$u_i \in C^6(\overline{Q}_T)$ and

$$\|u_i\|_{C^6(\overline{Q}_T)} \leq M_1, \tag{1.11}$$

where M_1 is a positive constant. Then there exists a positive constant $N = N(a, \mu(x_0), f, M, M_1, \Omega, x_0, T)$ such that the following Lipschitz stability estimate is valid:

$$\|p_1 - p_2\|_{H^2(\Omega)} + \|u_1 - u_2\|_{H^5(Q_T)} \leq N \left[\left\| D_t^4 (g_1 - g_2) \right\|_{H^1(S_T)} + \left\| D_t^4 (h_1 - h_2) \right\|_{L_2(S_T)} \right]. \tag{1.12}$$

In particular, suppose that there exist two pairs of functions $(p_1, u_1), (p_2, u_2)$ satisfying conditions (1.1)-(1.7) and such that functions $u_1, u_2 \in C^6(\overline{Q}_T)$. Then $p_1(x) = p_2(x)$ in Ω and $u_1(x, t) = u_2(x, t)$ in Q_T.

Remark 1.2. To guarantee that functions $u_1, u_2 \in C^6(\overline{Q}_T)$, one needs to impose some compatibility and smoothness conditions on the function $f(x)$ and coefficients p_1, p_2, see, e.g., Ladyzhenskaya [2], Chapter 4. We are not formulating such conditions in this paper, since we are focused exclusively on the inverse problem. To simplify the presentation, we are not concerned here with weakening smoothness conditions. However, the latter might well be a topic of our next publication.

Remark 1.3. The assumption $\|p_1\|_{C^3(\overline{\Omega})} \leq M, \|p_2\|_{C^3(\overline{\Omega})} \leq M$ is going along well with Tikhonov's concept of finding a solution of an ill-posed/inverse problem on *a priori* given compact set, see Tikhonov and Arsenin [3], Chapter 2. The inequality involving $\nabla(p^{-1})$ in (1.5) is a sufficient condition for the validity of the Carleman estimate for the operator $p^{-1}\partial_t^2 - \Delta$, see Lemma 2.1. It is unknown whether or not that inequality can be significantly relaxed. Analogously, inequality (1.8) guarantees validity of the Carleman estimates for a differential operator of the first order, and for an operator of the third order, see Lemmata 3.2 and 3.3.

To prove Theorem 1, we use a modified Bukhgeim-Klibanov (BK) method of Carleman estimates, see [4-11] and references cited in [11] for this method. Such a modification is necessary because the unknown coefficient $p(x)$ is involved together with its derivatives. The latter represents the major difficulty here as compared with the majority of previous works, in which derivatives of unknown coefficients were not involved. In addition, we explore in this paper an idea, which was first proposed by Klibanov and Malinsky [12] and modified by Kazemi and Klibanov [13] then, also see [11], Chapter 2 for more details. This idea, which is also based on Carleman estimates, allows one to obtain Lipschitz stability estimates for hyperbolic equations and inequalities with the lateral Cauchy data.

The Lipschitz stability for a similar inverse problem for the equation $u_{tt} = \Delta u + a(x)u$ with the unknown coefficient $a(x)$ was addressed in two papers of Imanuvilov and Yamamoto [14], [15] via a modification of the BK method. In addition to the absence of derivatives of the unknown coefficient $a(x)$ in this equation, another important difference with the current work was that the zero Neumann boundary condition on the function $u(x,t)$ was imposed in [14], [15]. This is because in [14], [15], Lemma 2.3 is not used, see more details in Subsection 2.3.

There are two publications, which have also addressed this problem, see Imanuvilov and Yamamoto [16] and Imanuvilov, Isakov and Yamamoto [17]. The data in these references were assumed to be given in $\omega \times (0,T)$, where $\omega \subset \Omega$ is a boundary layer. In [16] equation (1.1) was considered. It was assumed in [16] that functions p_1, p_2 satisfy conditions similar with (1.9), $p_1, p_2 \in C^2(\overline{\Omega})$ and $u_1, u_2 \in W^{4,\infty}(Q_T)$. Condition (1.8) was also imposed in [16]. Under these assumptions, a Hölder stability estimate was proven in this reference. A similar result was obtained in [17] for the Lamé system. Note that turning (1.12) in a Hölder stability estimate would mean that the right hand side of this formula would have the form$\|D_t^4(g_1 - g_2)\|_{H^1(S_T)}^{\alpha} + \|D_t^4(h_1 - h_2)\|_{L_2(S_T)}^{\alpha}$ for an $\alpha \in (0,1)$. This is weaker than the Lipshcitz stability estimate (1.12). Although the BK method was used in [16], [17], but H^{-1} norms were explored. However, since Carleman estimates are established in L_2 norms, it is natural to use only L_2 norms when working with these estimates. As a consequence of our L_2 approach, stability estimate (1.12) of the present paper is stronger than one in [16] at the expense of a slightly stronger condition (1.6).

The rest of this paper is devoted to the proof of Theorem 1. We assume below that conditions of this theorem hold. We prove it in several stages. First, we obtain equations for functions $D_t^k(u_1 - u_2)$, $k = 3, 4$, because these equations are more convenient to work with. This is done in Section 2. Also, in this section we formulate our key Carleman estimate for a hyperbolic operator (Lemma 2.1) and two more lemmata. Next, we prove three new Carleman estimates. The first one is for the Laplacian (Section 3) and two others are for differential operators of the first and third orders (Section 4). In Section 5 we complete the proof of this theorem.

2 Beginning of the proof

2.1 Equations for functions $D_t^3(u_1 - u_2)$ and $D_t^4(u_1 - u_2)$

In this proof, $N = N(a, \mu(x_0), f, M, M_1, \Omega, x_0, T)$ denotes different positive constants which are dependent on $a, \mu(x_0), f, M, M_1, \Omega, x_0, T$, but independent of respective choices of $p_1, p_2 \in W(a, M, x_0)$. Also let $x = (x_1, ..., x_n), x_0 = (x_{01}, ..., x_{0n}) \in R^n$ and, for a smooth function $F(x)$, let us denote $F_i = \partial_i F$, $F_{ij} = \partial_j \partial_i F$. Let $U(x,t) = p(x)u(x,t)$. Then conditions (1.1)-(1.4) imply that

$$\frac{1}{p}U_{tt} = \Delta U - \nabla(\ln p) \cdot \nabla U - \left(\frac{\Delta p}{p} - \frac{|\nabla p|^2}{p^2}\right) U \quad \text{in } Q_T, \tag{2.1}$$

$$U(x,0) = p(x)f(x), \quad U_t(x,0) = 0, \tag{2.2}$$

$$U\mid_{S_T} = p(x)\mid_{\partial\Omega} \cdot g(x,t), \tag{2.3}$$

$$\frac{\partial U}{\partial \nu}\mid_{S_T} = \frac{\partial p}{\partial \nu}\mid_{\partial\Omega} \cdot g(x,t) + p(x)\mid_{\partial\Omega} \cdot h(x,t). \tag{2.4}$$

Let $U^{(3)}(x,t) = D_t^3 U(x,t)$. Then using (2.1)-(2.4) and simple algebraic manipulations, we obtain

$$\frac{1}{p}U_{tt}^{(3)} = \Delta U^{(3)} - \nabla(\ln p) \cdot \nabla U^{(3)} - \left(\frac{\Delta p}{p} - \frac{|\nabla p|^2}{p^2}\right) U^{(3)} \text{ in } Q_T, \tag{2.5}$$

$$U^{(3)}(x,0) = 0, \tag{2.6}$$

$$\begin{aligned} U_t^{(3)}(x,0) &= p^2\Delta(\nabla p \cdot \nabla f) + p[\Delta p(\nabla p \cdot \nabla f) + 2\nabla p \cdot \nabla(\nabla p \cdot \nabla f)] \\ &+ p\Delta(p^2\Delta f) - \nabla p \cdot \nabla(p\nabla p \cdot \nabla f + p^2\Delta f) \\ &-(p\Delta p - p|\nabla p|^2)(\nabla p \cdot \nabla f + p\Delta f) \\ &= p^2\Delta(\nabla p \cdot \nabla f) + low\ order\ terms, \end{aligned} \tag{2.7}$$

$$U^{(3)}\mid_{S_T} = p(x)\mid_{\partial\Omega} \cdot D_t^3 g(x,t), \tag{2.8}$$

$$\frac{\partial U^{(3)}}{\partial \nu}\mid_{S_T} = \frac{\partial p}{\partial \nu}\mid_{\partial\Omega} \cdot D_t^3 g(x,t) + p(x)\mid_{\partial\Omega} \cdot D_t^3 h(x,t). \tag{2.9}$$

In (2.7) “low order terms” denote polynomials with respect to the function p and its derivatives up to the second order.

Denote

$$v = D_t^3 u_1 - D_t^3 u_2, \quad (x,t) \in Q_T, \tag{2.10}$$

$$\widetilde{g}(x,t) = p(x)\mid_{\partial\Omega} \cdot D_t^3(g_1 - g_2)(x,t), \quad (x,t) \in S_T, \tag{2.11}$$

$$\widetilde{h}(x,t) = \frac{\partial p}{\partial \nu}\mid_{\partial\Omega} \cdot D_t^3(g_1 - g_2)(x,t) + p(x)\mid_{\partial\Omega} \cdot D_t^3(h_1 - h_2), \quad (x,t) \in S_T, \tag{2.12}$$

$$q(x)=p_1(x)-p_2(x). \tag{2.13}$$

By (2.6)

$$v\left(x,0\right)\mid_{\partial\Omega}=\frac{\partial v(x,0)}{\partial\nu}\mid_{\partial\Omega}=0.$$

Hence,

$$\widetilde{g}(x,t)=\int\limits_0^t\widetilde{g}_t(x,\tau)d\tau,\quad \widetilde{h}(x,t)=\int\limits_0^t\widetilde{h}_t(x,\tau)d\tau,$$

which imply that

$$\left\|\widetilde{g}\right\|_{H^1(S_T)}\leq N\left\|\widetilde{g}_t\right\|_{H^1(S_T)},\quad \left\|\widetilde{h}\right\|_{L_2(S_T)}\leq N\left\|\widetilde{h}_t\right\|_{L_2(S_T)}. \tag{2.14}$$

Let a_1,a_2,b_1 and b_2 be four numbers, $\widetilde{a}=a_1-a_2,\widetilde{b}=b_1-b_2$. Then $a_1b_1-a_2b_2=\widetilde{a}b_1+\widetilde{b}a_2$. Using this formula and relations (2.5)-(2.13), we obtain

$$p_1^{-1}(x)v_{tt}=\Delta v-\nabla\left(\ln p_1\right)\cdot\nabla v-\left(\frac{\Delta p_1}{p_1}-\frac{\left|\nabla p_1\right|^2}{p_1^2}\right)v+h^{(2)}(x,t)\Delta q \tag{2.15}$$

$$+\sum_{i=1}^n h^{(1i)}(x,t)q_i+h^{(0)}(x,t)q\quad\text{in } Q_T,$$

$$v(x,0)=0, \tag{2.16}$$

$$v_t(x,0)=p_1^2\Delta\left(\nabla q\cdot\nabla f\right)+\sum_{i,j=1}^n\alpha^{(ij)}(x)q_{ij}+\sum_{i=1}^n\alpha^{(i)}(x)q_i+\alpha^{(0)}(x)q, \tag{2.17}$$

$$v\mid_{S_T}=\widetilde{g}\left(x,t\right),\quad \frac{\partial v}{\partial\nu}\mid_{S_T}=\widetilde{h}\left(x,t\right), \tag{2.18}$$

where functions

$$h^{(2)},h^{(1i)},h^{(0)}\in C^1\left(\overline{Q}_T\right);\alpha^{(ij)},\alpha^{(i)},\alpha^{(0)}\in C\left(\overline{\Omega}\right). \tag{2.19}$$

Also,

$$\left\|h^{(2)}\right\|_{C^1\left(\overline{Q}_T\right)},\left\|h^{(1i)}\right\|_{C^1\left(\overline{Q}_T\right)},\left\|h^{(0)}\right\|_{C^1\left(\overline{Q}_T\right)}\leq N \tag{2.20}$$

and

$$\left\|\alpha^{(ij)}\right\|_{C\left(\overline{\Omega}\right)},\left\|\alpha^{(i)}\right\|_{C\left(\overline{\Omega}\right)},\left\|\alpha^{(0)}\right\|_{C\left(\overline{\Omega}\right)}\leq N. \tag{2.21}$$

Denote $w(x,t)=v_t(x,t)$. Differentiating (2.15) and (2.18) with respect to t and using (2.16), we obtain

$$p_1^{-1}(x)w_{tt}=\Delta w-\nabla\left(\ln p_1\right)\cdot\nabla w-\left(\frac{\Delta p_1}{p_1}-\frac{\left|\nabla p_1\right|^2}{p_1^2}\right)w+h_t^{(2)}(x,t)\Delta q \tag{2.22}$$

$$+\sum_{i=1}^n h_t^{(1,i)}(x,t)q_i+h_t^{(0)}(x,t)q\quad\text{in } Q_T,$$

$$w_t(x,0)=0, \tag{2.23}$$

$$w\mid_{S_T}=\widetilde{g}_t\left(x,t\right),\quad \frac{\partial w}{\partial\nu}\mid_{S_T}=\widetilde{h}_t\left(x,t\right). \tag{2.24}$$

2.2 Carleman estimate for a hyperbolic operator

Consider the Carleman Weight Function (CWF) for the hyperbolic operator

$$\mathcal{C}(x,t) = \exp\left[\lambda\left(|x-x_0|^2 - \eta t^2\right)\right] = \exp\left[\lambda\psi\left(x,t\right)\right], \quad \eta = const \in \left(0, \eta_0\right], \tag{2.25}$$

where λ is a positive parameter. Let

$$c^2\left(x_0\right) = \min_{x\in\overline{\Omega}} |x-x_0|^2$$

For a number $c^2 \in \left(0, c^2\left(x_0\right)\right)$, consider the domain G_{c^2} defined by

$$G_{c^2} = \left\{(x,t) \in Q_T : |x-x_0|^2 - \eta t^2 > c^2\right\}.$$

Choose a sufficiently small positive number $\delta = \delta\left(0, c^2\left(x_0\right), T\right) \in \left(0, c^2\left(x_0\right)\right)$ such that

$$\overline{G_{c^2}} \cap \{t = T\} = \varnothing \text{ for } c^2 \in \left(c^2\left(x_0\right) - \delta, c^2\left(x_0\right) - \delta/2\right), \eta \in \left(0, \eta_0\right]. \tag{2.26}$$

Inequality (1.10) guarantees relation (2.26) for sufficiently small δ. Also, there exists a positive number $\xi = \xi\left(\Omega, x_0, T\right)$ such that

$$\Omega \times (0,\xi) \subset G_{c^2} \text{ for } c^2 \in \left(c^2\left(x_0\right) - \delta, c^2\left(x_0\right) - \delta/2\right), \quad \eta \in \left(0, \eta_0\right]. \tag{2.27}$$

Thus, the boundary of the domain G_{c^2} consists of three parts

$$\partial G_{c^2} = \bigcup_{i=1}^{3} \partial_i G_{c^2} \text{ for } c^2 \in \left(c^2\left(x_0\right) - \delta, c^2\left(x_0\right) - \delta/2\right), \tag{2.28}$$

where

$$\partial_1 G_{c^2} = \left\{(x,t) \in Q_T : \psi\left(x,t\right) = c^2\right\}, \tag{2.29}$$

$$\partial_1 G_{c^2} = \left\{(x,t) \in S_T : \psi\left(x,t\right) > c^2\right\}, \tag{2.30}$$

$$\partial_3 G_{c^2} = \left\{(x,t) : x \in \Omega, t = 0\right\}. \tag{2.31}$$

These parts of the boundary play an important role in the method of Carleman estimates, as always.

The following lemma is a reformulation of Theorem 2.2.4 of [11].

Lemma 2.1. *Suppose that the function $r(x) \in C^1\left(\overline{\Omega}\right)$ satisfies the following conditions*

$$1 \le r(x) \le a, \quad \frac{1}{2} + \left(x - x_0, \nabla r(x)\right) \ge 0 \quad \text{in } \Omega. \tag{2.32}$$

Let the $\delta\left(\Omega, x_0, T\right)$ be a sufficiently small positive constant such that condition (2.26) is fulfilled. Suppose that the number $c^2 \in \left(c^2\left(x_0\right) - \delta, c^2\left(x_0\right) - \delta/2\right)$. Then there exist a sufficiently large positive constant $\lambda_0 = \lambda_0\left(\Omega, x_0, \eta\right) > 1$ and a positive number $C = C\left(\Omega, x_0, \eta\right)$ such that the following pointwise Carleman estimate is valid for all functions $u \in C^2\left(\overline{Q}_T\right)$:

$$\left(r(x)u_{tt} - \Delta u\right)^2 \mathcal{C}^2 \ge C\lambda\left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2\right)\mathcal{C}^2 + \nabla\cdot\mathcal{U} + V_t \quad \text{in } G_{c^2},$$

where the vector function $(\mathcal{U}, V)$ *satisfies the following estimate:*

$$|(\mathcal{U}, V)| \leq C\lambda \left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2\right) \mathcal{C}^2 \quad in\ G_{c^2}.$$

In addition, the function $V(x,t)$ *can be estimated as*

$$|V(x,t)| \leq C\lambda^3 \left[t\left(|\nabla u|^2 + u_t^2 + u^2\right) + (|\nabla u| + |u|) \cdot |u_t|\right]. \tag{2.33}$$

Thus, we have

$$V(x,0) \equiv 0, \tag{2.34}$$

if either

$$u(x,0) \equiv 0 \tag{2.35}$$

or

$$u_t(x,0) \equiv 0. \tag{2.36}$$

Relations (2.34)-(2.36) represent a single new element of this lemma as compared with previously known Carleman estimates for hyperbolic operators. These relations allow us not to use the odd extension with respect to t of the function $v(x,t)$ in $\Omega \times (-T,0)$, as it was done in all previous works on the BK method. In some cases of inverse problems, we need the even extension, but here we need the odd extension for the function $U(x,t)$, the solution to (2.1) - (2.4), because of $U_t(x,0) = 0$ in (2.2).

The assumption $r(x) \geq 1$ is made for brevity only. Actually, it is sufficient to assume that $r(x) \geq const$, which would lead to a small change of (2.32).

2.3 Two more lemmata

Lemma 2.2 provides an estimate from the above for an integral, in which the CWF (2.25) is involved. This lemma was proven in Chapter 3 of [11].

Lemma 2.2. *For all functions* $s \in C\left(\overline{G_{c^2}}\right)$ *and for all* $\lambda \geq 1$, *the following estimate holds:*

$$\int_{G_{c^2}} \left[\int_0^t s(x,\tau)d\tau\right]^2 \mathcal{C}^2(x,t)dxdt \leq \frac{1}{\lambda\eta} \int_{G_{c^2}} \left(s^2\mathcal{C}^2\right)(x,t)dxdt.$$

Lemma 2.3 is an analogue of the classical energy estimate for hyperbolic operators of the second order, see Ladyzhenskaya [2], Chapter 4. A problem with such estimates arises when a non-homogeneous boundary condition (Dirichlet or Neumann) is in place. This is why the zero Neumann boundary condition is imposed in references [14], [15]. We handle this problem in this lemma via using both Dirichlet and Neumann boundary condition simultaneously. While such a use is unacceptable for the forward problem, it is sufficient for our goal, because both these conditions are given in our case. Although Lemma 2.3 can certainly be extended to the case of a general hyperbolic operator of the second order, we restrict our attention to a specific case of our current interest, for brevity.

Lemma 2.3. *Let the function* $s(x) \in C\left(\overline{\Omega}\right)$ *and* $s(x) \in (s_0, s_1)$, *where* $s_0, s_1 = const >$ 0. *Also, suppose that the function* $u(x,t) \in C^2\left(\overline{Q}_T\right)$ *satisfies the hyperbolic inequality*

$$|s(x)u_{tt} - \Delta u| \leq B\left(|\nabla u| + |u_t| + |u| + |z(x,t)|\right), \text{ in } Q_T, \tag{2.37}$$

as well as initial and boundary conditions

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x),$$

$$u\mid_{S_T}=\varphi_0(x,t), \quad \frac{\partial u}{\partial \nu}\mid_{S_T}=\varphi_1(x,t),$$

where B *is a positive constant, the function* $z\in L_2(Q_T)$ *and*

$$u_0\in H^1(\Omega), \quad u_1\in L_2(\Omega), \quad \varphi_0\in H^1(S_T), \quad \varphi_1\in L_2(S_T).$$

Then there exists a positive constant $B_1=B_1(\Omega,T,B,s_0,s_1)$ *depending on* Ω,T,B,s_0 *and* s_1 *such that*

$$\|u\|_{H^1(Q_T)}\leq B_1\left(\|u_0\|_{H^1(\Omega)}+\|u_0\|_{L_2(\Omega)}+\|\varphi_0\|_{H^1(S_T)}+\|\varphi_1\|_{L_2(S_T)}+\|z\|_{L_2(Q_T)}\right). \qquad (2.38)$$

Proof. In this proof, $B_1=B_1(\Omega,T,B,s_0,s_1)$ denotes different positive constants depending on these parameters. Denote

$$s(x)\,u_{tt}-\Delta u=Y(x,t). \qquad (2.39)$$

Multiply the both sides of (2.39) by $2u_t$ and integrate over the cylinder $Q_t=\Omega\times(0,t)$ for an arbitrary $t\in(0,T)$. We obtain

$$\begin{aligned}&\int_0^t\left\{\int_\Omega\frac{\partial}{\partial\tau}\left[s(x)\,u_\tau^2(x,\tau)\right]dx\right\}d\tau+\int_0^t\left[\int_\Omega\frac{\partial}{\partial\tau}\left(|\nabla u(x,\tau)|^2\right)dx\right]d\tau\\ =\;&2\int_0^t\left[\int_{\partial\Omega}\varphi_{0t}\varphi_1 dS\right]dt+2\int_{Q_t}Yu_t dxdt.\end{aligned}$$

Hence, using the Gauss-Ostrogradsky formula and the Cauchy-Bunyakovsky inequality, we obtain

$$\int_\Omega\left(u_t^2+|\nabla u|^2\right)(x,t)\,dx\leq B_1\left(\|u_0\|_{H^1(\Omega)}^2+\|u_1\|_{L_2(\Omega)}^2+\|\varphi_0\|_{H^1(S_T)}^2+\|\varphi_1\|_{L_2(S_T)}^2\right) \qquad (2.40)$$

$$+\|u_t\|_{L_2(Q_t)}^2+\|Y\|_{L_2(Q_t)}^2.$$

We also have

$$u(x,t)=u_0(x)+\int_0^t u_t(x,\tau)\,d\tau.$$

Hence,

$$\int_\Omega u^2(x,t)dx\leq B_1\left(\|u_0\|_{L_2(\Omega)}^2+\|u_t\|_{L_2(Q_t)}^2\right).$$

Thus, this inequality, (2.37) and (2.40) lead to

$$\int_\Omega\left(u_t^2+|\nabla u|^2+u^2\right)(x,t)\,dx\leq B_1\left(\|\nabla u\|_{L_2(Q_t)}^2+\|u_t\|_{L_2(Q_t)}^2+\|u\|_{L_2(Q_t)}^2\right) \qquad (2.41)$$

$$+B_1\left(\|u_0\|_{H^1(\Omega)}^2+\|u_1\|_{L_2(\Omega)}^2+\|\varphi_0\|_{H^1(S_T)}^2+\|\varphi_1\|_{L_2(S_T)}^2+\|z\|_{L_2(Q_T)}\right).$$

Finally, the Gronwall inequality and (2.41) lead to (2.38). $\square$

3 A new Carleman estimate for the Laplace operator

Lemma 3.2 of this section is a new Carleman estimate for the Laplace operator. The corresponding CWF can be obtained from the CWF (2.25) by setting $\eta := 0$. The difference with the conventional CWF for the elliptic operators is that the level sets of the latter are paraboloids (see, e.g., [11], Chapter 2), whereas the level sets of our CWF are spheres $\{|x - x_0| = const > 0\}$. It seems that this result can be extended to more general elliptic operators of the second order. However, we do not need such an extension in this paper. We first formulate lemma 3.1, which was proven in [10]. In this lemma, only lower order derivatives are involved in the Carleman estimate (3.1). A result similar to Lemma 3.1 was recently proven by Hrycak and Isakov [18].

Lemma 3.1. *Let $n \in [1, 13]$, λ be a positive parameter and*

$$\widetilde{\mathcal{C}}(x) = \exp\left(\lambda |x - x_0|^2\right), \quad x \in \Omega. \tag{3.1}$$

Then there exist a sufficiently large positive constant $\lambda_0 = \lambda_0(\Omega, x_0) > 1$ and a positive number $C = C(\Omega, x_0)$ such that the following pointwise Carleman estimate is valid for all functions $u \in C^2\left(\overline{\Omega}\right)$:

$$(\Delta u)^2 \widetilde{\mathcal{C}}^2 \geq C\lambda\left(|\nabla u|^2 + \lambda^2 u^2\right)\widetilde{\mathcal{C}}^2 + \nabla \cdot \mathcal{U} \quad \text{in } \Omega, \tag{3.2}$$

where the vector function U satisfies the inequality

$$|\mathcal{U}| \leq C\lambda\left(|\nabla u|^2 + \lambda^2 u^2\right)\widetilde{\mathcal{C}}^2. \tag{3.3}$$

To include the second order derivatives in the Carleman estimate, we prove

Lemma 3.2. *Let $n \in [1, 13]$ and $\widetilde{\mathcal{C}}(x)$ be the function defined by (3.1). Then there exist a sufficiently large positive constant $\lambda_0 = \lambda_0(\Omega, x_0) > 1$ and a positive number $C = C(\Omega, x_0)$ such that the following pointwise Carleman estimate is valid for all functions $u \in C^3\left(\overline{\Omega}\right)$:*

$$(\Delta u)^2 \widetilde{\mathcal{C}}^2 \geq \frac{C}{\lambda}\sum_{i,j=1}^{n} u_{ij}^2 \widetilde{\mathcal{C}}^2 + C\lambda\left(|\nabla u|^2 + \lambda^2 u^2\right)\widetilde{\mathcal{C}}^2 + \nabla \cdot \mathcal{U} \quad \text{in } \Omega, \tag{3.4}$$

where the vector function U satisfies the inequality

$$|\mathcal{U}| \leq \frac{C}{\lambda}\left(\sum_{i,j=1}^{n} |\nabla u| \, |u_{ij}|\right)\widetilde{\mathcal{C}}^2 + C\lambda\left(|\nabla u|^2 + \lambda^2 u^2\right)\widetilde{\mathcal{C}}^2 \quad \text{in } \Omega. \tag{3.5}$$

In particular,

$$\int_\Omega (\Delta u)^2 \widetilde{\mathcal{C}}^2 dx \geq \frac{C}{\lambda}\sum_{i,j=1}^{n}\int_\Omega u_{ij}^2 \widetilde{\mathcal{C}}^2 dx + C\lambda\int_\Omega \left(|\nabla u|^2 + \lambda^2 u^2\right)\widetilde{\mathcal{C}}^2 dx \tag{3.6}$$

for all real valued functions $u \in H^3(\Omega)$ satisfying

$$u \mid_{\partial\Omega} = \frac{\partial u}{\partial \nu} \mid_{\partial\Omega} = 0.$$

Proof. Note that since the set of functions $u \in C^3(\overline{\Omega})$ is dense in the space $H^3(\Omega)$, estimate (3.6) follows from (3.4), (3.5) and the Gauss-Ostrogradsky formula. Let the function $u \in C^3(\overline{\Omega})$. By (3.1)

$$(\Delta u)^2 \widetilde{\mathcal{C}}^2 = \sum_{i=1}^{n} u_{ii}^2 \widetilde{\mathcal{C}}^2 + \sum_{i,j=1,i\neq j}^{n} u_{ii}u_{jj}\widetilde{\mathcal{C}}^2$$

$$= \sum_{i=1}^{n} u_{ii}^2 \widetilde{\mathcal{C}}^2 + \sum_{i,j=1,i\neq j}^{n} \left(u_i u_{jj}\widetilde{\mathcal{C}}^2\right)_i - \sum_{i,j=1,i\neq j}^{n} u_i u_{ijj}\widetilde{\mathcal{C}}^2 - 4\lambda \sum_{i,j=1,i\neq j}^{n} (x_i - x_{0i})\, u_i u_{jj}\widetilde{\mathcal{C}}^2$$

$$= \sum_{i,j=1}^{n} u_{ij}^2 \widetilde{\mathcal{C}}^2 - 4\lambda \sum_{i,j=1,i\neq j}^{n} (x_i - x_{0i})\, u_i u_{jj}\widetilde{\mathcal{C}}^2 + 4\lambda \sum_{i,j=1,i\neq j}^{n} (x_j - x_{0j})\, u_i u_{ij}\widetilde{\mathcal{C}}^2$$

$$+ \sum_{i,j=1,i\neq j}^{n} \left(u_i u_{jj}\widetilde{\mathcal{C}}^2\right)_i - \left(\sum_{i,j=1,i\neq j}^{n} u_i u_{ij}\widetilde{\mathcal{C}}^2\right)_j .$$

Thus,

$$(\Delta u)^2 \widetilde{\mathcal{C}}^2 = \sum_{i,j=1}^{n} u_{ij}^2 \widetilde{\mathcal{C}}^2 - 4\lambda \sum_{i,j=1,i\neq j}^{n} (x_i - x_{0i})\, u_i u_{jj}\widetilde{\mathcal{C}}^2 + 4\lambda \sum_{i,j=1,i\neq j}^{n} (x_j - x_{0j})\, u_i u_{ij}\widetilde{\mathcal{C}}^2 + \nabla \cdot \widetilde{\mathcal{U}}, \tag{3.7}$$

where the vector function $\widetilde{\mathcal{U}}$ satisfies

$$\left|\widetilde{\mathcal{U}}\right| \leq C \left(\sum_{i,j=1}^{n} |\nabla u|\,|u_{ij}|\right) \widetilde{\mathcal{C}}^2. \tag{3.8}$$

Applying the Cauchy-Bunyakovsky inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2, \varepsilon > 0$ to (3.7), we obtain

$$(\Delta u)^2 \widetilde{\mathcal{C}}^2 \geq \sum_{i,j=1}^{n} u_{ij}^2 \widetilde{\mathcal{C}}^2 - \frac{C}{2\varepsilon}\lambda^2 |\nabla u|^2 \widetilde{\mathcal{C}}^2 - C\varepsilon \sum_{i,j=1}^{n} u_{ij}^2 \widetilde{\mathcal{C}}^2 + \nabla \cdot \widetilde{\mathcal{U}}.$$

Choose $\varepsilon = (2C)^{-1}$. Then

$$(\Delta u)^2 \widetilde{\mathcal{C}}^2 \geq \frac{1}{2} \sum_{i,j=1}^{n} u_{ij}^2 \widetilde{\mathcal{C}}^2 - C\lambda^2 |\nabla u|^2 \widetilde{\mathcal{C}}^2 + \nabla \cdot \widetilde{\mathcal{U}}.$$

Divide this inequality by λ, sum up with (3.2) and take into account (3.3) and (3.8). Then we obtain (3.4) and (3.5). $\square$

4 Two additional Carleman estimates

4.1 Carleman estimate for a first order operator

In this subsection we prove

Lemma 4.1. *Let* $f \in C^2(\overline{\Omega})$ *satisfy (1.8). Then there exist a positive constant* $C = C\left(\Omega, x_0, \|f\|_{C^2(\overline{\Omega})}, \mu(x_0)\right)$ *and a sufficiently large positive constant* $\lambda_0 = \lambda_0(\Omega, x_0, \mu(x_0)) > 1$ *such that for all functions* $u \in C^1\left(\overline{\Omega}\right)$ *the following pointwise Carleman estimate is valid:*

$$\left(\sum_{i=1}^{n} u_i f_i\right)^2 \widetilde{\mathcal{C}}^2 \geq C\lambda^2 u^2 \widetilde{\mathcal{C}}^2 + \nabla \cdot W_1, \tag{4.1}$$

where the vector function W_1 *satisfies*

$$|W_1| \leq C\lambda u^2 \widetilde{\mathcal{C}}^2. \tag{4.2}$$

Remark 4.1. A similar Carleman estimate can be found in [16], [17]. However, in [16] the term $C\lambda u^2 \widetilde{\mathcal{C}}^2$ is used instead of $C\lambda^2 u^2 \widetilde{\mathcal{C}}^2$. This additional power of the large parameter λ is crucial for the proof of Theorem 1: compare (5.30) and (5.32) with (5.33).

Proof. Denote $\widetilde{u} = u\widetilde{\mathcal{C}}$. Then

$$u_i = \left(\widetilde{u}_i - 2\lambda(x_i - x_{0i})\widetilde{u}\right)\widetilde{\mathcal{C}}^{-1}.$$

Hence,

$$\left(\sum_{i=1}^{n} u_i f_i\right)^2 \widetilde{\mathcal{C}}^2 = \left[\sum_{i=1}^{n} \widetilde{u}_i f_i - 2\lambda(x - x_0, \nabla f)\widetilde{u}\right]^2$$

$$\geq -4\lambda \sum_{i=1}^{n} \widetilde{u}_i \widetilde{u} f_i (x - x_0, \nabla f) + 4\lambda^2 (x - x_0, \nabla f)^2 \widetilde{u}^2$$

$$= \sum_{i=1}^{n} \left[-2\lambda \widetilde{u}^2 f_i \cdot (x - x_0, \nabla f)\right]_i + 2\lambda \widetilde{u}^2 \sum_{i=1}^{n} \left[f_i \cdot (x - x_0, \nabla f)\right]_i + 4\lambda^2 (x - x_0, \nabla f)^2 \widetilde{u}^2.$$

Hence, for a sufficiently large λ_0 and $\lambda \geq \lambda_0$, by (1.8) we can absorb the second term into the third term to obtain

$$\left(\sum_{i=1}^{n} u_i f_i\right)^2 \widetilde{\mathcal{C}}^2 \geq \mu^2(x_0)\lambda^2 u^2 \widetilde{\mathcal{C}}^2 + \nabla \cdot W_1,$$

where the vector function W satisfies estimate (4.2). □

4.2 Carleman estimate for a third order operator

Lemma 4.2. *Let the function* $f \in C^2\left(\overline{\Omega}\right)$ *satisfy condition (1.8). Then there exist a positive constant* $C = C\left(\Omega, x_0, \|f\|_{C^3(\overline{\Omega})}, \mu(x_0)\right)$ *and a sufficiently large positive constant* $\lambda_0 = \lambda_0(\Omega, x_0, \mu(x_0)) > 1$ *such that for all functions* $u \in C^3\left(\overline{\Omega}\right)$ *the following pointwise Carleman estimate is valid:*

$$\left[\Delta(\nabla u \cdot \nabla f)\right]^2 \widetilde{\mathcal{C}}^2 \geq C\lambda \sum_{i,j=1}^{n} u_{ij}^2 \widetilde{\mathcal{C}}^2 + C\lambda^3\left(|\nabla u|^2 + \lambda^2 u^2\right)\widetilde{\mathcal{C}}^2 + \nabla \cdot W_2, \tag{4.3}$$

where the vector function W_2 satisfies

$$|W_2| \leq C\lambda \sum_{i,j=1}^{n} u_{ij}^2 \widetilde{\mathcal{C}}^2 + C\lambda^3 \left(|\nabla u|^2 + \lambda^2 u^2 \right) \widetilde{\mathcal{C}}^2. \tag{4.4}$$

In particular,

$$\int\limits_{\Omega} [\Delta (\nabla u \cdot \nabla f)]^2 \widetilde{\mathcal{C}}^2 \geq C\lambda \sum_{i,j=1}^{n} \int\limits_{\Omega} u_{ij}^2 \widetilde{\mathcal{C}}^2 dx + C\lambda^2 \int\limits_{\Omega} \left(|\nabla u|^2 + \lambda^2 u^2 \right) \widetilde{\mathcal{C}}^2 dx \tag{4.5}$$

for all real valued functions $u \in H^3(\Omega)$ satisfying

$$u \mid_{\partial\Omega} = \frac{\partial^k u}{\partial \nu^k} \mid_{\partial\Omega} = 0, \quad k = 1, 2.$$

Proof. Again, since the set of functions $u \in C^3\left(\overline{\Omega}\right)$ is dense in the space $H^3(\Omega)$, inequality (4.5) follows from (4.3), (4.4) and the Gauss-Ostrogradsky formula. We have

$$[\Delta (\nabla u \cdot \nabla f)]^2 \widetilde{\mathcal{C}}^2 = \left\{ \sum_{i=1}^{n} \left[(\Delta u)_i \cdot f_i + 2\nabla (u_i) \cdot \nabla (f_i) + u_i \cdot (\Delta f)_i \right] \right\}^2 \widetilde{\mathcal{C}}^2$$

$$\geq \frac{1}{2} \left[\sum_{i=1}^{n} (\Delta u)_i \cdot f_i \right]^2 \widetilde{\mathcal{C}}^2 - C \sum_{i,j=1}^{n} u_{ij}^2 \widetilde{\mathcal{C}}^2 - C |\nabla u|^2 \widetilde{\mathcal{C}}^2.$$

Applying Lemma 4.1 to the first term, we obtain

$$[\Delta (\nabla u \cdot \nabla f)]^2 \widetilde{\mathcal{C}}^2 \geq C\lambda^2 (\Delta u)^2 \widetilde{\mathcal{C}}^2 + \nabla \cdot W_1$$

$$-C \sum_{i,j=1}^{n} u_{ij}^2 \widetilde{\mathcal{C}}^2 - C |\nabla u|^2 \widetilde{\mathcal{C}}^2.$$

The rest of the proof follows from Lemma 3.2. □

5 Completion of the proof of Theorem 1

Recall that the number $c^2 \in (c^2(x_0) - \delta, c^2(x_0) - \delta/2)$. Choose a sufficiently small positive number σ such that $c^2 + 3\sigma \in (c^2(x_0) - \delta, c^2(x_0) - \delta/2)$. Obviously, $G_{c^2+3\sigma} \subset G_{c^2+2\sigma} \subset G_{c^2+\sigma} \subset G_{c^2}$. Conditions (2.26) and (2.27) imply that $G_{c^2+3\sigma} \neq \varnothing$,

$$\Omega \times (0, \xi) = Q_\xi \subset G_{c^2+3\sigma} \tag{5.1}$$

and

$$G_{c^2+3\sigma} \cap \{t = T\} = G_{c^2} \cap \{t = T\} = \varnothing.$$

Also,

$$\mathcal{C}^2(x,t) \begin{cases} \geq \exp\left[2\lambda \left(c^2 + 3\sigma\right)\right], \text{ if } (x,t) \in G_{c^2+3\sigma}, \\ \leq \exp\left[2\lambda \left(c^2 + 2\sigma\right)\right], \text{ if } (x,t) \in G_{c^2} \setminus G_{c^2+2\sigma}. \end{cases} \tag{5.2}$$

Introduce the cut-off function $\chi_\sigma(x,t) \in C^3\left(\overline{Q}_T\right)$ such that

$$\chi_\sigma(x,t) = \begin{cases} 1, \text{ if } (x,t) \in G_{c^2+2\sigma}, \\ 0, \text{ if } (x,t) \in G_{c^2} \backslash G_{c^2+\sigma}, \\ \text{between 0 and 1 otherwise.} \end{cases} \tag{5.3}$$

Denote $\overline{v}(x,t) = \chi_\sigma(x,t) \cdot v(x,t)$. Then (2.29) and (5.2) imply that

$$\overline{v} \mid_{\partial_1 G_{c^2}} = \overline{v}_t \mid_{\partial_1 G_{c^2}} = |\nabla \overline{v}| \mid_{\partial_1 G_{c^2}} = 0. \tag{5.4}$$

By (5.1) and (5.3), $\chi_\sigma(x,t) = 1$ for $(x,t) \in \Omega \times (0,\xi)$. Hence, $\overline{v}(x,0) = v(x,0)$ and $\overline{v}_t(x,0) = v_t(x,0)$. Multiplying (2.15)-(2.18) by the function $\chi_\sigma(x,t)$, using $\chi_\sigma v_{tt} = \overline{v}_{tt} - 2\chi_{\sigma t} v_t - \chi_{\sigma tt} v$ and similar formulas for $\chi_\sigma \Delta v$ and $\chi_\sigma v_i$, we obtain

$$p_1^{-1}\overline{v}_{tt} = \Delta \overline{v} - \nabla(\ln p_1) \cdot \nabla \overline{v} - \left(\frac{\Delta p_1}{p_1} - \frac{|\nabla p_1|^2}{p_1^2}\right)\overline{v} \tag{5.5}$$

$$+2p_1^{-1} \cdot \chi_{\sigma t} \cdot v_t + p_1^{-1} \cdot \chi_{\sigma tt} \cdot v - 2\nabla \chi_\sigma \cdot \nabla v - \Delta \chi_\sigma \cdot v + (\nabla(\ln p_1) \cdot \nabla \chi_\sigma) \cdot v$$

$$+\chi_\sigma h^{(2)} \Delta q + \sum_{i=1}^{n} \chi_\sigma h^{(1i)} q_i + \chi_\sigma h^{(0)} q \quad \text{in } Q_T,$$

$$\overline{v}(x,0) = 0, \tag{5.6}$$

$$\overline{v}_t(x,0) = v_t(x,0) = p_1^2 \Delta(\nabla q \cdot \nabla f) + \sum_{i,j=1}^{n} \alpha^{(ij)} q_{ij} + \sum_{i=1}^{n} \alpha^{(i)} q_i + \alpha^{(0)} q, \tag{5.7}$$

$$\overline{v} \mid_{S_T} = \chi_\sigma \cdot \widetilde{g}(x,t), \quad \frac{\partial \overline{v}}{\partial \nu} \mid_{S_T} = \chi_\sigma \cdot \widetilde{h}(x,t) + \frac{\partial \chi_\sigma}{\partial \nu} \cdot \widetilde{g}(x,t). \tag{5.8}$$

Also, denote $\overline{w}(x,t) = \chi_\sigma(x,t) \cdot w(x,t)$. Hence, functions

$$\overline{v}, \overline{w} \in C^2\left(\overline{Q}_T\right). \tag{5.9}$$

Using (2.22)-(2.24), we obtain similarly to (5.5)-(5.8)

$$p_1^{-1}\overline{w}_{tt} = \Delta \overline{w} - \nabla(\ln p_1) \cdot \nabla \overline{w} - \left(\frac{\Delta p_1}{p_1} - \frac{|\nabla p_1|^2}{p_1^2}\right)\overline{w} \tag{5.10}$$

$$+2p_1^{-1} \cdot \chi_{\sigma t} \cdot w_t + p_1^{-1} \cdot \chi_{\sigma tt} \cdot w - 2\nabla \chi_\sigma \cdot \nabla w - \Delta \chi_\sigma \cdot w + w \cdot \nabla(\ln p_1) \cdot \nabla \chi_\sigma$$

$$+h_t^{(2)} \chi_\sigma \Delta q + \sum_{i=1}^{n} \chi_\sigma h_t^{(1i)} q_i + \chi_\sigma h_t^{(0)} q \quad \text{in } Q_T,$$

$$\overline{w}_t(x,0) = 0, \tag{5.11}$$

$$\overline{w} \mid_{S_T} = \chi_\sigma \cdot \widetilde{g}_t(x,t), \quad \frac{\partial \overline{w}}{\partial \nu} \mid_{S_T} = \chi_\sigma \cdot \widetilde{h}_t(x,t) + \frac{\partial \chi_\sigma}{\partial \nu} \cdot \widetilde{g}_t(x,t). \tag{5.12}$$

Note that by (5.3) the derivatives $\chi_{\sigma t} = \chi_{\sigma tt} = \chi_{\sigma i} = \chi_{\sigma ij} = 0$ in $G_{c^2+2\sigma}$. Also, when applying the method of Carleman estimates, it is often convenient to replace differential equations with differential inequalities, for brevity, see, e.g., [11], Chapters 2 and 3. Thus, we obtain from (1.5), (2.19)-(2.21) and (5.5)-(5.8)

$$\left|p^{-1}(x)\overline{v}_{tt}-\Delta\overline{v}\right|\leq N\left(|\nabla\overline{v}|+|\overline{v}|+|\Delta q|+|\nabla q|+|q|\right) \tag{5.13}$$

$$+N\left(1-\chi_{\sigma}\right)\left(|\nabla v|+|v_t|+|v|\right) \text{ in } Q_T,$$

$$\overline{v}(x,0)=0, \tag{5.14}$$

$$|\overline{v}_t(x,0)|=|v_t(x,0)|\geq a^{-2}\left|\Delta\left(\nabla q\cdot\nabla f\right)\right|-N\left(\sum_{i,j=1}^{n}|q_{ij}|+|\nabla q|+|q|\right), \tag{5.15}$$

$$\overline{v}\mid_{S_T}=\chi_{\sigma}\cdot\widetilde{g}\left(x,t\right),\quad \frac{\partial\overline{v}}{\partial\nu}\mid_{S_T}=\chi_{\sigma}\cdot\widetilde{h}\left(x,t\right)+\frac{\partial\chi_{\sigma}}{\partial\nu}\cdot\widetilde{g}\left(x,t\right). \tag{5.16}$$

Also, relations (5.10)-(5.12) lead to

$$\left|p^{-1}(x)\overline{w}_{tt}-\Delta\overline{w}\right|\leq N\left(|\nabla\overline{w}|+|\overline{w}|+|\Delta q|+|\nabla q|+|q|\right) \tag{5.17}$$

$$+N\left(1-\chi_{\sigma}\right)\left(|\nabla\overline{w}|+|\overline{w}_t|+|\overline{w}|\right) \quad \text{in } Q_T,$$

$$\overline{w}_t(x,0)=0, \tag{5.18}$$

$$\overline{w}\mid_{S_T}=\chi_{\sigma}\cdot\widetilde{g}_t\left(x,t\right),\frac{\partial\overline{w}}{\partial\nu}\mid_{S_T}=\chi_{\sigma}\cdot\widetilde{h}_t\left(x,t\right)+\frac{\partial\chi_{\sigma}}{\partial\nu}\cdot\widetilde{g}_t\left(x,t\right). \tag{5.19}$$

Multiply the both sides of (5.13) by the CWF $\mathcal{C}(x,t)$, square and integrate over the domain G_{c^2}. Applying Lemma 2.1, (5.4), (5.9), (5.14) and the Gauss-Ostrogradsky formula, we obtain for $\lambda\geq\lambda_0\left(\Omega,x_0,\eta\right)>1$

$$N\int_{S_T}\left(\widetilde{g}^2+|\nabla\widetilde{g}|^2+\widetilde{g}_t^2+\widetilde{h}^2\right)\mathcal{C}^2dS+N\int_{G_{c^2}\diagdown G_{c^2+2\sigma}}\left(|\nabla v|^2+v_t^2+v^2\right)\mathcal{C}^2dxdt$$

$$+N\int_{G_{c^2}}\left(|\nabla\overline{v}|^2+\overline{v}^2\right)\mathcal{C}^2dxdt+N\int_{G_{c^2}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt \tag{5.20}$$

$$\geq C\lambda\int_{G_{c^2}}\left(|\nabla\overline{v}|^2+\overline{v}_t^2+\lambda^2\overline{v}^2\right)\mathcal{C}^2dxdt.$$

Choose a sufficiently large $\lambda_1=\lambda_1\left(\Omega,x_0,\eta,N\right)>\lambda_0\left(\Omega,x_0,\eta\right)$, and set below $\lambda\geq\lambda_1$. Since $G_{c^2+2\sigma}\subset G_{c^2}$ and $\overline{v}(x,t)=v(x,t)$ in $G_{c^2+2\sigma}$, (5.20) leads to

$$N\int_{S_T}\left(\widetilde{g}^2+|\nabla\widetilde{g}|^2+\widetilde{g}_t^2+\widetilde{h}^2\right)\mathcal{C}^2dS+N\int_{G_{c^2}\diagdown G_{c^2+2\sigma}}\left(|\nabla v|^2+v_t^2+v^2\right)\mathcal{C}^2dxdt$$

$$+N\int_{G_{c^2}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt \tag{5.21}$$

$$\geq C\lambda\int_{G_{c^2+2\sigma}}\left(|\nabla v|^2+v_t^2+\lambda^2v^2\right)\mathcal{C}^2dxdt.$$

Similarly we obtain from (5.17)-(5.19)

$$N\int_{S_T}\left(\widetilde{g}_t^2+|\nabla\widetilde{g}_t|^2+\widetilde{g}_{tt}^2+\widetilde{h}_t^2\right)\mathcal{C}^2dS+N\int_{G_{c^2}\diagdown G_{c^2+2\sigma}}\left(|\nabla w|^2+w_t^2+w^2\right)\mathcal{C}^2dxdt$$

$$+N\int_{G_{c^2}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt \tag{5.22}$$

$$\geq C\lambda\int_{G_{c^2+2\sigma}}\left(|\nabla w|^2+w_t^2+\lambda^2w^2\right)\mathcal{C}^2dxdt.$$

For each $x\in\Omega$, denote

$$t_{c^2+2\sigma}(x)=\frac{\sqrt{|x-x_0|^2-(c^2+2\sigma)}}{\sqrt{\eta}}.$$

Then

$$\int_{G_{c^2+2\sigma}}s(x,t)dxdt=\int\limits_{\Omega}\left[\int\limits_0^{t_{c^2+2\sigma}(x)}s(x,t)dt\right]dx,\quad \forall s\in C\left(\overline{G_{c^2+2\sigma}}\right). \tag{5.23}$$

Hence,

$$\int_{G_{c^2}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt=\int_{G_{c^2}\diagdown G_{c^2+2\sigma}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt$$

$$+\int_{G_{c^2+2\sigma}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt$$

$$=\int_{G_{c^2}\diagdown G_{c^2+2\sigma}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt \tag{5.24}$$

$$+\int\limits_{\Omega}\int\limits_0^{\xi}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)(x)\,\mathcal{C}^2dtdx+\int\limits_{\Omega}\int\limits_{\xi}^{t_{c^2+2\sigma}(x)}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)(x)\,\mathcal{C}^2dtdx.$$

Note that

$$\int\limits_{\Omega}\int\limits_0^{\xi}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)(x)\,\mathcal{C}^2dtdx+\int\limits_{\Omega}\int\limits_{\xi}^{t_{c^2+2\sigma}(x)}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)(x)\,\mathcal{C}^2dtdx \tag{5.25}$$

$$=\int\limits_{\Omega}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)(x)\left(\int\limits_0^{\xi}\mathcal{C}^2dt\right)dx+\int\limits_{\Omega}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)(x)\left(\int\limits_{\xi}^{t_{c^2+2\sigma}(x)}\mathcal{C}^2dt\right)dx.$$

Also, since the function $\theta(t) = \exp\left(-2\lambda\eta t^2\right)$ is decreasing for $t > 0$, we have

$$\int\limits_{\xi}^{t_{c^2+2\sigma}(x)} \mathcal{C}^2 dt = \exp\left(2\lambda\left|x - x_0\right|^2\right) \int\limits_{\xi}^{t_{c^2+2\sigma}(x)} \exp\left(-2\lambda\eta t^2\right) dt$$

$$\leq \ \left(t_{c^2+2\sigma}(x) - \xi\right) \exp\left(2\lambda\left|x - x_0\right|^2\right) \exp\left(-2\lambda\eta\xi^2\right)$$

$$= \ \left(t_{c^2+2\sigma}(x) - \xi\right) \exp\left(2\lambda\left|x - x_0\right|^2\right) \xi^{-1} \int\limits_{0}^{\xi} \exp\left(-2\lambda\eta\xi^2\right) dt$$

$$\leq \left(t_{c^2+2\sigma}(x) - \xi\right) \exp\left(2\lambda\left|x - x_0\right|^2\right) \xi^{-1} \int\limits_{0}^{\xi} \exp\left(-2\lambda\eta t^2\right) dt \leq N \int\limits_{0}^{\xi} \mathcal{C}^2 dt.$$

Hence, (5.24) and (5.25) imply that

$$\int_{G_{c^2}} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right) \mathcal{C}^2 dxdt \leq \int_{G_{c^2} \diagdown G_{c^2+2\sigma}} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right) \mathcal{C}^2 dxdt \tag{5.26}$$

$$+N \int\limits_{Q_\xi} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)(x)\, \mathcal{C}^2 dxdt.$$

Note that

$$\mathcal{C}^2(x,t) \leq \exp\left[2\lambda\left(c^2 + 2\sigma\right)\right] \text{ in } G_{c^2} \diagdown G_{c^2+2\sigma}. \tag{5.27}$$

Hence, applying (5.23), we obtain

$$\int_{G_{c^2} \diagdown G_{c^2+2\sigma}} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right) \mathcal{C}^2 dxdt$$

$$\leq \exp\left[2\lambda\left(c^2 + 2\sigma\right)\right] \int_{G_{c^2} \diagdown G_{c^2+2\sigma}} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right) dxdt$$

$$\leq \exp\left[2\lambda\left(c^2 + 2\sigma\right)\right] \int_{G_{c^2}} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right) dxdt$$

$$= \exp\left[2\lambda\left(c^2 + 2\sigma\right)\right] \int_{\Omega} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right) dx \int_{0}^{t_{c^2}(x)} dt$$

$$\leq N \exp\left[2\lambda\left(c^2 + 2\sigma\right)\right] \int_{\Omega} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right) dx \int\limits_{0}^{\xi} dt$$

$$= N \exp\left[2\lambda\left(c^2 + 2\sigma\right)\right] \int\limits_{Q_\xi} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right) dxdt.$$

Since

$$\exp[2\lambda(c^2+2\sigma)] < \exp[2\lambda(c^2+3\sigma)] \leq \mathcal{C}^2(x,t) \quad \text{in } Q_\xi, \tag{5.28}$$

the latter estimate implies that

$$\int_{G_{c^2}\diagdown G_{c^2+2\sigma}} \left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2 dxdt$$

$$\leq N\int_{Q_\xi} \left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2 dxdt.$$

This and (5.26) lead to

$$\int_{G_{c^2}} \left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2 dxdt \leq N\int_{Q_\xi} \left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2 dxdt. \tag{5.29}$$

Using (5.21), (5.27) and (5.29), we obtain

$$\begin{aligned} &N\int_{S_T}\left(\widetilde{g}^2+|\nabla\widetilde{g}|^2+\widetilde{g}_t^2+\widetilde{h}^2\right)\mathcal{C}^2 dS + N\exp\left[2\lambda\left(c^2+2\sigma\right)\right]\|v\|^2_{H^1(Q_T)} \\ &+N\int_{Q_\xi}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2 dxdt \\ &\geq C\lambda\int_{G_{c^2+2\sigma}}\left(|\nabla v|^2+v_t^2+\lambda^2 v^2\right)\mathcal{C}^2 dxdt. \end{aligned} \tag{5.30}$$

Similarly, using (5.22), (5.27) and (5.29), we obtain for the function w

$$\begin{aligned} &N\int_{S_T}\left(\widetilde{g}_t^2+|\nabla\widetilde{g}_t|^2+\widetilde{g}_{tt}^2+\widetilde{h}_t^2\right)\mathcal{C}^2 dS + N\exp\left[2\lambda\left(c^2+2\sigma\right)\right]\|w\|^2_{H^1(Q_T)} \\ &+N\int_{Q_\xi}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2 dxdt \\ &\geq C\lambda\int_{G_{c^2+2\sigma}}\left(|\nabla w|^2+w_t^2+\lambda^2 w^2\right)\mathcal{C}^2 dxdt. \end{aligned} \tag{5.31}$$

We have

$$v_t(x,t) = v_t(x,0) + \int_0^t v_{tt}(x,\tau)d\tau = v_t(x,0) + \int_0^t w_t(x,\tau)d\tau.$$

Hence,

$$v_t^2(x,t) \geq \frac{1}{2}v_t^2(x,0) - \left(\int_0^t w_t(x,\tau)d\tau\right)^2.$$

This and (5.15) imply that

$$v_t^2(x,t) \geq \frac{a^{-4}}{2} \left|\Delta\left(\nabla q \cdot \nabla f\right)\right|^2 - N\left(\sum_{i,j=1}^{n} q_{ij}^2 + |\nabla q|^2 + q^2\right) - \left(\int\limits_0^t w_t(x,\tau)d\tau\right)^2.$$

Apply Lemma 2.2 to the last term to obtain:

$$\int_{G_{c^2+2\sigma}} \left(\int_0^t w_t(x,\tau)d\tau\right)^2 \mathcal{C}^2 dxdt \leq \frac{1}{\lambda\eta}\int_{G_{c^2+2\sigma}} w_t^2(x,\tau)\mathcal{C}^2 dxdt.$$

Hence, using (5.29), we obtain

$$\frac{C}{2}\lambda \int\limits_{G_{c^2+2\sigma}} v_t^2 \mathcal{C}^2 dxdt \geq N\lambda \int\limits_{Q_\xi} \left|\Delta\left(\nabla q \cdot \nabla f\right)\right|^2 \mathcal{C}^2 dxdt \tag{5.32}$$

$$-N\lambda \int\limits_{Q_\xi} \left(\sum_{i,j=1}^{n} q_{ij}^2 + |\nabla q|^2 + q^2\right)\mathcal{C}^2 dxdt - N \int\limits_{G_{c^2+2\sigma}} w_t^2 \mathcal{C}^2 dxdt.$$

Note that

$$\int\limits_{Q_\xi} \left|\Delta\left(\nabla q \cdot \nabla f\right)\right|^2 \mathcal{C}^2 dxdt = \int\limits_0^\xi \exp\left(-2\lambda\eta t^2\right)dt \int\limits_\Omega \left|\Delta\left(\nabla q \cdot \nabla f\right)\right|^2 \widetilde{\mathcal{C}}^2(x)dx.$$

Also, by (1.6) and (2.13)

$$q\mid_{\partial\Omega} = \frac{\partial^k q}{\partial \nu^k}\mid_{\partial\Omega} = 0, \quad k = 1,2.$$

Thus, Lemma 4.2 and (5.32) imply that

$$\frac{C}{2}\lambda \int\limits_{G_{c^2+2\sigma}} v_t^2 \mathcal{C}^2 dxdt \geq N\lambda^2 \sum_{i,j=1}^{n} \int\limits_{Q_\xi} q_{ij}^2 \mathcal{C}^2 dxdt + N\lambda^3 \int\limits_{Q_\xi} \left(|\nabla q|^2 + \lambda^2 q^2\right)\mathcal{C}^2 dxdt$$

$$-N \int\limits_{G_{c^2+2\sigma}} w_t^2 \mathcal{C}^2 dxdt.$$

Therefore, (5.30) implies that

$$N\int_{S_T} \left(\widetilde{g}^2 + |\nabla \widetilde{g}|^2 + \widetilde{g}_t^2 + \widetilde{h}^2\right)\mathcal{C}^2 dS + N\exp\left[2\lambda\left(c^2+2\sigma\right)\right]\|v\|_{H^1(Q_T)}^2$$

$$\geq C\lambda \int_{G_{c^2+2\sigma}} \left(|\nabla v|^2 + v_t^2 + \lambda^2 v^2\right)\mathcal{C}^2 dxdt \tag{5.33}$$

$$+N\lambda^2\sum_{i,j=1}^{n}\int_{Q_\xi} q_{ij}^2\mathcal{C}^2dxdt+N\lambda^3\int_{Q_\xi}\left(|\nabla q|^2+\lambda^2q^2\right)\mathcal{C}^2dxdt-N\int_{G_{c^2+2\sigma}} w_t^2\mathcal{C}^2dxdt.$$

Let $d=d(x_0,\Omega)=\max_{x\in\overline{\Omega}}|x-x_0|^2$. Then by (2.25) $\mathcal{C}^2(x,t)\leq\exp(2\lambda d)$ in $\overline{Q}_T$. Hence, summing up (5.31) and (5.33) and taking into account (2.14), we obtain for $\lambda\geq\lambda_1(\Omega,x_0,\eta,N)$

$$Ne^{2\lambda d}\left(\|\widetilde{g}_t\|_{H^1(S_T)}^2+\left\|\widetilde{h}_t\right\|_{L_2(S_T)}^2\right)+N\exp\left[2\lambda\left(c^2+2\sigma\right)\right]\left(\|v\|_{H^1(Q_T)}^2+\|w\|_{H^1(Q_T)}^2\right)$$
$$\geq C\lambda\int_{G_{c^2+2\sigma}}\left(|\nabla v|^2+v_t^2+|\nabla w|^2+w_t^2+\lambda^2v^2+\lambda^2w^2\right)\mathcal{C}^2dxdt$$
$$+N\lambda^2\sum_{i,j=1}^{n}\int_{Q_\xi} q_{ij}^2\mathcal{C}^2dxdt+N\lambda^3\int_{Q_\xi}\left(|\nabla q|^2+\lambda^2q^2\right)\mathcal{C}^2dxdt.$$

Since by (5.1) $Q_\xi\subset G_{c^2+3\sigma}\subset G_{c^2+2\sigma}$ and $\mathcal{C}^2(x,t)\geq\exp\left[2\lambda\left(c^2+3\sigma\right)\right]$ in $G_{c^2+3\sigma}$, the latter estimate and (5.28) imply that

$$Ne^{2\lambda d}\left(\|\widetilde{g}_t\|_{H^1(S_T)}^2+\left\|\widetilde{h}_t\right\|_{L_2(S_T)}^2\right)+N\exp\left[2\lambda\left(c^2+2\sigma\right)\right]\left(\|v\|_{H^1(Q_T)}^2+\|w\|_{H^1(Q_T)}^2\right)$$
$$\geq\lambda\exp\left[2\lambda\left(c^2+3\sigma\right)\right]\left(\|v\|_{H^1\left(G_{c^2+3\sigma}\right)}^2+\|w\|_{H^1\left(G_{c^2+3\sigma}\right)}^2\right)$$

and

$$Ne^{2\lambda d}\left(\|\widetilde{g}_t\|_{H^1(S_T)}^2+\left\|\widetilde{h}_t\right\|_{L_2(S_T)}^2\right)+N\exp\left[2\lambda\left(c^2+2\sigma\right)\right]\left(\|v\|_{H^1(Q_T)}^2+\|w\|_{H^1(Q_T)}^2\right)$$
$$\geq\lambda^2\exp\left[2\lambda\left(c^2+3\sigma\right)\right]\|q\|_{H^2(\Omega)}^2.$$

Dividing these inequalities respectively by $\lambda\exp\left[2\lambda\left(c^2+3\sigma\right)\right]$ and $\lambda^2\exp\left[2\lambda\left(c^2+3\sigma\right)\right]$, we obtain

$$\|v\|_{H^1\left(G_{c^2+3\sigma}\right)}^2+\|w\|_{H^1\left(G_{c^2+3\sigma}\right)}^2\leq N\exp\left[-2\lambda\sigma\right]\left(\|v\|_{H^1(Q_T)}^2+\|w\|_{H^1(Q_T)}^2\right)\tag{5.34}$$
$$+Ne^{2\lambda d}\left(\|\widetilde{g}_t\|_{H^1(S_T)}^2+\left\|\widetilde{h}_t\right\|_{L_2(S_T)}^2\right)$$

and

$$\|q\|_{H^2(\Omega)}^2\leq N\exp\left[-2\lambda\sigma\right]\left(\|v\|_{H^1(Q_T)}^2+\|w\|_{H^1(Q_T)}^2\right)\tag{5.35}$$
$$+Ne^{2\lambda d}\left(\|\widetilde{g}_t\|_{H^1(S_T)}^2+\left\|\widetilde{h}_t\right\|_{L_2(S_T)}^2\right).$$

We now use the design of [12] and [13]; also see [11], Chapter 2 for some details. By (5.1), we can replace the norms of the space $H^1\left(G_{c^2+3\sigma}\right)$ in the left hand side of (5.34) with the norms in the space $H^1\left(Q_\xi\right)$. We obtain

$$\|v\|_{H^1\left(Q_\xi\right)}^2+\|w\|_{H^1\left(Q_\xi\right)}^2\leq N\exp\left[-2\lambda\sigma\right]\left(\|v\|_{H^1(Q_T)}^2+\|w\|_{H^1(Q_T)}^2\right)\tag{5.36}$$

$$+Ne^{2\lambda d}\left(\|\widetilde{g}_t\|^2_{H^1(S_T)}+\left\|\widetilde{h}_t\right\|^2_{L_2(S_T)}\right).$$

On the other hand, there exists a number $t_1\in(0,\xi)$ such that

$$\|v(\cdot,t_1)\|^2_{H^1(\Omega)}+\|v_t(\cdot,t_1)\|^2_{L_2(\Omega)}+\|w(\cdot,t_1)\|^2_{H^1(\Omega)}+\|w_t(\cdot,t_1)\|^2_{L_2(\Omega)}$$

$$\leq\frac{1}{\xi}\left(\|v\|^2_{H^1(Q_\xi)}+\|w\|^2_{H^1(Q_\xi)}\right).$$

In fact, otherwise we have

$$\|v(\cdot,t)\|^2_{H^1(\Omega)}+\|v_t(\cdot,t)\|^2_{L_2(\Omega)}+\|w(\cdot,t)\|^2_{H^1(\Omega)}+\|w_t(\cdot,t)\|^2_{L_2(\Omega)}$$

$$>\frac{1}{\xi}\left(\|v\|^2_{H^1(Q_\xi)}+\|w\|^2_{H^1(Q_\xi)}\right)$$

for all $t\in(0,\xi)$. Here by (1.11) we note that the left hand side of the above inequality is continuous in t. Taking the integrals over $t\in(0,\xi)$, we reach a contradiction.

Hence, (5.36) implies that

$$\|v(\cdot,t_1)\|^2_{H^1(\Omega)}+\|v_t(\cdot,t_1)\|^2_{L_2(\Omega)}\leq N\xi^{-1}\exp\left[-2\lambda\sigma\right]\left(\|v\|^2_{H^1(Q_T)}+\|w\|^2_{H^1(Q_T)}\right)\tag{5.37}$$

$$+N\xi^{-1}e^{2\lambda d}\left(\|\widetilde{g}_t\|^2_{H^1(S_T)}+\left\|\widetilde{h}_t\right\|^2_{L_2(S_T)}\right)$$

and

$$\|w(\cdot,t_1)\|^2_{H^1(\Omega)}+\|w_t(\cdot,t_1)\|^2_{L_2(\Omega)}\leq N\xi^{-1}\exp\left[-2\lambda\sigma\right]\left(\|v\|^2_{H^1(Q_T)}+\|w\|^2_{H^1(Q_T)}\right)\tag{5.38}$$

$$+N\xi^{-1}e^{2\lambda d}\left(\|\widetilde{g}_t\|^2_{H^1(S_T)}+\left\|\widetilde{h}_t\right\|^2_{L_2(S_T)}\right).$$

Consider now hyperbolic equation (2.15) for the function $v(x,t)$ in the cylinder $\Omega\times(t_1,T)$ with the boundary conditions

$$v\mid_{\partial\Omega\times(t_1,T)}=\widetilde{g}(x,t),\quad \frac{\partial v}{\partial\nu}\mid_{\partial\Omega\times(t_1,T)}=\widetilde{h}(x,t)$$

and with the initial conditions

$$v\mid_{t=t_1}=v(x,t_1),\quad v_t\mid_{t=t_1}=v_t(x,t_1).$$

Then Lemma 2.3 and (5.37) imply that

$$\|v\|^2_{H^1(\Omega\times(t_1,T))}\leq N\exp\left[-2\lambda\sigma\right]\left(\|v\|^2_{H^1(Q_T)}+\|w\|^2_{H^1(Q_T)}\right)\tag{5.39}$$

$$+Ne^{2\lambda d}\left(\|\widetilde{g}_t\|^2_{H^1(S_T)}+\left\|\widetilde{h}_t\right\|^2_{L_2(S_T)}\right).$$

Change variables $(x,t) \Leftrightarrow (x,\tau = t_1 - t)$. Then we obtain the same hyperbolic equation (2.15), in which $v(x,\tau) := v(x,t_1-\tau)$ and t is replaced with $t_1-\tau$ in all coefficients. This reflects the fact that the hyperbolic equation can be solved in both the positive and negative directions of time. This new hyperbolic equation is satisfied in the cylinder $\Omega \times (0,t_1)$ with the boundary conditions

$$v \mid_{\partial\Omega\times(0,t_1)} = \widetilde{g}(x,t_1-\tau), \quad \frac{\partial v}{\partial \nu} \mid_{\partial\Omega\times(t_1,T)} = \widetilde{h}(x,t_1-\tau)$$

and with the initial conditions

$$v \mid_{\tau=0} = v(x,t_1), \quad v_\tau \mid_{\tau=0} = -v_t(x,t_1).$$

Therefore using again Lemma 2.3 and (5.37), we obtain

$$\|v\|^2_{H^1(\Omega\times(0,t_1))} \leq N\exp\left[-2\lambda\sigma\right]\left(\|v\|^2_{H^1(Q_T)} + \|w\|^2_{H^1(Q_T)}\right)$$
$$+Ne^{2\lambda d}\left(\|\widetilde{g}_t\|^2_{H^1(S_T)} + \left\|\widetilde{h}_t\right\|^2_{L_2(S_T)}\right).$$

Summing up this estimate with (5.39), we obtain

$$\|v\|^2_{H^1(Q_T)} \leq N\exp\left[-2\lambda\sigma\right]\left(\|v\|^2_{H^1(Q_T)} + \|w\|^2_{H^1(Q_T)}\right)$$
$$+Ne^{2\lambda d}\left(\|\widetilde{g}_t\|^2_{H^1(S_T)} + \left\|\widetilde{h}_t\right\|^2_{L_2(S_T)}\right).$$

Similarly

$$\|w\|^2_{H^1(Q_T)} \leq N\exp\left[-2\lambda\sigma\right]\left(\|v\|^2_{H^1(Q_T)} + \|w\|^2_{H^1(Q_T)}\right)$$
$$+Ne^{2\lambda d}\left(\|\widetilde{g}_t\|^2_{H^1(S_T)} + \left\|\widetilde{h}_t\right\|^2_{L_2(S_T)}\right).$$

Summing up two latter estimates, we obtain

$$\{1 - N\exp\left[-2\lambda\sigma\right]\}\left(\|v\|^2_{H^1(Q_T)} + \|w\|^2_{H^1(Q_T)}\right) \tag{5.40}$$
$$\leq Ne^{2\lambda d}\left(\|\widetilde{g}_t\|^2_{H^1(S_T)} + \left\|\widetilde{h}_t\right\|^2_{L_2(S_T)}\right).$$

Recall that $N = N(a,\mu(x_0),f,M,M_1,\Omega,x_0,T)$ denotes different positive constants depending on $a,\mu(x_0),f,M,M_1,\Omega,x_0,T$ and $\sigma,\eta,\mathcal{C}$. Choose a sufficiently large $\lambda_1 = \lambda_1(\Omega,x_0,\eta,N)$ such that

$$1 - N\exp\left[-2\lambda_1\sigma\right] \geq \frac{1}{2}.$$

Hence, (5.40) implies that

$$\left[\|v\|^2_{H^1(Q_T)} + \|w\|^2_{H^1(Q_T)}\right] \leq Ne^{2\lambda_1 d}\left(\|\widetilde{g}_t\|^2_{H^1(S_T)} + \left\|\widetilde{h}_t\right\|^2_{L_2(S_T)}\right). \tag{5.41}$$

Estimates (5.41) and (5.35) lead to

$$\|q\|^2_{H^2(\Omega)} \leq Ne^{2\lambda_1 d}\left(\|\widetilde{g}_t\|^2_{H^1(S_T)} + \left\|\widetilde{h}_t\right\|^2_{L_2(S_T)}\right),$$

which completes the proof. □

Acknowledgment

The work of M.V. Klibanov was supported in part by a grant provided by the University of North Carolina at Charlotte. The work of M. Yamamoto was supported in part by Grants 15340027 and 15654015 from the Japan Society for the Promotion of Sciences.

References

1. A.N. Tikhonov and A.A.Samarskii, *Parital Differential Equations of Mathematical Physics*, Dover Publications, New York, 1963.

2. O.A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics*, Springer-Verlag, Berlin, 1985.

3. A.N. Tikhonov and V.Ya. Arsenin, *Solutions of Ill-Posed Problems*, Winston & Sons, Washington, D.C., 1977.

4. A.L. Bukhgeim and M.V. Klibanov, Uniqueness in the large of a class of multidimensional inverse problems, *Soviet Mathematics Doklady*, 17, 244-247, 1981.

5. A.L. Bukhgeim, Carleman estimates for Volterra operators and uniqueness of inverse problems. In: *Non-Classical Problems of Mathematical Physics*, edited by M.M. Lavrent'ev. Computing Center of The Siberian Branch of the USSR Academy of Science, 55-64, 1981 (in Russian).

6. M.V. Klibanov, Uniqueness of solutions in the 'large' of some multidimensional inverse problems, In: *Non-Classical Problems of Mathematical Physics*, edited by M.M. Lavrent'ev. Computing Center of The Siberian Branch of the USSR Academy of Science, 101-114, 1981 (in Russian). Also, available in the pdf format at
http://www.math.uncc.edu/people/research/mklibanv.php3

7. A.L. Bukhgeim, *Introduction to the Theory of Inverse Problems*, VSP, Utrecht, The Netherlands, 2000.

8. M.V. Klibanov, Inverse problems in the 'large' and Carleman bounds, *Differential Equations*, 20, 755-760, 1984.

9. M.V. Klibanov, Inverse problems and Carleman estimates, *Inverse Problems*, 8, 575-596, 1992.

10. M.V. Klibanov and A. Timonov, Global uniqueness of a non-overdetermined inverse conductivity problem in unbounded domains, *Preprint No 2004-01 of The Department of Mathematics of The University of North Carolina at Charlotte.* Available in the pdf format at http://www.math.uncc.edu/people/research/mklibanv.php3 and at http://www.math.uncc.edu/preprin

11. M.V. Klibanov and A. Timonov, *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications*, VSP, Utrecht, The Netherlands, 2004.

12. M.V. Klibanov and J. Malinsky, Newton-Kantorovich method for 3-dimensional potential inverse scattering problem and stability of the hyperbolic Cauchy problem with time dependent data, *Inverse Problems*, 7, 577-595, 1991.

13. M. Kazemi and M.V. Klibanov, Stability estimates for ill-posed Cauchy problems involving hyperbolic equations and inequalities, *Applicable Analysis*, 50, 93-102, 1993.
14. O.Yu. Imanuvilov and M. Yamamoto, Global uniqueness and stability in determining coefficients of wave equations, *Commun. Part. Diff. Equat.*, 26, 1409-1425, 2001.
15. O.Yu. Imanuvilov and M. Yamamoto, Global Lipschitz stability in an inverse hyperbolic problem by interior observations, *Inverse Problems*, 17, 717-728, 2001.
16. O.Yu. Imanuvilov and M. Yamamoto, Determination of a coefficient in an acoustic equation with a single measurement, *Inverse Problems*, 19, 157-171, 2003.
17. O.Yu. Imanuvilov, V. Isakov and M. Yamamoto, An inverse problem for the dynamical Lamé system, *Comm. Pure Appl. Math.*, 56, 1-17, 2003.
18. T. Hrycak and V. Isakov, Increased stability in the continuation of solutions to the Helmholtz equation, *Inverse Problems*, 20, 697-712, 2004.