Global uniqueness for a 3-D/2-D inverse conductivity problem in tube domains via Carleman estimates

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Abstract.

We consider the problem of determining the electric conductivity in tube domains from the boundary measurements of the voltage potential. The data are assumed to be incomplete. The Tikhonov's formulation of the one-dimensional non-overdetermined inverse conductivity problem is extended to the case of the two-dimensional conductivity distribution. In this formulation, the number of independent variables in the Cauchy data equals the number of independent variables in the unknown conductivity distribution and the position of an electrode injecting the electric current into a tube is assumed to be fixed. The method of Carleman estimates combined with both the direct Fourier and inverse Laplace transforms is employed to establish the Lipschitz stability for an auxiliary inverse hyperbolic problem. The global uniqueness theorem for the original inverse conductivity problem follows from the Lipschitz stability estimate.

1. Introduction

In this paper we consider the problem of determining the electric conductivity in the infinite cylinders whose generatrixes are sufficiently smooth closed curves. Such a problem arises in medical electrical imaging (also known as electrical impedance tomography), electrical prospecting and nondestructive testing of structures elongated in one direction. In such structures, the variation of electric conductivity along the axis of a cylinder may often be neglected, so that the inverse problem consists of recovering the conductivity distribution in a cross-section of the cylinder from observations of the voltage potential performed on a part of its lateral surface.

Formally, let $F(x_1, x_3) = const$ is the equation of a cylinder whose generatrix $F(x_1, x_3)$ is supposed to be a sufficiently smooth closed curve. Let Ω be a cross-section of the cylinder by the x_1x_3-plane. We assume that the conductivity function σ does not depend on the x_2 variable, i.e., $\sigma = \sigma(x_1, x_2)$ and the voltage potential is measured on a set $\Gamma \times (-\infty, \infty),\ \Gamma \subseteq \partial\Omega$. Since the voltage potential is modelled by an elliptic boundary value problem in the cylinder lying in R^3 and the conductivity distribution depends on two variables, we refer below to the 3-D/2-D inverse conductivity problem. The goal of this paper is to establish the global uniqueness result for this problem. As far as the authors are aware, there is no such a result published in the mathematics literature.

If the electrodes injecting the electric currents into the cylinder are not elongated in its axis, the electric steady-state field induced by such currents depends on three spatial coordinates. In this case, the two-dimensional elliptic equation, which is widely exploited to model electrical impedance tomography (see, e.g., the overviews [1], [2]), is inadequate. In the framework of popular Dirichlet-to-Neumann map (D-t-N) techniques, knowledge of all possible voltage potential - current density patterns on the entire boundary of a bounded domain is required to determine uniquely the conductivity distribution [3], [4]. Providing such patterns in practice is, however, a difficult task (see, e.g., [5]).

A non-overdetermined inverse conductivity problem of electrical prospecting was first formulated by Tikhonov in [6] (see also the textbook [7], pp. 426-33). A coefficient inverse problems is said to be non-overdetermined if the number of independent variables in the Cauchy data equals the number of independent variables in an unknown coefficient (see [8], Chapter 1 for details). Sometimes, this is also referred as to an inverse problem with a single measurement. For completeness, we indicate here the Tikhonov's formulation. A steady-state electric field induced by a pointlike source on the plane $x_3 = 0$ was considered in the half-space $R^3_+ = \{x = (x_1, x_2, x_3) \in R^3 : x_3 > 0\}$ filled with the conductivity depending on the x_3 variable. The half-space $R^3_- = \{x = (x_1, x_2, x_3) \in R^3 : x_3 < 0\}$ was assumed to be filled with the dielectric. The steady-state electric field $u(x_1, x_2, x_3)$ was modelled by the problem

$$\Delta_{x_1x_2} u + \sigma^{-1}(x_3) \frac{\partial}{\partial x_3} \left(\sigma(x_3) \frac{\partial u}{\partial x_3} \right) = 0 \ \text{ in } \ R^3_+, \tag{1.1}$$

$$\frac{\partial u}{\partial x_3}\, |_{x_3=0} = 0, \tag{1.2}$$

$$\lim_{|x|\to\infty} u(x) = 0, \tag{1.3}$$

where the voltage potential was represented in the form

$$u(x) = \frac{I}{2\pi\sigma(0)} \cdot \frac{1}{|x|} + \overline{u}(x). \tag{1.4}$$

Here, $\overline{u}(x)$ is a regular component of the voltage potential bounded at the origin and I is the current injected into the half-space R^3_+ by a pointlike ground situated at the origin. In Tikhonov's formulation, given the radial dependence of the voltage potential on the plane $x_3 = 0$, find the conductivity distribution $\sigma(x_3)$ of a layered medium. The position of a pointlike source was fixed and the steady-state electric field was depending on three spatial variables. Thus, one may speak about the 3-D/1-D inverse conductivity problem. In [6], the uniqueness was established by transforming the original problem to the Sturm-Liouville inverse problem via the Fourier-Bessel transform. In this paper, we extend the Tikhonov's formulation to the 3-D/2-D inverse conductivity problem in tube domains.

To derive the 3-D/2-D conductivity model, we first consider Maxwell's equations governing the time-harmonic electric $\mathbf{E}$ and magnetic $\mathbf{H}$ components of the electromagnetic field (e^{iwt} is a time factor) in a conductive medium

$$\nabla \times \mathbf{H} = \sigma(x)\mathbf{E} + \mathbf{j}_{ex}, \tag{1.5}$$

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}, \tag{1.6}$$

where $\mu > 0$ is the magnetic permeability, which is assumed to be constant, ω is an angular frequency, $\mathbf{j}_{ex}$ is the current density of extrinsic sources. The displacement currents are neglected. In the case of direct currents, the angular frequency equals zero. From (1.6) we obtain the relation $\nabla \times \mathbf{E} = 0$ expressing the second Kirchhoff's law in the differential form. Acting by the operator $\nabla\cdot$ on both parts of the equation (1.5) and using the vector identity $\nabla \cdot (\nabla \times \mathbf{a}) = 0$, we obtain

$$\nabla \cdot (\sigma(x)\mathbf{E} + \mathbf{j}_{ex}) = 0, \tag{1.7}$$

It follows from the second Kirchhoff's law that there exists the voltage potential $u(x)$, such that $\mathbf{E} = -\nabla u$. We then obtain from (1.7)

$$\nabla \cdot (\sigma(x)\nabla u) = \nabla \cdot \mathbf{j}_{ex}. \tag{1.8}$$

The extrinsic sources and boundary conditions are specified by the physical nature of the problem to be solved. As an example, we indicate a metallic ball with a negligibly small radius placed at the origin of Cartesian coordinate system. Let the electric current with the magnitude I be flowing uniformly through the surface of the ball into the conductive medium, which is assumed to be homogeneous in a sufficiently small vicinity of the origin. In this vicinity, the voltage potential is given by

$$u(x) = \frac{I}{4\pi\sigma_0} \cdot \frac{1}{|x|}, \tag{1.9}$$

where σ_0 is the conductivity at the origin. Then the relation $\Delta(|x|^{-1}) = -4\pi\delta(x)$ in R^3 implies that $\nabla \cdot \mathbf{j}_{ex} = -I\delta(x)$. Thus, a pointlike ground can be modelled by a δ -like source.

In many real setups, the electrodes are metallic solids, such as the rods (in geophysics of exploration), thin patches (in medical imaging), etc. If the surface of an object is a poor conductor (e.g., a dry soil or the skin of a human body), the electrolyte is placed on the surface of an electrode to improve the electrical contact. As a result, a double layer appears at the interface. It follows from the theory (see, e.g., [10]) that this leads to two types of boundary conditions at the interface. Denote u and u_m the voltage potentials in the electrolyte and electrode in the vicinity of the interface. Then the first boundary condition reflects the discontinuity of the voltage potential, i.e., $u - u_m = \mathcal{E}$, where $\mathcal{E}$ is the electromotive force. Conversely, the second boundary conditions states the continuity of the normal component of the current density, i.e., $\sigma\nabla u \cdot \nu - \sigma_m \nabla u_m \cdot \nu = 0$, where ν is an outward normal unit vector. Defining the impedance Z of the metallic electrode as

$$Z = -\frac{\sigma}{\sigma_m}\frac{u_m}{\nabla u_m \cdot \nu},$$

expressing the voltage potential u_m in terms of Z, we obtain that the Robin condition $u + Z\nabla u \cdot \nu = \mathcal{E}$ holds at the interface. If the dielectric bounds a conductive object, the electric current cannot flow through the boundary into the dielectric. In this case, the normal component of the current density must be equal to zero at the interface. This provides the zero Neumann boundary condition.

Thus, a mathematical model describing the steady-state electric field in tube domains induced by realistic electrodes can be constructed as follows. In the space R^3 filled by a dielectric, consider a cylinder $\mathcal{T}$ whose generatrix is assumed to be closed and sufficiently smooth. Without loss of generality, assume that the axis of this cylinder coincides with the x_2-axis of the system of Cartesian coordinates $x_1x_2x_3$, and it is filled with an isotropic conductive medium whose conductivity is $\sigma(x_1, x_3)$. Let the patches-like electrodes S_i, $(i = 1, 2, ..., n)$ be placed along the generatrix to form a current pattern $(I_1, I_2, ..., I_n)$, which is assumed to be fixed. Then, the voltage potential $u(x_1, x_2, x_3)$ satisfies the problem

$$\begin{aligned}
&\nabla \cdot (\sigma(x_1, x_3)\nabla u) = 0 \ \text{ in } \ \mathcal{T}, \\
&u + Z_i \nabla u \cdot \nu = \mathcal{E}_i \ \text{ on } \ S_i, \ (i = 1, 2, ..., n), \\
&\nabla u \cdot \nu = 0 \ \text{ on } \ \partial\mathcal{T} \setminus \bigcup_{i=1}^{n} S_i, \\
&\int_{S_i} \nabla u \cdot \nu \, ds = I_i, \ (i = 1, 2, ..., n), \\
&\lim_{|x|\to\infty} u = 0.
\end{aligned} \tag{1.10}$$

In the forward conductivity problem, the electromotive forces $\mathcal{E}_i$ are computed for all ‘measuring’ electrodes given the current pattern, whereas they are considered as the measured voltage potentials in the inverse conductivity problem. Note that an analogous model was indicated in [9]. It is, however, unclear how to establish the global uniqueness result for such a problem. Therefore, we simplify the mathematical model indicated above.

Denote $x' = (x_1, x_3)$. Let $\Omega \subset R^2$ be a bounded domain, such that $\partial\Omega \in C^\infty$, and $\mathcal{T} = \Omega \times (-\infty, \infty)$ is a cylinder. Assume that the electric current is injected into the cylinder $\mathcal{T}$ by a fixed electrode situated on its lateral surface. For convenience, the origin of the system of Cartesian coordinates is placed at the electrode location. Then the voltage potential satisfies the following problem

$$\nabla \cdot (\sigma(x')\nabla u) = -f(x_2)S_2(x') \quad \text{in } \mathcal{T}, \tag{1.11}$$

$$\nabla u \cdot \nu = 0 \quad \text{on} \quad \partial\mathcal{T}, \tag{1.12}$$

where

$$u \in H^2(\mathcal{T}) \tag{1.13}$$

and $f(x_2)S_2(x')$ is a source function.

Denote $\Gamma_\infty = \{(x_1, x_2, x_3) \in R^3 : x' \in \Gamma, x_2 \in (-\infty, \infty)\}$. Let $\Omega_1 \subset \Omega$ be a subdomain, such that $\partial\Omega_1 \bigcap \partial\Omega = \emptyset$. To demonstrate that the simplified mathematical model (1.11), (1.12) is substantive, we indicate below the following lemmata.

Lemma 1.1. *Let the following conditions be fulfilled*

$$\sigma(x') \in H^5(\Omega), \ \sigma(x') \geq const > 0 \quad in \quad \Omega, \tag{1.14}$$

$$\sigma(x') = 1 \quad in \quad \Omega \setminus \Omega_1, \tag{1.15}$$

$$S_2(x') \in C^\infty(\overline{\Omega}), \ S_2(x') = 0 \quad in \quad \{x' \in \Omega : \rho(x', \partial\Omega) < \varepsilon\}, \tag{1.16}$$

$$S_2(x') \neq 0 \ in \ \overline{\Omega}_1, \tag{1.17}$$

$$f(x_2) \in C_0^\infty(-\infty, \infty), \ \|f\|_{L_2(-\infty,\infty)} \neq 0, \tag{1.18}$$

where $f(x_2) = -\varphi''(x_2)$, $\varphi(x_2) \in C_0^\infty(-\infty, \infty)$, $\varepsilon > 0$ is a small number, and $p(x', \partial\Omega)$ is the distance from the point $x' \in \Omega$ to the boundary $\partial\Omega$. Then there exists a unique solution $u(x) \in H^2(\mathcal{T})$ of the problem (1.11)-(1.13).

Remark 1.1. It may seem that imposing the conditions (1.17), (1.18) precludes the δ-like source generated by a pointlike ground. However, one may consider the form indicated above as a sort of regularization ('spreading') of the 3-D δ-function (see, e.g., [11]), such that it is negligibly small outside of the vicinity of a pointlike ground. In this case, the perturbations of the electric field caused by such a regularization are small as well. Generally, it is not required to regularize the function $\delta(x_2)$ when establishing the global uniqueness theorem. We impose the regularity conditions on the function $f(x_2)$ for the purpose of establishing the existence of a unique solution of the forward problem. They may be omitted in the 3-D/2-D inverse conductivity problem, i.e., the function $f(x_2)$ can be replaced with $\delta(x_2)$.

Remark 1.2. The condition (1.16) means that the function $S_2(x')$ vanishes near the generatrix of the cylinder $\mathcal{T}$. It is imposed to provide the compatibility conditions (see e.g., [12], Chapter 4) for an auxiliary hyperbolic initial boundary value problem indicated below. The relation $f(x_2) = -\varphi''(x_2)$ means that we first approximate the distribution $\delta(x_2)$ by the function $\varphi(x_2) \in C_0^\infty(-\infty,\infty)$. After this, we differentiate the approximating function in terms of distributions.

It is known that the metric of a cylinder is Eucledian. This allows for introducing the Eucledian coordinates (x_2, l) on its surface. Here, l is a natural parameter of the generatrix (e.g., the arc lenght). Denote $w(x', s)$ and $\tilde{f}(s)$ the Fourier transforms of the functions $u(x)$ and $f(x_2)$ with respect to the x_2 variable and $Y(x', s) = \tilde{f}^{-1}(s)w(x', s)$. By virtue of (1.18), the function $\tilde{f}(s)$ can be extended into the complex plane as an entire analytic function. Therefore, such an extension may have only a finite number of zeros in every finite interval of the real axis. Applying the operator of the Fourier transform with respect to the x_2 variable to the problem (1.11) - (1.13), we then obtain that the function $Y(x', s)$ satisfies the following problem for almost all $s \in (-\infty, \infty)$

$$\nabla \cdot (\sigma \nabla Y) - s^2 \sigma Y = -S_2(x') \text{ in } \Omega, \tag{1.19}$$

$$\nabla Y(x', s) \cdot \nu = 0 \text{ on } \partial\Omega, \tag{1.20}$$

$$Y \in H^2(\Omega). \tag{1.21}$$

Lemma 1.2. *Let the conditions (1.14) - (1.18) be fulfilled. Then the problem (1.11)-(1.13) has at most one solution. Moreover, for every $s \neq 0$, there exists at most one solution of the problem (1.19)- (1.21).*

Proofs of lemmata 1.1 and 1.2 are based on the conventional techniques. Specifically, to prove the lemma 1.1, one may apply the Fourier transform to the equations (1.11) and (1.12) with respect to the x_2 variable. As a result, one may obtain the analogues of equations (1.19) and (1.20) for the function $w(x', s)$. After this, one may use the standard energy-like estimates for elliptic equations to complete the proof. The proof of lemma 1.2 is also based on such estimates.

Since $\int_{S_i} \nabla u \cdot \nu \, ds = 0$ for every 'measuring' electrode, then integrating the Robin condition over the patch S_i, applying the mean value theorem and shrinking the patch to a point, we obtain that $u = \mathcal{E}$ for all 'measuring' pointlike electrodes. This motivates the following formulation.

Inverse Conductivity Problem. *Let the conditions (1.14)-(1.18) be fulfilled, and the function $u(x) \in H^2(\mathcal{T})$ be a solution of the boundary value problem (1.11)- (1.13). Given the function*

$$u = \psi(x_2, l) \quad in \quad \Gamma_\infty, \tag{1.22}$$

determine the conductivity distribution $\sigma(x')$ in the domain Ω_1.

In [4], the inverse scattering theory for a first order elliptic system was used to establish the global uniqueness result for the 2-D/2-D inverse conductivity problem. However, such a theory is not applicable to our 3-D/2-D inverse problem formulated above. To prove the global uniqueness theorem, we formulate an auxiliary inverse problem for a hyperbolic equation. Let $P \subset R^n$ be a bounded domain, such that

$$\partial P \in C^\infty, T = const > 0, P_T = P \times (0,T), Z_T = \partial P \times (0,T),$$

the functions $p(x) \in C^3\left(\overline{P}\right), f(x) \in C^4\left(\overline{P}\right), p \geq m = const > 0$. Consider the following hyperbolic initial boundary value problem in the cylinder P_T

$$\partial_t^2 u = \Delta u - \frac{\nabla p}{p} \cdot \nabla u - F\left(\Delta p, \nabla p, p\right) u \text{ in } P_T, \tag{1.23}$$

$$u \mid_{t=0} = f(x), u_t \mid_{t=0} = 0, \tag{1.24}$$

$$u \mid_{Z_T} = g(x,t), \tag{1.25}$$

where

$$F(\Delta p, \nabla p, p) = \frac{\Delta p}{p} - \frac{|\nabla p|^2}{p^2}. \tag{1.26}$$

Auxiliary Inverse Problem *Let $F \in C^3\left(R^{n+2}\right)$. Suppose that the function $u(x,t) \in C^4\left(\overline{P}_T\right)$ is the soluton of the problem (1.23)-(1.25). Given the function $h(x,t)$, such that*

$$\frac{\partial u}{\partial \nu} \mid_{Z_T} = h(x,t), \tag{1.27}$$

determine the function $p(x)$ for $x \in P$.

The main results are formulated in the following theorems. Let $B(M) = \left\{p(x) \in C^3(\overline{P}) : \|p\|_{C^2(\overline{\Omega})} \leq M,\ M = const > 0\right\}$.

Theorem 1.1 (Lipschitz stability). *Let $n \in [1,13]$. Suppose that there exists two pairs of functions $(p_1,u_1), (p_2,u_2)$ satisfying the equations (1.23) and (1.24) and the conditions*

$$p_1 \mid_{\partial P} = p_2 \mid_{\partial P}, \quad \frac{\partial p_1}{\partial \nu} \mid_{\partial P} = \frac{\partial p_2}{\partial \nu} \mid_{\partial P}, \tag{1.28}$$

$$u_i \mid_{Z_T} = g_i(x,t), \quad \frac{\partial u_i}{\partial \nu} \mid_{Z_T} = h_i(x,t), (i = 1,2) \tag{1.29}$$

for some functions g_i, h_i. Also, assume that $p_1, p_2 \in B(M)$, $f \in C^4\left(\overline{P}\right)$,

$$\min_{\overline{P}} |f(x)| \geq \beta, \quad \left|\frac{\partial}{\partial(\Delta p)} F(y)\right| \geq \beta = const > 0,\ \forall y \in R^{n+2}, \tag{1.30}$$

$$T > diam\,(P)\,, \tag{1.31}$$

$u_i \in C^4\left(\overline{P}_T\right)$ and

$$\|u_i\|_{C^3\left(\overline{P}_T\right)} \leq M_1, \tag{1.32}$$

where M_1 is a positive constant. Then there exists a positive constant $N = N\left(f, M, M_1, P, T, \beta\right)$, such that the following Lipschitz stability estimate holds

$$\|p_1 - p_2\|_{H^2(P)} \leq N\left[\left\|\partial_t^2\left(g_1 - g_2\right)\right\|_{H^1(Z_T)} + \left\|\partial_t^2\left(h_1 - h_2\right)\right\|_{L_2(Z_T)}\right]. \tag{1.33}$$

In particular, suppose that there exists two pairs of functions $(p_1,u_1), (p_2,u_2)$ satisfying the conditions (1.23)-(1.28), such that $u_1, u_2 \in C^4\left(\overline{P}_T\right)$. Then $p_1(x) = p_2(x)$ in P and $u_1(x,t) = u_2(x,t)$ in P_T.

Theorem 1.2 (Global uniqueness). *Let the conditions (1.14)-(1.18) be fulfilled. Then there exists at most one pair of functions* $(\sigma(x'), u(x))$ *satisfying the relations (1.11) - (1.13) and (1.22).*

Remark 1.3. To guarantee that $u_1, u_2 \in C^4\left(\overline{P}_T\right)$, the compatibility conditions and smoothness of the function $f(x)$ and coefficients p_1, p_2 are required (see, e.g., [12], Chapter 4). To simplify the presentation, we do not indicate such conditions in this paper. Also, we are not concerned here with weakening smoothness conditions in the theorems 1.1 and 1.2.

Note that the problem of establishing the Lipschitz stability estimate for the auxiliary inverse problem is interesting by itself. To establish the Lipschitz stability estimate indicated in the theorem 1.1, the modified method of Carleman estimates is used (see, e.g., [8], Chapter 3 for details). We note that this method was also exploited in [13] and [14] to establish the Lipschitz stability for an inverse problem for the equation $\partial_t^2 u = \Delta u + p(x)u$. Once the Lipschitz stability is established, the theorem 1.2 follows directly from the stability estimate. Unlike [13] and [14], the unknown coefficient $\sigma(x')$ is contained in the equation (1.23) together with its derivatives up to the second order. This motivates the modification of the method of Carleman estimates. A similar issue was recently addressed in [15], in which the equation $u_{tt} = \nabla \cdot (p(x)\nabla u)$ was considered. Although introducing the function $\hat{u} = p(x)u$ reduces this equation to one similar to the equation (1.23), the initial condition $\hat{u}(x,0) = p(x)f(x)$ appears instead (1.24). Moreover, we do not impose the zero Neumann boundary condition as in [13] and [14]. Finally, we exploit a new technical feature of the Carleman estimate for the hyperbolic operator introduced recently in [8]. This feature allows one to avoid the use of the odd extension of a temporal function in $\{t < 0\}$.

The paper is organized as follows. In section 2, we indicate some a priori estimates including new Carleman estimates for both the Laplacian and D'Alembertian needed for proofs. In section 3, we complete proofs of the main results.

2. A priori estimates

We first introduce some notations and estimates that shall be used in the proof of theorem 1.1. Let $G(x) \in C^2(R^n)$. We denote $G_i = \partial_i G, G_{ij} = \partial_i \partial_j G$.

2.1. Some hyperbolic inequalities and estimates

Denote

$$v = u_{1t} - u_{2t}, \quad (x,t) \in P_T, \tag{2.1}$$

$$\tilde{g}(x,t) = \partial_t \left(g_1 - g_2\right)(x,t), \quad (x,t) \in Z_T, \tag{2.2}$$

$$\tilde{h}(x,t) = \partial_t \left(h_1 - h_2\right), \quad (x,t) \in Z_T, \tag{2.3}$$

$$q(x) = p_1(x) - p_2(x). \tag{2.4}$$

By virtue of (1.24), we have

$$v\,(x,0)\mid_{\partial P}=\frac{\partial v(x,0)}{\partial \nu}\mid_{\partial P}=0.$$

Hence,

$$\tilde{g}(x,t)=\int_0^t \partial_t \tilde{g}(x,\tau)d\tau,\quad \tilde{h}(x,t)=\int\limits_0^t \partial_t \tilde{h}(x,\tau)d\tau.$$

This implies that

$$\|\tilde{g}\|_{H^1(Z_T)}\leq N\,\|\partial_t\tilde{g}\|_{H^1(Z_T)}\,,\quad \left\|\tilde{h}\right\|_{L_2(Z_T)}\leq N\left\|\partial_t\tilde{h}\right\|_{L_2(Z_T)}. \tag{2.5}$$

Let a_1, a_2, b_1 and b_2 be four numbers, $\tilde{a}=a_1-a_2, \tilde{b}=b_1-b_2$. Then, $a_1b_1-a_2b_2=\tilde{a}b_1+\tilde{b}a_2$. We note that an n-dimensional analogue of Lagrange's formula

$$\rho(\xi_1)-\rho(\xi_2)=(\xi_1-\xi_2)\int_0^1 \rho'((\xi_1-\xi_2)\,\kappa+\xi_2)d\kappa,\ \forall\rho\in C^1\,(-\infty,\infty)$$

is valid. From the last two formulas and the relations (1.23)-(1.30), (1.32), and (2.1)-(2.4), we obtain

$$\partial_t^2 v=\Delta v+\sum_{j=1}^n c^j(x,t)v_j+c^0(x,t)v+l(x,t)\Delta q(x) \tag{2.6}$$

$$+\,(m(x,t),\nabla q(x))+m^0(x,t)q(x)\ \text{ in }\ P_T,$$

$$v\mid_{t=0}=0\ \text{ in }\ P, \tag{2.7}$$

$$|\partial_t v|\mid_{t=0}\geq N\,|\Delta q\,(x)|-N\,(|\nabla q|+|q|)\,(x)\quad\text{in}\quad P, \tag{2.8}$$

$$v\mid_{Z_T}=\tilde{g},\frac{\partial v}{\partial \nu}\mid_{Z_T}=\tilde{h}, \tag{2.9}$$

where $c^j, c^0, l, m, m^0 \in C^2\left(\overline{P}_T\right)$. If $b(x,t)$ is any of these functions, then $\|b\|_{C^1\left(\overline{P}_T\right)}\leq N$. It is sometimes advantageous for the method of Carleman estimates to replace the equations with the corresponding inequalities. The inequality (2.8) follows from the equation $\partial_t v_t(x,0)=\partial_t^2\,(u_1-u_2)$, (1.23) and (1.30). In just the same way, we replace the equation (2.6) with the inequality

$$\left|\partial_t^2 v-\Delta v\right|\leq N\,(|\nabla v|+|v|+|\Delta q|+|\nabla q|+|q|)\quad\text{in}\quad P_T. \tag{2.10}$$

Denote $w=\partial_t v$. Then the equations (2.6) and (2.7) imply that

$$\partial_t w(x,0)=0. \tag{2.11}$$

Differentiating (2.6) and (2.9), we obtain

$$\left|\partial_t^2 w-\Delta w\right|\leq N\,(|\nabla w|+|w|+|\Delta q|+|\nabla q|+|q|)\quad\text{in}\quad P_T, \tag{2.12}$$

$$w\mid_{Z_T}=\partial_t\tilde{g},\quad \frac{\partial w}{\partial \nu}\mid_{Z_T}=\partial_t\tilde{h}. \tag{2.13}$$

Also, we shall need the Lipschitz stability estimate for the Cauchy problem with the lateral data for a hyperbolic inequality. For this purpose, we indicate the following lemma (see [8], Theorem 2.4.1 for details).

Lemma 2.1. *Suppose that the condition (1.31) is fulfilled, the function* $u(x,t) \in C^2\left(\overline{P}_T\right)$ *satisfies the hyperbolic inequality*

$$\left|\partial_{tt}^2 u - \Delta u\right| \le K\left(|\nabla u| + |\partial_t u| + |u| + |z(x,t)|\right) \quad in \quad P_T,$$

and the boundary conditions

$$u\mid_{Z_T} = \varphi_0(x,t), \ \frac{\partial u}{\partial \nu}\mid_{Z_T} = \varphi_1(x,t),$$

where K *is a positive constant and*

$$z \in L_2(P_T), \ \varphi_0 \in H^1\left(Z_T\right), \ \varphi_1 \in L_2\left(Z_T\right).$$

Then there exists a positive constant $K_1 = K_1\left(P,T,K\right)$ *depending on* P,T,K*, such that*

$$\|u\|_{H^1(P_T)} \le K_1\left(\|\varphi_0\|_{H^1(Z_T)} + \|\varphi_1\|_{L_2(Z_T)} + \|z\|_{L_2(P_T)}\right).$$

2.2. New Carleman estimates

Choose a point $x_0 \notin \overline{P}$, such that $\rho\left(x_0,P\right) < \xi$, where ξ is a small positive number and $\rho\left(x_0,P\right)$ is the distance between the point x_0 and the domain P. Consider the Carleman Weight Function (CWF) for the hyperbolic operator

$$\mathcal{C}(x,t) = \exp\left[\lambda\left(|x-x_0|^2 - \eta t^2\right)\right] = \exp\left[\lambda\psi\left(x,t\right)\right], \ 0 < \eta < 1, \tag{2.14}$$

where λ is a positive parameter. Let

$$c^2\left(x_0\right) = \min_{x \in \overline{\Omega}} |x-x_0|^2. \tag{2.15}$$

For a number $c^2 \in \left(0, c^2\left(x_0\right)\right)$, consider the domain

$$G_{c^2} = \left\{(x,t) \in P_T : |x-x_0|^2 - \eta t^2 > c^2\right\}. \tag{2.16}$$

Choose a sufficiently small positive number $\delta = \delta\left(P, c^2\left(x_0\right), T, \xi\right) \in \left(0, c^2\left(x_0\right)\right)$, such that

$$\overline{G_{c^2}} \cap \{t = T\} = \emptyset \quad \text{for } c^2 \in \left(c^2\left(x_0\right) - \delta, c^2\left(x_0\right) - \delta/2\right), \eta \in (0,1). \tag{2.17}$$

The inequality (1.31) implies the relation (2.17) for sufficiently small ξ and $\delta = \delta\left(P, c^2\left(x_0\right), T, \xi\right)$. By virtue of (2.14)-(2.16), there exists a positive number $a = a\left(P, c^2\left(x_0\right), T, \xi\right)$, such that

$$P \times (0,a) \subset G_{c^2} \quad for \quad c^2 \in \left(c^2\left(x_0\right) - \delta, c^2\left(x_0\right) - \delta/2\right), \ \eta \in (0,1). \tag{2.18}$$

Thus, the boundary of the domain G_{c^2} consists of three parts

$$\partial G_{c^2} = \bigcup_{i=1}^{3} \partial_i G_{c^2} \quad \text{for} \quad c^2 \in \left(c^2\left(x_0\right) - \delta, c^2\left(x_0\right) - \delta/2\right), \tag{2.19}$$

where

$$\partial_1 G_{c^2} = \left\{(x,t) \in P_T : \psi\left(x,t\right) = c^2\right\}, \tag{2.20}$$

$$\partial_1 G_{c^2} = \left\{(x,t) \in Z_T : \psi\left(x,t\right) > c^2\right\}, \tag{2.21}$$

$$\partial_3 G_{c^2} = \left\{(x,t) : x \in P, t = 0\right\}. \tag{2.22}$$

These parts of the boundary play an important role in the method of Carleman estimates, as always.

Lemma 2.2. *Suppose that* $c^2 \in (c^2(x_0) - \delta, c^2(x_0) - \delta/2)$. *Then there exists a sufficiently large positive constant* $\lambda_0 = \lambda_0(P, x_0, \delta, \eta) > 1$ *and a positive number* $C = C(P, x_0, \delta, \eta)$, *such that the following pointwise Carleman estimate is valid for all functions* $u \in C^2\left(\overline{P}_T\right)$

$$(u_{tt} - \Delta u)^2 \mathcal{C}^2 \geq C\lambda\left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2\right)\mathcal{C}^2 + \nabla \cdot \mathcal{U} + V_t \quad in \quad G_{c^2},$$

where the vector function (U, V) *satisfies the following estimate*

$$|(U, V)| \leq C\lambda\left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2\right)\mathcal{C}^2 \quad in \quad G_{c^2}.$$

In addition, the function $V(x, t)$ *can be estimated as*

$$|V(x,t)| \leq C\lambda^3\left[t\left(|\nabla u|^2 + u_t^2 + u^2\right) + (|\nabla u| + |u|) \cdot |u_t|\right]. \tag{2.23}$$

Thus, the identity

$$V(x, 0) \equiv 0, \tag{2.24}$$

holds if

$$\text{either} \quad u(x,0) \equiv 0 \quad \text{or} \quad u_t(x,0) \equiv 0. \tag{2.25}$$

See the theorem 2.2.4 in [8] for the proof. The relations (2.23)-(2.25) represent a new feature of the Carleman estimate for a hyperbolic operator. These relations allow us to avoid using the odd extension of the function $v(x,t)$ with respect to t in $P \times (-T, 0)$ in contrast with the previous works.

The following lemma provides an estimate from above for an integral containing the CWF (2.14).

Lemma 2.3. *For all functions* $s \in C\left(\overline{G_{c^2}}\right)$ *and for all* $\lambda \geq 1$, *the following estimate holds*

$$\int_{G_{c^2}} \left[\int_0^t s(x,\tau)d\tau\right]^2 \mathcal{C}^2(x,t)dxdt \leq \frac{1}{\lambda\eta}\int_{G_{c^2}} \left(s^2\mathcal{C}^2\right)(x,t)dxdt.$$

See Chapter 3 in [8] for proof.

Following [15], we indicate a new Carleman estimate for the Laplace operator. It is implied by the CWF (2.14) by setting $\eta = 0$. Unlike the conventional CWF for the elliptic operators, whose level sets are paraboloids (see, e.g., [8], Chapter 2), the level sets of the proposed CWF are spheres $\{|x - x_0| = const > 0\}$. For the lower order derivatives, a similar estimate has been recently published in [16].

Lemma 2.4. *Let* $n \in [1, 13]$, λ *be a positive parameter and*

$$\tilde{\mathcal{C}}(x) = \exp\left(\lambda|x - x_0|^2\right), \quad x \in P.$$

Then there exists a sufficiently large positive constant $\lambda_0 = \lambda_0(P, x_0) > 1$ *and a positive number* $C = C(P, x_0)$, *such that the following pointwise Carleman estimate is valid for all functions* $u \in C^3\left(\overline{P}\right)$

$$(\Delta u)^2 \tilde{\mathcal{C}}^2 \geq \frac{C}{\lambda}\sum_{i,j=1}^n u_{ij}^2\tilde{\mathcal{C}}^2 + C\lambda\left(|\nabla u|^2 + \lambda^2 u^2\right)\tilde{\mathcal{C}}^2 + \nabla \cdot U \quad in \quad P, \tag{2.26}$$

where the vector function U satisfies the inequality

$$|U| \leq \frac{C}{\lambda}\left(\sum_{i,j=1}^{n}|\nabla u|\,|u_{ij}|\right)\tilde{\mathcal{C}}^2 + C\lambda\left(|\nabla u|^2+\lambda^2u^2\right)\tilde{\mathcal{C}}^2 \quad in \quad P.$$

In particular,

$$\int_P (\Delta u)^2\,\tilde{\mathcal{C}}^2 dx \geq \frac{C}{\lambda}\sum_{i,j=1}^{n}\int_P u_{ij}^2\tilde{\mathcal{C}}^2 dx + C\lambda\int_P\left(|\nabla u|^2+\lambda^2u^2\right)\tilde{\mathcal{C}}^2 dx$$

for all real valued functions $u \in H^2(P)$ satisfying the conditions

$$u\mid_{\partial P} = \frac{\partial u}{\partial \nu}\mid_{\partial P} = 0.$$

3. Proofs of theorems 1.1 and 1.2

We first prove the theorem 1.1. Recall that $c^2 \in (c^2(x_0)-\delta, c^2(x_0)-\delta/2)$. Choose a sufficiently small positive number μ such that $c^2+3\mu \in (c^2(x_0)-\delta, c^2(x_0)-\delta/2)$. Clearly, $G_{c^2+3\mu} \subset G_{c^2+2\mu} \subset G_{c^2+\mu} \subset G_{c^2}$. The conditions (2.15)-(2.17) imply that $G_{c^2+3\mu} \neq \emptyset$. By virtue of (2.17) and (2.18), there exists a sufficiently small positive number $a = a(P, c^2(x_0), T, \xi, \mu)$, such that

$$P\times(0,a) = P_a \subset G_{c^2+3\mu} \tag{3.1}$$

and

$$G_{c^2+3\mu}\bigcap\{t=T\} = G_{c^2}\bigcap\{t=T\} = \emptyset.$$

Also, we have

$$\mathcal{C}^2(x,t)\begin{cases} \geq \exp\left[2\lambda\left(c^2+3\mu\right)\right] & \text{if} \quad (x,t)\in G_{c^2+3\mu}, \\ \leq \exp\left[2\lambda\left(c^2+2\mu\right)\right] & \text{if} \quad (x,t)\in G_{c^2}\setminus G_{c^2+2\mu} \end{cases}. \tag{3.2}$$

Introduce the cutoff function $\chi_\mu(x,t) \in C^3\left(\overline{P}_T\right)$, such that

$$\chi_\mu(x,t) = \begin{cases} 1 & \text{if} \quad (x,t)\in G_{c^2+2\mu}, \\ 0 & \text{if} \quad (x,t)\in G_{c^2}\setminus G_{c^2+\mu}, \\ \text{between} \quad 0 \quad \text{and} \quad 1, & \text{otherwise}. \end{cases} \tag{3.3}$$

Denote $\overline{v}(x,t) = \chi_\mu(x,t)\cdot v(x,t)$, $\overline{w}(x,t) = \chi_\mu(x,t)\cdot w(x,t)$. Hence,

$$\overline{v}, \overline{w} \in C^2\left(\overline{P}_T\right). \tag{3.4}$$

By virtue of (2.20) and (3.3), we have

$$\overline{v}\mid_{\partial_1 G_{c^2}} = \partial_t\overline{v}\mid_{\partial_1 G_{c^2}} = \nabla\overline{v}\mid_{\partial_1 G_{c^2}} = \overline{w}\mid_{\partial_1 G_{c^2}} = \partial_t\overline{w}\mid_{\partial_1 G_{c^2}} = \nabla\overline{w}\mid_{\partial_1 G_{c^2}} = 0. \tag{3.5}$$

It follows from (3.1) and (3.3) that $\chi_\mu(x,t) = 1$ for $(x,t)\in P_a$. Hence, for $x \in P$ we obtain

$$\overline{v}(x,0) = v(x,0), \quad \partial_t\overline{v}(x,0) = \partial_t v(x,0), \tag{3.6}$$
$$\overline{w}(x,0) = w(x,0), \quad \partial_t\overline{w}(x,0) = \partial_t w(x,0).$$

It follows from (3.3) that all derivatives of the function χ_μ equal zero in the domain $G_{c^2+2\mu}$. Multiplying (2.10) and (2.12) by the function χ_μ, using (2.7) - (2.13), the relations $\chi_\mu \partial_{tt}^2 v = \partial_{tt}^2 \overline{v} - 2\partial_t \chi_\mu \partial_t v - \partial_{tt}^2 \chi_\mu \cdot v$ and similar formulas for $\chi_\mu \Delta v$ and $\chi_\mu v_i$, we obtain

$$\left|\partial_{tt}^2 \overline{v} - \Delta \overline{v}\right| \leq N\left(|\nabla \overline{v}| + |\overline{v}| + |\Delta q| + |\nabla q| + |q|\right) \qquad (3.7)$$
$$+ N\left(1-\chi_\mu\right)\left(|\nabla v| + |\partial_t v| + |v|\right) \quad \text{in} \quad P_T,$$

$$\overline{v}(x,0) = 0 \quad \text{in} \quad P, \qquad (3.8)$$

$$\left|\partial_t \overline{v}(x,0)\right| = \left|\partial_t v(x,0)\right| \geq N\left|\Delta q\,(x)\right| - N\left(|\nabla q| + |q|\right)(x) \quad \text{in} \quad P, \qquad (3.9)$$

$$\overline{v}\mid_{Z_T} = \chi_\mu \cdot \tilde{g}\,(x,t)\,, \quad \frac{\partial \overline{v}}{\partial \nu}\mid_{Z_T} = \chi_\mu \cdot \tilde{h}\,(x,t) + \frac{\partial \chi_\mu}{\partial \nu} \cdot \tilde{g}\,(x,t)\,. \qquad (3.10)$$

Also, we obtain the following relations for the function $\overline{w}$

$$\left|\partial_{tt}^2 \overline{w} - \Delta \overline{w}\right| \leq N\left(|\nabla \overline{w}| + |\overline{w}| + |\Delta q| + |\nabla q| + |q|\right) \qquad (3.11)$$
$$+N\left(1-\chi_\mu\right)\left(|\nabla \overline{w}| + |\partial_t \overline{w}| + |\overline{w}|\right) \quad in \quad P_T,$$

$$\partial_t \overline{w}(x,0) = 0, \qquad (3.12)$$

$$\overline{w}\mid_{Z_T} = \chi_\mu \cdot \partial_t \tilde{g}\,(x,t)\,, \; \frac{\partial \overline{w}}{\partial \nu}\mid_{S_T} = \chi_\mu \cdot \partial_t \tilde{h}\,(x,t) + \frac{\partial \chi_\mu}{\partial \nu} \cdot \partial_t \tilde{g}\,(x,t)\,. \qquad (3.13)$$

Multiplying both sides of the inequality (3.7) by the CWF $\mathcal{C}(x,t)$, squaring them, integrating over the domain G_{c^2} and applying the lemma 2.1, (2.20)-(2.22), (3.6)-(3.10) and the Gauss-Ostrogradsky formula, we obtain for $\lambda \geq \lambda_0\left(P, x_0, \eta\right) > 1$

$$N\int_{Z_T}\left(\tilde{g}^2 + |\nabla \tilde{g}|^2 + \tilde{g}_t^2 + \tilde{h}^2\right)\mathcal{C}^2 dS +$$
$$N\int_{G_{c^2}\setminus G_{c^2+2\mu}}\left(|\nabla v|^2 + v_t^2 + v^2\right)\mathcal{C}^2 dxdt +$$
$$N\int_{G_{c^2}}\left(|\nabla \overline{v}|^2 + \overline{v}^2\right)\mathcal{C}^2 dxdt + \qquad (3.14)$$
$$N\int_{G_{c^2}}\left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)\mathcal{C}^2 dxdt \geq$$
$$C\lambda\int_{G_{c^2}}\left(|\nabla \overline{v}|^2 + \overline{v}_t^2 + \lambda^2 \overline{v}^2\right)\mathcal{C}^2 dxdt.$$

Choosing a sufficiently large $\lambda_1 = \lambda_1\left(N\right) > \lambda_0\left(P, x_0, \eta\right)$ and assuming that $\lambda \geq \lambda_1$, we obtain from (3.14)

$$N\int_{Z_T}\left(\tilde{g}^2 + |\nabla \tilde{g}|^2 + \left(\partial_t \tilde{g}\right)^2 + \tilde{h}^2\right)\mathcal{C}^2 dS +$$
$$N\int_{G_{c^2}\setminus G_{c^2+2\mu}}\left(|\nabla v|^2 + \left(\partial_t v\right)^2 + v^2\right)\mathcal{C}^2 dxdt +$$
$$N\int_{G_{c^2}}\left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)\mathcal{C}^2 dxdt \geq$$

$$C\lambda\int_{G_{c^2}}\left(|\nabla\overline{v}|^2+(\partial_t\overline{v})^2+\lambda^2\overline{v}^2\right)\mathcal{C}^2dxdt.$$

Since $G_{c^2+2\mu}\subset G_{c^2}$ and $\overline{v}(x,t)=v(x,t)$ in $G_{c^2+2\mu}$, then the latter estimate leads to

$$\begin{aligned}&N\int_{Z_T}\left(\tilde{g}^2+|\nabla\tilde{g}|^2+(\partial_t\tilde{g})^2+\tilde{h}^2\right)\mathcal{C}^2dS+\\&N\int_{G_{c^2}\setminus G_{c^2+2\mu}}\left(|\nabla v|^2+(\partial_t v)^2+v^2\right)\mathcal{C}^2dxdt+\\&N\int_{G_{c^2}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt\geq\\&C\lambda\int_{G_{c^2+2\mu}}\left(|\nabla v|^2+(\partial_t v)^2+\lambda^2v^2\right)\mathcal{C}^2dxdt.\end{aligned}\tag{3.15}$$

Similarly, we obtain from (3.11)-(3.13)

$$\begin{aligned}&N\int_{Z_T}\left((\partial_t\tilde{g})^2+|\nabla(\partial_t\tilde{g})|^2+\left(\partial_t^2\tilde{g}\right)^2+\left(\partial_t\tilde{h}\right)^2\right)\mathcal{C}^2dS\\&+N\int_{G_{c^2}\setminus G_{c^2+2\mu}}\left(|\nabla w|^2+(\partial_t w)^2+w^2\right)\mathcal{C}^2dxdt\\&+N\int_{G_{c^2}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt\\&\geq C\lambda\int_{G_{c^2+2\mu}}\left(|\nabla w|^2+(\partial_t w)^2+\lambda^2w^2\right)\mathcal{C}^2dxdt.\end{aligned}\tag{3.16}$$

For each $x\in P$ denote

$$t_{c^2+2\mu}(x)=\frac{\sqrt{|x-x_0|^2-(c^2+2\mu)}}{\sqrt{\eta}}.$$

Then we have

$$\int_{G_{c^2+2\mu}}s(x,t)dxdt=\int_\Omega\left[\int_0^{t_{c^2+2\mu}(x)}s(x,t)dt\right]dx,\ \forall s\in C\left(\overline{G}_{c^2+2\mu}\right).\tag{3.17}$$

Hence,

$$\begin{aligned}&\int_{G_{c^2}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt=\\&\int_{G_{c^2}\setminus G_{c^2+2\mu}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt\\&+\int_{G_{c^2+2\mu}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt\\&=\int_{G_{c^2}\setminus G_{c^2+2\mu}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt\\&+\int_P\int_0^a\left(|\Delta q|^2+|\nabla q|^2+q^2\right)(x)\,\mathcal{C}^2dtdx+\\&\int_P\int_a^{t_{c^2+2\mu}(x)}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)(x)\,\mathcal{C}^2dtdx.\end{aligned}\tag{3.18}$$

Note that

$$\int_P \int_0^a \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)(x)\,\mathcal{C}^2 dtdx +$$
$$\int_P \int_a^{t_{c^2+2\mu}(x)} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)(x)\,\mathcal{C}^2 dtdx \tag{3.19}$$
$$= \int_P \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)(x)\left(\int_0^a \mathcal{C}^2 dt\right)dx$$
$$+ \int_P \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)(x)\left(\int\limits_a^{t_{c^2+2\mu}(x)} \mathcal{C}^2 dt\right)dx.$$

Since the function $\theta(t) = \exp\left(-2\lambda\eta t^2\right)$ decreases for $t > 0$, then

$$\int_a^{t_{c^2+2\mu}(x)} \mathcal{C}^2 dt = \exp\left(2\lambda\left|x - x_0\right|^2\right)\int_a^{t_{c^2+2\mu}(x)} \exp\left(-2\lambda\eta t^2\right)dt$$
$$\leq\ (t_{c^2+2\mu}(x) - a)\exp\left(2\lambda\left|x - x_0\right|^2\right)\exp\left(-2\lambda\eta a^2\right)$$
$$=\ (t_{c^2+2\mu}(x) - a)\exp\left(2\lambda\left|x - x_0\right|^2\right)a^{-1}\int_0^a \exp\left(-2\lambda\eta a^2\right)dt$$
$$\leq (t_{c^2+2\mu}(x) - a)\exp\left(2\lambda\left|x - x_0\right|^2\right)a^{-1}\int_0^a \exp\left(-2\lambda\eta t^2\right)dt \leq N\int_0^a \mathcal{C}^2 dt.$$

Hence, the relations (3.1), (3.18), and (3.19) imply that

$$\int_{G_{c^2}} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)\mathcal{C}^2 dxdt \leq$$
$$\int_{G_{c^2}\setminus G_{c^2+2\mu}} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)\mathcal{C}^2 dxdt \tag{3.20}$$
$$+N\int_{P_a} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)(x)\,\mathcal{C}^2 dxdt.$$

Note that

$$\mathcal{C}^2(x,t) \leq \exp\left[2\lambda\left(c^2 + 2\mu\right)\right] \quad \text{in} \quad G_{c^2}\setminus G_{c^2+2\mu}. \tag{3.21}$$

Hence, the relations (3.17), (3.20), and (3.21) imply that

$$\int_{G_{c^2}\setminus G_{c^2+2\mu}} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)\mathcal{C}^2 dxdt$$
$$\leq \exp\left[2\lambda\left(c^2 + 2\mu\right)\right]\int_{G_{c^2}\setminus G_{c^2+2\mu}} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)dxdt$$
$$\leq \exp\left[2\lambda\left(c^2 + 2\mu\right)\right]\int_{G_{c^2}} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)dxdt$$
$$= \exp\left[2\lambda\left(c^2 + 2\mu\right)\right]\int_P \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)dx\int_0^{t_{c^2}(x)} dt$$
$$\leq N\exp\left[2\lambda\left(c^2 + 2\mu\right)\right]\int_P \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)dx\int_0^a dt$$
$$= N\exp\left[2\lambda\left(c^2 + 2\mu\right)\right]\int_{P_a} \left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)dxdt.$$

Thus, we have

$$\int_{G_{c^2}\setminus G_{c^2+2\mu}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt \tag{3.22}$$

$$\leq N\exp\left[2\lambda\left(c^2+2\mu\right)\right]\int_{P_a}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)dxdt.$$

Since by virtue of (3.1), we obtain

$$\exp\left[2\lambda\left(c^2+2\mu\right)\right]<\exp\left[2\lambda\left(c^2+3\mu\right)\right]\leq\mathcal{C}^2(x,t)\quad\text{in}\quad P_a. \tag{3.23}$$

Then, the estimate (3.22) implies that

$$\int_{G_{c^2}\setminus G_{c^2+2\mu}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt$$

$$\leq N\int_{P_a}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt.$$

This and (3.20) imply that

$$\int_{G_{c^2}}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt\leq N\int_{P_a}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt. \tag{3.24}$$

Using (3.15), (3.21) and (3.24), we obtain

$$N\int_{Z_T}\left(\tilde{g}^2+|\nabla\tilde{g}|^2+(\partial_t\tilde{g})^2+\tilde{h}^2\right)\mathcal{C}^2dS+$$

$$N\exp\left[2\lambda\left(c^2+2\mu\right)\right]\|v\|^2_{H^1(P_T)}+$$

$$N\int_{P_a}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt\geq \tag{3.25}$$

$$C\lambda\int_{G_{c^2+2\mu}}\left(|\nabla v|^2+(\partial_t v)^2+\lambda^2v^2\right)\mathcal{C}^2dxdt,$$

Similarly we obtain for the function w

$$N\int_{Z_T}\left((\partial_t\tilde{g})^2+|\nabla(\partial_t\tilde{g})|^2+\left(\partial_t^2\tilde{g}\right)^2+\tilde{h}_t^2\right)\mathcal{C}^2dS+$$

$$N\exp\left[2\lambda\left(c^2+2\mu\right)\right]\|w\|^2_{H^1(P_T)}+$$

$$N\int_{P_a}\left(|\Delta q|^2+|\nabla q|^2+q^2\right)\mathcal{C}^2dxdt\geq \tag{3.26}$$

$$C\lambda\int_{G_{c^2+2\mu}}\left(|\nabla w|^2+w_t^2+\lambda^2w^2\right)\mathcal{C}^2dxdt.$$

We have

$$\partial_t v(x,t)=\partial_t v(x,0)+\int_0^t\partial_t^2v(x,\tau)d\tau=\partial_t v(x,0)+\int_0^t\partial_t w(x,\tau)d\tau.$$

Hence,

$$[\partial_t v(x,t)]^2\geq\frac{1}{2}[\partial_t v(x,0)]^2-\left(\int_0^t\partial_t w(x,\tau)d\tau\right)^2.$$

This and (3.9) imply that

$$[\partial_t v(x,t)]^2 \geq N(\Delta q)^2 - N\left(|\nabla q|^2 + q^2\right) - N\left(\int_0^t \partial_t w(x,\tau)d\tau\right)^2. \quad (3.27)$$

It follows from the lemma 2.2 that

$$\int_{G^2_{c+2\mu}} \left(\int_0^t \partial_t w(x,\tau)d\tau\right)^2 \mathcal{C}^2 dxdt \leq \frac{1}{\lambda\eta}\int_{G^2_{c+2\mu}} (\partial_t w)^2 \mathcal{C}^2 dxdt$$

Hence, the relations (3.1) and (3.27) imply that

$$\frac{C}{2}\lambda\int_{G^2_{c+2\mu}} (\partial_t v)^2 \mathcal{C}^2 dxdt \geq N\lambda\int_{P_a} (\Delta q)^2 \mathcal{C}^2 dxdt \quad (3.28)$$

$$-N\lambda\int_{P_a}\left(|\nabla q|^2 + q^2\right)\mathcal{C}^2 dxdt - N\int_{G^2_{c+2\mu}} (\partial_t w)^2 C^2 dxdt.$$

Note that

$$\int_P (\Delta q)^2 \mathcal{C}^2 dxdt = \int_0^a \exp\left(-2\lambda\eta t^2\right)\int_P (\Delta q)^2 \mathcal{C}^2 dx$$

It follows from (1.28) and (2.4) that

$$q\mid_{\partial P} = \frac{\partial q}{\partial \nu}\mid_{\partial P} = 0.$$

Hence, the lemma 2.4 and (3.28) imply that

$$\frac{C}{2}\lambda\int_{G^2_{c+2\sigma}} (\partial_t v)^2 \mathcal{C}^2 dxdt \geq \frac{N}{2}\lambda\int_{P_a} (\Delta q)^2 \mathcal{C}^2 dxdt$$

$$+N\sum_{i,j=1}^n \int_{P_a} q_{ij}^2 \mathcal{C}^2 dxdt + N\lambda^2\int_{P_a}\left(|\nabla q|^2 + \lambda^2 q^2\right)\mathcal{C}^2 dxdt$$

$$-N\lambda\int_{P_a}\left(|\nabla q|^2 + q^2\right)\mathcal{C}^2 dxdt - N\int_{G^2_{c+2\mu}} (\partial_t w)^2 \mathcal{C}^2 dxdt.$$

Since λ is sufficiently large, then this formula leads to

$$\frac{C}{2}\lambda\int_{G^2_{c+2\mu}} (\partial_t v)^2 \mathcal{C}^2 dxdt \geq \frac{N}{2}\lambda\int_{P_a} (\Delta q)^2 \mathcal{C}^2 dxdt +$$

$$N\sum_{i,j=1}^n \int_{P_a} q_{ij}^2 \mathcal{C}^2 dxdt + N\lambda^2\int_{P_a}\left(|\nabla q|^2 + \lambda^2 q^2\right)\mathcal{C}^2 dxdt - \quad (3.29)$$

$$N\int_{G^2_{c+2\mu}} (\partial_t w)^2 \mathcal{C}^2 dxdt.$$

For sufficiently large $\lambda \geq \lambda_1$

$$\frac{N}{2}\lambda\int_{P_a} (\Delta q)^2 \mathcal{C}^2 dxdt + N\sum_{i,j=1}^n \int_{P_a} q_{ij}^2 \mathcal{C}^2 dxdt +$$

$$N\lambda^2\int_{P_a}\left(|\nabla q|^2 + \lambda^2 q^2\right)\mathcal{C}^2 dxdt -$$

$$N\int_{P_a}\left(|\Delta q|^2 + |\nabla q|^2 + q^2\right)\mathcal{C}^2 dxdt \geq$$

$$\frac{N}{4}\lambda \int_{P_a} (\Delta q)^2 \mathcal{C}^2 dxdt + N \sum_{i,j=1}^{n} \int_{P_a} q_{ij}^2 \mathcal{C}^2 dxdt +$$

$$\frac{N}{2}\lambda^2 \int_{P_a} \left(|\nabla q|^2 + \lambda^2 q^2 \right) \mathcal{C}^2 dxdt. \tag{3.30}$$

Hence, the relations (3.25) and (3.29) imply that

$$N \int_{Z_T} \left(\tilde{g}^2 + |\nabla \tilde{g}|^2 + (\partial_t \tilde{g})^2 + \tilde{h}^2 \right) \mathcal{C}^2 dS +$$

$$N \exp\left[2\lambda \left(c^2 + 2\mu \right) \right] \|v\|^2_{H^1(P_T)} \geq$$

$$C\lambda \int_{G_{c^2+2\mu}} \left(|\nabla v|^2 + (\partial_t v)^2 + \lambda^2 v^2 \right) \mathcal{C}^2 dxdt +$$

$$N\lambda \int_{P_a} (\Delta q)^2 \mathcal{C}^2 dxdt + N \sum_{i,j=1}^{n} \int_{P_a} q_{ij}^2 \mathcal{C}^2 dxdt + \tag{3.31}$$

$$N\lambda^2 \int_{P_a} \left(|\nabla q|^2 + \lambda^2 q^2 \right) \mathcal{C}^2 dxdt - N \int_{G^2_{c+2\mu}} (\partial_t w)^2 \mathcal{C}^2 dxdt.$$

Let $d = d(x_0, \Omega) = \max_{x \in \overline{\Omega}} |x - x_0|^2$. Then it follows from (2.14) that $\mathcal{C}^2(x,t) \leq \exp(2\lambda d)$ in $\overline{P}_T$. Hence, summing up (3.26) and (3.31) and taking into account the relations (2.5) and (3.30), we obtain for $\lambda \geq \lambda_1(N)$

$$N e^{2\lambda d} \left(\|\partial_t \tilde{g}\|^2_{H^1(Z_T)} + \left\| \partial_t \tilde{h} \right\|^2_{L_2(Z_T)} \right) +$$

$$N \exp\left[2\lambda \left(c^2 + 2\mu \right) \right] \left(\|v\|^2_{H^1(P_T)} + \|w\|^2_{H^1(P_T)} \right) \geq$$

$$C\lambda \int_{G_{c^2+2\mu}} \left(|\nabla v|^2 + (\partial_t v)^2 + |\nabla w|^2 + (\partial_t v)^2 + \lambda^2 v^2 + \lambda^2 w^2 \right) \mathcal{C}^2 dxdt +$$

$$N \sum_{i,j=1}^{n} \int_{P_a} q_{ij}^2 \mathcal{C}^2 dxdt + N\lambda^2 \int_{P_a} \left(|\nabla q|^2 + \lambda^2 q^2 \right) \mathcal{C}^2 dxdt. \tag{3.32}$$

The lemma 2.1 and the relations (2.5) and (2.9)-(2.13) lead to the estimate

$$\|v\|^2_{H^1(P_T)} + \|w\|^2_{H^1(P_T)} \leq N \left(\|\partial_t \tilde{g}\|^2_{H^1(Z_T)} + \left\| \partial_t \tilde{h} \right\|^2_{L_2(Z_T)} \right) \tag{3.33}$$

$$+ N \int_{P_a} \left(|\Delta q|^2 + |\nabla q|^2 + q^2 \right) dxdt.$$

Hence, the relation (3.32) implies that

$$N e^{2\lambda d} \left(\|\partial_t \tilde{g}\|^2_{H^1(Z_T)} + \left\| \partial_t \tilde{h} \right\|^2_{L_2(Z_T)} \right) +$$

$$N \exp\left[2\lambda \left(c^2 + 2\mu \right) \right] \int_{P_a} \left(|\Delta q|^2 + |\nabla q|^2 + q^2 \right) dxdt \geq$$

$$N \sum_{i,j=1}^{n} \int_{P_a} q_{ij}^2 \mathcal{C}^2 dxdt + N\lambda^2 \int_{P_a} \left(|\nabla q|^2 + \lambda^2 q^2 \right) \mathcal{C}^2 dxdt. \tag{3.34}$$

By virtue of (3.1), we have $P_a \subset G_{c^2+3\mu} \subset G_{c^2+2\mu}$ and $\mathcal{C}^2(x,t) \geq \exp\left[2\lambda\left(c^2+3\mu\right)\right]$ in $G_{c^2+3\mu}$. Since $\exp\left[2\lambda\left(c^2+3\mu\right)\right] > \exp\left[2\lambda\left(c^2+2\mu\right)\right]$, we obtain from (3.34)

$$\exp\left[2\lambda\left(c^2+3\mu\right)\right]\|q\|^2_{H^2(P)} \leq Ne^{2\lambda d}\left(\|\partial_t\tilde{g}\|^2_{H^1(Z_T)} + \left\|\partial_t\tilde{h}\right\|^2_{L_2(Z_T)}\right).$$

Dividing this inequality by $\exp\left[2\lambda\left(c^2+3\mu\right)\right]$ and setting $\lambda = \lambda_1(N)$, we obtain the estimate

$$\|q\|^2_{H^2(P)} \leq N\left(\|\partial_t\tilde{g}\|^2_{H^1(Z_T)} + \left\|\partial_t\tilde{h}\right\|^2_{L_2(Z_T)}\right).$$

Combining this estimate with (3.33), we complete the proof. □

Based on the Lipschitz stability estimate, we complete now the proof of theorem 1.2. Since by (1.14) the function $\sigma(x')$ is known in $\Omega \setminus \Omega_1$, the uniqueness of the Cauchy problem for the elliptic equation and the relations (1.19)-(1.21) imply that the function $Y(x',s)$ is uniquely determined for $x' \in \Omega \setminus \Omega_1$ for almost all $s \in (-\infty,\infty)$. Hence, the following two functions $\psi_1(x',s)$ and $\psi_2(x',s)$ are determined for almost all $s \in (-\infty,\infty)$, so that

$$Y(x',s) = \psi_1(x',s), \quad \frac{\partial Y(x',s)}{\partial \nu} = \psi_2(x',s), \ \forall x' \in \partial\Omega_1, \tag{3.35}$$

where ν is the outward normal unit vector on $\partial\Omega_1$.

Denote

$$\Omega_\infty = \{(x,t) : x \in \Omega, t \in (0,\infty)\}, S_\infty = \{(x,t) : x \in \partial\Omega, t \in (0,\infty)\}.$$

and consider the hyperbolic initial boundary value problem

$$\sigma\partial^2_{tt}V = \nabla\cdot(\sigma\nabla V), \ (x',t) \in \Omega\times(0,\infty), \tag{3.36}$$

$$V\mid_{t=0} = 0, \ \ \partial_t V\mid_{t=0} = S_2(x')\sigma^{-1}(x'), \tag{3.37}$$

$$\frac{\partial V}{\partial \nu}\mid_{S_\infty} = 0. \tag{3.38}$$

Using the standard energy estimates and conditions (1.14)-(1.16) (see, e.g., [12], Chapter 4), one can prove that for each $T > 0$ there exists a unique solution $V \in H^7(\Omega\times(0,T))$ of this problem. It follows from the Sobolev embedding theorem that $V \in C^5\left(\overline{\Omega}\times[0,T]\right)$. Furthermore, there exists the positive numbers C and s_0 depending only on the functions $S_2(x'), \sigma(x')$ and domain Ω, so that

$$|D^\alpha V(x',t)| \leq Ce^{s_0 t} \quad \text{in} \quad \Omega_\infty, |\alpha| \leq 5.$$

Therefore, the Laplace transform

$$\tilde{V}(x',s) = \int_0^\infty V(x',t)e^{-st}dt, s > s_0$$

exists for all $s > s_0$ and $\tilde{V}(x',s) \in H^5(\Omega)$. Hence, the lemma 1.2, the relations (1.19)-(1.20) and (3.36)-(3.38) imply that $\tilde{V}(x',s) = Y(x',s)$ for almost all $s > s_0$. Since the Laplace transform is a one-to-one operator, then the relation (3.35) implies that the functions $\psi_3(x',t)$ and $\psi_4(x',t)$ can be uniquely determined. Here,

$$V(x',t)\mid_{\partial\Omega_1} = \psi_3(x',t), \quad \frac{\partial V(x',t)}{\partial \nu}\mid_{\partial\Omega_1} = \psi_4(x',t), \tag{3.39}$$

$$\forall (x', t) \in \partial\Omega \times (0, \infty) . \tag{3.40}$$

Denote $W(x', t) = \sigma(x') \cdot \partial_t V(x', t)$. Then, the relations (3.36)-(3.40) imply that

$$\partial_t^2 W = \Delta W - \frac{\nabla \sigma}{\sigma} \cdot \nabla w - F(\Delta\sigma, \nabla\sigma, \sigma) W, \tag{3.41}$$

$$W \mid_{t=0} = S_2(x'), \quad \partial_t W \mid_{t=0} = 0, \tag{3.42}$$

$$W(x', t) \mid_{S_{1\infty}} = \partial_t \psi_3(x', t), \quad \frac{\partial W(x', t)}{\partial \nu} \mid_{S_{1\infty}} = \partial_t \psi_4(x', t), \tag{3.43}$$

where $S_{1\infty} = \partial\Omega_1 \times (0, \infty)$, and

$$F(\Delta\sigma, \nabla\sigma, \sigma) = \left[\frac{\Delta\sigma}{\sigma} - \frac{|\nabla\sigma|^2}{\sigma^2} \right] (x') . \tag{3.44}$$

The completion of the proof of theorem 1.2 follows directly from the theorem 1.1. □

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