On the General Intertwining Lifting Problem – I

A. Biswas, C. Foias

We consider the general intertwining lifting problem as formulated in [F1] and which is connected to interpolation problems in reproducing kernel Hilbert spaces. We reduce this general problem to the case where the operators involved are $n \times n$ block upper-triangular. As a consequence, we show that the causal commutant lifting (see [FT]) and the general intertwining lifting (or extension) problems are equivalent. We also obtain a seemingly new commutant lifting result for the case where one of the operators involved is nilpotent and the other canonical block Jordan. Finally, as an application, we obtain a completely new proof for the Ceausescu-Carswell-Schubert result (see [Ce], [CaS]).

1 Introduction

The commutant lifting theorem of Sarason ([Sa]) and Sz.-Nagy–Foias (see [Sz.-NF]) formed the basis of a new method, known as the commutant lifting approach, to solve various metric constrained interpolation problems in the Hardy space setting. These interpolation problems arise in diverse fields such as Electrical Engineering, Prediction Theory and Geophysics (see [FF]). In recent years, the commutant lifting approach was further developed to solve more complex metric constrained interpolation problems (like the ones involving operator-valued functions with operator arguments and the two sided Nudelman problem) and found new area of application in connection with H^∞–control problems for the time-variant systems (see [FFGK]).

Recently, there has been a surge of interest in considering interpolation problems, for instance the Nevanlinna-Pick, in reproducing kernel Hilbert function spaces which are more general than the Hardy space (see [Ag], [AgM]). Although the results of Sarason and Sz.-Nagy–Foias are adequate to solve a wide variety of interpolation problems which can be cast in the Hardy space framework, such is not the case in the more general function spaces. However, a unifying feature in these cases is that the interpolation problems can still be viewed as a problem of "intertwining lifting" of an operator A. The important difference now with the classical Sarason or Sz.-Nagy–Foias theorem is that A is now required to intertwine operators much more general than a co-isometry.

The outline of this paper is as follows. In this section, we establish some notation and describe the general intertwining lifting problem as given in [F1]. In Section 2, we obtain a reduction of the general problem to the case where the operators involved have a triangular structure. The techniques introduced in this section allows one to pass from the "$C_{\cdot 0}$" or "$C_{0\cdot}$" case to the general case in the intertwining lifting problem. As an application, we give an alternate proof of Ceausescu's generalization of the Carswell-Schubert result, concerning the intertwining extension of an operator intertwining restrictions of unilateral shifts to their respective invariant subspaces, to the case of general isometries. In Section 3, we introduce the “three chain structure” in the general intertwining lifting problem. Subsequently, we obtain further reduction of the problem to a certain “finite block” version of it. Our results in Section 3 allow us to show in Section 4 that the general intertwining extension problem can be regarded as a special case of the *causal commutant lifting* theory introduced in [FT]. The causal commutant lifting theorem incorporates the nest algebra approach of [Arv] and allows one to address several nonlinear interpolation/optimization problems (in several variables) arising in Control Theory (see [FT, [FGT], [FG]) in the commutant lifting framework. In [FGT], it was already shown that the causal commutant lifting problem can be regarded as a special case of the general intertwining lifting problem. Our result in Section 4 thus establishes the equivalence of these two seemingly different problems. In Section 5, we obtain a commutant lifting result for nilpotent operators. Finally, in Section 6, as an application, we obtain a completely new proof of the Carswell-Schubert theorem mentioned before.

Let V and V' be two operators on Hilbert spaces $\mathcal{K}$ and $\mathcal{K}'$ respectively and let $\mathcal{H}' \subset \mathcal{K}'$ be invariant to $V^{'*}$, that is, $V^{'*}\mathcal{H}' \subset \mathcal{H}'$. Defining the operator T' on $\mathcal{H}'$ by $T^{'*} = V^{'*}|\mathcal{H}'$, it is easy to see that

$$P_{\mathcal{H}'}V' = T'P_{\mathcal{H}'}, \tag{1.1}$$

where, for any two Hilbert spaces $\mathcal{L}_1,\ \mathcal{L}_2$ with $\mathcal{L}_1 \subset \mathcal{L}_2$, we denote by $P_{\mathcal{L}_1}$ the orthogonal projection onto the smaller subspace $\mathcal{L}_1$. Recall that for an operator T' on $\mathcal{H}'$, an operator V' on $\mathcal{K}' \supset \mathcal{H}'$ is called a lifting of T' if it satisfies (1.1). Assume that we are also given an operator

$$A : \mathcal{K} \to \mathcal{H}' \text{ such that } AV = T'A. \tag{1.2}$$

An operator $B : \mathcal{K} \to \mathcal{K}'$ is said to be an *intertwining lifting of* A if it satisfies

$$P_{\mathcal{H}'}B = A,\ \ BV = V'B.$$

An intertwining lifting B of A need not always exists. For example, if we take $\mathcal{K} = \mathbb{C}$, $\mathcal{K}^{'} = \mathbb{C} \oplus \mathbb{C}$, $\mathcal{H}' = \mathbb{C} \oplus \{0\}$ and $A = V = T' = 1$, $V' = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, it is easy to check that an intertwining lifting of A does not exists. Indeed, $P'B = A$ forces $B = \begin{bmatrix} 1 \\ x \end{bmatrix}$. Then $BV = \begin{bmatrix} 1 \\ x \end{bmatrix}$, $V'B = \begin{bmatrix} 1 \\ 1+x \end{bmatrix}$ and clearly $BV \neq V'B$.

The *general intertwining lifting problem* as formulated in [F1], can be described as follows:
Find necessary and sufficient condition(s) for the existence of an operator $B : \mathcal{K} \to \mathcal{K}'$ *such that*

$$BV = V'B,\ \ P_{\mathcal{H}'}B = A,\ \ \|B\| \leq \gamma, \tag{1.3}$$

where $\gamma > 0$ *is a given fixed constant.*
Henceforth, we will say that the ordered tuple of operators $\{V, V', T', A\}$, where the operators satisfy (1.1) and (1.2), constitutes intertwining lifting data.

As previously mentioned, any B satisfying the first two relations in (1.3) is called an *intertwining lifting* of A. In case the necessary and sufficient condition(s) for the existence of an intertwining lifting of A is satisfied, we are also interested in obtaining an explicit description of all intertwining liftings B satisfying (1.3) as well as the ones of minimum norm.

It will sometimes be more convenient to consider the general intertwining extension problem which is the dual version of the general intertwining lifting problem. The data in the general intertwining extension problem comprises of an ordered tuple of operators $\{V, V', T, A\}$ where V and V' are operators on Hilbert spaces $\mathcal{K}$ and $\mathcal{K}'$ respectively. We are also given an invariant subspace $\mathcal{H} \subset \mathcal{K}$ of V and $T = V|\mathcal{H}$. The operator $A : \mathcal{H} \to \mathcal{K}'$ satisfies

$$AT = V'A.$$

The *general intertwining extension problem* with data $\{V, V', T, A\}$ is to find necessary and sufficient condition(s) for the existence of an operator $B : \mathcal{K} \to \mathcal{K}'$ satisfying

$$B|\mathcal{H} = A,\ BV = V'B,\ \|B\| \leq \gamma,$$

where $\gamma > 0$ is a given fixed constant.

Clearly, by taking adjoints, it is easy to see that the general intertwining lifting problem with data $\{V, V', T', A\}$ is equivalent to the general intertwining extension problem with data $\{V'^{*}, V^*, T'^{*}, A^*\}$. In what follows, we will reduce the general intertwining lifting problem (or equivalently, the general intertwining extension problem) to a much more special case. We first make the following easy remark.

Remark 1 In the general intertwining lifting problem, we can without loss of generality assume that the operators V, T' and V' are contractions. To see this, set $1/c = \max\{1, \|V\|, \|T'\|, \|V'\|, \gamma\}$ and note that cV' is a contractive lifting of the contraction cT' and moreover, $A(cV) = (cT')A$. An operator B satisfies (1.3) if and only if it satisfies

$$B(cV) = (cV')B,\ P_{\mathcal{H}'}B = A,\ \|B\| \leq 1.$$

Consequently, by replacing the operators V, T' and V' by cV, cT' and cV' respectively, we may without loss of generality assume that in the general intertwining lifting problem, all operators are contractions.

Henceforth we will assume that the operators V, V', T' and A are contractions. We will be interested in formulating a necessary and sufficient condition(s) for the existence of a contractive intertwining lifting B of A.

2 Further Reduction

In this section, we will show that the general intertwining lifting problem can be reduced to the particular case when the operator V' is a co-isometry and the operator V has *big kernel property,* that is, $\bigvee_{n=0}^{\infty} \ker V^n = \mathcal{K}$. Here and throughout, if $\mathcal{H}_\alpha \subset \mathcal{H}$ form a collection of subspaces of a Hilbert space $\mathcal{H}$, $\bigvee \mathcal{H}_\alpha$ denotes the smaller closed linear subspace of $\mathcal{H}$ containing $\mathcal{H}_\alpha$ for all α.

First, let us introduce some notations. For a Hilbert space $\mathcal{L}$, the space $\ell^2_+(\mathcal{L})$ denotes the space of all norm square summable sequence with entries in $\mathcal{L}$, that is,

$$\ell^2_+(\mathcal{L}) = \bigoplus_{n=0}^{\infty} \mathcal{L} = \{\vec{l} = (l_n)_{n=0}^{\infty} : \|\vec{l}\|^2 := \sum_{n=0}^{\infty} \|l_n\|^2 < \infty\}.$$

Define the spaces

$$\mathcal{K}_\infty = \ell^2_+(\mathcal{K}),\ \mathcal{K}'_\infty = \ell^2_+(\mathcal{K}'),\ \text{and } \mathcal{H}'_\infty = \ell^2_+(\mathcal{H}') \tag{2.1}$$

and operators V_∞, V'_∞ and A_∞ by

$$V_\infty \vec{x} = (V x_{n+1})_0^\infty,\ A_\infty \vec{x} = (A x_n)_0^\infty,\ \text{and}\ \ V'_\infty \vec{y} = (V' y_{n+1})_0^\infty, \tag{2.2}$$

where $\vec{x} = (x_n)_0^\infty \in \mathcal{K}_\infty$ and $\vec{y} = (y_n)_0^\infty \in \mathcal{K}'_\infty$. Since $\mathcal{H}'$ is invariant for $V^{'*}$, it follows that $V_\infty^{'*}|\mathcal{H}'_\infty \subset \mathcal{H}'_\infty$. Moreover, noting that $P_{\mathcal{H}'_\infty} \vec{y} = (P_{\mathcal{H}'} y_n)_0^\infty$ for $\vec{y} = (y_n)_0^\infty \in \mathcal{K}'_\infty$, and using (1.1), one obtains

$$P_{\mathcal{H}'_\infty} V'_\infty \vec{y} = (P_{\mathcal{H}'} V' y_{n+1})_0^\infty = (T' P_{\mathcal{H}'} y_{n+1})_0^\infty = T'_\infty P_{\mathcal{H}'_\infty} \vec{y},$$

where the operator $T'_\infty : \mathcal{H}'_\infty \to \mathcal{H}'_\infty$ is defined by

$$T'_\infty \vec{h} = (T' h_{n+1})_0^\infty,\ \vec{h} = (h_n)_0^\infty \in \mathcal{H}'_\infty. \tag{2.3}$$

This implies the relation $P_{\mathcal{H}'_\infty} V_\infty^{'} = T'_\infty P_{\mathcal{H}'_\infty}$. Moreover, using the definition of the operators involved and the intertwining relation $AV = T'A$, it is also easy to see that the relation $A_\infty V_\infty = T'_\infty A_\infty$ holds as well.

Remark 2 The operator V_∞ has the big kernel property, that is, $\bigvee_{n=0}^{\infty} ker\ V_\infty^n = \mathcal{K}_\infty$. Indeed, note that $ker\ V_\infty^n \supset \mathcal{H}_n$, where

$$\mathcal{H}_n = \{\vec{h} = (h_k)_0^\infty \in \mathcal{K}_\infty : h_k = 0 \,\forall\, k > n\}.$$

We now prove a result which shows that the general intertwining lifting problem can be reduced to the case when the operator V has big kernel property.

Theorem 1 *There exists a contraction $B : \mathcal{K} \to \mathcal{K}'$ satisfying (1.3) if and only if there exists a contraction $B_\infty : \mathcal{K}_\infty \to \mathcal{K}'_\infty$ such that*

$$B_\infty V_\infty = V'_\infty,\ P_{\mathcal{H}'_\infty} B_\infty = A_\infty \ \textit{and}\ \|B_\infty\| \le 1. \tag{2.4}$$

Proof: First assume that there exists a contraction B satisfying (1.3). Define $B_\infty : \mathcal{K}_\infty \to \mathcal{K}'_\infty$ by $B_\infty \vec{k} = (B k_n)_0^\infty$, $\vec{k} = (k_n)_0^\infty \in \mathcal{K}_\infty$. Clearly, B_∞ is a contraction with $P_{\mathcal{H}'_\infty} B_\infty = A_\infty$. Moreover, for $\vec{k} = (k_n)_0^\infty \in \mathcal{K}_\infty$, we have

$$B_\infty V_\infty \vec{k} = (B V k_{n+1})_0^\infty = (V^{'} B k_{n+1})_0^\infty = V'_\infty B_\infty \vec{k}.$$

This proves that the general lifting problem with data $(V_\infty, V'_\infty, T'_\infty, A_\infty)$, is solvable provided the general intertwining lifting problem with data $\{V, V', T', A\}$ has a solution.

Conversely, now assume that the general intertwining lifting problem with data $(V_\infty, V'_\infty, T'_\infty, A_\infty)$ has a solution, that is, there exists a contractive $B_\infty : \mathcal{K}_\infty \to \mathcal{K}'_\infty$ satisfying (2.4). Let B_∞ be represented by the matrix $B_\infty = (B_{n,m})_{n,m=0}^\infty$ with respect to the natural decomposition of the space $\mathcal{K}_\infty$. Using the notation $\delta_{i,j}$ for the Kronecker delta, notice that for $k \in \mathcal{K}$, and any integer $n \geq 0$, we have

$$B_\infty V_\infty (\delta_{n+1,p} k)_{p=0}^\infty = B_\infty (\delta_{n,p} V k)_{p=0}^\infty = (B_{p,n} V k)_{p=0}^\infty. \tag{2.5}$$

On the other hand,

$$B_\infty V_\infty (\delta_{n+1,p} k)_{p=0}^\infty = V'_\infty B_\infty (\delta_{n+1,p} k)_{p=0}^\infty = V'_\infty (B_{p,n+1} k)_{p=0}^\infty = (V^{'} B_{p+1,n+1} k)_{p=0}^\infty. \tag{2.6}$$

Using the computations in (2.5) and (2.6) and taking the $n-$th coordinates of the vectors, it follows that we have the relation

$$B_{n,n} V k = V' B_{n+1,n+1} k, \ \ k \in \mathcal{K}, \ \ n \geq 0.$$

Furthermore, we claim that $P_{\mathcal{H}'} B_{n,n} = A$ for all $n \geq 0$. Indeed, for $k \in \mathcal{K}$ using the relation $P_{\mathcal{H}'_\infty} B_\infty = A_\infty$, it follows that

$$(\delta_{n,p} A k)_{p=0}^\infty = A_\infty (\delta_{n,p} k)_{p=0}^\infty = P_{\mathcal{H}'_\infty} B_\infty (\delta_{n,p} k)_{p=0}^\infty = P_{\mathcal{H}'_\infty} (B_{p,n} k)_{p=0}^\infty = (P_{\mathcal{H}'} B_{p,n} k)_{p=0}^\infty.$$

Taking the $n-$th coordinates of all the vectors involved in the calculation above gives $P_{\mathcal{H}'} B_{n,n} = A$ for all $n \geq 0$.

Consequently, setting $B_n = B_{n.n}$, we see that the sequence of operators $\{B_n\}_0^\infty$ satisfy

$$V' B_{n+1} = B_n V, \ \ P_{\mathcal{H}'} B_n = A, \ \text{and} \ \|B_n\| \leq 1. \tag{2.7}$$

Define the operator $B : \mathcal{K} \to \mathcal{K}'$ by setting

$$(Bk, k') = LIM_{n\to\infty} (B_n k, k'), \ \ k \in \mathcal{K}, \ \ k' \in \mathcal{K}',$$

where LIM denotes a generalized Banach limit on ℓ^∞ (see [Con]). Using (2.7) and the shift invariance of a generalized Banach limit, it can be easily checked that B is a contractive intertwining lifting of A.

Remark 3 Due to Proposition 3.2 in [FGT], we may also without loss of generality assume that V' is a co-isometry. Note that in Theorem 1, V' is a co-isometry then V'_∞ is a co-shift. As a result, *in the general intertwining lifting problem, we can without loss of generality assume that the operator V has big kernel property, and the operator V' is a co-shift. Henceforth we will make this assumption without any further mention.*

Remark 4 Recall that the Carswell-Schubert theorem (see [CS]) states that in case V and V' are both co-shifts, a contractive intertwining lifting B of A exists if and only if

$$\|P_n A^* h'\| \leq \|P'_n h'\| \qquad (n \geq 1, \ h' \in \mathcal{H}') \tag{2.8}$$

where P_n and P'_n denote the orthogonal projection on ker V^n and ker V'^n respectively. The condition in (2.8) was previously identified by LePage (see [LeP]) as a necessary condition for the existence of an intertwining lifting B and we will henceforth refer to it as the LePage condition. Later, Ceausescu (see [Ce]) extended the Carswell-Schubert result to the case when V and V' are co-isometries, the same necessary and sufficient condition, namely (2.8) still remaining valid. Assuming the Carswell-Schubert result, we will now give a proof of Ceausescu's generalization of it by employing the reduction achieved in Theorem 1 thus providing an alternate proof of Ceausescu's generalization of Carswell-Schubert result showing that the latter result implies its generalization.

Theorem 2 *Consider the intertwining lifting problem with data $\{V, V', T', A\}$ where V and V' are both co-isometries. In this case, there exists a contractive intertwining lifting B of A if and only if (2.8) holds.*

Proof: Notice that in our construction in Theorem 1, if we start with V and V' being co-isometries, the operators V_∞ and V'_∞ turn out to be co-shifts. Let $\mathcal{E}_n$ and $\mathcal{E}'_n$ denote the kernels of V^n and V'^n respectively and as before, denote $P_{\mathcal{E}_n} = P_n$ and $P_{\mathcal{E}'_n} = P'_n$. Let $\mathcal{K}_{\infty,n}$ ($\mathcal{K}'_{\infty,n}$) be the space of all vectors in $\mathcal{K}_\infty$ ($\mathcal{K}'_\infty$) which are supported on the first n components. Then it is easy to see that the orthogonal projections on the kernel of V_∞^n and $V_\infty^{'n}$ are given by $P_{ker\, V_\infty^n} = P_{\mathcal{K}_{\infty,n}} \oplus P_{\mathcal{E}_n} \oplus P_{\mathcal{E}_n} \oplus \cdots$ and $P_{ker\, V_\infty^{'n}} = P_{\mathcal{K}'_{\infty,n}} \oplus P_{\mathcal{E}'_n} \oplus P_{\mathcal{E}'_n} \oplus \cdots$. Recall that $A_\infty = A \oplus A \oplus \cdots$ and note that for $\vec{h}' = (h'_p)_{p=0}^\infty \in \mathcal{H}'_\infty$, we have

$$\|P_{ker\, V_\infty^n} A_\infty^* \vec{h}'\|^2 =$$

$$\|P_{ker\, V_\infty^n}(A^* h'_p)_{p=0}^\infty\|^2 = \sum_{p=0}^{n-1} \|A^* h'_p\|^2 + \sum_{p=n}^{\infty} \|P_n A^* h'_p\|^2$$

and

$$\|P_{ker\, V_\infty^{'n}} \vec{h}'\|^2 = \sum_{p=0}^{n-1} \|h'_p\|^2 + \sum_{p=n}^{\infty} \|P'_n h'_p\|^2.$$

Since A is a contraction, it is now easy to see that A satisfies (2.8) if and only if A_∞ satisfies the appropriate analogous condition

$$\|P_{ker\, V_\infty^n} A_\infty^* h'\| \le \|P_{ker\, V_\infty^{'n}} h'\|, \qquad (n \ge 1, h' \in \mathcal{H}'_\infty).$$

The proof of the theorem follows by invoking Carswell-Schubert result and Theorem 1.

3 Connection with three chains completion

Due to Remark 3 henceforth we will assume that in the intertwining lifting problem V' is a co-shift and the contraction V has the big kernel property, that is, $\bigvee_{n=0}^\infty \ker V^n = \mathcal{K}$. Next we adapt the proof of the Carswell-Schubert theorem obtained via the three chains completion theorem (see Section XII.6 in [FFGK]) to our setup. In passing, we mention that although in that proof, both

V and V' are coisometries, we will use the same setting when V is a contraction, while V' is still assumed to be a co-isometry.

We start by introducing some supplementary notation. Let

$$\ker V^n = \mathcal{K}_n,\ \ker {V'}^n = \mathcal{K}'_n,\ \text{and } \mathcal{M}_n = \mathcal{K}'_n \cap (\mathcal{K}' \ominus \mathcal{H}'). \tag{3.1}$$

Notice that $\mathcal{M}_n \subset \mathcal{K}'_n$ and we also have the inclusion relations

$$\mathcal{K}_n \subset \mathcal{K}_{n+1},\ \mathcal{K}'_n \subset \mathcal{K}'_{n+1} \text{ and } \mathcal{M}_n \subset \mathcal{M}_{n+1}. \tag{3.2}$$

Remark 5 If an intertwining lifting B satisfying (1.3) exists, then from the intertwining relation $BV = V'B$ and the definitions of $\mathcal{K}_n$ and $\mathcal{K}'_n$, it immediately follows that $B\mathcal{K}_n \subset \mathcal{K}'_n$. In other words, denoting $\mathcal{E}_n = \ker V^n \ominus \ker V^{n-1}$ and $\mathcal{E}'_n = \ker {V'}^n \ominus \ker {V'}^{n-1} \approx \ker V'$, with respect to the decompositions of the spaces $\mathcal{K} = \bigoplus_{n=1}^{\infty} \mathcal{E}_n$ and $\mathcal{K}' = \bigoplus_{n=1}^{\infty} \mathcal{E}'_n$ any intertwining operator B must be upper-triangular. Since we are working under the assumption that that V has big kernel property and V' is a co-shift, we also have

$$\mathcal{K} = \bigvee_{n=0}^{\infty} \mathcal{K}_n \text{ and } \mathcal{K}' = \bigvee_{n=0}^{\infty} \mathcal{K}'_n. \tag{3.3}$$

Let P_n and P'_n acting on $\mathcal{K}$ and $\mathcal{K}'$ respectively denote the orthogonal projections on the subspaces $\mathcal{K}_n$ and $\mathcal{K}'_n$. In the following proposition we collect a number of facts required for further development.

Lemma 1 *The projections P_n and P'_n satisfy the relations*

$$VP_{n+1} = P_n V P_{n+1},\ \textit{and } P'_n V' = V' P'_{n+1}. \tag{3.4}$$

Moreover, we also have

$$\mathcal{K}'_n \ominus \mathcal{M}_n = P'_n \mathcal{H}' \ \textit{ where } \mathcal{M}_n = \mathcal{K}'_n \cap (\mathcal{K}' \ominus \mathcal{H}'). \tag{3.5}$$

Proof. We briefly outline the proof here. For more details, refer to Section XII.4 in [FFGK]. The first relation in (3.4) follows from the observation that $V\mathcal{K}_{n+1} \subset \mathcal{K}_n$. For the second relation in (3.4), first note that since V' is a co-isometry, we have the formula

$$P'_n = I - {V'}^{*n} {V'}^n,\ n \geq 0, \tag{3.6}$$

and now the relation $P'_n V' = V' P'_{n+1}$ follows by an easy computation. In order to prove (3.5), we just need to observe that $y \in \mathcal{M}_n$ if and only if $y \in \mathcal{K}'_n$ and y is orthogonal to $P'_n \mathcal{H}'$. This finishes the proof of the lemma.

Remark 6 The operators V and V' have the following properties:

$$V\mathcal{K}_{n+1} \subset \mathcal{K}_n,\ V'\mathcal{K}'_{n+1} \subset \mathcal{K}_n,\ {V'}^{*}\mathcal{K}'_n \subset \mathcal{K}_{n+1},\ V'\mathcal{M}_{n+1} \subset \mathcal{M}_n. \tag{3.7}$$

The first three relations in (3.7) easily follows from (3.4). In order to prove the inclusion $V'\mathcal{M}_{n+1} \subset \mathcal{M}_n$, recall that $\mathcal{M}_j = \mathcal{K}'_j \cap (\mathcal{K}' \ominus \mathcal{H}')$, $j = n, n+1$, and note that since $\mathcal{H}'$ is invariant to ${V'}^*$, we must have $V'(\mathcal{K}' \ominus \mathcal{H}') \subset (\mathcal{K}' \ominus \mathcal{H}')$. This, along with the fact that $V'\mathcal{K}'_{n+1} \subset \mathcal{K}_n^{'}$ implies the desired inclusion.

In the lemma below, it is shown that the LePage condition (2.8) is necessary for the existence of a contractive intertwining lifting in the general intertwining lifting problem.

Proposition 1 *Assume that the general intertwining lifting problem with data $\{V, V', T', A\}$ has a solution. Then, for all integers $n \geq 0$ and for all $h' \in \mathcal{H}'$, we must have*

$$\|P_n A^* h'\| \leq \|P'_n h'\|. \tag{3.8}$$

Proof. Let B be a contractive intertwining lifting of A, that is, B is a contraction satisfying the first two relations in (1.3). Then, $P_{\mathcal{H}'} B = A$ implies that $B^*|\mathcal{H}' = A^*$ and in view of Remark 5, we also have the relation $P_n B^* P'_n = P_n B^*$. It follows that

$$\|P_n A^* h'\| = \|P_n B^* h'\| = \|P_n B^* P_n^{'} h'\| \leq \|P'_n h'\|,$$

since B is a contraction and this proves the proposition.

Remark 7 The condition in (3.8) is equivalent to the existence of uniquely defined contractions

$$A_n : \mathcal{K}_n \to \mathcal{K}_n^{'} \ominus \mathcal{M}_n, \text{ such that } A|\mathcal{K}_n = P_{\mathcal{H}'} A_n, \qquad (n \geq 1). \tag{3.9}$$

To see this, note that the condition in (3.8) is equivalent to the existence of (uniquely defined) contractions $C_n : \mathcal{K}_n^{'} \ominus \mathcal{M}_n \to \mathcal{K}_n$ satisfying the relation $C_n P'_n|\mathcal{H}' = P_n A^*$. Now, setting $A_n = C_n^*$, it follows that A_n $(n \geq 1)$ are contractions such that

$$A_n^* P'_n h' = P_n A^* h' \qquad (h' \in \mathcal{H}'). \tag{3.10}$$

Hence the remark follows.

Remark 8 *Since the LePage condition (3.8) is necessary for the existence of an intertwining lifting, henceforth we will consider only the case when A satisfies (3.8).*

We now prove an useful property of the contractions A_n defined in Remark 7.

Lemma 2 *The contractions A_n $(n \geq 1)$ satisfy the compatibility condition*

$$(I - P_{\mathcal{M}_{n+1}}) A_n = A_{n+1}|\mathcal{K}_n \tag{3.11}$$

and the intertwining condition

$$A_n V|\mathcal{K}_{n+1} = P_{\mathcal{K}'_n \ominus \mathcal{M}_n} V' A_{n+1} = (I - P_{\mathcal{M}_n}) V' A_{n+1}. \tag{3.12}$$

Proof: First, let us give a proof of the compatibility condition (3.11). Take a vector $h' \in \mathcal{H}'$. Then, using (3.10) we have

$$P_n A_{n+1}^* P'_{n+1} h' = P_n P_{n+1} A^* h' = P_n A^* h' = A_n^* P'_n h' = A_n^* P'_n P'_{n+1} h'$$

and therefore (using (3.5)

$$P_n A_{n+1}^* |(\mathcal{K}'_{n+1} \ominus \mathcal{M}_{n+1}) = A_n^* P'_n |(\mathcal{K}'_{n+1} \ominus \mathcal{M}_{n+1}).$$

By taking adjoints, we see that

$$A_{n+1}|\mathcal{K}_n = P_{\mathcal{K}'_{n+1}\ominus\mathcal{M}_{n+1}} P'_n A_n = (I - P_{\mathcal{M}_{n+1}})A_n.$$

This proves (3.11).

In order to prove (3.12) first note that due to (3.4) we have $P_{n+1}V^*P_n = P_{n+1}V^*$. Take vectors $k_{n+1} \in \mathcal{K}_{n+1}$ and $h' \in \mathcal{H}'$. Then,

$$\begin{aligned}(A_n V k_{n+1}, P'_n h') = \\ (k_{n+1}, P_{n+1}V^* A_n^* P'_n h') = (k_{n+1}, P_{n+1}V^* P_n A^* h') = (k_{n+1}, P_{n+1}A^* V'^* h')\end{aligned}$$

where in the second equality we used (3.10) and in the third equality we used the relation $P_{n+1}V^*P_n = P_{n+1}V^*$ as well as the intertwining relation $V^*A^* = A^*V'^*|\mathcal{H}'$. Now note that

$$\begin{aligned}(k_{n+1}, P_{n+1}A^*V'^*h') = \\ (k_{n+1}, A_{n+1}^* P'_{n+1} V'^* h') = (A_{n+1}k_{n+1}, V'^* P'_n h') = (V' A_{n+1} k_{n+1}, P'_n h')\end{aligned}$$

where in the in the first equality we used (3.10) along with the fact that V'^*h' belongs to $\mathcal{H}'$ and in the second equality we used the second relation in (3.4). Thus we conclude that $(A_n V k_{n+1}, P'_n h') = (V' A_{n+1} k_{n+1}, P'_n h')$ for all $k_{n+1} \in \mathcal{K}_{n+1}$ and $h' \in \mathcal{H}'$. Recalling (3.5) and observing that the range of A_n is contained in $\mathcal{K}'_n$ for all n this immediately implies

$$A_n V|\mathcal{K}_{n+1} = P_{\mathcal{K}'_n\ominus\mathcal{M}_n} V' A_{n+1} = (I - P_{\mathcal{M}_n})V' A_{n+1}.$$

This concludes the proof of the lemma.

We continue the process of reducing the general intertwining lifting problem. Let

$$\hat{\mathcal{M}} = \bigvee_{n=1}^{\infty} \mathcal{M}_n, \ \hat{\mathcal{H}}' = \mathcal{K}' \ominus \hat{\mathcal{M}}. \tag{3.13}$$

Remark 9 Recall that the space $\mathcal{M}_n = \mathcal{K}' \bigcap \mathcal{K}'_n$ $(n \geq 1)$. Consequently we have the relations

$$\hat{\mathcal{M}} \subset \mathcal{M} \text{ and } \mathcal{K}'_n \cap \hat{\mathcal{M}} = \mathcal{M}_n \qquad (n \geq 1). \tag{3.14}$$

Moreover, since $V'\mathcal{M}_n \subset \mathcal{M}_{n-1}$ it easily follows that $V'\hat{\mathcal{M}} \subset \hat{\mathcal{M}}$, that is,

$$P_{\hat{\mathcal{H}}'} V' = P_{\hat{\mathcal{H}}'} V' P_{\hat{\mathcal{H}}'}.$$

In other words, V' is a lifting of the contraction $\hat{T}' = P_{\hat{\mathcal{H}}'} V' P_{\hat{\mathcal{H}}'}$.

Remark 10 Let $x_n \in \mathcal{K}_n$ and $k \geq 1$ be an integer. By repeatedly using the compatibility relation (3.11) and the fact that the spaces $\mathcal{M}_j \subset \mathcal{M}_{j+1}$ for all $j \geq 1$ it follows that

$$A_{n+k} P_{n+k} x_n = (I - P_{\mathcal{M}_{n+k}}) A_n x_n.$$

Recall that $\mathcal{K} = \bigvee_{n=1}^{\infty} \mathcal{K}_n$, the projections $P_{\mathcal{M}_j}$ converges strongly to $P_{\hat{\mathcal{M}}}$ and the operators $A_j P_j$ are contractions.Thus the above relation implies that there exists a contraction $\hat{A}$ from $\mathcal{K}$ to $\mathcal{K}'$ with range included in $\hat{\mathcal{H}}'$ such that

$$A_n P_n \longrightarrow \hat{A} \text{ strongly}, \quad \hat{A}|\mathcal{K}_n = (I - P_{\hat{\mathcal{M}}})A_n. \tag{3.15}$$

Taking strong limits in the relation $AP_n = P_{\mathcal{H}'} A_n P_n$ (see (3.9)) and using the fact that $P_n \nearrow I$ strongly we obtain

$$A = P_{\mathcal{H}'} \hat{A}. \tag{3.16}$$

Let $x_j \in \mathcal{K}_j$ for some integer $j \geq 1$ and let $n \geq j$ be an integer. Then

$$A_n P_n V x_j = A_n V x_j = (I - P_{\mathcal{M}_n})V' A_{n+1} x_j = (I - P_{\mathcal{M}_n})V' A_{n+1} P_{n+1} x_j,$$

where the first and third equality in the above computation follows from (3.4) and the second equality follows from (3.12). Now taking strong limit as n goes to infinity and using the fact that the union of spaces $\mathcal{K}_j$ are dense in $\mathcal{K}$ one has

$$\hat{A} V = (I - P_{\hat{\mathcal{M}}})V' \hat{A} = \hat{T}' \hat{A}. \tag{3.17}$$

The last equality in (3.17) follows by recalling that the range of $\hat{A}$ is included in $\hat{\mathcal{H}}' \subset \hat{\mathcal{K}}'$. The relation in (3.17) implies that $\hat{A}$ intertwines V with $\hat{T}'$.

Proposition 2 *There exists a contractive intertwining lifting B of A if and only if there exists a contractive intertwining lifting of $\hat{A}$, that is, if and only if there exists an operator $\hat{B}$ from $\mathcal{K}$ to $\mathcal{K}'$ satisfying*

$$P_{\hat{\mathcal{H}}'} \hat{B} = \hat{A}, \quad \hat{B} V = V' \hat{B} \tag{3.18}$$

Proof: First assume that there exists an intertwining lifting B of A, that is, there exists an operator $B : \mathcal{K} \to \mathcal{K}'$ such that $P_{\mathcal{H}'} B = A$ and $BV = V'B$. Let $\hat{B} = B$. To complete the proof in one direction, it remains only to show that $P_{\hat{\mathcal{H}}'} B = \hat{A}$.

Recall that as in Remark 5, the intertwining relation $BV = V'B$ implies that $BP_n = P'_n B P_n$. For $h' \in \mathcal{H}'$ and $k_n \in \mathcal{K}_n$ we have

$$(P'_n h', Bk_n) = (h', Bk_n) = (B^* h', k_n) = (A^* h', k_n) = (A_n^* P'_n h', k_n)$$

where the last equality follows from (3.10). Recalling the fact that $P'_n \mathcal{H}' = \mathcal{K}'_n \ominus \mathcal{M}_n$ and the relation $BP_n = P'_n B P_n$ it follows that

$$(I - P_{\mathcal{M}_n}) B P_n = A_n P_n. \tag{3.19}$$

Moreover, due to Remarks 2 and 8, both P_n and P'_n converge strongly to the identity and by (3.13) $\hat{\mathcal{H}}' \subset \mathcal{K}' \ominus \mathcal{M}_n$ for all $n \geq 1$. These observations along with (3.19) yield the chain of equalities below (where all the convergences are strong):

$$P_{\hat{\mathcal{H}}'} B = \lim_{n\to\infty} P_{\hat{\mathcal{H}}'} B P_n = \lim_{n\to\infty} P_{\hat{\mathcal{H}}'} (I - P_{\mathcal{M}_n}) B P_n = \lim_{n\to\infty} P_{\hat{\mathcal{H}}'} A_n P_n = \hat{A}.$$

The last equality in the above chain of equalities follows from (3.15). This concludes the proof of the first half of the lemma.

For the converse, assume that there exists an operator $\hat{B}$ from $\mathcal{K}$ to $\mathcal{K}'$ satisfying (3.18). To complete the proof, it is enough to show that $P_{\mathcal{H}'}\hat{B} = A$. To that end, let $k \in \mathcal{K}$ and $h' \in \mathcal{H}'$ be two arbitrary vectors. We have

$$\begin{aligned}(Bk, h') &= \lim_{n\to\infty}(P'_n Bk, h') = \lim_{n\to\infty}(BP_n k, P'_n h') \qquad (3.20)\\ &= \lim_{n\to\infty}(A_n P_n k, h') = (\hat{A}k, h'). \qquad (3.21)\end{aligned}$$

In order to obtain (3.20) we used the fact that P_n and P'_n converge to the identity whereas, to obtain (3.21) we used the fact that $A_n P_n$ converges to $\hat{A}$ strongly. Recalling (3.16) the proof of the proposition is now complete.

Due to the previous proposition and the results in the previous section in the three chain setup of the general intertwining lifting problem we can (and henceforth shall) without loss of generality make the following additional assumption

$$P_{\mathcal{M}_n} \nearrow P_{\mathcal{M}} \text{ (strongly,) where } \mathcal{M} = \mathcal{K}' \ominus \mathcal{H}'. \qquad (3.22)$$

We show now that the general intertwining lifting problem can be reduced even further. First note that due to (3.4) $V\mathcal{K}_n \subset \mathcal{K}_{n-1}$ and $V'\mathcal{K}'_n \subset \mathcal{K}'_{n-1}$ and let

$$V_n : \mathcal{K}_n \to \mathcal{K}_n, V'_n : \mathcal{K}'_n \to \mathcal{K}'_n, V_n = V|\mathcal{K}_n, \qquad V'_n = V'|\mathcal{K}'_n. \qquad (3.23)$$

Also, using the fact that $\mathcal{M} = \mathcal{K}' \ominus \mathcal{H}'$ is invariant for V' and (3.4) it is easy to see that $V'_n\mathcal{M}_n \subset \mathcal{M}_n$, that is, $P_{\mathcal{F}_n}V'_n = P_{\mathcal{F}_n}V'_nP_{\mathcal{F}_n}$ where henceforth, we will denote $\mathcal{F}_n = \mathcal{K}'_n \ominus \mathcal{M}_n$. Let $T'_n = P_{\mathcal{F}_n}V'_n|\mathcal{F}_n$. Using (3.11) and (3.12) it follows that we have the intertwining relation

$$A_m V_m = P_{\mathcal{F}_m}V'_m A_m = T'_m A_m \qquad (3.24)$$

for all $m = n + 1 (n \geq 0)$, i.e., $m \geq 1$. For $m = 0$, (3.24) reduces to the trivial case $0 = 0$.
Remark 11 For later use, we note that

$$V'^*_n P_{\mathcal{F}_n} = P_{\mathcal{F}_n}V'^*_n P_{\mathcal{F}_n} \text{ or equivalently } V'^*_n\mathcal{F}_n \subset \mathcal{F}_n \quad (n \geq 1). \qquad (3.25)$$

Remark 12 Note that since $V_n = V|\mathcal{K}_n$, it follows that $V_n^n = 0$. Also, recalling that V' is a co-shift, $\mathcal{K}'_n$ can be identified with $\underbrace{\mathcal{E} \oplus \cdots \oplus \mathcal{E}}_{n\,times}$ where $\mathcal{E} = \ker V'$ and V' is the upper-triangular canonical Jordan operator, that is,

$$V' = \begin{bmatrix} 0 & I & 0 & 0 & \cdot & \cdot \\ 0 & 0 & I & 0 & \cdot & \cdot \\ 0 & 0 & 0 & I & 0 & \cdot \\ 0 & 0 & 0 & 0 & I & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}_{n\times n}. \qquad (3.26)$$

With these notations, we have the following theorem.

Proposition 3 *Let V be a contraction on $\mathcal{K}$ satisfying the big kernel property and let V' be a co-shift on $\mathcal{K}'$. Assume moreover that $P_{\mathcal{M}_n} \nearrow P_{\mathcal{M}}$. The general intertwining lifting problem with data $\{V, V', T', A\}$ has a solution if and only if the general intertwining lifting problems with data (A_n, V_n, V_n', T_n') has a solution for all $n \geq 1$, where the operators A_n are defined in (3.9) and V_n, V_n' are as in (3.23).*

Proof: First assume that the general intertwining lifting problem $\{V, V', T', A\}$ has a solution, that is, there exists a contraction B such that

$$P_{\mathcal{H}'}B = A, \qquad BV = V'B.$$

As mentioned earlier, from $BV = V'B$ it easily follows that $B\mathcal{K}_n \subset \mathcal{K}_n'$. Let $B_n = B|\mathcal{K}_n$. Then for $h' \in \mathcal{H}'$ and $k_n \in \mathcal{K}_n$ we have

$$(P_n'h', B_n k_n) = (P_n'h', Bk_n) = (h', Bk_n) = (B^*h', k_n) = (A^*h', k_n) = (A_n^*P_n'h', k_n)$$

where the last equality follows from (3.10). Thus, using (3.5), we obtain

$$P_{\mathcal{F}_n}B_n = A_n.$$

Also from the intertwining relation $BV = V'B$ it easily follows that

$$B_nV_n = V_n'B_n.$$

Consequently B_n is a solution to the intertwining lifting problem (A_n, V_n, V_n', T_n') for each $n \geq 1$.

Conversely assume that there exist contractions B_n solving the general intertwining lifting problems (A_n, V_n, V_n', T_n') for each $n \geq 1$, that is, B_n satisfy

$$\|B_n\| \leq 1, \quad P_{\mathcal{F}_n}B_n = A_n \text{ and } B_nV_n = V_n'B_n.$$

Recall that $LIM_{n\to\infty}$ denotes any fixed Banach generalized limit on ℓ^∞. Define

$$(Bk, k') = LIM_{n\to\infty}(B_nP_nk, k') \qquad (k \in \mathcal{K},\ k' \in \mathcal{K}'). \tag{3.27}$$

Clearly B is a contraction since each B_n is a contraction. Recall that from (3.10) we have $P_{\mathcal{H}'}A_nP_n = AP_n$. This implies that

$$P_{\mathcal{H}'}(I - P_{\mathcal{M}_n})B_nP_n = P_{\mathcal{H}'}P_{\mathcal{F}_n}B_nP_n = P_{\mathcal{H}'}A_nP_n = AP_n.$$

Thus for $k \in \mathcal{K}$ and $k' \in \mathcal{K}'$ we obtain

$$(B_nP_nk, (I - P_{\mathcal{M}_n})P_{\mathcal{H}'}k') = (AP_nk, k') \to (Ak, k') \text{ as } n \to \infty.$$

Consequently, $LIM_{n\to\infty}(B_nP_nk, (I - P_{\mathcal{M}_n})P_{\mathcal{H}'}k') = (Ak, k')$. Again, for $k \in \mathcal{K}$ and $k' \in \mathcal{K}'$ one gets

$$\begin{aligned}&(Bk, P_{\mathcal{H}'}k') = \\ &\quad LIM_{n\to\infty}(B_nP_nk, P_{\mathcal{H}'}k') = (Ak, k') + LIM_{n\to\infty}(B_nP_nk, P_{\mathcal{H}'}k' - (I - P_{\mathcal{M}_n})P_{\mathcal{H}'}k').\end{aligned}$$

However, due to our assumption $P_{\mathcal{M}_n} \nearrow P_{\mathcal{M}} = I - P_{\mathcal{H}'}$ (strongly) it follows that

$$(B_n P_n k, P_{\mathcal{H}'} k' - (I - P_{\mathcal{M}_n}) P_{\mathcal{H}'} k') \to 0.$$

Consequently, the contraction B satisfies $P_{\mathcal{H}'} B = A$.

To complete the proof we need to show that the relation $BV = V'B$ holds. To that end, let $k \in \mathcal{K}$ and $k' \in \mathcal{K}'$ be two arbitrary vectors. We have

$$\begin{array}{l} (Bk, V'^* k') = LIM_{n\to\infty}(B_n P_n k, V'^* k') = \\ LIM_{n\to\infty}(B_n P_n k, P'_{n+1} V'^* k') + LIM_{n\to\infty}(B_n P_n k, (I - P'_{n+1}) V'^* k') \quad = \\ LIM_{n\to\infty}(B_n P_n k, V'^* P'_n k') + LIM_{n\to\infty}(B_n P_n k, (I - P'_{n+1}) V'^* k') \quad = \\ LIM_{n\to\infty}(V' B_n P_n k, P'_n k') + LIM_{n\to\infty}(B_n P_n k, (I - P'_{n+1}) V'^* k'). \end{array}$$

Now notice that $LIM_{n\to\infty}(B_n P_n k, (I - P'_{n+1}) V'^* k') = 0$ since $P'_{n+1} \nearrow I$. Also using the intertwining relation $V'_n B_n = B_n V_n$ we get

$$\begin{array}{rcl} & & (V' B_n P_n k, P'_n k') = \\ (V'_n B_n P_n k, P'_n k') & = & (B_n V_n P_n k, P'_n k') = (B_n P_n V P_n k, P'_n k') = \\ & & (B_n P_n V k, k') + (B_n P_n V (I - P_n) k, P'_n k') + (B_n P_n V k, (I - P'_n) k'). \end{array}$$

Since the last two terms in the above line goes to zero as n tends to infinity, we have that

$$LIM_{n\to\infty}(V' B_n P_n k, P'_n k') = LIM_{n\to\infty}(B_n P_n V k, k') = (BVk, k').$$

Therefore, $BV = V'B$ as asserted. This concludes the proof of the proposition.

We conclude this section by stating a theorem which summarizes our discussion in this section. The proof of this theorem follows immediately from Propositions 2 and 3.

Theorem 3 *The general intertwining lifting problem with data $\{V, V', T', A\}$ has a contractive intertwining lifting B if and only if there exists an adequate sequence of intertwining lifting data $\{V_n, V'_n, T'_n, A_n\}$ each admitting a contractive intertwining lifting B_n, $n \geq 1$. Furthermore, the lifting data $\{V_n, V'_n, T'_n, A_n\}$ additionally satisfies $V_n^n = 0$ and V'_n is unitarily equivalent to the $n \times n$ upper-triangular canonical block Jordan operator.*

The significance of this result is that one may now legitimately hope to solve the general intertwining lifting problem by a "one step lifting procedure" which is used in Chapter VII of [FF] to prove the Sz.-Nagy–Foias commutant lifting theorem. Note also that in Proposition 3, in case the fiber of the co-shift V' and the spaces $\mathcal{E}_n = \ker V^n \ominus \ker V^{n-1}, n \geq 1$ are finite dimensional, the general intertwining lifting problem in that special case essentially reduces to a problem concerning finite matrices.

4 Causal Commutant Lifting and Intertwining Extension

The notion of causality in the context of commutant lifting was first introduced by Foias and Tannenbaum in [FT]. This was motivated by certain problems in nonlinear systems theory and the causal commutant lifting developed in [FT] allows one to incorporate the nest algebra structure of [Arv] in the commutant lifting framework. As mentioned before, the causal commutant lifting theorem has applications to certain nonlinear interpolation problems arising in Control Theory. See [FT], [FGT], [FG] for more details. It was shown in [FGT] that in fact, the causal commutant lifting problem can be regarded as a special case of the general intertwining extension problem, or equivalently, the general intertwining lifting problem. Using our reduction in the previous section, we will now show that conversely the general intertwining lifting problem can be regarded as a special case of the causal commutant lifting problem.

We will describe here the notion of causality in commutant lifting. Consider the lifting problem with data $\{V, V', T', A\}$ where V and V' are both isometries. The Sz.-Nagy–Foias commutant lifting (see [FF]) says that there exists a contractive intertwining lifting for this data, i.e., there exists a contraction $B : \mathcal{K} \to \mathcal{K}'$ satisfying (1.3) with $\gamma = 1$.

A *causal structure* $\mathcal{C}$ on $\mathcal{K}$ is a sequence of projections $\{P_n\}_{n=1}^{\infty}$ on $\mathcal{K}$ satisfying

(i) $P_1 \leq P_2 \leq \cdots$,

(ii) $P_n \leq I - V^n V^{*\,n}$, i.e., $\operatorname{Ran} P_n \subset \ker V^{*\,n}$ and

(iii) $P_{n+1} V (I - P_n) = 0$.

An operator $B : \mathcal{K} \to \mathcal{K}'$ is said to be a *causal intertwining lifting* with respect to the given causal structure if in addition to (1.3), it also satisfies

$$(I - V'^n V'^{*\,n}) B = (I - V'^n V'^{*\,n}) B P_n, n = 1, 2, \cdots . \tag{4.28}$$

It is easy to see that $\mathcal{C} = \{I - V^n V^{*\,n}\}_{n=1}^{\infty}$ is a causal structure on $\mathcal{K}$ and any B satisfying (1.3) is automatically causal with respect to this causal structure. It is known that (see [FT]) that a causal intertwining lifting need not always exist. The *causal intertwining lifting problem* is to determine when a contractive causal intertwining lifting B of A exists.

Consider now the general intertwining extension problem (which is the dual of the general intertwining lifting problem) with data $\{V, V', T, A\}$ where the operator V is an isometry but V' is an arbitrary contraction. Also, recall that $\mathcal{H}$ is an invariant subspace of V and $T = V|\mathcal{H}$ is an isometry and A satisfies $AT = V'A$. Let U' on $\mathcal{U}' \supset \mathcal{K}'$ be the minimal isometric dilation of V' and $\hat{V}$ on $\hat{\mathcal{K}} \supset \mathcal{K}$ and $\hat{U}'$ on $\hat{\mathcal{U}}'$ be the minimal unitary extensions of the isometries V and U' respectively. Let the spaces $\mathcal{H}_n \subset \mathcal{H}$ and projections P_n be defined by

$$\mathcal{H}_n := V^n \mathcal{K} \cap \mathcal{H}, n = 1, 2, \cdots , \quad \text{and projections } P_n = I - P_{\mathcal{H}_n}. \tag{4.29}$$

By Lemma 5.1 of [FGT], the projections P_n define a causal structure on $\mathcal{H}$ for $T = V|\mathcal{H}$. Define the spaces

$$\mathcal{H}_\infty = \bigvee_{n=0}^{\infty} \hat{V}^{*\,n} \mathcal{H}, \quad \mathcal{H}_{-\infty} = \bigvee_{n=1}^{\infty} \hat{V}^{*\,n} \mathcal{H}_n.$$

Clearly, $\mathcal{H}_{-\infty} \subset \mathcal{K} \cap \mathcal{H}_{\infty}$. Moreover, $\mathcal{H}_{\infty}$ is invariant for $\hat{V}$ and $\hat{V}_{\infty} = \hat{V}|\mathcal{H}_{\infty}$ is the minimal unitary extension of the isometry T. For any operator $B' : \mathcal{H} \to \mathcal{U}'$ satisfying $B'T = U'B'$ define

$$B'_{\infty} : \mathcal{H}_{\infty} \to \hat{\mathcal{U}}', \; B'_{\infty} = \text{ strong } \lim_{n\to\infty} \hat{U}'^{*\,n} B P_{\mathcal{H}} \hat{V}^{n}_{\infty}. \tag{4.30}$$

It is well known (see [FF]) that the strong limit in (4.30) exists and the operator B'_{∞} is an extension of B' (i.e., $B'_{\infty}|\mathcal{H} = B'$) satisfying $B'_{\infty}\hat{V}_{\infty} = \hat{U}'B'_{\infty}$.

Remark 13 Suppose that $\mathcal{K}$ satisfies the condition

$$\overline{\{k \in \mathcal{K} : \exists m = m(k) \ni V^n k \in \mathcal{H}\}} = \mathcal{K}. \tag{4.31}$$

Then, it can be easily checked that $\mathcal{K} \subset \mathcal{H}_{-\infty}$.

We are now ready to prove a theorem (see [F]) which will allow us to reformulate a particular case of the general intertwining extension problem as a causal commutant lifting problem.

Theorem 4 *Let $\{V, V', T, A\}$ be the general intertwining extension data where V is an isometry on $\mathcal{K}$ and such that (4.31) holds. The intertwining extension problem with data $\{V, V', T, A\}$ described above has a contractive solution if and only if there exists a contractive causal intertwining lifting B' of A intertwining T and U', i.e., a contraction $B' : \mathcal{H} \to \mathcal{U}'$ satisfying*

$$P_{\mathcal{K}'}B' = A, B'T = U'B', (I - U'^{n}U'^{*\,n})B' = (I - U'^{n}U'^{*\,n})B'P_n, \tag{4.32}$$

where P_n is as in (4.29).

Proof: By Lemma 5.2 in [FGT], a contraction B' intertwining T and U' satisfies the third equality in (4.32) if and only if B'_{∞} satisfies

$$B'_{\infty}\mathcal{H}_{-\infty} \subset \mathcal{U}'. \tag{4.33}$$

Assume now that a contraction $B' : \mathcal{H} \to \mathcal{U}'$ satisfying (4.32) exists. Due to the assumption in (4.31), by Remark 13 we have $\mathcal{K} \subset \mathcal{H}_{-\infty}$. Consider the contraction $B = P_{\mathcal{K}'}B'_{\infty}|\mathcal{K}$. Then, using the fact that B'_{∞} extends B' it is easy to check that

$$\begin{aligned} B|\mathcal{H} &= P_{\mathcal{K}'}B'_{\infty}|\mathcal{H} \\ &= P_{\mathcal{K}'}B' = A \end{aligned} \tag{4.34}$$

where the equalities in (4.34) follows from the fact that B'_{∞} extends B' and B' is a lifting of A. Moreover, since $\mathcal{K} \subset \mathcal{H}_{-\infty} \subset \mathcal{K} \cap \mathcal{H}_{\infty}$, we have $\hat{V}_{\infty}|\mathcal{K} = \hat{V}|\mathcal{K} = V$ and

$$\begin{aligned} BV &= P_{\mathcal{K}'}B'_{\infty}V = P_{\mathcal{K}'}B'_{\infty}\hat{V}_{\infty}|\mathcal{K} \\ = P_{\mathcal{K}'}\hat{U}'B'_{\infty}|\mathcal{K} &= P_{\mathcal{K}'}U'B'_{\infty}|\mathcal{K} = V'P_{\mathcal{K}'}B'_{\infty}|\mathcal{K} = V'B. \end{aligned} \tag{4.35}$$

Conversely, assume that a contractive intertwining extension B of A exists, i.e., B satisfies

$$B : \mathcal{K} \to \mathcal{K}', B|\mathcal{H} = A, BV = V'B.$$

Since U' is the minimal isometric dilation of V', applying the Sz.-Nagy–Foias commutant lifting theorem, there exists a contraction B'_1 satisfying

$$P_{\mathcal{K}'}B'_1 = B, B'_1 V = U' B'_1.$$

Define $B' = B'_1|\mathcal{H}$. Clearly, B' satisfies

$$B'T = U'B', P_{\mathcal{K}'}B' = A.$$

We will now show that B' is also causal. To that end, recall that the range of P_n is orthogonal to $\mathcal{H}_n = V^n\mathcal{K} \cap \mathcal{H}$. We only need to verify the third equality in (4.32), or equivalently $B'\mathcal{H}_n \subset \operatorname{Ran} U'^n$. This immediately follows from the definition of the contraction B', the space $\mathcal{H}_n$ and the intertwining relation $B'_1 V = U' B'_1$. This finishes the proof of the theorem.

We will now show that the general intertwining lifting problem can be regarded as a causal commutant lifting problem. To do this we will need the following proposition.

Proposition 4 *Let $\{V, V', T', A\}$ be an intertwining lifting data where V and V' acts on spaces $\mathcal{K}$ and $\mathcal{K}'$, $\mathcal{H}' \subset \mathcal{K}'$ is an invariant subspace of V'^* and $T'^* = V'^*|\mathcal{H}'$. Assume moreover that $V^n = 0$ for some $n \in \mathbb{N}$ and V' is the $n \times n$ upper-triangular canonical block Jordan operator, i.e., V' on $\mathcal{K}' = \bigoplus_{i=1}^{n} \mathcal{E}$ is the operator defined by*

$$V'(e_1 \oplus e_2 \oplus \cdots e_n) = e_2 \oplus e_3 \oplus \cdots \oplus e_{n-1} \oplus 0.$$

Let $\hat{V}'$ on $\hat{\mathcal{K}}' = \bigoplus_{i=1}^{\infty} \mathcal{E} = \mathcal{K}' \oplus (\bigoplus_{i=n+1}^{\infty} \mathcal{E})$ be the backward shift, i.e.,

$$\hat{V}'(e_1 \oplus e_2 \oplus \cdots) = e_2 \oplus e_3 \oplus \cdots .$$

Define

$$\hat{\mathcal{H}}' = \mathcal{H}' + (\hat{\mathcal{K}}' \ominus \mathcal{K}') \text{ and } \hat{T}' = P_{\hat{\mathcal{H}}'}\hat{V}'|\hat{\mathcal{H}}'. \tag{4.36}$$

Then, $\{V, \hat{V}', \hat{T}', A\}$ constitutes a lifting data and moreover, the data $\{V, V', T', A\}$ has a contractive intertwining lifting if and only if the data $\{\hat{V}', V, \hat{T}', A\}$ has a contractive intertwining lifting as well. Furthermore, $\hat{\mathcal{K}}'$ satisfies

$$\hat{\mathcal{K}}' = \{k' \in \hat{\mathcal{K}}' : \exists\, m = m(k') \text{ such that } \hat{V}'^{*m}k' \in \hat{\mathcal{H}}'\}. \tag{4.37}$$

Proof: In (4.36) the sum is actually an orthogonal sum and since $\hat{V}'^{*n}\hat{\mathcal{K}}' \subset (\hat{\mathcal{K}}' \ominus \mathcal{K}')$, clearly (4.37) is satisfied. Also, since $\mathcal{H}' \subset \mathcal{K}'$ is invariant to V'^*, it is easy to see that $\hat{\mathcal{H}}'$ is invariant to $\hat{V}'^*$. Moreover, using the fact that $AV = T'A = P_{\mathcal{H}'}V'A$ and $P_{\hat{\mathcal{H}}'}|\mathcal{K}' = P_{\mathcal{H}'}|\mathcal{K}'$ one can verify that $AV = P_{\hat{\mathcal{H}}'}\hat{V}'A = \hat{T}'A$. Thus, $\{\{\hat{V}', V, \hat{T}', A\}$ constitutes an intertwining lifting data.

It is now easy to check that the intertwining lifting problem with data $\{V, V', T', A\}$ has a solution if and only if the data $\{\hat{V}', V, \hat{T}', A\}$ has a solution. Indeed, if B solves the intertwining lifting problem with data $\{V, V', T', A\}$ then since $\hat{V}'|\mathcal{K}' = V'$, the contraction $\hat{B} = B$ also solves the intertwining lifting problem with data $\{\hat{V}', V, \hat{T}', A\}$. Conversely, if $\hat{B}$ solves $\{\hat{V}', V, \hat{T}', A\}$, then $P_{\mathcal{K}'}\hat{B}$ solves $\{V, V', T', A\}$. This finishes the proof.

The intertwining lifting problem with data $\{V, V', T', A\}$ satisfying the conditions of Proposition 4 can be regarded as a intertwining extension problem with data $\{V'^*, V^*, T'^*, A^*\}$ and by (4.37) the operator V'^* is an isometry on which satisfies condition (4.31). By Theorem 4, this extension problem can thus be reformulated as a causal commutant lifting problem.

5 Commutant Lifting for Nilpotent Operators

As we have seen before, in view of Theorem 3, the general intertwining lifting problem is reduced to a case where the lifting data $\{V, V', T', A\}$ has the property that V is nilpotent (say of order n) and V' is the $n \times n$ canonical block Jordan operator. Our next result shows that in case V is also unitarily equivalent to $n \times n$ canonical block Jordan operator, then the lifting problem above has a solution, i.e., there exists a contractive intertwining lifting B of A. This can be regarded as a "finite block" version of the commutant lifting theorem. Before stating the result, we remark that the $n \times n$ upper-triangular block Jordan operator is unitarily equivalent to the $n \times n$ lower-triangular block Jordan operator.

Theorem 5 *Let V on $\mathcal{K} = \underbrace{\mathcal{E} \oplus \cdots \oplus \mathcal{E}}_{n}$ denote the $n \times n$ upper-triangular canonical block Jordan operator on $\mathcal{K}$ and let V' on $\mathcal{K}'$ be a nilpotent contraction of order n, that is, ${V'}^n = 0$. In this case, any lifting problem $\{V, V', T', A\}$ has a solution.*

Proof: Recall that in our set up, the space $\mathcal{K}' \ominus \mathcal{H}'$ is invariant to V'^* and $T' = P_{\mathcal{H}'} V' | \mathcal{H}'$. We are given a contraction A satisfying $AV = P_{\mathcal{H}'} V' A = T' A$ and we want to find a contraction $B : \mathcal{K} \to \mathcal{K}'$ such that

$$BV = V'B, \; P_{\mathcal{H}'} B = A.$$

Let $\hat{V}$ on $\hat{\mathcal{K}}$ and $\hat{V}'$ on $\hat{\mathcal{K}}'$ be the Sz.-Nagy - Schaffer minimal isometric dilations of V and V' respectively. Recall that the minimal isometric dilation $\hat{V}$ and $\hat{V}'$ satisfy

$$VP_{\mathcal{K}} = P_{\mathcal{K}} \hat{V}, \quad V' P_{\mathcal{K}'} = P_{\mathcal{K}'} \hat{V}'. \tag{5.38}$$

Define the contraction $\hat{A} : \hat{\mathcal{K}} \to \mathcal{H}'$ by

$$\hat{A} = AP_{\mathcal{K}}.$$

Then,

$$P_{\mathcal{H}'} \hat{V}' \hat{A} = P_{\mathcal{H}'} \hat{V}' A P_{\hat{\mathcal{K}}} = P_{\mathcal{H}'} P_{\hat{\mathcal{K}}'} \hat{V}' A P_{\hat{\mathcal{K}}} = P_{\mathcal{H}'} V' A P_{\mathcal{K}} = AVP_{\mathcal{K}} = \hat{A} \hat{V}. \tag{5.39}$$

Note that since V' is a lifting of T' and $\hat{V}'$ is a lifting of V', it follows that $\hat{V}'$ is a lifting of T'. In other words, we have a new lifting data $(\hat{A}, \hat{V}, \hat{V}', T')$. Since $\hat{V}'$ is an isometric lifting of T', by the commutant lifting theorem, there exists a contraction $\hat{B}$ for this lifting data, i.e.,

$$\hat{B} \hat{V} = \hat{V}' \hat{B}, \; P_{\mathcal{H}'} \hat{B} = \hat{A}.$$

Now define the contraction $B = P_{\mathcal{K}'}\hat{B}|\mathcal{K}$ and note that

$$P_{\mathcal{H}'}B = P_{\mathcal{H}'}\hat{B}|\mathcal{K} = \hat{A}|\mathcal{K} = A.$$

Since V is canonical Jordan operator and V' satisfies $V'^n = 0$, we can check (for instance, by using the Sz.-Nagy-Schaeffer construction of the minimal isometric dilation) that

$$\hat{V}^n(\hat{\mathcal{K}} \ominus \mathcal{K}) = \hat{\mathcal{K}} \ominus \mathcal{K}, \quad \text{and} \quad \hat{V'}^n(\hat{\mathcal{K}}' \ominus \mathcal{K}') \subset \hat{\mathcal{K}}' \ominus \mathcal{K}'.$$

Now since $\hat{B}\hat{V} = \hat{V}'\hat{B}$, the above observation implies the relation

$$\hat{B}(\hat{\mathcal{K}} \ominus \mathcal{K}) \subset \hat{\mathcal{K}}' \ominus \mathcal{K}'.$$

Therefore, $\hat{B}$ satisfies

$$P_{\mathcal{K}'}\hat{B} = P_{\mathcal{K}'}\hat{B}P_{\mathcal{K}} = BP_{\mathcal{K}}. \tag{5.40}$$

Consequently,

$$\begin{aligned} &V'B = V'P_{\mathcal{K}'}\hat{B}|\mathcal{K} = P_{\mathcal{K}'}\hat{V}'\hat{B}|\mathcal{K} = \\ &P_{\mathcal{K}'}\hat{B}\hat{V}|\mathcal{K} = BP_{\mathcal{K}}\hat{V}|\mathcal{K} = BV, \end{aligned}$$

where we have used (5.38) and (5.40). Thus, B is a contractive intertwining lifting for the lifting data $\{V, V', T', A\}$. This concludes the proof of the theorem.

Below, we also show the converse of the theorem. To be precise, let V be a nilpotent contraction of order n, that is, $V^n = 0$ and let V' be the canonical block Jordan operator. Then, as mentioned before, with respect to the decomposition $\mathcal{K} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$, $\mathcal{E}_i = \mathcal{K}_i \ominus \mathcal{K}_{i-1}, (i = 1, \cdots, n)$, the matrix of V is strictly upper-triangular. Moreover, $\mathcal{K}' = \underbrace{\mathcal{E}' \oplus \cdots \oplus \mathcal{E}'}_{n}$ and the operator $V' = ((V'_{ij}))$, $V'_{ij} = \delta_{i+1,j}I$ where I denotes identity on $\mathcal{E}'$ and $\delta_{m,n}$ is the Kronecker delta.

Theorem 6 *Let V be a nilpotent contraction of order n and V' be the $n \times n$ upper-triangular canonical block Jordan operator as described in the preceding paragraph. If for all T' and A, such that $\{V, V', T', A\}$ constitutes a lifting data, there exists a contractive intertwining lifting, then V is unitarily equivalent to a canonical Jordan operator.*

Proof: It is enough to show that the contraction $V = ((V_{ij}))$ satisfies

$$V_{ij} = \delta_{i+1,j}V_i, \ V_i : \mathcal{E}_{i+1} \to \mathcal{E}_i \text{ are unitary } (i = 1, \cdots, n-1). \tag{5.41}$$

The proof will be by induction. Let $1 \le m \le n-1$. In case $m < n-1$, assume that we have proven that (5.41) holds for $i \in \{m+1, \cdots, n-1\}$. In this case, $V_{i(i+1)}$ is unitary for $i \in \{m+1, \cdots, n-1\}$. Since V is a contraction, it immediately follows that for each $i \in \{m+1, \cdots, n-1\}$, $V_{l(i+1)} = 0$, $l = 1, 2, \cdots, n$ such that $l \neq i$. Using a unitary equivalence, without loss of generality, we may assume that $\mathcal{E}_i = \mathcal{E},\ V_{i(i+1)} = I,\ i \in \{m+1, \cdots, n-1\}$. To do the inductive step, we need to show that $V_{m(m+1)}$ is unitary.

First we show that $V_{m(m+1)} : \mathcal{E}_{m+1}(= \mathcal{E}) \to \mathcal{E}_m$ is an isometry. Assume that it is not. Then there exists a vector $\tilde{e} \in \mathcal{E}_{m+1}$ such that $D_{V_{m(m+1)}} e \neq 0$. Let $\tilde{A} : \mathcal{E}_{m+1} \to \mathcal{E}'$ be any contraction satisfying $\|\tilde{A} D_{V_{m(m+1)}} \tilde{e}\| = \|D_{V_{m(m+1)}} \tilde{e}\|$. Let $\mathcal{H}' = \mathcal{K}' \ominus \mathcal{K}'_m$ and note that $\mathcal{H}'$ is clearly invariant to V'^*. Let $T' = P_{\mathcal{H}'} V' | \mathcal{H}'$ and let A be the block diagonal operator $A = \tilde{A}_1 \oplus \cdots \oplus \tilde{A}_n$ where

$$\tilde{A}_i = \begin{cases} 0, & i \leq m \\ \tilde{A}, & m+1 \leq i \leq n \end{cases}$$

Using the definition of the spaces and operators, it is now a simple computation to check that $AV = P_{\mathcal{H}'} V' A$.

We claim that the intertwining lifting problem with data $\{V, V', T', A\}$ has no contractive intertwining lifting B. To show this, assume by way of contradiction that B is a contractive intertwining lifting of A. Recall that in our set up

$$\mathcal{K}_m = ker\ V^m = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m \ \text{ and } \mathcal{K}'_m = ker\ V'^{\,m} = \underbrace{\mathcal{E}' \oplus \cdots \mathcal{E}'}_{m \text{ times}} \ \text{ and } \mathcal{H}' = \mathcal{K}' \ominus \mathcal{K}'_m.$$

Then since $P_{\mathcal{H}'} B = A$ with respect to the decomposition $\mathcal{K} = \mathcal{K}_m \oplus (\mathcal{K} \ominus \mathcal{K}_m)$ and $\mathcal{K}' = \mathcal{K}'_m \oplus \mathcal{H}'$ the contraction B has a block diagonal representation given by $B = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & A \end{bmatrix}$. Let e be the vector in $\mathcal{K} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$ defined by $e = (0, \cdots, 0, \underbrace{D_{V_{m(m+1)}} \tilde{e}}_{(m+1)-th}, 0, \cdots, 0)$ and let $Q'_i,\ i = 1, \cdots, n$ be the projection on $i-$th component in $\mathcal{K}'$. Then $Q'_i V' B e = (0, \cdots, 0, \underbrace{\tilde{A} D_{V_{m(m+1)}} \tilde{e}}_{m-th}, 0, \cdots, 0)$. On the other hand, recall that by the induction hypothesis, $i \in \{m+1, \cdots, n-1\},\ V_{l(i+1)} = 0,\ l = 1, \cdots i-1$ and moreover, as mentioned before, without of loss of generality, we are working under the assumption that $\mathcal{E}_{i+1} = \mathcal{E},\ V_{i(i+1)} = I,\ i \in \{m+1, \cdots, n-1\}$. So, $Ve = (0, \cdots, 0, \underbrace{V_{m(m+1)} D_{V_{m(m+1)}} \tilde{e}}_{m-th}, 0, \cdots, 0)$.

Comparing components in the relation $BV = V'B$ we get

$$CV_{m(m+1)} D_{V_{m(m+1)}} \tilde{e} = \tilde{A} D_{V_{m(m+1)}} \tilde{e} \tag{5.42}$$

where the contraction C is an entry of B. Recall that by construction of $\tilde{A}$ we have

$$\begin{aligned} \|\tilde{A} D_{V_{m(m+1)}} \tilde{e}\| &= \|D_{V_{m(m+1)}} \tilde{e}\| \\ = \|CV_{m(m+1)} D_{V_{m(m+1)}} \tilde{e}\| &\leq \|V_{m(m+1)} D_{V_{m(m+1)}} \tilde{e}\| < \|D_{V_{m(m+1)}} \tilde{e}\| \end{aligned} \tag{5.43}$$

where the last inequality is strict, since for any contraction T, if $D_T h \neq 0$ for some vector h then $\|T D_T h\| < \|D_T h\|$. The contradiction in (5.43) came from the assumption that there exists a vector $\tilde{e}$ such that $D_{V_{m(m+1)}} \tilde{e} \neq 0$. This forces $V_{m(m+1)}$ to be an isometry and since V is a contraction, this automatically implies that $V_{l(m+1)} = 0,\ l = 1, \cdots, m-1$.

In order to prove now that $V_{m(m+1)}$ is onto, we argue once again by contradiction. Suppose that $V_{m(m+1)}$ is not onto. Then, there exists a contraction $\tilde{A} : \mathcal{E}_{m+1} \to \mathcal{E}'$ such that $\tilde{A} \neq 0$ and

$\tilde{A}V_{m(m+1)} = 0$. Define $A = ((A_{ij}))$ by

$$A_{ij} = \begin{cases} \tilde{A}, \ i = m+1, \text{ and } j = m \\ 0 \text{ otherwise.} \end{cases}$$

With $\mathcal{H}' = \mathcal{K}' \ominus \mathcal{K}'_m$ it is easy to see that $AV = 0 = P_{\mathcal{H}'}V'A$. This A does not admit an intertwining lifting B. To see this note that, if B is a lifting of A, by the definition of $\mathcal{H}'$, it follows that the $(m+1, m) - th$ block entry of B must be $\tilde{A}$. On the other hand, as we have seen before, $BV = V'B$ forces B to be upper triangular. This is a contradiction since $\tilde{A} \neq 0$. Thus $V_{m,m+1}$ is onto as well. This concludes the proof of the proposition.

The convenient feature of Theorem 5 is that it can be reformulated as a result concerning extension of intertwining operators instead of a result concerning its lifting. Indeed Theorem 5 has the following corollary.

Theorem 7 *Let $\{V, V', T, A\}$ constitute an intertwining extension data where V and V' are both unitarily equivalent to the $n \times n$ canonical block Jordan operator. Then there exists a contractive intertwining extension B of A.*

Proof: The proof follows by noting that the intertwining extension problem can be reformulated as an intertwining lifting problem with data $\{V'^*, V^*, T^*, A^*\}$ and the operators V^* and V'^* are also unitarily equivalent to $n \times n$ canonical block Jordan operator. The proof now follows by applying Theorem 5.

6 Carswell-Schubert Result

We will first recall here the setup of the Carswell-Schubert result in the extension formulation. Let V and V' be isometries acting on spaces $\mathcal{K}$ and $\mathcal{K}'$ respectively and $\mathcal{H} \subset \mathcal{K}$ invariant to V (i.e., $V\mathcal{H} \subset \mathcal{H}$). We are given a contraction $A : \mathcal{H} \to \mathcal{K}'$ which satisfies $AT = V'A$ where $T = V|\mathcal{H}$. We seek an operator $B : \mathcal{K} \to \mathcal{K}'$ such that

$$BV = V'B, \ B|\mathcal{H} = A \text{ and } \|B\| \leq 1. \tag{6.44}$$

To this end, we have the following theorem.

Theorem 8 *There exists an operator B as in (6.44) if and only if*

$$\|(1 - V'^n V'^{n*})Ah\| \leq \|(1 - V^n V^{*n})h\|, \quad (h \in \mathcal{H}). \tag{6.45}$$

We will give a proof of the sufficiency of the conditions (6.45) based on Theorem 5. The necessity of (6.45) was proven in[LeP]. We include the proof here for the sake of completeness. **Proof:** Assume that B as in (6.44) exists. Recall that for any isometry U, the projection on the range of U^n is given by U^nU^{*n} and $I-U^nU^{*n}$ is the projection on the kernel of U^{*n}. Using the intertwining relation $BV^n\mathcal{K}\subset V'^n\mathcal{K}'$. Thus, $V'^nV'^{*n}BV^nV^{*n}=BV^nV^{*n}$, or equivalently,

$$(I-V'^nV'^{*n})B=(I-V'^nV'^{*n})B(I-V^nV^{*n}).$$

Consequently, using $A=B|\mathcal{H}$ along with the above equality we have for all $h\in\mathcal{H}$

$$\|(I-V'^nV'^{*n})Ah\|=\|(I-V'^nV'^{*n})B(I-V^nV^{*n})h\|\le\|(I-V^nV^{*n})h\|.$$

Conversely, assume that (6.45) holds. Using Theorem 5, we will show that there exists B as in (6.44). Due to Remark 4, we may assume that V and V' are both unilateral (backward) shifts. Let $\mathcal{K}_n=\ker V^{*n}$, $\mathcal{K}'_n=\ker V'^{*n}$ and $\mathcal{H}_n=P_n\mathcal{H}$, where P_n and P'_n denote the projections on $\mathcal{K}_n$ and $\mathcal{K}'_n$ respectively. Since V and V' are shifts, up to a unitary equivalence, we may assume that $\mathcal{K}_n=\underbrace{\mathcal{E}\oplus\cdots\oplus\mathcal{E}}_{n}$, $\mathcal{K}_n=\underbrace{\mathcal{E}'\oplus\cdots\oplus\mathcal{E}'}_{n}$ where $\mathcal{E}=\ker V$, $\mathcal{E}'=\ker V'$. Now define the operators V_n on $\mathcal{K}_n$, V'_n on $\mathcal{K}'_n$ and $A_n:\mathcal{H}_n\to\mathcal{K}'_n$ by

$$V_n=P_nV|\mathcal{K}_n,\ V'_n=P'_nV'|\mathcal{K}'_n.$$

Clearly, V_n and V'_n are canonical block Jordan operators, i.e.,

$$V_n(e_0\oplus\cdots\oplus e_n)=0\oplus e_0\oplus\cdots\oplus e_{n-1},\ V'_n(e'_0\oplus\cdots\oplus e'_n)=0\oplus e'_0\oplus\cdots\oplus e'_{n-1}.$$

Moreover, since $\mathcal{K}_n=\ker V^{*n}$ and $\mathcal{K}'_n=\ker V'^{*n}$ are invariant to V^* and V'^* respectively and $\mathcal{H}$ is invariant to V, it readily follows that

$$P_nV=V_nP_n,\ P'_nV'_n=V'_nP'_n \text{ and } V_n\mathcal{H}_n\subset\mathcal{H}_n\quad(n\ge 1).\tag{6.46}$$

Assuming (6.45) holds and recalling $P_n=I-V^nV^{*n}$, $P'_n=I-V'^nV'^{*n}$, we see that there exists contractions A_n satisfying

$$A_n:\mathcal{H}_n\to\mathcal{K}'_n,\ A_nP_n|\mathcal{H}=P'_nA,\ (n\ge 1).\tag{6.47}$$

Let $h\in\mathcal{H}$. Then, using (6.46) and (6.47),

$$\begin{aligned}A_nV_nP_nh=A_nP_nVh &= A_nP_nTh=P'_nATh=\\ P'_nV'Ah=V'_nP'_nAh &= V'_nA_nP_nh,\end{aligned}$$

which implies that $A_nV_n|\mathcal{H}_n=V'_nA_n$ for all $n\ge 1$. By Theorem 7, there exist contractions $B_n:\mathcal{K}_n\to\mathcal{K}'_n$ satisfying

$$B_nV_n=V'_nB_n,\ B_n|\mathcal{H}_n=A_n\qquad(n\ge 1).\tag{6.48}$$

Define $B:\mathcal{K}\to\mathcal{K}'$ by

$$(Bk,k')=LIM_{n\to\infty}(B_nP_nk,k')$$

where, LIM denotes a generalized Banach limit on ℓ^∞. Clearly, since $\|B_n\|\le 1$ for all $n\ge 1$, we have $\|B\|\le 1$. Note that from (6.47) and (6.48), for $h\in\mathcal{H}$ and $k'\in\mathcal{K}'$

$$\begin{aligned}(Bh,k')=LIM_{n\to\infty}(B_nP_nh,k') &= LIM_{n\to\infty}(A_nP_nh,k')=\\ LIM_{n\to\infty}(P'_nAh,k') &= LIM_{n\to\infty}(Ah,P'_nk')=LIM_{n\to\infty}(Ah,k'),\end{aligned}$$

where the last equality is obtained by noting that $P_n' \to I$ strongly.

Finally, using (6.46) and (6.48) and recalling once again $P_n \to I$ strongly, for $k \in \mathcal{K}$ and $k' \in \mathcal{K}'$ we have

$$
\begin{aligned}
&(BVk - V'Bk, k') = \\
&\quad = (BVk, k') - (Bk, V^{'*}k') = LIM\ [(B_nP_nVk, k') - (B_nP_nk, V^{'*}k')] = \\
&\quad LIM\ [(B_nV_nP_nk, k') - (B_nP_nk, V^{'*}P_n'k')] + LIM\ (B_nP_nk, V^{'*}(P_n' - I)k') = \\
&\quad LIM\ ((B_nV_n - V_n'B_n)P_nk, k') = 0,
\end{aligned}
$$

which shows that $BV = V'B$. This finishes the proof of the theorem.

References

[Ag] Agler, J. Nevanlinna-Pick interpolation on Sopbolev spaces, *Proc. Amer. Math. Soc.* **108** (1990), 341 - 351.

[AgM] Agler, J. and McCarthy J. E. *Pick Interpolation and Hilbert Function Spaces,* American Mathematical Society, Providence, Rhode Island.

[Arv] Arveson, W. Interpolation problems in nest algebras, *J. Funct. Anal.* **20** (1975), 208 - 233.

[CaS] Carswell, J. G. W. and Schubert, C. F. Lifting of operators that commute with shifts, *Michigan Math. Journal,* **22** (1975), 65-69.

[Ce] Ceausescu, Z. Lifting of a contraction intertwining two isometries, *Michigan Math. J.*, **22** 1975, 65-69.

[Con] Conway, J. B. *A course in functional analysis, Second Edition,* GTM, Springer Verlag (1990).

[F1] Foias, C. On the extension of intertwining operators, *Contemporary Mathematics,* vol. 189, American Mathematical Society, 1995, 227-235.

[F] Foias, C. The general commutant extension problem, *NAW*, 43 - 47, 2001.

[FF] Foias, C., Frazho, A. E. *The commutant lifting approach to interpolation problems,* Operator Theory: Advances and Applications **44**, Birkhäuser Verlag, Basel, 1990.

[FFGK] Foias, C., Frazho, A. E., Gohberg, I. and Kaashoek, M. A. *Metric Constrined Interpolation, Commutant Lifting and Systems,* Operator Theory: Advances and Applications **100**, Birkhäuser Verlag, Basel, 1998.

[FT] Foias, C. and Tannenbaum, A. Causality in commutant lifting theory, *Journal of Functional Analysis*, **vol. 118**, (1993), 407 - 441.

[FGT] Foias, C., Gu, C. and Tannenbaum, A. Intertwining dilations, intertwining extensions and causality, *Acta Sci. Math. (Szeged)*, **57** (1993), 101 - 123.

[FG] Foias, C. and Gu, C. On a causal linear optimization theorem, *Journal of Mathematical Analysis and Applications*, **Vol. 182** (1994), 555 - 565.

[LeP] Operator that commute with an unilateral shift on an invariant subspace, *Pacific J. of Math.,* **36** (1971), 787 - 794.

[Sa] Sarason, D., Generalized interpolation in H^∞, *Trans. American Math. Soc.,* **127** (1967) 179-203.

[Sz.-NF] B. Sz.-Nagy and C. Foias, Dilation des commutants d'opérateurs, *C. R. Acad. Sci. Paris, série A,* **266** (1968), 493-495.