Weyl-Heisenberg Frame Wavelets with Basic Supports

Xunxiang Guo, Yuanan Diao and Xingde Dai
Department of Mathematics

August 25, 2005

Abstract

Let a, b be two fixed non-zero constants. A measurable set $E \subset \mathbb{R}$ is called a Weyl-Heisenberg frame set for (a, b) if the function $g = \chi_E$ generates a Weyl-Heisenberg frame for $L^2(\mathbb{R})$ under modulates by b and translates by a, i.e., $\{e^{imbt}g(t - na)\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. It is an open question on how to characterize all frame sets for a given pair (a, b) in general. In the case that $a = 2\pi$ and $b = 1$, a result due to Casazza and Kalton shows that the condition that the set $F = \bigcup_{j=1}^{k}([0, 2\pi) + 2n_j\pi)$ (where $\{n_1 < n_2 < \cdots < n_k\}$ are integers) is a Weyl-Heisenberg frame set for $(2\pi, 1)$ is equivalent to the condition that the polynomial $f(z) = \sum_{j=1}^{k} z^{n_j}$ does not have any unit roots in the complex plane. In this paper, we show that this result can be generalized to a class of more general measurable sets (called basic support sets) and to set theoretical functions and continuous functions defined on such sets.

1 Introduction

Let $\mathbb{H}$ be a separable complex Hilbert space. Let $B(\mathbb{H})$ denote the algebra of all bounded linear operators on $\mathbb{H}$. Let $\mathbb{N}$ denote the set of natural numbers, and $\mathbb{Z}$ be the set of all integers. A collection of elements$\{x_j : j \in \jmath\}$ in $\mathbb{H}$ is called a *frame* (of $\mathbb{H}$) if there exist constants A and B, $0 < A \le B < \infty$, such that

$$A\|f\|^2 \le \sum_{j\in\jmath} |\langle f,\ x_j\rangle|^2 \le B\|f\|^2 \tag{1.1}$$

for all $f \in \mathbb{H}$. The supremum of all such numbers A and the infimum of all such numbers B are called the *frame bounds* of the frame and are denoted by A_0 and B_0 respectively. A frame is called a *tight frame* if $A_0 = B_0$ and is called a *normalized tight frame* if $A_0 = B_0 = 1$. Any orthonormal basis in a Hilbert space is a normalized tight frame. However a normalized tight frame is not necessarily an orthonormal basis. Frames can be regarded as the generalizations of orthogonal bases of Hilbert spaces. The concept of frames was introduced a long time ago ([14, 19]) and have received much attention recently due to the development and study of wavelet theory [3, 12, 13, 17]. Among those widely studied lately are the Weyl-Heisenberg frames (also called Gabor frames). Let a, b be two fixed positive constants and let T_a, M_b be the *translation operator by* a and *modulation operator by* b respectively, i.e., $T_af(t) = f(t-a)$ and $M_bf(t) = e^{ibt}f(t)$ for any $f \in L^2(\mathbb{R})$. For a fixed $g \in L^2(\mathbb{R})$, we say that (g, a, b) generates a *Weyl-Heisenberg frame* if $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$. We also say that the function g is a *mother Weyl-Heisenberg frame wavelet* for (a, b) in this case. Furthermore, a measurable set $E \subset \mathbb{R}$ is called a *Weyl-Heisenberg frame set* for $(a,\ b)$ if the function $g = \chi_E$ generates a Weyl-Heisenberg frame for $L^2(\mathbb{R})$ under modulates by b and translates by a, i.e., $\{e^{imbt}g(t-na)\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Characterizing the mother Weyl-Heisenberg frame wavelets in general is a difficult and open question. In fact, even in the special case of $a = 2\pi$ and $b = 1$, this question remains unsolved. There are many works related to this subject, for more information please refer to [2, 7, 8, 9, 10, 11, 15, 18].

In 2002, Casazza and Kalton proved the following theorem [6]:

Theorem 1.1. *For fixed integers* $n_1 < n_2 < \cdots < n_k$*, the set*

$$E = \cup_{j=1}^{k}\big([0, 2\pi) + 2\pi n_j\big) \tag{1.2}$$

is a Weyl-Heisenberg frame set for $(2\pi, 1)$ *if and only if the polynomial* $p(z) = \sum_{j=1}^{k} z^{n_j}$ *has no unit roots.*

This means that characterizing a Weyl-Heisenberg frame set E with the special form (1.2) is equivalent to the following problem, which was proposed by Littlewood in 1968 [16]:

Problem 1.2. Classify the integer sets $\{n_1 < n_2 < \cdots < n_k\}$ such that the polynomial $p(z) = \sum_{1\leq j\leq k} z^{n_j}$ does not have any unit roots.

While it is unfortunate that Theorem 1.1 fails to provide a definite answer to the characterization problem of Weyl-Heisenberg frame sets since Problem 1.2 is also an open question (see [4, 5, 6, 16] and the references therein), it does reveal a deep connection between the two seemingly irrelevant subjects. In this paper, we will further investigate Weyl-Heisenberg frames defined by set theoretical functions and continuous functions with restricted domain. In Section 2, we will introduce some concepts, definitions, preliminary lemmas. In Section 3, we will state and prove our main theorems. Some examples are given in Section 4.

2 Definitions and Preliminary Lemmas

Definition 2.1. The *Zak transform* of a function $f \in L^2(\mathbb{R})$ is defined as

$$Zf(t, w) = \sqrt{2\pi}\sum_{n\in\mathbb{Z}} f(t + 2\pi n)e^{inw},\ \forall\ t,\ w \in [0, 2\pi). \tag{2.1}$$

The above definition of Zak transform is actually slightly different from its original definition [1]. We have the following lemma.

Lemma 2.2. *[1] The Zak transform is a unitary map from $L^2(\mathbb{R})$ onto $L^2(Q)$, where $Q = [0, 2\pi) \times [0, 2\pi)$ and the inner product on $L^2(Q)$ is defined by*

$$\langle f(t,w), g(t,w)\rangle = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(t,w)\overline{g(t,w)}dtdw.$$

This can be easily checked and we leave the proof to our reader.

Lemma 2.3.

$$Z(M_m T_{2n\pi} g)(t,w) = e^{i(mt+nw)} Zg(t,w),\ \forall\ g \in L^2(\mathbb{R}). \tag{2.2}$$

Proof. Since $M_m T_{2n\pi} g(x) = e^{imx} g(x - 2n\pi)$,

$$\begin{aligned}
& Z(M_m T_{2n\pi} g)(t,w) \\
= & \sqrt{2\pi}\sum_{\ell\in\mathbb{Z}} e^{im(t+2\ell\pi)} g\big(t + 2(\ell - n)\pi\big)e^{i\ell w} \\
= & \sqrt{2\pi}\sum_{\ell\in\mathbb{Z}} e^{imt} g\big(t + 2(\ell - n)\pi\big)e^{i\ell w} \\
= & \sqrt{2\pi}\sum_{\ell\in\mathbb{Z}} e^{i(mt+nw)} g\big(t + 2(\ell - n)\pi\big)e^{i\ell w - inw} \\
= & \sqrt{2\pi}e^{i(mt+nw)}\sum_{\ell\in\mathbb{Z}} g\big(t + 2(\ell - n)\pi\big)e^{i(\ell - n)w} \\
= & \sqrt{2\pi}e^{i(mt+nw)}\sum_{k\in\mathbb{Z}} g(t + 2k\pi)e^{ikw} \\
= & e^{i(mt+nw)}\sqrt{2\pi}\sum_{k\in\mathbb{Z}} g(t + 2k\pi)e^{ikw} \\
= & e^{i(mt+nw)}(Zg)(t,w).
\end{aligned}$$

□

Lemma 2.4. *[6] Let E be a measurable subset of $[0, 2\pi)$, $F = \bigcup_{n\in\mathbb{Z}}(E + 2n\pi)$ and $g \in L^2(F)$. The following statements are equivalent:*
(1) $(M_m T_{2n\pi} g)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(F)$ with frame bounds A_0, B_0.
(2) $0 < A_0 = essinf_{(t,w)\in E\times[0,2\pi)}|Zg(t,w)|^2 \le esssup_{(t,w)\in E\times[0,2\pi)}|Zg(t,w)|^2 = B_0 < \infty$.

Let $E = \bigcup_{k=1}^m A_k$, where each $A_k = [a_k,\ b_k]$ is a closed interval and $A_i \cap A_j = \emptyset$ if $i \ne j$. Let $\tau_{2\pi}(x) : \mathbb{R} \longrightarrow [0, 2\pi)$ be the function defined by $\tau_{2\pi}(x) = x - 2\pi[\frac{x}{2\pi}]$, where $[.]$ is the integer function. Let us arrange the numbers

$$0,\ \tau_{2\pi}(a_1),\ \tau_{2\pi}(b_1),\ \cdots,\ \tau_{2\pi}(a_m),\ \tau_{2\pi}(b_m),\ 2\pi$$

in the ascending order and write the resulting numbers as $0 = t_0 < t_1 < \cdots < t_{j_0} = 2\pi$. Then for each $0 \le j \le j_0 - 1$ and each k, we have either

$(t_j,\ t_{j+1}) \subset \tau_{2\pi}(A_k)$ or $(t_j,\ t_{j+1}) \bigcap \tau_{2\pi}(A_k) = \emptyset$. Based on this observation we have:

Lemma 2.5. *Let $\{A_i\}_{i=1}^m$ be a sequence of finite and non-overlapping intervals and $E = \bigcup_{i=1}^m A_i$, then there exists a finite sequence of disjoint intervals $\{E_i\}_{i=1}^k$ with $E_i \subset [0, 2\pi)$, and an integer sequence $\{n_{ij}\}_{j=1}^{j_i}$ for each i, such that*

$$E = \bigcup_{i=1}^{k} F_i, \text{ where } \mathrm{F_i} = \bigcup_{\mathrm{j=1}}^{\mathrm{j_i}} (\mathrm{E_i} + 2\pi \mathrm{n_{ij}}).$$

We will call the set E defined in the above lemma a *basic support set*, which is just a finite disjoint union of finite intervals. The sequence $\{E_i\}_{i=1}^k$ associated with the set E will be called the *2π-translation generators* of E. Notice that $\bigcup_{j=1}^{j_i}(E_i+2\pi n_{ij})$ is simply the pre-image of the function $\tau_{2\pi}$ restricted to E. We will call the sequence $\{n_{ij}\}_{j=1}^{j_i}$ the *step-widths* of the corresponding generator E_i.

Definition 2.6. Let $g \in L^2(\mathbb{R})$ be a continuous function and E be a basic support set. For each $\xi \in E$, the sequence $\{g(\xi + 2\pi n_{ij})\}_{j=1}^{j_i}$ is called a *characteristic chain* of the function g associated with the set E, where $\xi \in E_i$ and E_i, $\{n_{ij}\}$ are as defined in Lemma 2.5.

If $\sum_{j=0}^k a_{n_j} z^{n_j} = 0$ has zeros on the unit circle $\mathbb{T}$, then the coefficient sequence $\{a_{n_j}\}_{j=0}^k$ is called a *root sequence* of $\{n_j\}$. For instance, $\{-2, -1, 1\}$ is a root sequence for $\{0, 1, 2\}$ since the polynomial $-2 - z + z^2$ has a unit root. But $\{-2, -1, 1\}$ is not a root sequence for $\{0, 1, 3\}$ since the polynomial $-2 - z + z^3$ does not have a unit root. Furthermore, the sequence $\{n_j\}$ may contain negative integers as well, in which case $\sum_{j=0}^k a_{n_j} z^{n_j}$ is a Laurent polynomial, but our results will still hold for such a polynomial.

3 Main Results and their Proofs

We first outline the main results obtained in this paper below. The first theorem generalizes the result of [6] to step like functions.

Theorem 3.1. *Let $n_0 < n_1 < n_2 < \cdots < n_k$ be $k+1$ fixed integers, a_0, a_1, a_2, $\cdots$, a_k be $k+1$ given complex numbers and let $F_j = [0, 2\pi) + 2\pi n_j$ ($j = 0$, 1, $2, \cdots, k$), $g = \sum_{j=0}^k a_j \chi_{F_j}$. The following statements are equivalent:*

(1) g is a mother Weyl-Heisenberg frame wavelet for $(2\pi, 1)$ with frame bounds A_0, B_0.

(2) $\frac{1}{2\pi} A_0 = \inf_{|z|=1} |\sum_{j=0}^k a_j z^{n_j}|^2$ and $\frac{1}{2\pi} B_0 = \sup_{|z|=1} |\sum_{j=0}^k a_j z^{n_j}|^2$.

(3) For every measurable set $E \subset [0, 2\pi)$ of positive measure, let $E_j = E + 2n_j\pi$, $j = 0, 1, 2, ..., k$, the function $g_E = \sum_{j=0}^k a_j \chi_{E_j}$ is a mother Weyl-Heisenberg frame wavelet of $L^2(\Omega)$ for $(2\pi, 1)$ with frame bounds $A \geq A_0$,

$B \le B_0$, *where* $\Omega = \cup_{n\in\mathbb{Z}}(E+2n\pi)$ *and* A_0, $B = B_0$ *are the frame bounds for* g_E *with* $E = [0, 2\pi)$.

Theorem 3.2. *Let* $E \subset [0, 2\pi)$ *be a measurable set of positive measure and* $n_0 < n_1 < n_2 < \cdots < n_k$ *be* $k+1$ *given integers. Let* $F = \bigcup_{j=0}^{k}(E + 2\pi n_j)$, $\Omega = \bigcup_{n\in\mathbb{Z}}(E+2\pi n)$. *Then for any continuous function* $g \in L^2(\mathbb{R})$, $(g\cdot\chi_F, 2\pi, 1)$ *generates a Weyl-Heisenberg frame of* $L^2(\Omega)$ *if and only if for any* $\xi \in \overline{E}$, $\{g(\xi + 2\pi n_j)\}$ *is not a root sequence for* $\{n_j\}$ *where* $\overline{E}$ *is the closure of* E. *In particular, if* $E = [0, 2\pi)$, *then* $(g\cdot\chi_F, 2\pi, 1)$ *generates a Weyl-Heisenberg frame of* $L^2(\mathbb{R})$ *if and only if for any* $\xi \in [0, 2\pi]$, $\{g(\xi + 2\pi n_j)\}$ *is not a root sequence for* $\{n_j\}$.

Theorem 3.2 further generalizes Theorem 3.1 to continuous functions.

Theorem 3.3. *Let* E *be a basic support set with generator and step-width pairs* $\{(E_i,\ \{n_{ij}\}_{j=0}^{j_i})\}_{i=0}^{k}$, $\Omega = \bigcup_{n\in\mathbb{Z}}(E + 2n\pi)$ *and* $g \in L^2(\mathbb{R})$ *be a continuous function, then* $(g\cdot\chi_E, 2\pi, 1)$ *generates a Weyl-Heisenberg frame for* $L^2(\Omega)$ *with frame bounds* A_0 *and* B_0 *if and only if*

$$\frac{1}{2\pi}A_0 = \min_{0\le i\le k}\ \inf_{\xi\in\overline{E}, z\in\mathbb{T}} |\sum_{j=0}^{j_i} g(\xi + 2\pi n_{ij}) z^{n_{ij}}|^2$$

and

$$\frac{1}{2\pi}B_0 = \max_{0\le i\le k}\ \sup_{\xi\in\overline{E}, z\in\mathbb{T}} |\sum_{j=0}^{j_i} g(\xi + 2\pi n_{ij}) z^{n_{ij}}|^2.$$

In other word, $g_E = g\cdot\chi_E$ *is a mother Weyl-Heisenberg frame wavelet of* $L^2(\Omega)$ *for* $(2\pi, 1)$ *if and only if no characteristic chains of* g *associated with* $\overline{E}$ *are root sequences (of the corresponding step-width sequences).*

Notice that the special case $g = 1$ of Theorem 3.3 will relate the characterization of a Weyl-Heisenberg frame set within the basic support sets (which is more general than the sets considered in [6]) to the classification of corresponding polynomials with unit roots. The result in the following corollary to Theorem 3.3 is immediate, which can also be obtained from the definition of mother Weyl-Heisenberg frames directly without difficulty.

Corollary 3.4. *If* $g \in L^2(\mathbb{R})$ *is a continuous function with a zero characteristic chain associated with a closed basic support set* E, *then* $g\cdot\chi_E$ *is not a mother Weyl-Heisenberg frame wavelet of* $L^2(\Omega)$ *for* $(2\pi, 1)$, *where* $\Omega = \bigcup_{n\in\mathbb{Z}}(E+2n\pi)$.

Proof of Theorem 3.1. For all $x,\ y \in [0, 2\pi)$, we have

$$\begin{aligned} & Zg(x,y) = \sqrt{2\pi}\sum_{n\in\mathbb{Z}} g(x+2n\pi)e^{iny} \\ = \ & \sqrt{2\pi}\sum_{n\in\mathbb{Z}}\sum_{j=0}^{k} a_j\chi_{F_j}(x+2n\pi)e^{iny} \\ = \ & \sqrt{2\pi}\chi_{[0,2\pi)}(x)\sum_{j=0}^{k} a_j e^{in_j y}. \end{aligned}$$

Thus

$$\begin{aligned} & \text{ess}\inf_{(x,y)\in[0,2\pi)^2} |Zg(x,y)|^2 \\ = \ & 2\pi\text{ess}\inf_{y\in[0,2\pi)} |\sum_{j=0}^{k} a_j e^{in_j y}|^2 \\ = \ & 2\pi\text{ess}\inf_{z\in\mathbb{T}} |\sum_{j=0}^{k} a_j z^{n_j}|^2 \end{aligned}$$

and

$$\begin{aligned} & \text{ess}\sup_{(x,y)\in[0,2\pi)^2} |Zg(x,y)|^2 \\ = \ & 2\pi\text{ess}\sup_{y\in[0,2\pi)} |\sum_{j=0}^{k} a_j e^{in_j y}|^2 \\ = \ & 2\pi\text{ess}\sup_{z\in\mathbb{T}} |\sum_{j=0}^{k} a_j z^{n_j}|^2, \end{aligned}$$

where $\mathbb{T}$ is the unit circle. By Lemma 2.4, it follows that A_0 and B_0 are the frame bounds of g if and only if $\frac{1}{2\pi}A_0 = \inf_{|z|=1}|\sum_{j=0}^{k} a_j z^{n_j}|^2$ and $\frac{1}{2\pi}B_0 = \sup_{|z|=1}|\sum_{j=0}^{k} a_j z^{n_j}|^2$. This proves the equivalence of (1) and (2). The special case of $E = [0, 2\pi)$ in (3) implies (1). By (1), $(g, 2\pi, 1)$ generates a Weyl-Heisenberg frame for $L^2(\mathbb{R})$ with frame bounds A, B. Let P be the orthogonal projection of

$$L^2(\mathbb{R}) \to L^2(\Omega)$$

defined by

$$Pf = f|_{\Omega} = f \cdot \chi_{\Omega}.$$

Then

$$\begin{aligned} & P(E_m T_{2n\pi} g) = P\big(e^{imt} g(t-2n\pi)\big) \\ = \ & \big(e^{imt} g(t-2n\pi)\big)\cdot\chi_{\Omega} \\ = \ & e^{imt} g_E(t-2n\pi) = E_m T_{2n\pi} g_E. \end{aligned}$$

So $g_E = \sum_{j=0}^{k} a_j \chi_{E_j}$ is a mother Weyl-Heisenberg frame for $L^2(\Omega)$ with frame bounds $A \geq A_0$, $B \leq B_0$.

□

Proof of Theorem 3.2. Since g is continuous on $\mathbb{R}$, $g_F = g \cdot \chi_F$ is continuous on F and it follows that $Zg_F(x,y)$ is continuous on $E \times [0, 2\pi)$. So there exist $0 < A \leq B$ such that

$$A = \text{ess} \inf_{(x,y)\in E\times[0,2\pi)} |Zg(x,y)|^2 \leq \text{ess} \sup_{(x,y)\in E\times[0,2\pi)} |Zg(x,y)|^2 = B$$

if and only if

$$0 < A = \inf_{x\in E,\ z\in\mathbb{T}} |\sqrt{2\pi} \sum_{j=0}^{k} g(x + 2\pi n_j) z^{n_j}|^2$$

and

$$B = \sup_{x\in E,\ z\in\mathbb{T}} |\sqrt{2\pi} \sum_{j=0}^{k} g(x + 2\pi n_j) z^{n_j}|^2$$

by Lemma 2.4. The result of the Theorem then follows since A and B are actually attained on the set $\overline{E} \times \mathbb{T}$ when g is continuous. □

Proof of Theorem 3.3. By the definition of basic support sets, we have $E = \bigcup_{i=1}^{k} F_i$, where

$$F_i = \bigcup_{j=1}^{j_i} (E_i + 2n_{ij}\pi)$$

and the E_i's are disjoint intervals in $[0, 2\pi)$. Let $M_i = \bigcup_{n\in\mathbb{Z}} (E_i + 2n\pi)$, then $M_i \bigcap M_j = \phi \ for\ any\ i \neq j,$

$$\Omega = \bigcup_{n\in\mathbb{Z}} (E + 2n\pi) = \bigcup_{i=1}^{k} M_i$$

and

$$L^2(\Omega) = \bigoplus_{i=1}^{k} L^2(M_i).$$

It is easy to see that g is a mother Weyl-Heisenberg frame wavelet of $L^2(\Omega)$ if and only if for each i, $g \cdot \chi_{F_i}$ is a mother Weyl-Heisenberg frame wavelet of $L^2(M_i)$. By Theorem 3.2, $g \cdot \chi_{F_i}$ is a mother Weyl-Heisenberg frame wavelet of $L^2(M_i)$ if and only if no characteristic chains of g associated with $\overline{F_i}$ are root sequences (of their corresponding step-width sequences). We leave it to our reader to check that

$$A_0 = 2\pi \cdot \min_{0\leq i\leq k} \inf_{\xi\in\overline{E}, z\in\mathbb{T}} |\sum_{j=0}^{j_i} g(\xi + 2\pi n_{ij}) z^{n_{ij}}|^2$$

and

$$B_0 = 2\pi \cdot \max_{0\le i\le k} \sup_{\xi\in\overline{E}, z\in\mathbb{T}} |\sum_{j=0}^{j_i} g(\xi + 2\pi n_{ij}) z^{n_{ij}}|^2$$

are the frame bounds of $g \cdot \chi_E$ in case that $A_0 > 0$ (hence $g \cdot \chi_E$ is a mother Weyl-Heisenberg frame wavelet of $L^2(\Omega)$). □

4 Examples

Example 4.1. The set $[3\pi, 7\pi)$ is a Weyl-Heisenberg frame set: We have $E_1 = [0, \pi)$, $E_2 = [\pi, 2\pi)$, $n_{11} = 2$, $n_{12} = 3$, $n_{21} = 1$, $n_{22} = 2$ since

$$[3\pi, 7\pi) = \big(E_1 + 4\pi\big) \cup \big(E_1 + 6\pi\big) \cup \big(E_2 + 2\pi\big) \cup \big(E_2 + 4\pi\big).$$

Furthermore, the corresponding polynomials $2 + 3z$ and $1 + 2z$ have no unit roots.

Example 4.2. The set

$$(\frac{5}{2}\pi, \frac{7}{2}\pi] \cup (4\pi, \frac{11}{2}\pi] = \big((0, \frac{1}{2}\pi] + 4\pi\big) \cup \big((\frac{1}{2}\pi, \frac{3}{2}\pi] + 2\pi\big) \cup \big((\frac{1}{2}\pi, \frac{3}{2}\pi] + 4\pi\big)$$

is a WH-frame set since $E_1 = (0, \frac{1}{2}\pi]$, $E_2 = (\frac{1}{2}\pi, \frac{3}{2}\pi]$, $n_{11} = 2$, $n_{21} = 1$, $n_{22} = 2$ and the polynomials 2 and $1 + 2z$ have no unit roots.

Example 4.3. On the unit circle of complex plane,

$$|4 + 3z + 2z^3| = |2 + 3z^2 + 4z^3|.$$

The roots of $p(z) = 4+3z+2z^3$ are: $r_1 = 0.4398+1.4423i$, $r_2 = 0.4398-1.4423i$, $r_3 = -0.8796$. So $p(z)$ doesn't have unit zeros on the complex plane. It follows that the set functions

$$g_1(\xi) = 4\chi_{[0,2\pi)} + 3\chi_{[2\pi,4\pi)} + 2\chi_{[6\pi,8\pi)}$$

and

$$g_2(\xi) = 2\chi_{[0,2\pi)} + 3\chi_{[4\pi,6\pi)} + 4\chi_{[6\pi,8\pi)}$$

are both mother Weyl-Heisenberg frame wavelets for $(2\pi, 1)$ with the same frame bounds.

Example 4.4. Similarly, using the fact that $|1-2z+3z^3-5z^5| = |-5+3z^2-2z^4+z^5|$ on the unit circle and that $p(z) = 1-2z+3z^3-5z^5$ doesn't have unit zeros, we see that the following set functions

$$g_3(\xi) = \chi_{[0,2\pi)} - 2\chi_{[2\pi,4\pi)} + 3\chi_{[6\pi,8\pi)} - 5\chi_{[10\pi,12\pi)}$$

and

$$g_4(\xi) = -5\chi_{[0,2\pi)} + 3\chi_{[4\pi,6\pi)} - 2\chi_{[8\pi,10\pi)} + \chi_{[10\pi,12\pi)}$$

are both mother Weyl-Heisenberg frame wavelets for $(2\pi, 1)$ with the same frame bounds.

Example 4.5. The function $f(t) = \sin t$ is not a mother Weyl-Heisenberg frame wavelet of $L^2(\mathbb{R})$ for $(2\pi, 1)$ on any basic support set E (in fact any measurable set) since for any set E such that $\mathbb{R} = \cup_{n\in\mathbb{Z}}(E + 2n\pi)$, $\overline{E}$ must contain a subsequence of $\{k\pi\}_{k\in\mathbb{Z}}$. Consequently, the function $\sin t$ will always have a zero characteristic chain associated with the set $\overline{E}$. However, if E is a WH-frame set such that $\overline{E}$ is disjoint from $\{k\pi\}_{k\in\mathbb{Z}}$, then $\sin t \cdot \chi_E$ is a mother WH-frame wavelet of $L^2(\Omega)$ for $(2\pi, 1)$ where $\Omega = \cup_{n\in\mathbb{Z}}(E + 2n\pi)$ since any characteristic chain of $\sin t$ associated with $\overline{E}$ is a constant sequence. This example can be generalized to other 2π-periodic functions. In particular, given any continuous 2π-periodic function $g(t)$ that is bounded away from 0 and a WH-frame set E, $g \cdot \chi_E$ is a mother WH-frame wavelet of $L^2(\mathbb{R})$ for $(2\pi, 1)$. For instance, the function

$$g(t) = \begin{cases} |\sin(t)| & \text{if } t \in [\frac{\pi}{6}, \frac{5\pi}{6}) + k\pi, \ \forall k \in \mathbb{Z}, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

is a mother Weyl-Heisenberg frame wavelet of $L^2(\mathbb{R})$ for $(2\pi, 1)$ when it is restricted to the set $E = [3\pi, 7\pi)$. See Figure 1 below.

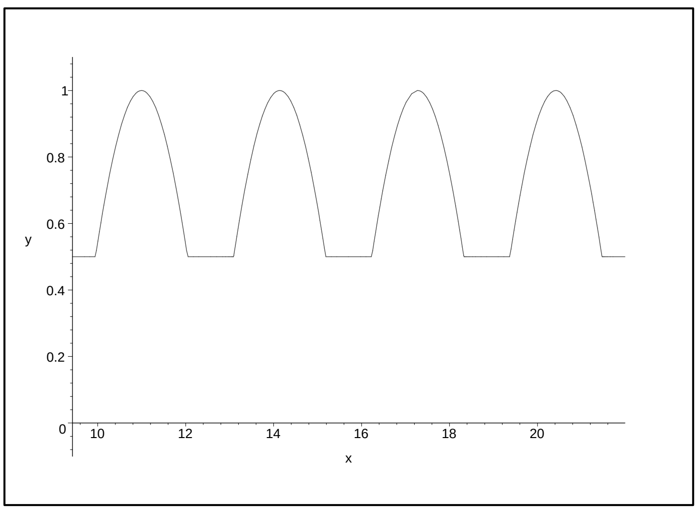

Figure 1: A continuous periodic function bounded away from 0 on a WH-frame set.

Example 4.6. Show that the following function $g(t)$ (see Figure 2) is a mother Weyl-Heisenberg frame wavelet of $L^2(\mathbb{R})$ for $(2\pi, 1)$ when it is restricted to the set $E = [0, 2\pi) \cup [4\pi, 6\pi) \cup [8\pi, 10\pi)$. Notice that the set E is not a WH-frame set. The function $g(t)$ is defined piecewisely by

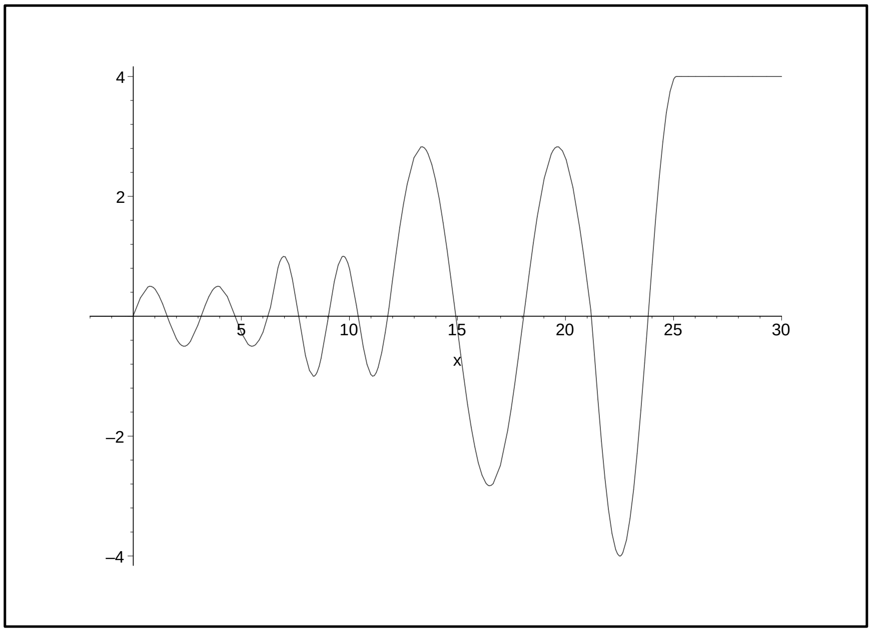

Figure 2: A continuous non-periodic function that is not bounded away from 0 which defines a mother Weyl-Heisenberg frame wavelet of $L^2(\mathbb{R})$ for $(2\pi, 1)$ when it is restricted to a set that is not a WH-frame set.

$$g(t) = \begin{cases} 0 & t < 0, \\ \sin(2t)/2 & t \in [0, 2\pi), \\ \sin(\frac{16}{7}(t - 2\pi)) & t \in [2\pi, \frac{15}{4}\pi), \\ 2(\sin t + \cos t), & t \in [\frac{15}{4}\pi, \frac{27}{4}\pi), \\ -4\sin(\frac{6}{5}(t - \frac{27}{4}\pi)), & t \in [\frac{27}{4}\pi, 8\pi), \\ 4 & t \geq 8\pi. \end{cases}$$

In this case, $E_1 = [0, 2\pi)$ and $n_{11} = 0$, $n_{12} = 2$, $n_{13} = 4$ since $E = E_1 \cup (E_1 + 2 \cdot 2\pi) \cup (E_1 + 4 \cdot 2\pi)$. So for any given $t \in E_1$, the corresponding characteristic chain of $g(t)$ with respect to E is $\{\sin(2t)/2, 2(\sin t + \cos t), 4\}$. It is not a root sequence of $\{0, 2, 4\}$ since $\sin(2t)/2 + 2(\sin t + \cos t)z^2 + 4z^4 = (\sin t + 2z^2)(\cos t + 2z^2)$ apparently have no unit zeros for any given t.

References

[1] A. J. E. M. Janssen, *Bergmann transform, Zak transform, and coherent states*, J. Math. Phys, 23(1982), pp. 720-731.

[2] K. Bittner, and C. K. Chui, *Gabor frames with arbitrary windows.* Approximation Theorie X, C. K. Chui, L. L. Schumakes and J. Steckler Eds., Vanderbilt University Press (2002).

[3] L. Baggett, H. Medina and K. Merrill, *Generalized Multiresolution Analyses, and a Construction Procedure for All Wavelet Sets in* $\mathbb{R}^n$, Preprint.

[4] P. Borwein and T. Erdélyi, *On the zeros of polynomials with restricted coefficients*, Illinois Jour. Math. **41** (1997) pp. 667-675.

[5] P. Borwein, T. Erdélyi and G. Kós, *Littlewood-type problems on* $[0, 1]$, Proc. London Math. Soc. **3** 79 (1999), pp. 22-46.

[6] P. Casazza and N. Kalton, *Roots of complex polynomials and Weyl-Heisenberg frame sets*, Proc. AMS, Vol. **130** (2002), no. 8, pp. 2313-2318.

[7] O. Christensen, *Perturbations of frames and applications to Gabor frames.* In "Gabor Analysis and Algorithms: Theory and Applications", Eds. H. G. Feichtinger and T. Strohmer. Birkhauser (1997), pp. 193- 209.

[8] P. G. Casazza, and O. Christensen, *Weyl-Heisenberg frames for subspaces of* $L^2(\mathbb{R})$. Proc. Amer. Math. Soc. 129 (2001), pp. 145-154.

[9] P. G. Casazza, and M. C. Lammers, *Analyzing the Gabor frame identity*, Appl. Comput. Harmon. Anal. 12, no. 2 (2002), pp. 171-178.

[10] P. G. Casazza, O. Christensen, and A. J. E. M. Janssen, *Weyl-Heisenberg frames, translation invariant systems and the Walnut representation.* Journal of Functional Analysis, 180 (2001), pp. 85-147.

[11] O. Christensen, C. Deng, and C. Heil, *Density of Gabor frames.* Applied and Computational Harmonic Analysis 7 (1999), pp. 292-304.

[12] I. Daubechies, *The wavelet transform, time-freqeency localization and signal analysis,* IEEE Trans. Inform. Theory, 36 (1990), pp. 961-1005.

[13] I. Daubechies, *Ten Lectures on Wavelets*,CBS-NSF Regional Conferences in Applied Mathematics, 61, SIAM,(1992).MR 93e:42045

[14] R. J. Duffin and A. C. Shaffer, *A class of nonharmonic Fourier Series,* Trans. Amer. Math. Soc., 72 (1952), pp. 341-366.

[15] H. G. Feichtinger, and A. J. E. M. Janssen *Validity of WH-frame conditions depends on lattice parameters.* Appl. Comp. Harm. Anal. 8 (2000), pp. 104-112.

[16] J. E. Littlewood, *Some problems in Real and complex Analysis*, Heath Mathematical Monographs, Lexington, Massachusetts, 1968.

[17] Y. Meyer, *Ondelettes et operateurs I, Hermann editeurs des sciences et des arts*, 1990; Eng. trsansl, *Wavelets and Operators*, Camb. Studies in Adv. Math., 37, 1992. MR 93i:42002

[18] W. Sun and X. Zhou, *On the stability of Gabor frames* Adv. in Appl. Math. 26 (2001), pp. 181-191.

[19] B. Sz-Nagy, *Expansion theorems of Paley-Wiener type,* Duke Math. J., 14 (1947), pp. 975-978.