

# Total Curvature, Ropelength and Crossing Number of Thick Knots

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ABSTRACT. We first study the minimum total curvature of a knot when it is embedded on the cubic lattice. Let  $\mathcal{K}$  be a knot or link with a lattice embedding of minimum total curvature  $\tau(\mathcal{K})$  among all possible lattice embeddings of  $\mathcal{K}$ . We show that there exist positive constants  $c_1$  and  $c_2$  such that  $c_1 \sqrt{Cr(\mathcal{K})} \leq \tau(\mathcal{K}) \leq c_2 Cr(\mathcal{K})$  for any knot type  $\mathcal{K}$ . Furthermore we show that the powers of  $Cr(\mathcal{K})$  in the above inequalities are sharp hence cannot be improved in general. Our results and observations show that lattice embeddings with minimum total curvature are quite different from those with minimum or near minimum lattice embedding length. In addition, we discuss the relationship between minimal total curvature and minimal ropelength for a given knot type. At the end of the paper, we study the total curvatures of smooth thick knots and show that there are some essential differences between the total curvatures of smooth thick knots and lattice knots.

## 1. Introduction

An essential field in the area of geometric knot theory concerns questions about the behavior of thick knots. One such question is about the ropelength of a knot. That is, if we are to tie a given knot with a rope of unit thickness, how long does the rope needs to be? In particular, for a given knot  $\mathcal{K}$  whose crossing number is  $Cr(\mathcal{K})$ , can we express the ropelength  $L(\mathcal{K})$  of  $\mathcal{K}$  as a function of  $Cr(\mathcal{K})$ ? Or alternatively, can we find a lower and upper bound of  $L(\mathcal{K})$  in terms of  $Cr(\mathcal{K})$ ? For a general lower bound, it is shown in [3, 4] that there exists a constant  $a > 0$  such that for any knot type  $\mathcal{K}$ ,  $L(\mathcal{K}) \geq a \cdot (Cr(\mathcal{K}))^{3/4}$ . The constant  $a$  is estimated to be at least 2.135 in [25]. For a general upper bound on the ropelength of a knot, it is known that there exists a constant  $c > 0$  such that for any knot  $\mathcal{K}$  the ropelength of  $\mathcal{K}$  is at most  $c \cdot (Cr(\mathcal{K}))^{3/2}$  [15]. However, it is not known whether there exist knot families whose ropelength grows faster than  $O(Cr(\mathcal{K}))$ . It is only known [11] that for any real number  $p$  such that  $3/4 \leq p \leq 1$ , there exists a family of knots  $\{\mathcal{K}_n\}$  with the property that  $Cr(\mathcal{K}_n) \rightarrow \infty$  (as  $n \rightarrow \infty$ ) such that  $L(\mathcal{K}_n) = O(Cr(\mathcal{K}_n)^p)$ .

In this paper we ask similar questions about the relationship of the minimal total curvatures of thick knots and their crossing numbers. First we study this

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question for knots embedded on the cubic lattice. We show that there exist positive constants  $c_1$  and  $c_2$  such that  $c_1\sqrt{Cr(\mathcal{K})} \leq \tau(\mathcal{K}) \leq c_2Cr(\mathcal{K})$  for any knot type  $\mathcal{K}$ , among a few other results. We will then study the minimal total curvature of smooth thick knots. In the case of smooth knots the question about minimal total curvature can be answered in terms of the bridge index of  $\mathcal{K}$  if the length of the embedding is of no concern. In this respect there is an essential difference between the case of smooth thick embeddings and lattice embeddings of knots. However, in the case that the length of the thick embedding is limited, the minimum total curvature of the thick embeddings becomes a difficult problem and its general behavior remains largely unknown.

Questions about thick knots are motivated by applications of knot theory in fields such as biology and physics since such information plays an important role in studying the effect of topological entanglement in subjects such as circular DNAs and long chain polymers, where knots occur and cannot be treated as volumeless curves [7, 16, 17, 18, 20, 21].

## 2. Bounds on the Total Curvature of Lattice Knots

The simple cubic lattice is a graph in  $R^3$  whose vertices are all points with coordinates  $(x, y, z)$  where  $x, y,$  and  $z$  are integers and whose edges are the unit length line segments connecting the adjacent vertices. Let  $\mathcal{K}$  be a knot type and let  $P_{\mathcal{K}}$  be a lattice realization of  $\mathcal{K}$ . Let  $L(P_{\mathcal{K}})$  be the length of  $P_{\mathcal{K}}$  and let  $S(P_{\mathcal{K}})$  be the number of straight segments in  $P_{\mathcal{K}}$ . In other word, there are  $S(P_{\mathcal{K}})$  right angles in  $P_{\mathcal{K}}$ . If we replace each right angle by a suitable quarter circle, then  $P_{\mathcal{K}}$  becomes a smooth knot with thickness at least  $1/2$  and the total curvature of this smooth representation of  $\mathcal{K}$  has total curvature  $\frac{\pi}{2}S(P_{\mathcal{K}})$ . This leads to the following definition.

$$\tau_m(\mathcal{K}) = \min\left\{\frac{\pi}{2}S(P_{\mathcal{K}}) : L(P_{\mathcal{K}}) \leq m\right\}.$$

In particular, we let

$$\tau(\mathcal{K}) = \min\left\{\frac{\pi}{2}S(P_{\mathcal{K}}) : L(P_{\mathcal{K}}) < \infty\right\}.$$

In the rest of the paper, whenever we use the term  $\tau_m(\mathcal{K})$ , it is always assumed that  $m$  is big enough so that  $\mathcal{K}$  may be realized on the simple cubic lattice with a total length of  $m$  and  $\tau_m(\mathcal{K})$  is defined. It is immediate from the definitions above that  $\tau(\mathcal{K}) \leq \tau_m(\mathcal{K}) \leq \tau_n(\mathcal{K})$  for any  $n < m$ .

**2.1. The case of lower bounds.** In [26], it is shown that if  $Cr(\mathcal{K})$  is the crossing number of  $\mathcal{K}$  and  $b(\mathcal{K})$  is the bridge index of  $\mathcal{K}$ , then we have  $S(P_{\mathcal{K}}) \geq (3 + \sqrt{9 + 8Cr(\mathcal{K})})/2$  and  $S(P_{\mathcal{K}}) \geq 6b(P_{\mathcal{K}})$  (for any given  $m$  for which at least one  $P_{\mathcal{K}}$  can be made with length  $m$ ). While the bound  $6b(P_{\mathcal{K}})$  is shown to be sharp in [26] hence cannot be improved in general, the following theorem slightly improves the bound  $(3 + \sqrt{9 + 8Cr(\mathcal{K})})/2$ , but only for the coefficients in the bound, not the  $O(\sqrt{Cr(\mathcal{K})})$  order.

**THEOREM 2.1.** *If  $\mathcal{K}$  is a non-trivial knot type, then we have  $\tau_m(\mathcal{K}) \geq \frac{3\pi}{2}(1 + \sqrt{Cr(\mathcal{K}) + 1})$  for any  $m$  for which  $\tau_m(\mathcal{K})$  is defined.*

REMARK 2.2. Notice that the bound given in Theorem 2.1 is larger than the  $(3 + \sqrt{9 + 8Cr(\mathcal{K})})\pi/4$  bound given in [26] in general. Furthermore, if  $\mathcal{K}$  is such that  $b(\mathcal{K}) < (1 + \sqrt{Cr(\mathcal{K}) + 1})/2$  (and there are plenty of knots with this property), then the result above would provide a larger lower bound than the  $3b(\mathcal{K})\pi$  bound.

PROOF. Let  $P_{\mathcal{K}}$  be a lattice realization of  $\mathcal{K}$  with  $n$  straight segments of length  $m \geq n$ . Let  $x_0, y_0$  and  $z_0$  be the number of straight segments in  $P_{\mathcal{K}}$  that are parallel to the  $x, y$  and  $z$  axis respectively. Call a straight segment an  $x, y$  or  $z$  segment if it is parallel to the  $x, y$  or  $z$  axis respectively. W. l. o. g. assume that  $z_0$  is the maximum of the three values  $x_0, y_0$  and  $z_0$ , then we have  $x_0 + y_0 \leq \frac{2n}{3}$ . A suitable small deformation of  $P_{\mathcal{K}}$  will yield a projection of  $P_{\mathcal{K}}$  in which all the crossings are caused by the straight segments parallel to the  $x$  or  $y$  axis. Furthermore, since any  $x$  straight segment is connected to (either directly or by a sequence of  $z$  and  $x$  segments) two  $y$  segments (otherwise the embedding will only contain  $x$  and  $z$  segments and  $\mathcal{K}$  would be trivial since the embedding is then planar), it will not have any crossing points with these two  $y$  segments. In other word, each  $x$  segment may have at most  $y_0 - 2$  crossings. Therefore, the total number of crossings we may have is at most  $x_0(y_0 - 2)$ . Similarly, the total number of crossings is also at most  $y_0(x_0 - 2)$ . Combining these two bounds yields  $x_0y_0 - (x_0 + y_0) \geq Cr(\mathcal{K})$ . This is as large as possible if  $x_0 = y_0 = n/3$ , so  $(n/3 - 1)^2 \geq Cr(\mathcal{K}) + 1$ . Solving this for  $n$  yields  $n \geq 3(1 + \sqrt{Cr(\mathcal{K}) + 1})$ . The result now follows.  $\square$

While  $3b(\mathcal{K})\pi$  is shown to be a sharp lower bound for  $\tau_m(\mathcal{K})$  [26],  $\frac{3\pi}{2}(1 + \sqrt{Cr(\mathcal{K}) + 1})$  is probably not a sharp lower bound for  $\tau_m(\mathcal{K})$ . (This is itself a hard and interesting question.) However, we will show that it is a sharp lower bound in the  $O(\sqrt{Cr(\mathcal{K})})$  sense for  $\tau_m(\mathcal{K})$  provided that  $m$  is large enough. In particular, it is sharp for  $\tau(\mathcal{K})$ . This can be done by showing that there exists a family of (infinitely many) knots  $\{\mathcal{K}_n\}$  such that  $\lim_{n \rightarrow \infty} Cr(\mathcal{K}_n) = \infty$  and  $\tau(\mathcal{K}_n) \leq a\sqrt{Cr(\mathcal{K}_n)}$  for some constant  $a > 0$ . This is implied by the following theorem.

THEOREM 2.3. *Let  $n \geq 2$  be an integer, then the  $(n^2, n^2 + 1)$  torus knot  $\mathcal{T}_n$  can be realized by a lattice polygon whose total curvature is  $\frac{3\pi}{2}(1 + 2\sqrt{Cr(\mathcal{T}_n) + 1})$ . Moreover*

$$3\pi < \frac{\tau(\mathcal{T}_n)}{\sqrt{Cr(\mathcal{T}_n)}} \leq \frac{9\sqrt{3}\pi}{2\sqrt{5}} < 3.5\pi$$

PROOF. We prove the theorem by actually constructing a lattice embedding of  $\mathcal{T}_n$  and counting the number of right angle turns in the embedding. First notice that  $\mathcal{T}_n$  can be represented by a homogeneous closed braid of  $n^2 + 1$  strings. (A *homogeneous* braid is a braid where every generator of the braid either occurs always with positive or always with negative exponent.) Figure 1 shows a homogeneous braid with 5 strings and two layers.

We will now try to embed  $\mathcal{T}_n$  in the lattice using a small number of turns. We will first arrange the  $n^2 + 1$  strings so that their starting points are at the same  $z$  level ( $z = n^2 + 1$ ) and the strings are going downward in the  $z$  direction. Let  $(0, 0, n^2 + 1)$  be the starting point of the first string,  $(-1, 1, n^2 + 1)$  be the starting point of the second string,  $(-2, 2, n^2 + 1)$  be the starting point of the third string, and so on (so the starting point of the last string would be  $(-n^2, n^2, n^2 + 1)$ ). First

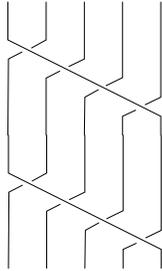


FIGURE 1. A homogeneous braid with 5 strings and two layers.

we let each string go down one unit in  $z$  direction. We will realize the first layer of the corresponding homogeneous braid by letting each string (except the first string) go down straight for one step. The first string will make a turn toward the positive  $y$  direction, go straight until it reaches the point  $(0, n^2 + 1, n^2)$ , from where it will take a turn towards the negative  $x$  direction, go straight until it reaches the point  $(-n^2 - 1, n^2 + 1, n^2)$ , then it will take a step downward to end at the point  $(-n^2 - 1, n^2 + 1, n^2 - 1)$ . Notice that this requires only three right angle turns. For the next layer, we will simply repeat the above operation since we are now starting at  $n^2 + 1$  starting points whose relative positions are exactly like the ones in the first step, i.e., all strings except the new left most string move one step down in negative  $z$  direction. The new leftmost string moves from  $(-1, 1, n^2 - 1)$  to  $(-1, n^2 + 2, n^2 - 1)$  in positive  $y$ -direction, from  $(-1, n^2 + 2, n^2 - 1)$  to  $(-n^2 - 2, n^2 + 2, n^2 - 1)$  in negative  $x$ -direction, and finally moves one step down to end at  $(-n^2 - 2, n^2 + 2, n^2 - 2)$ . After repeating this  $n^2$  times we now have a braid with endpoints  $(-n^2, n^2, 0)$ ,  $(-n^2 - 1, n^2 + 1, 0)$ ,  $(-n^2 - 2, n^2 + 2, 0)$ ,  $\dots$ ,  $(-2n^2, 2n^2, 0)$ . So far the total number of right angles in the construction (before the braid is closed) is  $3n^2$  since the last string does not involve any turns. This embedding in the case  $n = 2$  is shown in Figure 2.

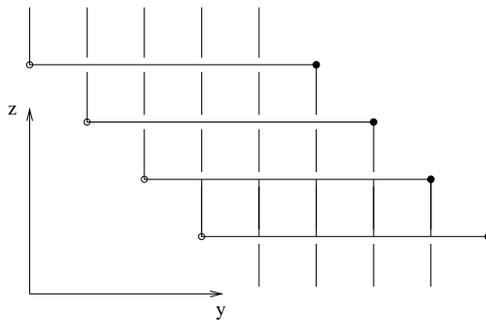


FIGURE 2. Realizing a homogeneous braid such that each layer uses only 3 right angle turns. The (solid) dot indicates where there is a right angle turn toward the negative  $x$  direction, which is the direction into the plane the figure is in.

We now close the braid as follows. From the point  $(-n^2 - k, n^2 + k, 0)$  we move in negative  $y$  direction to  $(-n^2 - k, k, 0)$ , then in  $z$  direction to  $(-n^2 - k, k, n^2 + 1)$ , followed by a segment in  $x$  direction to  $(-k, k, n^2 + 1)$ . Each of these path introduces 4 new right angles. We now have constructed the  $(n^2, n^2 + 1)$  torus knot with  $7n^2 + 4$  right angles, see Figure 3 for the case  $n = 2$  viewed in  $z$ -direction.

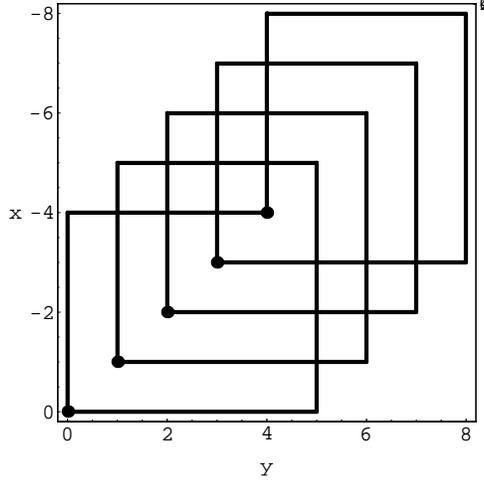


FIGURE 3. A projection into the  $xy$  plane of the embedding of the  $(4, 5)$  torus knot. The thick points are the initial  $n^2 + 1$  points in the construction.

However we can shorten the constructed lattice embedding of the  $(n^2, n^2 + 1)$  torus knot by moves that replace three consecutive sides of a rectangle in the embedding with the fourth side of the rectangle. During such a move one vertical segment is eliminated and another vertical segment is shortened. A view of such a move is shown in Figure 4. More precisely, the path

$$\begin{aligned} (-n^2 - k, n^2 + k, 0) &\longrightarrow (-n^2 - k, k, 0) \longrightarrow (-n^2 - k, k, n^2 + 1) \\ &\longrightarrow (-k, k, n^2 + 1) \longrightarrow (-k, k, n^2 - k) \end{aligned}$$

is replaced by

$$\begin{aligned} (-n^2 - k, n^2 + k, 0) &\longrightarrow (-n^2 - k, k, 0) \\ &\longrightarrow (-n^2 - k, k, n^2 - k) \longrightarrow (-k, k, n^2 - k) \end{aligned}$$

for  $k = 0, 1, 2, \dots, n^2 - 1$ . For the last path the move is slightly different: the path

$$\begin{aligned} (-n^2 - 1, 2n^2, 1) &\longrightarrow (-n^2 - 1, 2n^2, 0) \longrightarrow (2n^2, 2n^2, 0) \\ &\longrightarrow (-2n^2, n^2, 0) \longrightarrow (-2n^2, n^2, n^2) \end{aligned}$$

is replaced by

$$\begin{aligned} (-n^2 - 1, 2n^2, 1) &\longrightarrow (2n^2, 2n^2, 1) \\ &\longrightarrow (-2n^2, n^2, 1) \longrightarrow (-2n^2, n^2, n^2). \end{aligned}$$

The vertical segment eliminated is the vertical segment ending at a vertex marked by a circle in Figure 2. Note that these shortening moves do not change the projection in  $z$ -direction as shown in Figure 3. Each shortening move eliminates one right angle by eliminating one segment. Such a move can be carried out for all  $n^2 + 1$  paths used to close the braid and thus this will result in a lattice embedding of the  $(n^2, n^2 + 1)$  torus knot with  $6n^2 + 3$  segments and with  $6n^2 + 3$  right angles. Since  $Cr(\mathcal{T}_n) = n^4 - 1$ ,  $n^2 = \sqrt{Cr(\mathcal{T}_n) + 1}$ . This gives a total curvature for the lattice polygon of  $\frac{3\pi}{2}(1 + 2\sqrt{Cr(\mathcal{T}_n) + 1})$ . The inequality is derived as follows: For the upper bound note that the crossing number of the  $(n^2, n^2 + 1)$  torus knot is  $n^4 - 1$ , see [24]. Since  $n \geq 2$  the crossing number  $Cr(\mathcal{T}_n) \geq 15$ . Under these assumptions  $\frac{3\pi}{2} \frac{(1+2\sqrt{Cr(\mathcal{T}_n)+1})}{\sqrt{Cr(\mathcal{T}_n)}} \leq \frac{9\sqrt{3}\pi}{2\sqrt{5}}$ . To show the lower bound we use the fact that the bridge index of the  $(n^2, n^2 + 1)$  torus knot is  $b(\mathcal{T}_n) = n^2$ , see [24]. Thus  $\tau(\mathcal{T}_n) \geq 3\pi b(\mathcal{T}_n) = 3\pi n^2$ .  $\square$

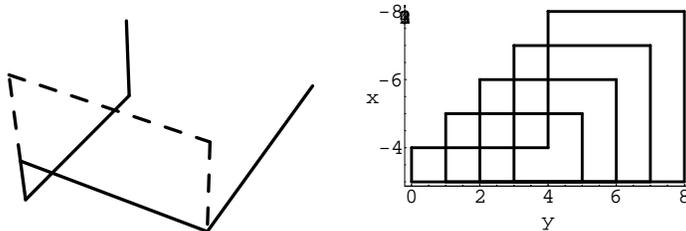


FIGURE 4. The move on the left reduces the lattice embedding by eliminating one segment (and a right angle turn). The dashed path is replaced by a single segment and the new path uses only 3 right angle turns. On the right a projection of a lattice embedding of the  $(4, 5)$  torus knot that is further reduced in length but has the same curvature as the embedding constructed in Theorem 2.3.

REMARK 2.4. The construction in Theorem 2.3 is almost optimal in realizing the minimal total curvature. Since  $Cr(\mathcal{T}_n) = n^4 - 1$  (see [24]),  $\frac{3\pi}{2}(1 + 2\sqrt{Cr(\mathcal{T}_n) + 1}) = \frac{3\pi}{2}(1 + 2n^2)$  and we have  $3\pi n^2 \leq \tau(\mathcal{T}_n) \leq 3\pi n^2 + \frac{3\pi}{2}$ . So the upper and lower bounds differ only by 3 right angles! If one uses Theorem 2.1 to obtain the lower bound in the inequality of Theorem 2.3 then we only obtain  $\frac{3\pi}{2} < \frac{\tau(\mathcal{T}_n)}{\sqrt{Cr(\mathcal{T}_n)}}$ . Therefore the lower bound  $\frac{3\pi}{2}(1 + \sqrt{Cr(\mathcal{T}_n) + 1})$  which holds for all lattice knots given in Theorem 2.1 is quite good and could only be improved by a factor of at most 2. The embedding of Theorem 2.3 is almost optimal with regards to the total curvature, however there are additional moves that can further shorten the length of the embedding of the  $(n^2, n^2 + 1)$  torus knot constructed in Theorem 2.3 without changing the number of segments. Therefore they have no effect on the

curvature. For example, one such move replaces the three segments

$$\begin{aligned} &(-n^2 - k, k, n^2 - k) \longrightarrow (-k, k, n^2 - k) \\ &\longrightarrow (-k, -n^2 - 1 - k, n^2 - k) \longrightarrow (n^2 + 1 + k, n^2 + 1 + k, n^2 - k) \end{aligned}$$

by the three segments

$$\begin{aligned} &(-n^2 - k, k, n^2 - k) \longrightarrow (-n^2 + 1, k, n^2 - k) \\ &\longrightarrow (-n^2 + 1, n^2 + 1 + k, n^2 - k) \longrightarrow (-n^2 - 1 - k, n^2 + 1 + k, n^2 - k) \end{aligned}$$

for  $k = 0, 1, 2, \dots, n^2 - 1$ . A view of such an embedding of the  $(4, 5)$  torus knot is shown in Figure 4 on the right.

Observe that in the construction of the embedding given in the proof of Theorem 2.3, the emphasis is on minimizing the total curvature hence the total length of the construction is much larger than needed to just embed the knot in the cubic lattice. It is easy to see that the total length of the segments in the  $y$ -direction is  $2n^4 + 2n^2$ , which already exceeds  $Cr(\mathcal{T}_n) = n^4 - 1$ . This can be compared to the length of the cubic lattice embedding of  $\mathcal{T}_n$  constructed in [10], which is at most  $16n^3$ . However the total curvature of the embedding constructed in [10] is about  $5\pi n^2$ , which is much larger than the total curvature  $3\pi n^2$  of the embedding constructed in Theorem 2.3. This leads us to the following question: if we try to minimize the length of the embedding, what would be the effect on the total curvature? In other words, what can we say about  $\tau_{m_0}(\mathcal{K})$  if  $m_0$  is the minimum embedding length of  $\mathcal{K}$ ? This is a difficult question in general since not much is known about minimum length embeddings of knots. In [8] it is shown that the minimum length of a lattice embedding of the trefoil knot is 24. One such lattice realization constructed there is shown at the left side of Figure 5. Notice that this

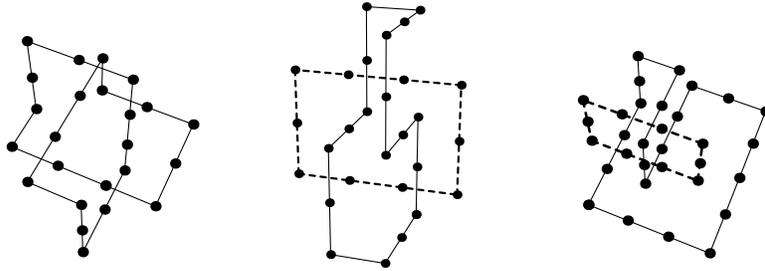


FIGURE 5. On the left is an embedding of a trefoil knot with 12 segments and length 24. In the middle is an embedding of the link  $4_1^2$  with 14 segments and length 28. On the right is an embedding of the link  $4_1^2$  with 13 segments and length 32.

lattice embedding has a total curvature  $6\pi$ , which is also the minimum total curvature of any lattice embedding of the trefoil. That is,  $\tau(\text{trefoil}) = \tau_{24}(\text{trefoil}) = 6\pi$ . This provides an example in which a minimum length lattice embedding of a knot is also a minimum total curvature lattice embedding of the knot. On the other hand, it is shown in [19] that the minimum length of a lattice embedding of  $4_1^2$  is 28. One such minimum length embedding is shown in the middle of Figure 5, which has a

total curvature  $7\pi$  since there are 14 straight segments there. In fact, one can check that all minimum length lattice embeddings of  $4_1^2$  have total curvature  $\geq 7\pi$  [19]. It follows that  $\tau_{28}(4_1^2) = 7\pi$ . However on the right in Figure 5 is a lattice embedding of  $4_1^2$  with length 32 and 13 segments. Thus  $\tau(4_1^2) \leq \tau_{32}(4_1^2) \leq 6.5\pi$ . (In fact we can prove that  $\tau(4_1^2) = 6.5\pi$ .) So no minimum length lattice embeddings of  $4_1^2$  can achieve the (overall) minimum total curvature at the same time. This provides an example in which no minimum length lattice embeddings of the (two-component) knot can achieve the (overall) minimum total curvature of the lattice embeddings of the knot. We suspect that this is the case for most knots, though this is obviously very difficult to prove.

**2.2. The case of upper bounds.** As we have seen in the last subsection, the general lower bound for  $\tau_m(\mathcal{K})$  is of the order  $O(\sqrt{Cr(\mathcal{K})})$ , hence one cannot expect an upper bound for  $\tau_m(\mathcal{K})$  that only involves  $b(\mathcal{K})$ . There are knots with arbitrarily large crossing numbers but with fixed bridge index, whose lattice embeddings must have large total curvature. Thus we will aim to prove an upper bound on curvature in terms of the crossing number of the knot. One immediate observation is that  $\tau_m(\mathcal{K})$  is bounded above by  $n\pi/2$  for any  $n$  such that a lattice embedding of  $\mathcal{K}$  with length  $n$  exists, since if the length of the embedding is  $n$ , then there are at most  $n$  right angle turns in the embedding. So an upper bound on the lattice embedding length of  $\mathcal{K}$  would lead to an upper bound of  $\tau(\mathcal{K})$ . Since the best known upper bound for the lattice embedding length of  $\mathcal{K}$  is of the order  $O((Cr(\mathcal{K}))^{3/2})$  [15], we can immediately conclude that  $\tau(\mathcal{K})$  is bounded above by  $O((Cr(\mathcal{K}))^{3/2})$ . However, we can actually do better than this as stated in the next theorem.

**THEOREM 2.5.** *For any knot  $\mathcal{K}$ ,  $\tau(\mathcal{K})$  is bounded above by  $O(Cr(\mathcal{K}))$ .*

**PROOF.** The proof of this theorem depends largely on the lattice realization of  $\mathcal{K}$  constructed in [15], which is based on the idea of finding a Hamiltonian cycle in a knot diagram, see Figure 6. The construction shown can be carried out for

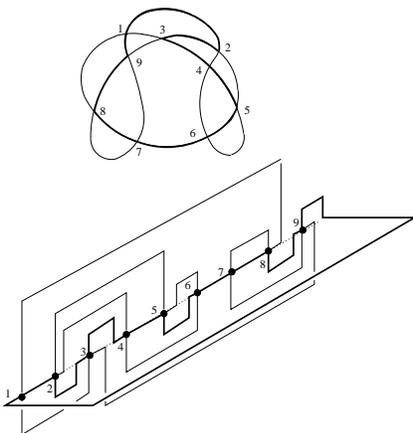


FIGURE 6. The top shows a projection of a 9 crossing knot which contains a Hamiltonian cycle (the thick curve). The bottom is an embedding of the diagram on the cubic lattice.

any knot diagrams of any size as long as there is a Hamiltonian cycle in the knot diagram. We can estimate the number of straight segments using Figure 6 as a guide. Let us assume that  $n = Cr(\mathcal{K})$  and we are given a minimum knot diagram  $D$  of  $\mathcal{K}$  that is Hamiltonian. The Hamiltonian cycle is then embedded into the cubic lattice as shown in Figure 6. Notice that this embedding will create at most  $3n + 4$  right angle turns (at most 3 right angle turns corresponding to each crossing points in  $D$ ). Each of the  $n$  arcs of  $D$  not on the Hamiltonian cycle is then embedded using 2, 3, or 4 additional right turns. These additional right turns can also be grouped into subsets of at most 3 per crossing point in the diagram  $D$ . Thus we have at most  $6n + 4$  right angle turns at points not corresponding to the crossing points in  $D$ . Figure 6 shows the knot diagram and the lattice embedding without the over/under information at the crossings of the diagram. As a final step we need to add the over- and under-passes at the crossings in order to recover the knot  $\mathcal{K}$ . This can be done by moving the vertical segment at each crossing either one unit forward or backward as needed as shown in Figure 7. This will add an additional

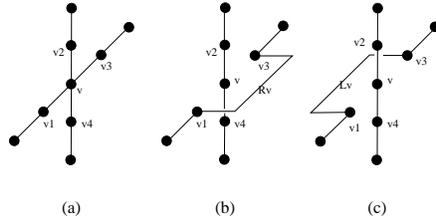


FIGURE 7. Assigning the over- or under-pass at a crossing in a lattice embedding.

4 right angles at each crossing. Thus the total number of turns will be bounded above by  $10n + 4$ . Although not every knot diagram exhibits a Hamiltonian cycle, it is shown in [15] that there exists a Hamiltonian diagram for every knot or link  $\mathcal{K}$  with at most  $4Cr(\mathcal{K})$  crossings. Thus we obtain an upper bound of at most  $40Cr(\mathcal{K}) + 4$  for the number of right turns needed to embed any knot  $\mathcal{K}$  on the cubic lattice. In other words we have  $\tau(\mathcal{K}) \leq 2\pi(10Cr(\mathcal{K}) + 1)$  in general.  $\square$

Our next theorem shows that this bound is sharp up to the order of  $Cr(K)$ . This result is implied by a theorem in [14], which says that there exists a family of infinitely many prime knots  $\{\mathcal{K}_n\}$  such that  $\lim Cr(\mathcal{K}_n) \rightarrow \infty$  as  $n \rightarrow \infty$  and  $b(\mathcal{K}_n) \geq Cr(\mathcal{K}_n)/3$ . Thus  $\tau(\mathcal{K}_n) \geq \pi Cr(\mathcal{K}_n)$ .

**THEOREM 2.6.** *Let  $\mathcal{K}_n$  denote the knot whose Conway symbol is  $(\underbrace{3, 3, \dots, 3}_n, 2)$  ( $n \geq 2$ ) (the case  $n = 3$  is shown in Figure 8). Then  $\mathcal{K}_n$  can be realized by a lattice polygon whose total curvature is  $2\pi(Cr(\mathcal{K}_n) + 1)$ . Moreover*

$$\pi < \frac{\tau(\mathcal{K}_n)}{Cr(\mathcal{K}_n)} \leq 9\pi/4$$

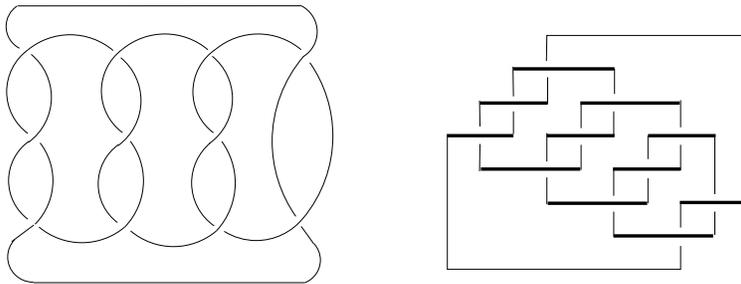


FIGURE 8. On the left is the Montesinos knot  $\mathcal{K}_3$ , on the right is the projection of a lattice embedding of it. The thick segments are in the plane  $z = 1$ , while the thin lines are in the plane  $z = 0$ .

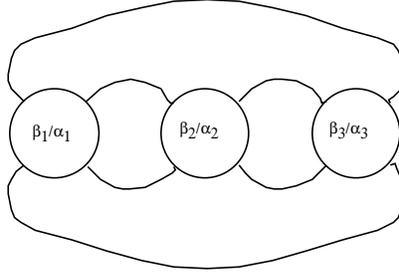
PROOF. The knot  $\mathcal{K}_3$  and a lattice embedding of it (projected onto the  $xy$ -plane) are shown in Figure 8. It is an alternating knot and in Figure 8 the thick segments are in the plane  $z = 1$ , while the thin lines are in the plane  $z = 0$ . The embedding scheme of Figure 8 can be extended to any member of the knot family  $\mathcal{K}_n$ . It is easy to see that this embedding has a total number of  $12(n + 1)$  straight segments. Therefore its total curvature is  $6\pi(n + 1)$ . On the other hand  $Cr(\mathcal{K}_n) = 3n + 2$  (since it is an alternating knot) and thus the total curvature of this embedding can be written as  $2\pi(Cr(\mathcal{K}_n) + 1)$ . In [14] it is shown that the bridge index of  $\mathcal{K}_n$  is  $n + 1$  and thus the total curvature of any embedding of  $\mathcal{K}_n$  must be at least  $3\pi(n + 1) = \pi(Cr(\mathcal{K}_n) + 1)$ . It follows that  $\pi(Cr(\mathcal{K}_n) + 1) \leq \tau(\mathcal{K}_n) \leq 2\pi(Cr(\mathcal{K}_n) + 1)$ . The inequalities of the theorem now follows since  $Cr(\mathcal{K}_n) \geq 8$  for any  $n \geq 2$ .  $\square$

REMARK 2.7. The knot  $\mathcal{K}_n$  in the above theorem is a particular case of Montesinos knots. A Montesinos knot is a knot (or link) obtained by concatenating  $n$  rational tangles, as shown in Figure 9 for the case of  $n = 3$ . In general  $n$  can be any integer greater or equal to two. The label  $\frac{\beta_i}{\alpha_i}$  ( $1 \leq i \leq n$ ) stands for the rational tangle classified by that same fraction  $\frac{\beta_i}{\alpha_i}$ . A treatment of Montesinos knots can be found in [1]. Assuming that  $n \geq 3$  and that for all  $i$ ,  $\alpha_i \geq 2$ , it is shown in [2] that the bridge index of any such Montesinos knot  $\mathcal{K}$  is exactly  $n$ . While the knot  $\mathcal{K}_n$  is alternating, Montesinos knots are not always alternating in general. If tangles of different signs are substituted into the diagram in Figure 9, then the diagram can be either non-alternating or non-minimum.

### 3. Minimum Total Curvature as a Power of the Crossing Number

Using the above examples of torus knots and Montesinos knots we can construct knot families whose lattice embedding minimum total curvatures behave as  $O((Cr(K))^p)$  for any given  $p$  between  $1/2$  and  $1$ . This is summarized in the following theorem:

THEOREM 3.1. *For any real number  $p$  such that  $1/2 \leq p \leq 1$ , there exists a family of infinitely many knots  $\{K_m\}$  with the property that  $Cr(K_m) \rightarrow \infty$  (as  $m \rightarrow \infty$ ) such that the total curvature of any realization of  $K_m$  on the cubic lattice*

FIGURE 9. The diagram of a Montesinos knot with  $n = 3$ .

will grow at least linearly with respect to  $(Cr(K_m))^p$ . Moreover there exists a lattice embedding  $P_m$  of  $K_m$  such that the total curvature of  $P_m$  grows linearly with respect to  $(Cr(K_m))^p$ .

PROOF. The construction of the knot  $K_m$  uses a connected sum of a Montesinos knot used in Theorem 2.6 and of a  $(m^2, m^2 + 1)$  torus knot used in Theorem 2.3. Let  $K_1$  be the  $(m^2, m^2 + 1)$  torus knot  $\mathcal{T}_m$ , then  $Cr(K_1) = c_1 = m^4 - 1$ , see [24]. Let  $p$  be a given number such that  $1/2 \leq p \leq 1$  and let  $K_2$  be the Montesinos knot  $\mathcal{K}_n = (\underbrace{3, 3, \dots, 3}_n, 2)$ , where  $n$  is so chosen that  $c_1^p - 3 \leq 3n + 2 \leq c_1^p$ . Let  $K_m =$

$K_1 \# K_2$ . Recall from Theorem 2.6 that  $Cr(K_2) = c_2 = 3n + 2$ . By a result from [9],  $Cr(K_m) = c_1 + c_2$ . It follows that  $c_1 < Cr(K_m) = c_1 + 3n + 2 \leq c_1 + c_1^p \leq 2c_1$  since  $p \leq 1$ . On the other hand, the bridge index of  $K_1$  is  $m^2$  since the bridge index of a torus knot  $T(a, b)$  is  $\min(a, b)$  and the bridge index of  $K_2$  is  $n + 1$  ([14]). Thus  $b(K_m) = b(K_1) + b(K_2) - 1 = m^2 + n$  since the bridge index is minus one additive. This shows that the total curvature of any lattice embedding of  $K_m$  is at least  $3\pi b(K_m) = 3\pi(m^2 + n) \geq 3\pi(\sqrt{c_1} + 1 + 1/3(c_1^p - 5)) > \pi c_1^p \geq \frac{\pi}{2^p} Cr(K_m)^p$ .

We will now prove the second half of the theorem. From the lattice representation of  $K_2$  as shown in Figure 8 we can estimate that  $\tau(K_2) \leq 2\pi(Cr(K_2) + 1) \leq 6\pi(n + 1)$ . On the other hand, we know from Theorem 2.3 that there exists a lattice embedding of  $K_1$  (the  $(m^2, m^2 + 1)$  torus knot) with a total curvature at most  $\frac{3\pi}{2}(1 + 2\sqrt{Cr(K_1) + 1})$ . So finally we have  $\tau(K_m) \leq \tau(K_1) + \tau(K_2) + 2\pi \leq \frac{3\pi}{2}(1 + 2\sqrt{Cr(K_1) + 1}) + 6\pi(n + 1) + 2\pi = O((Cr(K_m))^p)$ .  $\square$

REMARK 3.2. The above results can also be applied to a variety of other knots and links. For example,  $K_1$  can be replaced by other torus knots and links whose minimum lattice embedding total curvature follows the 1/2-power power as in Theorem 2.3 and the knot  $K_2$  can be replaced by any other *deficiency zero* (see [9]) knots whose bridge indices grow linearly with respect to its crossing numbers (see the remark at the end of the last section).

#### 4. Minimal Total Curvature and Minimum Lattice Embedding Length

We have observed that some total curvature minimizers of a particular knot type on the lattice are quite different from the length minimizers of the same knot

on the lattice. Let  $L(\mathcal{K})$  be the shortest length for which a lattice embedding of  $\mathcal{K}$  with length  $L(\mathcal{K})$  exists. In this section we will explore the relationship between  $L(\mathcal{K})$  and  $\tau(\mathcal{K})$ . (Note that  $L(\mathcal{K})$  and  $\tau(\mathcal{K})$  will most likely arise from different lattice embeddings of  $\mathcal{K}$ .) More specifically, we will study the ratio  $L(\mathcal{K})/\tau(\mathcal{K})$ . It is obvious that  $\tau(\mathcal{K}) \leq L(\mathcal{K}) \cdot \pi/2$  since the number of straight segments in any length minimizers of  $\mathcal{K}$  on the lattice is at most  $L(\mathcal{K})$ . It follows that  $2/\pi \leq L(\mathcal{K})/\tau(\mathcal{K})$ . On the other hand, we know that  $\tau(\mathcal{K}) > 3\pi\sqrt{Cr(\mathcal{K})}/2$  (from Theorem 2.1) and  $L(\mathcal{K}) \leq a(Cr(\mathcal{K}))^{3/2}$  for some positive constant  $a$ . Thus  $L(\mathcal{K})/\tau(\mathcal{K}) < (a/3\pi)Cr(\mathcal{K})$ . This is summarized in the following theorem.

**THEOREM 4.1.** *There exist constants  $c_3 \geq 2/\pi$  and  $c_4$  such that*

$$c_3 \leq L(\mathcal{K})/\tau(\mathcal{K}) \leq c_4 Cr(\mathcal{K})$$

for any given knot type  $\mathcal{K}$ .

Let us now again consider the family of torus knots  $\mathcal{T}_n = T(n^2, n^2 + 1)$  and the families of Montesinos knots  $\mathcal{K}_n$  studied in the previous sections. From [10, 25] we know that  $2.13(n^4 - 1)^{3/4} \leq L(\mathcal{T}_n) \leq 16n^3$  and by Theorem 2.3,  $3\pi n^2 \leq \tau(\mathcal{T}_n) \leq \frac{3\pi}{2} + 3\pi n^2$ . From [11, 14] we know that  $6n + 4 \leq L(\mathcal{K}_n) \leq 26n + 14$  and by Theorem 2.3,  $3\pi(n + 1) \leq \tau(\mathcal{K}_n) \leq 6\pi(n + 1)$ . We have shown the following:

**THEOREM 4.2.** *Let  $n \geq 2$  be an integer, then*

(i) *for the  $(n^2, n^2 + 1)$  torus knot  $\mathcal{T}_n$*

$$\frac{2.13}{7\pi} n \leq \frac{L(\mathcal{T}_n)}{\tau(\mathcal{T}_n)} \leq \frac{16}{3\pi} n;$$

(ii) *for the Montesinos knot  $\mathcal{K}_n$*

$$\frac{2}{3\pi} \leq \frac{L(\mathcal{K}_n)}{\tau(\mathcal{K}_n)} \leq \frac{26}{3\pi}.$$

These two families of knots show two different behaviors of the quotient  $\frac{L(\mathcal{K}_n)}{\tau(\mathcal{K}_n)}$ : one behaves as a power  $O(Cr(\mathcal{K}_n)^{1/4})$  and one as a constant. By taking suitable connected sums of knots we can construct knot families for which this quotient takes on any power of  $Cr(\mathcal{K})^p$  for  $0 \leq p \leq 1/4$ .

**THEOREM 4.3.** *For any real number  $p$  such that  $0 \leq p \leq 1/4$ , there exists a family of infinitely many knots  $\{K_m\}$  with the property that  $Cr(K_m) \rightarrow \infty$  (as  $m \rightarrow \infty$ ) such that the quotient  $\frac{L(K_m)}{\tau(K_m)}$  behaves as  $O(Cr(K_m)^p)$ .*

**PROOF.** Since the argument is similar to previous arguments we will only give a brief outline of the proof. Again the construction of the knot  $K_m$  uses a connected sum of a Montesinos knot used in Theorem 2.6 and of a  $(m^2, m^2 + 1)$  torus knot used in Theorem 2.3. Let  $K_1$  be the  $(m^2, m^2 + 1)$  torus knot  $\mathcal{T}_m$ , then  $Cr(K_1) = c_1 = m^4 - 1$ , see [24]. Let  $p$  be a given number such that  $0 \leq p \leq 1/4$  and let  $K_2$  be the Montesinos knot  $\mathcal{K}_n = \underbrace{(3, 3, \dots, 3)}_n, 2$ , where  $n$  is so chosen that

$c_1^{3-p} - 3 \leq 3n + 2 \leq c_1^{3-p}$ . Let  $K_m = K_1 \# K_2$ . The curvature of  $K_m$  will be an  $O(m^2 + n) = O(m^3 + n)$ . The length of  $K_m$  will be an  $O(m^3 + n) = O(m^3)$  and therefore the quotient of length over curvature will behave as an  $O(m^p)$ .  $\square$

Whether there exist a family of knots for which the quotient of minimum length over minimum total curvature will behave as an  $O(m^p)$  for some  $p > 1/4$  remains an open question. A positive answer to this question will require the knowledge of additional knot families whose minimum lattice embedding length can be estimated in terms of a power of  $Cr(\mathcal{K})$  as well as some estimate for the minimum total curvature of the lattice embeddings. For example there might be knot families whose ropelength is longer than the family of Montesinos knots but whose bridge index is much smaller so the minimum total curvature of its lattice embeddings may not be as large. On the other hand, a negative answer to this question will require the proof that a general upper bound on the ropelength of a knot  $\mathcal{K}$  is at most  $c \cdot (Cr(\mathcal{K}))^{5/4}$  first. While there are some optimistic signs that this may be true (for example, this has been proven to be true for all alternating braid knots in [12]), it remains an open question at this stage.

## 5. Minimum Total Curvature of Smooth Thick Knots

In this section, we explore the minimum total curvature of smooth thick knots. It turns out that the minimum total curvature of smooth thick knots behaves quite differently from that of the lattice knots.

There are different ways to define the thickness of a knot ([6], [13], [22]). In this paper, we will be using the so called *disk thickness* defined in [22] as follows: Let  $K$  be a  $C^2$  knot. A number  $r > 0$  is said to be “nice” if for any  $x, y$  on  $K$ , we have  $D(x, r) \cap D(y, r) = \emptyset$ , where  $D(x, r)$  and  $D(y, r)$  are the discs of radius  $r$  centered at  $x$  and  $y$  which are normal to  $K$ . The *disk thickness* of  $K$  is defined to be the supremum over all the nice  $r$ ’s. It is shown in [6] that the disk thickness definition can be extended to all  $C^{1,1}$  curves. For any knot with a given thickness, we can always re-scale the length of the knot so that the thickness becomes 1. For the sake of convenience, we will always deal with thick knots with unit thickness in this paper.

Let  $\mathcal{K}$  be a knot type and let  $K$  be a unit thickness knot of type  $\mathcal{K}$  and let  $L(K)$  be the length of  $K$ . The total curvature of  $K$  is denoted by  $t(K)$ . For a fixed positive number  $m$  that is large enough so that  $\mathcal{K}$  may be realized by a unit thickness knot  $K$  of length  $m$ , let

$$\tau'_m(\mathcal{K}) = \inf\{t(K) : L(K) \leq m\}.$$

In particular, we let

$$\tau'(\mathcal{K}) = \inf\{t(K) : L(K) < \infty\}.$$

As before, whenever the term  $\tau'_m(\mathcal{K})$  is used, it is assumed that  $m$  is large enough so that a *thick realization*  $K$  of  $\mathcal{K}$  (that is, a unit thickness knot with knot type  $\mathcal{K}$ ) with length at most  $m$  exists. Again, it is immediate from the definitions above that  $\tau'(\mathcal{K}) \leq \tau'_m(\mathcal{K}) \leq \tau'_n(\mathcal{K})$  for any  $n < m$ .

**5.1. The case of  $\tau'$ .** It turns out this is the most unsurprising case since the following result is implied by the following well known result [23, 27].

**THEOREM 5.1.** *For any knot type  $\mathcal{K}$ , we have  $\tau'(\mathcal{K}) = 2\pi b(\mathcal{K})$ , where  $b(\mathcal{K})$  is the bridge index of  $\mathcal{K}$ . Furthermore, if  $K$  is a ropelength minimizer of  $\mathcal{K}$ , then the total curvature of  $K$  is greater than  $2\pi b(\mathcal{K})$ . In other word, the ropelength minimizer of a smooth knot is never a total curvature minimizer at the same time.*

**PROOF.** It is well known that the total curvature of any smooth knot  $K$  of knot type  $\mathcal{K}$  is bounded below by  $2\pi b(\mathcal{K})$  [23, 27] and this lower bound cannot be attained at any given tame knot. Since a ropelength minimizer is certainly a tame knot, the second half of the theorem follows. On the other hand, for any  $\epsilon > 0$ , there exists a smooth closed curve  $C$  with knot type  $\mathcal{K}$  such that the total curvature of  $C$  is at most  $2\pi b(\mathcal{K}) + \epsilon$  (such a closed curve  $C$  is very close to a planar curve). Since scaling does not change the total curvature of a curve, we can re-scale  $C$  so that it becomes a unit thickness knot  $K$ . This proves the first part of the theorem. Notice that in order to achieve a total curvature very close to  $2\pi b(\mathcal{K})$ , the curve  $C$  needs to be very flat which means that the unit thickness curve obtained from  $C$  after re-scaling will have a very large length.  $\square$

**5.2. The case of  $\tau'_m$ .** On the other hand, the case of  $\tau'_m$  is much more complicated and harder to deal with. While  $\tau'_m(\mathcal{K})$  is always larger than  $2\pi b(\mathcal{K})$  as stated in Theorem 5.1, it is not known whether we can bound  $\tau'_m(\mathcal{K})$  below by a function of  $Cr(\mathcal{K})$  that is independent of  $b(\mathcal{K})$  even when  $m$  is close to the minimum ropelength of  $\mathcal{K}$ . Since the (local) curvature of a unit thickness knot is bounded above by 1 ([22]), its total curvature is bounded above by its length. It follows that  $\tau'_m(\mathcal{K})$  is bounded above by  $m$ . Therefore, in the case that  $m$  is small, this would give us a meaningful bound. On the other hand, if  $m$  is very large, then  $\tau'_m(\mathcal{K})$  is very close to  $2\pi b(\mathcal{K})$  as shown in the above theorem. The problem is when  $m$  is large, but not large enough for the construction depicted in the proof of the above theorem to work. The following theorem summarizes what we know about this case at this stage.

**THEOREM 5.2.** (1) *There exist constants  $b_1 > 0$  and  $b_2 > 0$  such that if  $m \geq b_1 \cdot (Cr(\mathcal{K}))^{3/2}$ , then  $\tau'_m(\mathcal{K}) \leq b_2 \cdot Cr(\mathcal{K})$ ;*

(2) *There exist constants  $0 < b_3 < b_4$  such that for any given knot type  $\mathcal{K}$  and any  $m$  such that  $\tau'_m(\mathcal{K})$  is defined, we can find a  $p$  such that  $0 \leq p \leq 3/2$  and  $b_3 \cdot (Cr(\mathcal{K}))^p \leq \tau'_m(\mathcal{K}) \leq b_4 \cdot (Cr(\mathcal{K}))^p$ .*

**PROOF.** The proof of (1) relies on the lattice embedding of  $\mathcal{K}$  on the cubic lattice as described in [15] and Theorem 2.5. It is proven in [15] that there exists a constant  $b'_1 > 0$  such that any knot type  $\mathcal{K}$  can be embedded in the cubic lattice with a length at most  $b'_1(Cr(\mathcal{K}))^{3/2}$ . Furthermore, the total curvature of such an embedding is of the order  $O(Cr(\mathcal{K}))$ . The construction in Theorem 2.5 uses a simplified version of the embedding given in [15], which would not achieve a ropelength bound of  $O(Cr(\mathcal{K}))^{3/2}$ . Although the Hamiltonian cycle is embedded in a more complicated way in the construction used in [15], it still has an embedding length of  $O(Cr(\mathcal{K}))$ . Therefore its curvature is still at most  $O(Cr(\mathcal{K}))$ . Now as in the proof of Theorem 2.5, we can show that the embedding of any of the  $n$  remaining edges has curvature bounded by a constant, giving a total curvature of

$O(Cr(\mathcal{K}))$  for all of these  $n$  edges. It follows that  $\mathcal{K}$  can be realized by a smooth knot of unit thickness with length at most  $b_1 \cdot (Cr(\mathcal{K}))^{3/2}$  whose total curvature is bounded above by  $b_2 \cdot Cr(\mathcal{K})$  for some constant  $b_2 > 0$ . For details of the embedding construction we refer to [15]. This proves (1).

Since  $\tau'_m(\mathcal{K}) \leq b_2 \cdot Cr(\mathcal{K})$  when  $m \geq b_1 \cdot (Cr(\mathcal{K}))^{3/2}$  and  $\tau'_m(\mathcal{K}) \leq m$  in general, we have  $\tau'_m(\mathcal{K}) \leq \min\{b_1 \cdot (Cr(\mathcal{K}))^{3/2}, b_2 \cdot Cr(\mathcal{K})\} \leq b_4 \cdot (Cr(\mathcal{K}))^{3/2}$  in general for some constant  $b_4 > 0$ . If  $\tau'_m(\mathcal{K}) > b_4$ , we have  $\tau'_m(\mathcal{K}) = b_4 \cdot Cr(\mathcal{K})^p$  where  $0 < p = \frac{\ln(\tau'_m(\mathcal{K})/b_4)}{\ln Cr(\mathcal{K})} \leq 3/2$  and we have obtained (2). If  $\tau'_m(\mathcal{K}) \leq b_4$ , then we can simply choose  $p = 0$  and  $b_3 = 4\pi$  since  $\tau'_m(\mathcal{K}) \geq 4\pi$ .  $\square$

An interesting and hard question is, if  $K$  is the ropelength minimizer of  $\mathcal{K}$  or if the length of  $K$  is very close to the ropelength of  $\mathcal{K}$ , how would the total curvature of  $K$  behave? This question is hard because very little is known about the geometric properties of ropelength minimizers. Based on our results in the lattice case, we suspect that the following conjecture may be true.

**CONJECTURE 5.3.** There exists a positive constant  $b$  such that if  $m$  is close enough to the minimal ropelength of  $\mathcal{K}$ , then  $\tau'_m(\mathcal{K}) \geq b \cdot (Cr(\mathcal{K}))^{1/2}$ .

**REMARK 5.4.** A weaker form of the above conjecture would be replacing the power  $1/2$  by just a positive power  $p$  in the above statement. Let  $K$  be a thick realization of  $\mathcal{K}$ ,  $L(K)$  be the length of  $K$  and  $t(K)$  be the total curvature of  $K$ , it is shown in [5] that  $4t(K)L(K) \geq Cr(\mathcal{K})$ . Thus, for any knot type  $\mathcal{K}$  that admits a ropelength minimizer  $K$  such that  $m = L(K) = O(Cr(\mathcal{K})^q)$  with  $q < 1$ , we would have  $\tau'_m(\mathcal{K}) \geq O(Cr(\mathcal{K})^p)$  with  $p = 1 - q > 0$ . The knot family discussed in the proof of Theorem 3.1 falls in this category, though in this case the formula  $4t(K)L(K) \geq Cr(\mathcal{K})$  does not lead us to any new information since we already knew that  $\tau'(\mathcal{K})$  is of the order of  $Cr(\mathcal{K})^p$  ( $1/2 \leq p \leq 1$ ).

We end our paper with the following theorem, which is analogous to Theorem 3.1.

**THEOREM 5.5.** *For any real number  $p$  such that  $1/2 \leq p \leq 1$ , there exists a family of infinitely many knots  $\{K_n\}$  with the property that  $Cr(K_n) \rightarrow \infty$  (as  $n \rightarrow \infty$ ) such that the total curvature of any smooth realization of  $K_n$  with unit thickness will grow at least linearly with respect to  $Cr(K_n)^p$ . Moreover there exists a smooth realization  $T_n$  of  $K_n$  such that the total curvature of  $T_n$  grows linearly with respect to  $Cr(K_n)^p$ .*

The construction and arguments in the proof of this Theorem are very similar to that used in the proof of Theorem 3.1. We leave it to our reader.

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