From Weyl-Heisenberg Frames to Infinite Quadratic Forms

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Abstract

Let a, b be two fixed positive constants. A function $g \in L^2(\mathbb{R})$ is called a *mother Weyl-Heisenberg frame wavelet* for (a, b) if g generates a frame for $L^2(\mathbb{R})$ under modulates by b and translates by a, i.e., $\{e^{imbt}g(t-na)\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. In this paper, we establish a connection between mother Weyl-Heisenberg frame wavelets of certain special forms and certain strongly positive definite quadratic forms of infinite dimension. Some examples of application in matrix algebra are provided.

1 Introduction

Let H be a separable complex Hilbert space. Let $B(\mathbb{H})$ denote the algebra of all bounded linear operators on $\mathbb{H}$. Let $\mathbb{N}$ denote the set of natural numbers, and $\mathbb{Z}$ be the set of all integers. A collection of elements $\{x_j : j \in \jmath\}$ in $\mathbb{H}$ is called a *frame* if there exist constants A and B, $0 < A \leq B < \infty$, such that

$$A\|f\|^2 \leq \sum_{j\in\jmath} |\langle f,\ x_j\rangle|^2 \leq B\|f\|^2 \tag{1.1}$$

for all $f \in \mathbb{H}$. The supremum of all such numbers A and the infimum of all such numbers B are called the *frame bounds* of the frame and are denoted by A_0 and B_0 respectively. The frame is called a *tight* frame if $A_0 = B_0$ and is called a *normalized tight* frame if $A_0 = B_0 = 1$. Any orthonormal basis in a Hilbert space is a normalized tight frame. However, a normalized tight frame is not necessary an orthonormal basis. Frames can be regarded as the generalizations of orthogonal bases of Hilbert spaces. Although the concept of frames was introduced a long time ago ([14, 18]), it is only in recent years that many mathematicians have started to study them extensively. This is largely due to the development and study of wavelet theory and the close connections between wavelets and frames. For a glance of the recent development and work on frames and related topics, see [1, 11, 12, 16]. Among those widely studied lately are the Weyl-Heisenberg frames (or Gabor frames). Let a, b be two fixed positive constants and let T_a and M_b be the *translation operator by* a and *modulation operator by* b respectively, i.e., $T_a g(t) = g(t-a)$ and $M_b g(t) = e^{ibt} g(t)$ for any $g \in L^2(\mathbb{R})$. For a fixed $g \in L^2(\mathbb{R})$, we say that (g, a, b) generates a *Weyl-Heisenberg frame* if $\{M_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$. We also say that the function g is a *mother Weyl-Heisenberg frame wavelet* for (a, b) in this case. Furthermore, a measurable set $E \subset \mathbb{R}$ is called a *Weyl-Heisenberg frame set* for $(a,\ b)$ if the function $g = \chi_E$ generates a Weyl-Heisenberg frame for $L^2(\mathbb{R})$ under modulates by b and translates by a, i.e., $\{e^{imbt} g(t - na)\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. It is known that if $ab > 2\pi$, then (g, a, b) cannot generate a Weyl-Heisenberg frame for any $g \in L^2(\mathbb{R})$. On the other hand, for any $a > 0$, $b > 0$ such that $ab \leq 2\pi$, there always exists a function $g \in L^2(\mathbb{R})$ such that (g, a, b) generates a Weyl-Heisenberg frame [11]. However, in general, for any given $a > 0$, $b > 0$ with $ab \leq 2\pi$, characterizing the mother Weyl-Heisenberg frame wavelets g for (a, b) is a difficult problem. In fact, even for the special case, i.e., for $a = 2\pi$, $b = 1$ and $g = \chi_{_E}$ for some measurable set E, it is still an unsolved open question. In [3], Casazza and Kalton was able to establish an equivalence relation between this problem and a classical problem in complex analysis regarding the unit roots of a special kind of polynomials. There are many other works related to this subject, for more information please refer to [2, 4, 5, 6, 7, 8, 15, 17].

In this paper, we will establish an equivalence relation between mother Weyl-Heisenberg frame wavelets of certain special forms and certain strongly positive definite quadratic forms of infinite dimension and show some examples of application in matrix algebras. In Section 2, we will introduce some basic concepts and preliminary lemmas regarding frames and frame sets. In Section 3, we will briefly discuss quadratic forms of infinite dimension. In Section 4 we will state our main results and give the proofs. Some examples are given in Section 5.

2 Preliminary Lemmas on Frames

Let E be a Lebesgue measurable set with finite measure, g be a function in $L^2(\mathbb{R})$ and E_g be the support of g. Following the notations used in [9] and [10], for any $f \in L^2(\mathbb{R})$, we will let $H_E^0(f)$ and $H_g^0(f)$ be the following formal summations:

$$H_E^0(f)(\xi) = \sum_{n \in \mathbb{Z}} \langle M_n \frac{1}{\sqrt{2\pi}} \chi_E, f \rangle M_n \frac{1}{\sqrt{2\pi}} \chi_E(\xi), \tag{2.1}$$

$$H_g^0(f) = \sum_{n \in \mathbb{Z}} \langle M_n g, f \rangle M_n g. \tag{2.2}$$

Two points $x, y \in E$ are said to be 2π-translation equivalent if $x - y = 2j\pi$ for some integer j. The set of all points in E that are 2π-translation equivalent to a point x is called the 2π-translation equivalent class of x and the number of elements in this class is denoted by $\tau(x)$.

Lemma 2.1. *[10] Let E be a Lebesgue measurable set of positive measure such that $\tau(x) \le M$ for all $x \in E$ for some constant $M > 0$, then for any $f \in L^2(\mathbb{R})$, $H_E^0(f)$ converges to a function in $L^2(\mathbb{R})$ under the $L^2(\mathbb{R})$ norm topology. Moreover,*

$$\begin{aligned} H_E^0(f)(\xi) &= \chi_E(\xi) \cdot \sum_{j\in\mathbb{Z}} (f \cdot \chi_E)(\xi + 2\pi j) \\ &= \chi_E(\xi) \cdot \sum_{j\in\mathbb{Z}} f(\xi + 2\pi j) \cdot \chi_E(\xi + 2\pi j). \end{aligned} \tag{2.3}$$

An immediate consequence of Lemma 2.1 is the following corollary.

Corollary 2.2. *Let $0 = n_0 < n_1 < n_2 < \cdots < n_k$ be $k+1$ fixed integers and let $E = \cup_{j=0}^{k}([0, 2\pi) + 2\pi n_j)$. For any $f \in L^2(\mathbb{R})$, let F be the 2π-periodical extension of the function $\sum_{j=0}^{k} f(\xi + 2\pi n_j), \forall \xi \in [0, 2\pi)$. Then we have*

$$H_E^0(f) = F \cdot \chi_E. \tag{2.4}$$

It follows that for any $f \in L^2(\mathbb{R})$, we have

$$\langle H_E^0 f, f\rangle = \frac{1}{k+1}\|F \cdot \chi_E\|^2. \tag{2.5}$$

Proof. The first part of the corollary is straight forward and is left to the reader to verify. To see the second part, observe first that

$$\|F \cdot \chi_E\|^2 = (k+1)\int_0^{2\pi} |\sum_{j=0}^{k} f(\xi + 2\pi n_j)|^2 d\xi.$$

On the other hand,

$$\langle H_E^0 f, f\rangle = \sum_{j=0}^{k} \int_{[0,2\pi)+2\pi n_j} F(\xi) \cdot \overline{f}(\xi) d\xi.$$

For each j, with a suitable substitution, we have

$$\int_{[0,2\pi)+2\pi n_j} F(\xi) \cdot \overline{f}(\xi) d\xi = \int_{[0,2\pi)} F(\xi) \cdot \overline{f}(\xi + 2\pi n_j) d\xi.$$

Therefore,

$$\begin{aligned} \langle H_E^0 f, f\rangle &= \int_{[0,2\pi)} F(\xi) \sum_{j=0}^{k} \overline{f}(\xi + 2\pi n_j) d\xi \\ &= \int_{[0,2\pi)} F(\xi) \cdot \overline{F}(\xi) d\xi \\ &= \frac{1}{k+1}\|F \cdot \chi_E\|^2. \end{aligned}$$

□

Corollary 2.3. *For fixed integers* $0 = n_0 < n_1 < n_2 < \cdots < n_k$, *let* $E = \cup_{j=0}^{k}([0, 2\pi) + 2\pi n_j)$, $E_n = E + 2n\pi$ *and* $g = \frac{1}{\sqrt{2\pi}}\chi_E$. *For any* $f \in L^2(\mathbb{R})$, *let* $F_m(\xi)$ *be the* 2π*-periodical function over* $\mathbb{R}$ *defined by* $F_m(\xi) = \sum_{j=0}^{k} f(\xi + 2\pi(n_j + m))$, $\forall \xi \in [0, 2\pi)$. *We have*

$$\sum_{m,n\in\mathbb{Z}} |\langle M_m T_{2n\pi} g, f\rangle|^2 = \frac{1}{k+1} \sum_{n\in\mathbb{Z}} \|F_n \cdot \chi_{E_n}\|^2. \tag{2.6}$$

We leave the proof of Corollary 2.3 to our reader. The following more general results are from [9] and will be needed in proving our main result.

Lemma 2.4. *If* E_g *is a Lebesgue measurable set of positive measure such that* $\tau(x) \leq M$ *for all* $x \in E_g$ *for some constant* $M > 0$ *and* $|g| \leq b$ *for some constant* $b > 0$ *on* E_g, *then* H_g^0 *defines a bounded linear operator. Furthermore, we have*

$$H_g^0(f) = g(\xi) \sum_{j\in\mathbb{Z}} f(\xi + 2\pi j)\overline{g}(\xi + 2\pi j).$$

Lemma 2.5. *Let* $0 = n_0 < n_1 < n_2 < \cdots < n_k$ *be* $k+1$ *integers and* a_0, a_1, a_2, $\cdots$, a_k *be* $k+1$ *complex numbers. Let* $E_j = [0, 2\pi) + 2n_j\pi$, $G_n = \bigcup_{j=0}^{k} \big(E_j + 2n\pi\big)$, $g = \sum_{j=0}^{k} a_j \chi_{E_j}$ *and* $g_n = \sum_{j=0}^{k} a_j \chi_{(E_j + 2n\pi)}$. *For any* $f \in L^2(\mathbb{R})$, *let* F_n *be the* 2π*-periodical function defined by*

$$\begin{aligned} F_n(\xi) &= \sum_{j=0}^{k} f\big(\xi + 2\pi(n_j + n)\big)\overline{g_n}\big(\xi + 2\pi(n_j + n)\big) \\ &= \sum_{j=0}^{k} f(\xi + 2\pi(n_j + n))\overline{a_j}, \ \forall \xi \in [0, 2\pi), \end{aligned}$$

then we have

$$\sum_{m,n\in\mathbb{Z}} |\langle M_m T_{2n\pi} g, f\rangle|^2 = \sum_{n\in\mathbb{Z}} \|F_n \cdot \chi_{[0,2\pi)}\|^2.$$

Proof. By Lemma 2.4,

$$H_{g_n}^0 f = \sum_{m\in\mathbb{Z}} \langle M_m g_n, f\rangle M_m g_n$$

converges to

$$g_n(\xi) \sum_{m\in\mathbb{Z}} f(\xi + 2m\pi)\overline{g_n}(\xi + 2m\pi) = F_n \cdot g_n.$$

It follows that

$$
\begin{aligned}
& \sum_{m\in\mathbb{Z}} |\langle M_m g_n, f\rangle|^2 \\
= \;& \langle H^0_{g_n} f, f\rangle \\
= \;& \int_{G_n} F_n g_n \overline{f} d\xi \\
= \;& \sum_{j=0}^{k} \int_{[0,2\pi)+2\pi(n+n_j)} F_n g_n \overline{f} d\xi \\
= \;& \sum_{j=0}^{k} a_j \int_{[0,2\pi)+2\pi(n+n_j)} F_n \overline{f} d\xi \\
= \;& \sum_{j=0}^{k} a_j \int_{[0,2\pi)} F_n \cdot \overline{f}(\xi + 2\pi(n+n_j)) d\xi \\
= \;& \int_{[0,2\pi)} F_n \cdot \sum_{j=0}^{k} a_j \overline{f}(\xi + 2\pi(n+n_j)) d\xi \\
= \;& \int_{[0,2\pi)} F_n \cdot \overline{F_n} d\xi \\
= \;& \int_{[0,2\pi)} |F_n|^2 d\xi = \|F_n \cdot \chi_{[0,2\pi)}\|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \sum_{m,n\in\mathbb{Z}} |\langle M_m T_{2n\pi} g, f\rangle|^2 \\
= \;& \sum_{m,n\in\mathbb{Z}} |\langle M_m g_n, f\rangle|^2 \\
= \;& \sum_{n\in\mathbb{Z}} \Big\langle \sum_{m\in\mathbb{Z}} \langle M_m g_n, f\rangle M_m g_n, f \Big\rangle \\
= \;& \sum_{n\in\mathbb{Z}} \langle H^0_{g_n} f, f\rangle \\
= \;& \sum_{n\in\mathbb{Z}} \|F_n \cdot \chi_{[0,2\pi)}\|^2.
\end{aligned}
$$

□

3 A Special Quadratic Form of Infinite Dimension

Let $\{x_n\}$ be a real sequence in $\ell^2(\mathbb{Z})$, i.e., $\{x_n\}$ is a real valued sequence such that the series $\sum_{n\in\mathbb{Z}} x_n^2$ is convergent. Let $\{a_{ij}\}_{i,j\in\mathbb{Z}}$ be a sequence of real numbers with $a_{ij} = a_{ji}$. We can formally write $A = \{a_{ij}\}$ and think of A as an infinite dimensional symmetric matrix. Similarly, we will write $x = \{x_n\}$ and xAx^t for the formal sum $\sum_{i,j\in\mathbb{Z}} a_{ij}x_ix_j$. If this formal sum is convergent for all $x = \{x_n\} \in \ell^2(\mathbb{Z})$, then we will call xAx^t an *infinite quadratic form.* Notice that it is easy to come up with examples of A such that xAx^t is not defined for some x (that is, the series $\sum_{i,j\in\mathbb{Z}} a_{ij}x_ix_j$ is not convergent).

Definition 3.1. We say that an infinite quadratic form xAx^t is strongly positive definite if there exists a constant $c > 0$ such that $xAx^t \geq c\|x\|^2$ for all $x \in \ell^2(\mathbb{Z})$, where $\|x\|^2 = \sum_{n\in\mathbb{Z}} x_n^2$. Similarly, one can define negative definiteness.

In this paper, we will be mainly dealing with a special type of quadratic forms of infinite dimension, which we will describe below.

Let $0 = n_0 < n_1 < n_2 < \cdots < n_k$ be $k+1$ given integers, a_0, a_1, a_2, $\cdots$, a_k be $k+1$ given nonzero real numbers and $p(z) = \sum_{j=0}^{k} a_j z^{n_j}$. Then we can associate with $p(z)$ the following quadratic form of infinite dimension:

$$\sum_{n\in\mathbb{Z}}(a_0x_n + a_1x_{n+n_1} + a_2x_{n+n_2} + \cdots + a_kx_{n+n_k})^2 \tag{3.1}$$

We may write the above quadratic form formally as $x^t(A^tA)x$ by formally viewing x as a column vector (of infinite dimension), where A is the upper triangular matrix constructed by the $k+1$ given nonzero real numbers $\{a_i\}_{i=0}^{k}$. The main diagonal of A consists of a_0, the n_1^{th} sub-diagonal consists of a_1, ..., the n_k^{th} sub-diagonal consists of a_k and all other entries are zeroes. A^t is the transpose of A. We leave it to our reader to verify that for any $x = \{x_n\} \in \ell^2(\mathbb{Z})$, $x^t(A^tA)x$ is always convergent hence is a well-defined infinite dimensional quadratic form. Apparently, such a quadratic form is always semi-positive definite.

4 Main Results and Proofs

Theorem 4.1. *Let* $0 = n_0 < n_1 < n_2 < \cdots < n_k$ *be* $k+1$ *given integers and* a_0, a_1, a_2, $\cdots$, a_k *be* $k+1$ *given nonzero real numbers, then the following statements are equivalent:*

1. The infinite dimensional quadratic form

$$\sum_{n\in\mathbb{Z}}(a_0x_n + a_1x_{n+n_1} + a_2x_{n+n_2} + \cdots + a_kx_{n+n_k})^2$$

is strongly positive definite;

2. The function $g = \sum_{j=0}^{k} a_j\chi_{F_j}$ *is a mother Weyl-Heisenberg frame wavelet for* $(2\pi, 1)$*, where* $F_j = [0, 2\pi) + 2n_j\pi$, $j = 0, 1, ..., k$.

Proof. 2 $\Longrightarrow$ 1: Let $x = \{x_n\}$ be a sequence in $\ell^2(\mathbb{Z})$ and define $f(\xi) = x_n$ if $\xi \in [0, 2\pi) + 2\pi n$, then $f \in L^2(\mathbb{R})$ by its definition. Furthermore, it is easy to see that $\|f\|^2 = 2\pi \cdot \|x\|^2 = 2\pi \sum_{m\in\mathbb{Z}} x_m^2$. Thus, if $g = \sum_{j=1}^{k} a_j \chi_{F_j}$ is a mother Weyl-Heisenberg frame wavelet for $(2\pi, 1)$, then there exists a constant $A > 0$ such that

$$\sum_{m,n\in\mathbb{Z}} |\langle M_m T_{2n\pi} g, f\rangle|^2 \geq A\|f\|^2 = 2\pi A\|x\|^2 \tag{4.1}$$

by the definition of the mother Weyl-Heisenberg frame wavelet. On the other hand, by Lemma 2.5, we also have

$$\sum_{m,n\in\mathbb{Z}} |\langle M_m T_{2n\pi} g, f\rangle|^2 = \sum_{n\in\mathbb{Z}} \|F_n \cdot \chi_{[0,2\pi)}\|^2,$$

where F_n is the 2π-periodic function defined by

$$F_n(\xi) = \sum_{j=0}^{k} f(\xi + 2\pi(n + n_j))\overline{g}(\xi + 2\pi(n + n_j)), \forall \xi \in [0, 2\pi), \tag{4.2}$$

which is simply

$$\sum_{j=0}^{k} f(\xi + 2\pi(n + n_j))\overline{a_j} = \sum_{j=0}^{k} a_j x_{n+n_j} \tag{4.3}$$

as one can easily check. It follows that

$$\|F_n \cdot \chi_{[0,2\pi)}\|^2 = 2\pi \left(\sum_{j=0}^{k} a_j x_{n+n_j}\right)^2 = 2\pi \left(\sum_{j=0}^{k} a_j x_{n+n_j}\right)^2,$$

and (4.1) becomes

$$2\pi \sum_{n\in\mathbb{Z}} \left(\sum_{j=0}^{k} a_j x_{n+n_j}\right)^2 \geq 2\pi A\|x\|^2. \tag{4.4}$$

That is

$$\sum_{n\in\mathbb{Z}} \left(\sum_{j=0}^{k} a_j x_{n+n_j}\right)^2 \geq A\|x\|^2. \tag{4.5}$$

Therefore, the infinite dimensional quadratic form $\sum_{n\in\mathbb{Z}} \left(\sum_{j=0}^{k} a_j x_{n+n_j}\right)^2$ is strongly positive definite. This proves 2 $\Longrightarrow$ 1.

We will now prove 1 $\Longrightarrow$ 2. That is, assuming that the quadratic form $\sum_{n\in\mathbb{Z}} \left(\sum_{j=0}^{k} a_j x_{n+n_j}\right)^2$ is strongly positive definite, we need to prove that the function $g = \sum_{j=0}^{k} a_j \chi_{([0,2\pi)+2n_j\pi)}$ is a mother Weyl-Heisenberg frame wavelet for $(2\pi, 1)$. In other word, we need to prove that there exists a constant $c_0 > 0$ such that

$$\sum_{m,n\in\mathbb{Z}} |\langle M_m T_{2n\pi} g, f\rangle|^2 \geq c_0 \|f\|^2 \tag{4.6}$$

for any $f \in L^2(\mathbb{R})$. For any given $f \in L^2(\mathbb{R})$ and $0 < \epsilon < 1$, there exists $M_1 > 0$ such that

$$\|f \cdot \chi_{[-2N\pi, 2M\pi]}\|^2 \geq \epsilon \|f\|^2 \tag{4.7}$$

for any N, M such that $N \geq M_1$ and $M \geq M_1$. Let F_n be as defined in Lemma 2.5. By Lemma 2.5, we have

$$\sum_{m,n\in\mathbb{Z}} |\langle M_m T_{2n\pi} g, f\rangle|^2 = \sum_{n\in\mathbb{Z}} \|F_n \cdot \chi_{[0,2\pi)}\|^2. \tag{4.8}$$

Now choose $M = M_1 + n_k$. We have

$$\begin{aligned}
& \sum_{m,n\in\mathbb{Z}} |\langle M_m T_{2n\pi} g, f\rangle|^2 \\
= & \sum_{n\in\mathbb{Z}} \|F_n \cdot \chi_{[0,2\pi)}\|^2 \\
\geq & \sum_{-M\leq n\leq M} \|F_n \cdot \chi_{[0,2\pi)}\|^2 \\
= & \sum_{-M\leq n\leq M} \int_0^{2\pi} \big(\sum_{j=0}^{k} a_j f(\xi + 2\pi(n_j + n))\big)^2 d\xi.
\end{aligned} \tag{4.9}$$

Since the quadratic form $\sum_{n\in\mathbb{Z}} \big(\sum_{j=0}^{k} a_j x_{(n+n_j)}\big)\big)^2$ is strongly positive definite, there exists a constant $c_1 > 0$ such that

$$\sum_{n\in\mathbb{Z}} \big(\sum_{j=0}^{k} a_j x_{(n+n_j)}\big)\big)^2 \geq c_1 \|x\|^2. \tag{4.10}$$

In particular, for each fixed $\xi \in [0, 2\pi)$, we may define $x \in \ell^2(\mathbb{Z})$ by $x_{n+n_j} = f(\xi + 2\pi(n_j + n))$ if $-M \leq n \leq M + n_k$ and $x_{n+n_j} = 0$ otherwise. Then we have

$$\begin{aligned}
& \sum_{-M\leq n\leq M} \big(\sum_{j=0}^{k} a_j f(\xi + 2\pi(n_j + n))\big)^2 \\
= & \sum_{n\in\mathbb{Z}} \big(\sum_{j=0}^{k} a_j x_{n+n_j}\big)\big)^2 \geq c_1 \|x\|^2 \\
= & c_1 \sum_{-M\leq n\leq M+n_k} |f(\xi + 2\pi n)|^2.
\end{aligned} \tag{4.11}$$

It follows by (4.7) that

$$\begin{aligned}
& \sum_{-M\le n\le M}\int_0^{2\pi}\big(\sum_{j=0}^{k} a_j f(\xi+2\pi(n_j+n))\big)^2 d\xi \\
\ge\ & c_1 \sum_{-M\le n\le M+n_k}\int_0^{2\pi}|f(\xi+2\pi n)|^2 d\xi \\
=\ & c_1\|f\cdot\chi_{[-2M\pi,2(M+n_k+1)\pi)}\|^2 \\
\ge\ & c_1\epsilon\|f\|^2.
\end{aligned}$$

□

Notice that in the above proof, ϵ can be arbitrarily close to 1. It follows that the constant c_0 can be taken as c_1. With some additional but small modifications to the above proof, we can draw a stronger conclusion, which we state as the following corollary. The details are left to our reader.

Corollary 4.2. *Let $0 = n_0 < n_1 < n_2 < \cdots < n_k$ be $k+1$ given integers, and a_0, a_1, a_2, $\cdots$, a_k be $k+1$ given nonzero real numbers, then if the infinite dimensional quadratic form*

$$\sum_{n\in\mathbb{Z}}(a_0x_n + a_1x_{n+n_1} + a_2x_{n+n_2} + \cdots + a_kx_{n+n_k})^2$$

is strongly positive definite such that

$$c_1\|x\|^2 \le \sum_{n\in\mathbb{Z}}(a_0x_n + a_1x_{n+n_1} + a_2x_{n+n_2} + \cdots + a_kx_{n+n_k})^2 \le c_2\|x\|^2$$

for some positive constants $c_1 \le c_2$, then the function $g = \sum_{j=0}^{k} a_j\chi_{F_j}$ is a mother Weyl-Heisenberg frame wavelet for $(2\pi, 1)$ with c_1 as a lower frame bound and c_2 as an upper frame bound. Conversely, if the function $g = \sum_{j=0}^{k} a_j\chi_{F_j}$ is a mother Weyl-Heisenberg frame wavelet for $(2\pi, 1)$ with c_1 as a lower frame bound and c_2 as an upper frame bound, then

$$c_1\|x\|^2 \le \sum_{n\in\mathbb{Z}}(a_0x_n + a_1x_{n+n_1} + a_2x_{n+n_2} + \cdots + a_kx_{n+n_k})^2 \le c_2\|x\|^2$$

for any $x \in \ell^2(\mathbb{Z})$.

Let us point out that by a similar approach used in [3], we can show that the frame bounds of the Weil-Heisenberg frame generated by the function g (using $(2\pi, 1)$) defined above can be obtained by evaluating the minimum and maximum values of $|a_0 + a_1z^{n_1} + \cdots + a_kz^{n_k}|^2$ over the unit circle $\mathbb{T}$. Let $\ell^2[M_1, M_2]$ be the subset of $\ell^2(\mathbb{Z})$ which contain all the elements x such that $x_n = 0$ for any $n < M_1$ or $n > M_2$. If we restrict x to $\ell^2[M_1, M_2]$, then the quadratic form $\sum_{n\in\mathbb{Z}}(a_0x_n + a_1x_{n+n_1} + a_2x_{n+n_2} + \cdots + a_kx_{n+n_k})^2$ is finite and its corresponding matrix is a main diagonal block of A, where A is the (infinite dimensional) matrix of the quadratic form $\sum_{n\in\mathbb{Z}}(a_0x_n + a_1x_{n+n_1} + a_2x_{n+n_2} + \cdots + a_kx_{n+n_k})^2$. Conversely, any main diagonal block of A can be viewed as the matrix of a finite quadratic form with x chosen from some suitable $\ell^2[M_1, M_2]$. This observation leads to our last theorem.

Theorem 4.3. *Let $0 = n_0 < n_1 < n_2 < \cdots < n_k$ be $k+1$ given integers, a_0, a_1, a_2, $\cdots$, a_k be $k+1$ given nonzero real numbers, and A be the symmetrical infinite matrix corresponding to the infinite quadratic form*

$$\sum_{n\in\mathbb{Z}}(a_0x_n + a_1x_{n+n_1} + a_2x_{n+n_2} + \cdots + a_kx_{n+n_k})^2.$$

Let $\min_{z\in\mathbb{T}}|a_0 + a_1z^{n_1} + \cdots + a_kz^{n_k}|^2 = C_1$ and $\max_{z\in\mathbb{T}}|a_0 + a_1z^{n_1} + \cdots + a_kz^{n_k}|^2 = C_2$, then the eigenvalues of any main diagonal block of A are bounded between C_1 and C_2.

5 Examples

A symmetric infinite dimensional matrix $B = \{b_{ij}\}$ is called *periodic* if there exists a real sequence $\{b_k\} = \{b_0, b_1, ...$ such that $b_{ij} = b_k$ whenever $|i-j| = k$. Determining whether B is strongly positive definite is a hard question in general, since one would have to consider all the main diagonal blocks of B. However, if the sequence $\{b_k\} = \{b_0, b_1, ..., b_n\}$ is a finite sequence, B may be related to a polynomial $p(z) = a_0 + a_1z + \cdots + a_nz^n$ via the following equation

$$\begin{aligned}\sum_0^n a_i^2 &= b_0\\ \sum_0^{n-1} a_ia_{i+1} &= b_1\\ \sum_0^{n-2} a_ia_{i+2} &= b_2\\ ... &= ...\\ a_0a_n &= b_n,\end{aligned}$$

in which case whether B is strongly positive definite can be solved using Theorem 4.3.

Example 5.1. The infinite quadratic form $\sum_{n\in\mathbb{Z}}(x_n+x_{n+1}+x_{n+3})^2$ is strongly positive definite since its corresponding polynomial $p(z) = 1 + z + z^3$ has the property $min_{z\in\mathbb{T}}|p(z)|^2 \approx 0.3689$ and $max_{z\in\mathbb{T}}|p(z)|^2 = 9$. The symmetrical infinite matrix corresponding to $\sum_{n\in\mathbb{Z}}(x_n + x_{n+1} + x_{n+3})^2$ is

$$A = \begin{pmatrix} \cdot & & & \cdot & & & \cdot & & & \cdot & & & \cdot & & \\ \cdot & 0 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 0 & \cdot \\ \cdot & & & \cdot & & & \cdot & & & \cdot & & & \cdot & & \end{pmatrix}.$$

By Theorem 4.3, any eigenvalue λ of any main diagonal block of A must satisfy

$$0.3689 \leq \lambda \leq 9$$

The following are some main diagonal block matrices of the above infinite matrix along with their eigenvalues.

$A_2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, eigenvalues are: $\lambda_1 = 4$, $\lambda_2 = 2$.

$A_3 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$, eigenvalues are: $\lambda_1 = 2$, $\lambda_2 = 2$, $\lambda_3 = 5$.

$A_4 = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}$, eigenvalues are: $\lambda_1 = 2$, $\lambda_2 = 2$, $\lambda_3 = 2$, $\lambda_4 = 6$.

$A_5 = \begin{pmatrix} 3 & 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 & 3 \end{pmatrix}$, eigenvalues are: $\lambda_1 \approx 1.35$, $\lambda_2 = 2$, $\lambda_3 = 2$, $\lambda_4 = 3$, $\lambda_5 \approx 6.65$.

Example 5.2. Let $p(z) = 2+3z^2+4z^3$. We have $min_{z\in\mathbb{T}}|p(z)|^2 = 1$, $max_{z\in\mathbb{T}}|p(z)|^2 = 81$. Thus the infinite quadratic form $\sum_{n\in\mathbb{Z}}(2x_n + 3x_{n+2} + 4x_{n+3})^2$ is strongly positive definite. The corresponding symmetrical infinite matrix is

$$B = \begin{pmatrix} \cdot & & & \cdot & & & \cdot & & & \cdot & & & \cdot & & \\ \cdot & 0 & 8 & 6 & 12 & 29 & 12 & 6 & 8 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 8 & 6 & 12 & 29 & 12 & 6 & 8 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 8 & 6 & 12 & 29 & 12 & 6 & 8 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 8 & 6 & 12 & 29 & 12 & 6 & 8 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 8 & 6 & 12 & 29 & 12 & 6 & 8 & 0 & \cdot \\ \cdot & & & \cdot & & & \cdot & & & \cdot & & & \cdot & & \end{pmatrix}.$$

Again, any eigenvalue λ of any main diagonal block of B must satisfy

$$1 \leq \lambda \leq 81.$$

A few main diagonal blocks of the above infinite matrix along with their eigenvalues are listed below.

$B_2 = \begin{pmatrix} 29 & 12 \\ 12 & 29 \end{pmatrix}$, eigenvalues are: $\lambda_1 = 17$, $\lambda_2 = 41$.

$B_3 = \begin{pmatrix} 29 & 12 & 6 \\ 12 & 29 & 12 \\ 6 & 12 & 29 \end{pmatrix}$, eigenvalues are: $\lambda_1 = 14.77$, $\lambda_2 = 23$, $\lambda_3 = 49.23$.

$B_4 = \begin{pmatrix} 29 & 12 & 6 & 8 \\ 12 & 29 & 12 & 6 \\ 6 & 12 & 29 & 12 \\ 8 & 6 & 12 & 29 \end{pmatrix}$, eigenvalues are: $\lambda_1 \approx 12.68$, $\lambda_2 \approx 20.89$, $\lambda_3 \approx 25.32$, $\lambda_4 \approx 57.11$.

$B_5 = \begin{pmatrix} 29 & 12 & 6 & 8 & 0 \\ 12 & 29 & 12 & 6 & 8 \\ 6 & 12 & 29 & 12 & 6 \\ 8 & 6 & 12 & 29 & 12 \\ 0 & 8 & 6 & 12 & 29 \end{pmatrix}$, eigenvalues are: $\lambda_1 \approx 9.84$, $\lambda_2 \approx 20.69$, $\lambda_3 = 21$, $\lambda_4 = 31$, $\lambda_5 \approx 62.47$.

Example 5.3. Consider the infinite dimensional matrix

$$C = \begin{pmatrix} \cdot & & & \cdot & & & \cdot & & & \cdot & & & \cdot & & \\ \cdot & 0 & -2 & 1 & -2 & 4 & -2 & 1 & -2 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & -2 & 1 & -2 & 4 & -2 & 1 & -2 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & -2 & 1 & -2 & 4 & -2 & 1 & -2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & -2 & 1 & -2 & 4 & -2 & 1 & -2 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -2 & 1 & -2 & 4 & -2 & 1 & -2 & 0 & \cdot \\ \cdot & & & \cdot & & & \cdot & & & \cdot & & & \cdot & & \end{pmatrix}.$$

One can show that it is related to the polynomial $p(z) = -2 + z + z^3$, which has a unit zero. It follows that the the infinite quadratic form $\sum_{n\in\mathbb{Z}}(-2x_n + x_{n+2} + x_{n+3})^2$ is not strongly positive definite. Therefore, there must exist a sequence of main diagonal blocks of C whose least eigenvalues will approach 0 as the dimensions of the blocks go to infinity. The following is the list of the first few main diagonal blocks and their corresponding eigenvalues.

$C_2 = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$, eigenvalues are: $\lambda_1 = 2$, $\lambda_2 = 6$.

$C_3 = \begin{pmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{pmatrix}$, eigenvalues are: $\lambda_1 \approx 1.63$, $\lambda_2 = 3$, $\lambda_3 \approx 7.37$.

$C_4 = \begin{pmatrix} 4 & -2 & 1 & -2 \\ -2 & 4 & -2 & 1 \\ 1 & -2 & 4 & -2 \\ -2 & 1 & -2 & 4 \end{pmatrix}$, eigenvalues are: $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 3$, $\lambda_4 = 9$.

$$C_5 = \begin{pmatrix} 4 & -2 & 1 & -2 & 0 \\ -2 & 4 & -2 & 1 & -2 \\ 1 & -2 & 4 & -2 & 1 \\ -2 & 1 & -2 & 4 & -2 \\ 0 & -2 & 1 & -2 & 4 \end{pmatrix}$$, eigenvalues are: $\lambda_1 \approx 0.22$, $\lambda_2 \approx 2.70$, $\lambda_3 = 3$, $\lambda_4 = 4$, $\lambda_5 \approx 10.08$.

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