

The Closed Form Reproducing Kernel Particle Shape Functions: Part 1. Basic Constructions: Uniformly Distributed Particles

by

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Abstract

It has been known that Reproducing Kernel Particle (RKP) shape functions with Kronecker delta property are not available in simple forms. Thus, in this paper, we first construct highly regular piecewise polynomial RKP shape functions that are reproducing of order k for any given integer $k \geq 0$ and satisfy the Kronecker Delta Property. Second, we construct flexible Partition of Unity (PU) shape functions that make it possible for the closed form RKP shape functions to be used for locally uniformly distributed particles.

Keywords: The Closed form reproducing kernel particle shape functions; Reproducing kernel particle method(RKPM); The convolution partition of unity shape functions; Translation invariant particles; Partition of unity finite element method.

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1 Introduction

The Finite Element Method (FEM) has been a powerful tool in solving many difficult science and engineering problems, especially when the solution domains have complex geometry ([5], [19]). However, there are several difficulties arising in implementing this method. The prominent difficulties include the mesh refinements and constructing higher order interpolation fields.

In order to relax the constraints of the conventional FEM, several generalized finite element methods (GFEM), that use the meshes minimally or do not use the meshes at all, were recently introduced. Among many names of GFEM ([1],[2],[3],[4]), those methods related to this paper are Element Free Galerkin Method (EFGM) ([1],[9],[10],[11]), Reproducing Kernel Particle Method (RKPM)([8],[11]), h-p Cloud Method([6],[7]), Partition of Unity Finite Element Method (PUFEM)([14],[17],[18]), and Reproducing Kernel Element Method (RKEM) ([11],[12],[13]).

It was shown ([2],[3],[8]) that RKP shape functions with the reproducing property of high order ensure a good approximability. However, these RKP shape functions have not been given in a simple form, but usually known as the product of window functions and the solutions of the functional linear systems ([11],[12]). Moreover, in application of PU shape functions, the Shepard-type PU functions have been employed. In other words, neither the RKP shape functions nor PU shape functions are in a simple form. Thus, integrations related to RKP shape functions are not very accurate and require lengthy function evaluations. Furthermore, RKP shape functions do not satisfy the Kronecker delta property in general. Thus, they have difficulties in dealing with Dirichlet boundary conditions.

In this paper, we have constructed highly regular piecewise polynomial RKP shape functions that satisfy the reproducing property of any given order and also satisfy the Kronecker delta property.

Even though our constructions are valid only for the uniformly distributed particles, by introducing piecewise polynomial PU functions, we can make it possible for the closed form RKP basic shape functions to be applied for the particles that are locally uniformly distributed in such a way that the distances among particles in different patches are different.

This paper is organized as follows. In section two, the terminologies and notations used in this paper are introduced. A simple example for motivation of our constructions is provided. In section three, we have the followings.

- 1 After proving a key lemma for our construction, we prove the existence and uniqueness theorem for continuous (\mathcal{C}^0) piecewise polynomial RKP shape functions of maximal reproducing order that satisfy the Kronecker delta property.
- 2 These (\mathcal{C}^0) piecewise polynomial RKP shape functions of reproducing order $k = 1, 3, 5$ are tested with a model second order differential equation.
- 3 By sacrificing one order of reproducing property or extending the support from the \mathcal{C}^0 RKP shape functions, we constructed highly regular ($\mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3$) RKP basic shape functions

of reproducing order $1 \leq k \leq 7$, that satisfy the Kronecker delta property.

4 The \mathcal{C}^2 RKP shape functions of reproducing order $k = 2, 4, 6$ are tested to a model fourth order equation.

5 Two dimensional extensions of these RKP shape functions are briefly discussed.

Next, in section four, we introduced the convolution partition of unity shape functions that are piecewise polynomials and are as smooth as a chosen conical window function. PUFEM with respect to these convolution PU functions are tested with a model second order equation as well as a model fourth order equation. Finally, we discuss an application of the translation invariant RKP shape functions combined with the convolution PU functions to non-uniformly distributed particles.

2 Definitions and Notations

There are two slightly different definitions for RKP shape functions: one in [3] (B-B-O RKP) and another in [8] (H-M RKP). In this paper, for the meaning of the RKP shape functions, we follow the definition in [3] (B-B-O RKP).

Throughout this paper, $\alpha, \beta \in \mathbb{Z}^n$ are multi indices and $x = ({}^1x, {}^2x, \dots, {}^nx)$, $x_j = ({}^1x_j, {}^2x_j, \dots, {}^nx_j)$ denote points in \mathbb{R}^n . However, if there are no confusions, we also use the conventional notation for the points in \mathbb{R}^n or \mathbb{Z}^n as

$$x = (x_1, x_2, \dots, x_n) \text{ and } \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n).$$

By $\alpha \leq \beta$, we mean

$$\alpha_1 \leq \beta_1, \dots, \alpha_n \leq \beta_n.$$

We also use the following notations:

$$\begin{aligned} (x - x_j)^\alpha &:= ({}^1x - {}^1x_j)^{\alpha_1} \dots ({}^nx - {}^nx_j)^{\alpha_n}, \\ |\alpha| &:= \alpha_1 + \alpha_2 + \dots + \alpha_n. \end{aligned}$$

Let Ω be a domain in \mathbb{R}^n . For any nonnegative integer m , $\mathcal{C}^m(\Omega)$ denotes the space of all functions ϕ such that ϕ together with all their derivatives $D^\alpha \phi$ of orders $|\alpha| \leq m$, are continuous on Ω . The support of ϕ is defined by

$$\text{supp } \phi = \overline{\{x \in \Omega : \phi(x) \neq 0\}}.$$

In the following, a function $\phi \in \mathcal{C}^m(\Omega)$ is said to be a \mathcal{C}^m - function.

A weight function (or window function) is a non negative continuous function with compact support and is denoted by $w(x)$. For $x \in \mathbb{R}$, the window functions used in this paper are the following:

(a) Conical:

$$w(x) = \begin{cases} (1 - x^2)^l, & |x| \leq 1 \\ 0, & |x| > 1, \end{cases} \quad (1)$$

(b) Gaussian:

$$w(x) = \begin{cases} (e^{-1/1-x^2}) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases} \quad (2)$$

(c) Partition of unity ([15]):

$$w(x) = \begin{cases} (1+x)^3 g(x) & \text{if } -1 \leq x \leq 0 \\ (1-x)^3 g(-x) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } |x| > 1, \end{cases} \quad (3)$$

where $g(x) = (1 - 3x + 6x^2)$.

In \mathbb{R}^n , the weight function $w(x)$ can be constructed from a one-dimensional weight function either as $w(x) = w(\|x\|)$ or as $w(x) = \prod_{i=1}^n w(x_i)$, where $\|x\|^2 = x_1^2 + \dots + x_n^2$.

Adopting those terminologies and notations of ([3]), we have the following: For $j = (j_1, j_2, \dots, j_n) \in \mathbb{Z}^n$, and the mesh size $0 < h \leq 1$, let

$$x_j^h = (j_1 h, \dots, j_n h) = hj.$$

Then the points x_j^h are called uniformly distributed particles. Let ϕ be a continuous function with compact support that contains the origin 0. Then the particle shape functions associated to the uniformly distributed particles is defined by

$$\phi_j^h(x) = \phi\left(\frac{x - jh}{h}\right) = \phi\left(\frac{x_1 - j_1 h}{h}, \dots, \frac{x_n - j_n h}{h}\right),$$

for $j \in \mathbb{Z}^n$ and $0 < h \leq 1$. Then these particle shape functions are **translation invariant** in the sense that

$$\begin{aligned} x_{i+j}^h &= x_i^h + x_j^h, \\ \phi_j^h(x - ih) &= \phi_{i+j}^h(x). \end{aligned}$$

In this paper, we assume that the basic shape functions are translation invariant on the **uniformly distributed particles**, unless stated otherwise.

Without loss of generality, in the theoretical developments and constructions of particle shape functions, we assume that

$$h = 1 \text{ and hence } \phi_j^h(x) = \phi(x - j).$$

Definition 2.1. Let k be a non negative integer. Then $\{\phi_j(x) : j \in \mathbb{Z}\}$ are called RKP shape functions with the reproducing property of order k (or simply, “of reproducing order k ”) if and only if it satisfies the following condition:

$$\sum_{j \in \mathbb{Z}} (x_j)^\alpha \phi_j(x) = x^\alpha, \text{ for } x \in \mathbb{R}^n \text{ and for } 0 \leq |\alpha| \leq k, \quad (4)$$

where $|\alpha| = \alpha_1 + \dots + \alpha_n$.

The RKP shape function, associated with the particle x_j , is constructed by

$$\phi_j(x) = w(x - x_j) \sum_{0 \leq |\alpha| \leq k} (x - x_j)^\alpha b_\alpha(x) \quad (5)$$

where $b_\alpha(x)$ are chosen so that (4) is satisfied and $w(x)$ is a window function. This gives rise to a linear system in $b_\alpha(x)$, namely

$$\sum_{0 \leq |\alpha| \leq k} m_{\alpha+\beta}(x) b_\alpha(x) = \delta_{|\beta|}^0 \text{ for } 0 \leq |\beta| \leq k, \quad (6)$$

where $\delta_{|\beta|}^0$ is the Kronecker delta, and

$$m_\alpha(x) = \sum_{j \in \mathbb{Z}} w(x - x_j) (x - x_j)^\alpha. \quad (7)$$

For one dimensional case, this system can be written as

$$M(x) \cdot [b_0(x), b_1(x), \dots, b_k(x)]^T = [1, 0, \dots, 0]^T,$$

where

$$M(x) = \sum_{j \in \mathbb{Z}} w(x - x_j) \begin{bmatrix} 1 \\ (x - x_j)^1 \\ (x - x_j)^2 \\ \vdots \\ (x - x_j)^k \end{bmatrix} [1, (x - x_j)^1, \dots, (x - x_j)^k].$$

The coefficient matrix $M(x)$ of the linear system (6) is called the moment matrix.

Let us construct a one dimensional RKP shape function of reproducing order 1 by using (5) and(6). For example, let $w(x)$ be a smooth window function whose support is $[-1, 1]$. Then for $x \in [0, 1]$, the moment matrix is

$$M(x) = \begin{bmatrix} w(x) + w(x - 1) & x \cdot w(x) + (x - 1) \cdot w(x - 1) \\ x \cdot w(x) + (x - 1) \cdot w(x - 1) & x^2 \cdot w(x) + (x - 1)^2 \cdot w(x - 1) \end{bmatrix}$$

and $\det M(x) = w(x)w(x - 1)$ is non zero except at $x = 0, 1$.

Thus, for $x \in (0, 1)$, the solutions of the linear system (6) are

$$\begin{aligned} b_0(x) &= [x^2w(x) + (x-1)^2w(x-1)]/w(x)w(x-1), \\ b_1(x) &= -[xw(x) + (x-1)w(x-1)]/w(x)w(x-1), \end{aligned}$$

and hence by the relation (5),

$$\phi(x) = w(x)[b_0(x) + xb_1(x)] = 1 - x, \quad (8)$$

$$\phi(x-1) = w(x-1)[b_0(x) + (x-1)b_1(x)] = x. \quad (9)$$

Now we extend $\phi(x)$ onto $[0, 1]$ by defining

$$\phi(0) = 1, \text{ and } \phi(1) = 0.$$

Then the translation invariant RKP shape function of reproducing order 1, associated to any weight function $w(x)$ with support $[-1, 1]$, is the hat function.

In this example, for the construction of the RKP shape function, we only used the support size of $w(x)$, but did not use the nature of $w(x)$. Hence the resulting RKP function $\phi(x)$ is independent of the smoothness of the weight function $w(x)$. This is because the interior of $\text{supp } w(x)$ contains only one particle and hence $M(x)$ is singular at $x = 0, 1$.

It is stated in ([3] and [8]) that

$$\text{if } w(x) \in \mathcal{C}^q, \text{ then } \phi(x) \in \mathcal{C}^q,$$

provided that for all $x \in \mathbb{R}^n$, $M(x)$ is nonsingular (that is, $\det M(x) \neq 0$). In fact, if $M_{(\alpha)}(x)$ is the matrix obtained by replacing the α the column of $M(x)$ with $(1, 0, 0, \dots, 0)^T$, then we have $b_\alpha(x) = [\det M_{(\alpha)}(x)/\det M(x)] \in \mathcal{C}^q$ whenever $w(x) \in \mathcal{C}^q$. Hence the relation (5) implies $\phi(x) \in \mathcal{C}^q$.

Now, we are motivated to find another method constructing RKP shape functions without getting weight functions involved. For this purpose, we consider the following characterization.

By applying a similar argument to [3], one can show that (4) is equivalent to

$$\sum_{j \in \mathbb{Z}} (x - x_j)^\beta \phi_j(x) = \delta_{|\beta|}^0, \quad \text{for } 0 \leq |\beta| \leq k \text{ and } x \in \mathbb{R}^n. \quad (10)$$

This characterization of the RKP shape functions have no direct relation with the weight functions. Thus, by using this equivalent definition, various closed form RKP shape functions are constructed in the following sections.

3 One dimensional closed form RKP shape functions

In this section, by using (10), we construct piecewise polynomial shape functions that have the reproducing property of any desired reproducing order and also satisfy the Kronecker delta property. Since we do not use any specific window functions in our construction, it may be more accurate to call our shape functions “Reproducing particle shape functions”. However, since the reproducing particle shape functions constructed in this paper satisfy the equivalent definition of the RKP shape functions of the reproducing property of order k , we simply call them RKP shape functions.

For the construction purpose, we prove the following key lemma.

Lemma 3.1. *Suppose K is a positive integer and $f(x)$ is a function defined on*

$$\Omega = [-K, -K + 1) \cup \dots \cup (-1, 0) \cup (0, 1) \cup \dots \cup (K - 2, K - 1) \cup (K - 1, K],$$

such that, if

$$f|_{(j,j+1)}(x), \quad j = K - 1, K - 2, \dots, 2, 1, 0, -1, -2, \dots, -K,$$

are shifted on $(0, 1)$, then they are the solutions of the following system of linear function equations:

$$V(x) \cdot [f(x + (K - 1)), \dots, f(x + 1), f(x), f(x - 1), \dots, f(x - K)]^T = e_i, \quad (11)$$

where $x \in (0, 1)$, $e_i = [0, 0, \dots, 0, 1, 0, \dots, 0]^T$, the i -th base vector, and $V(x)$ is the following Vandermonde matrix:

$$\begin{bmatrix} 1 & \dots & 1 & 1 & \dots & 1 & 1 \\ (x + (K - 1)) & \dots & x + 1 & x & \dots & (x - (K - 1)) & (x - K) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (x + (K - 1))^{2K-1} & \dots & (x + 1)^{2K-1} & x^{2K-1} & \dots & (x - (K - 1))^{2K-1} & (x - K)^{2K-1} \end{bmatrix}.$$

Then

- (1) *$f(x)$ can be extended to a continuous piecewise polynomial function $F(x)$ defined on $\overline{\Omega}$, if the right hand side vector is e_i for an odd number i .*

In particular, if the right hand side vector is e_1 , $F(x)$ satisfies the Kronecker delta property. Moreover, in this case, $f(x)$ is also extendable to a continuous function $F(x)$ that satisfies the Kronecker delta property even when $f(x)$ has a non symmetric support (that is, $\Omega = [-K, K + 1] \setminus \mathbb{Z}$ or $\Omega = [-K - 1, K] \setminus \mathbb{Z}$).

- (2) *No continuous extensions are possible, if the right side vector is e_i for an even number i .*

Proof. (1) It suffices to show that for $l = -(K-1), \dots, -1, 0, 1, \dots, (K-1)$, the following is satisfied:

$$\lim_{x \rightarrow 1} f(x-l) = \lim_{x \rightarrow 0} f(x-(l-1)).$$

The determinant of the Vandermonde matrix $V(x)$ is $D \equiv \prod_{m=0}^{2K-1} m!$ ([16]). For a notational brevity, we use l for the ordinal column number of $V(x)$. That is, the column number, $-(K-1), \dots, -1, 0, 1, \dots, (K-1)$, are the column number, $1, 2, \dots, K, \dots, (2K-1)$, respectively. Let $V_{(l)}(x)$ is the matrix obtained by replacing the l -th column of $V(x)$ with the base vector e_i . Then

$$\begin{aligned} \lim_{x \rightarrow 1} f(x-l) &= \frac{\det V_{(l)}(1)}{D}, \\ \lim_{x \rightarrow 0} f(x-(l-1)) &= \frac{\det V_{(l-1)}(0)}{D}. \end{aligned}$$

By evaluating the Vandermonde matrix $V(x)$ at $x = 0, 1$, we have the following.

$$\begin{aligned} V(0) &= \begin{bmatrix} 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ (K-1)^1 & \cdots & 2^1 & 1 & 0 & -1 & \cdots & (2-K)^1 & (1-K)^1 & (-K)^1 \\ (K-1)^2 & \cdots & 2^2 & 1 & 0 & 1 & \cdots & (2-K)^2 & (1-K)^2 & (-K)^2 \\ (K-1)^3 & \cdots & 2^3 & 1 & 0 & -1 & \cdots & (2-K)^3 & (1-K)^3 & (-K)^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \\ V(1) &= \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ K^1 & (K-1)^1 & \cdots & 2^1 & 1 & 0 & -1 & \cdots & (2-K)^1 & (1-K)^1 \\ K^2 & (K-1)^2 & \cdots & 2^2 & 1 & 0 & 1 & \cdots & (2-K)^2 & (1-K)^2 \\ K^3 & (K-1)^3 & \cdots & 2^3 & 1 & 0 & -1 & \cdots & (2-K)^3 & (1-K)^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \end{aligned}$$

Then the l -th column of $V(0)$ is the same as $(l+1)$ -th column of $V(1)$, for $l = 1, 2, \dots, 2K-1$. Moreover, for $l > 1$, $V_{(l)}(1)$ and $V_{(l-1)}(0)$, respectively, can be obtained by replacing the l -th column of $V(1)$ and the $(l-1)$ -th column of $V(0)$ with the right side vector e_i .

(a) If the right hand side of (11) is e_1 , the continuity of the extension $F(x)$ is straight forward. Suppose the right had side of (11) is e_1 and $l-1 \neq K$, then $l \neq K+1$ and hence two columns of $V_{(l-1)}(0)$ and two columns of $V_{(l)}(1)$, respectively, are the same as e_1 . Hence, $\det V_{(l-1)}(0) = \det V_{(l)}(1) = 0$.

If $l-1 = K$, then $\det V_{(K)}(0) = \det V(0)$ and $\det V_{(K+1)}(1) = \det V(1)$. Thus $\det V_{(K)}(0)/D = 1 = \det V_{(K+1)}(1)/D$.

Hence, $f(x)$ determined by (11) can be extended to a continuous function $F(x)$ which satisfy the Kronecker delta property.

This argument is independent of the dimension of Vandermonde matrix $V(x)$. Hence, this conclusion holds even when $\Omega = [-K, K + 1] \setminus \mathbb{Z}$ or $\Omega = [-K - 1, K] \setminus \mathbb{Z}$.

(b) Suppose the right hand side of (11) is e_j for an odd number $j > 0$. Without loss of generality, let us suppose the right hand side is $e_3 = [0, 0, 1, 0, \dots, 0]^T$. By the column operations (adding the m -th column to the $(2K - m)$ -th column, $m = 1, 2, \dots, K - 1$), and N times row or column interchanges (depends on the location of l) to $V_{(l-1)}(0)$, we get

$$\begin{bmatrix} (K-1)^1 & \cdots & 2 & 1 & (-K)^1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ (K-1)^3 & \cdots & 2^3 & 1 & (-K)^3 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdot & \cdot & \cdot & 0 & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 0 & 1 & 2 & \cdots & 2 & 2 \\ (K-1)^2 & \cdots & 2^2 & 1 & (-K)^2 & 1 & 0 & 2 & \cdots & 2(2-K)^2 & 2(1-K)^2 \\ (K-1)^4 & \cdots & 2^4 & 1 & (-K)^4 & 0 & 0 & 2 & \cdots & 2(2-K)^4 & 2(1-K)^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

By applying Laplace expansion (generalized cofactor expansion), we have

$$\det V_{(l-1)}(0) = (-1)^N \det A \cdot \det B, \text{ where}$$

$$A = \begin{bmatrix} (K-1)^1 & \cdots & 2 & 1 & (-K)^1 \\ (K-1)^3 & \cdots & 2^3 & 1 & (-K)^3 \\ (K-1)^5 & \cdots & 2^5 & 1 & (-K)^5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 0 & 2 & 2(2)^2 & \cdots & 2(2-K)^2 & 2(1-K)^2 \\ 0 & 0 & 2 & 2(2)^4 & \cdots & 2(2-K)^4 & 2(1-K)^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

(Let us note that if the right hand side is e_j for an even number j , for example $e_2 = [0, 1, 0, \dots, 0]^T$, then this decomposition of determinant of $V_{(l-1)}(0)$ is impossible. One can easily construct a counterexample).

Let $\tilde{V}_{(l)}(1)$ be the matrix resulted by $(2K - 1)$ column interchanges to $V_{(l)}(1)$. Then we have

$$\det V_{(l)}(1) = -\det \tilde{V}_{(l)}(1),$$

and $\tilde{V}_{(l)}(1)$ is the same as $V_{(l-1)}(0)$ except the fact that the $2K$ -th column has an opposite sign.

If one apply the same column operations and Laplace expansion to $\tilde{V}_{(l)}(1)$, the columns of the generalized cofactor matrix corresponding to A are the same as A except that all of the last column have opposite sign. Therefore, we have

$$\det V_{(l-1)}(0) = -\det \tilde{V}_{(l)}(1) = \det V_{(l)}(1).$$

(2) Suppose the right side of (11) is the even-th base vector e_{2j} . Without loss of generality, we assume $e_{2j} = e_2$. Then by the column operations (adding (-1) times m -th column to $(2K - m)$ -th column, $m = 1, 2, \dots, K - 1$), and N' times row and column interchanges to $V_{(l-1)}(0)$, we get

$$\begin{bmatrix} (K-1)^2 & \dots & 2^2 & 1 & (-K)^2 & 0 & 0 & 0 & \dots & 0 & 0 \\ (K-1)^4 & \dots & 2^4 & 1 & (-K)^4 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 & 0 & \dots & 0 & 0 \\ 1 & \dots & 1 & 1 & 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ (K-1)^1 & \dots & 2^1 & 1 & (-K)^1 & 1 & 0 & -2 & \dots & -2(2-K)^1 & -2(1-K)^1 \\ (K-1)^3 & \dots & 2^3 & 1 & (-K)^3 & 0 & 0 & -2 & \dots & -2(2-K)^3 & -2(1-K)^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

By generalized cofactor expansion, we get the following

$$\begin{aligned} \det V_{(l-1)}(0) &= (-1)^{N'} \det A' \cdot \det B', \text{ where} \\ A' &= \begin{bmatrix} (K-1)^2 & \dots & 2^2 & 1 & (-K)^2 \\ (K-1)^4 & \dots & 2^4 & 1 & (-K)^4 \\ (K-1)^8 & \dots & 2^8 & 1 & (-K)^8 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\ B' &= \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & -2 & -2(2)^1 & \dots & -2(2-K)^1 & -2(1-K)^1 \\ 0 & 0 & -2 & -2(2)^3 & \dots & -2(2-K)^3 & -2(1-K)^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \end{aligned}$$

Hence, we have

$$\det V_{(l-1)}(0) = \det \tilde{V}_{(l)}(1) = -\det V_{(l)}(1).$$

Therefore, $f(x)$ cannot be made to be continuous at integers in $(-K, K)$. □

3.1 Existence and uniqueness of a closed form C^0 - RKP shape function of maximum reproducing order.

Theorem 3.1. *There exists a unique translation invariant RKP shape function $\phi(x)$, which satisfies the following conditions:*

- (a) *The support of $\phi(x)$ is $[-K, K]$, where K is a positive integer,*
- (b) *The reproducing order of $\phi(x)$ is $2K - 1$,*

(c) $\phi(x)$ is continuous.

(d) This satisfies the Kronecker delta property: $\phi_j(i) = \delta_i^j$.

The unique RKP basic shape function is a piecewise polynomial of degree $2K - 1$.

Moreover, this RKP shape function is independent of choice of weight functions. Since $M(x)$ is singular in general for some $x \in [-K, K]$, it is not possible to construct a RKP shape function of reproducing order $2K - 1$ through the standard construction (5) and (6).

Proof. For $x \in (0, 1)$, a translation invariant RKP shape function of order $2K - 1$ satisfies the following system of equations:

$$\sum_{k=-K+1}^K (x-k)^\alpha \phi(x-k) = \delta_0^\alpha, \alpha = 0, 1, 2, \dots, 2K-1$$

This linear system can be written in matrix form as follow.

$$V(x) \cdot [\phi(x+(K-1)), \phi(x+(K-2)), \dots, \phi(x), \dots, \phi(x-(K-1)), \phi(x-K)]^T = e_1. \quad (12)$$

Lemma 3.1 shows that the shape function $\phi(x)$ determined as the solutions of this system can be continuously extended to $[-K, K]$ so that the extended function satisfies the Kronecker delta property. Moreover, such a unique extension makes $\phi(x)$ to be also satisfied the definition (4) at all integer points. Since the system (12) has a unique solution and the extension of the solution to $[-K, K]$ is also unique, this is the only RKP basic shape function of order $2K - 1$ whose support is $[-K, K]$.

On each interval by two consecutive integers, $\phi(x)$ is a constant times the determinant of $(2K-1) \times (2K-1)$ minor matrix of $V(x)$, $\phi(x)$ is a piecewise polynomial of degree $2K - 1$. \square

Remark 3.1. It follows from lemma 3.1 that Theorem 3.1 holds for non-symmetric support $[-K, K+1]$ or $[-K-1, K]$.

Remark 3.2. If the support of $\phi(x)$ is exactly $[-1, 1]$, then the unique RKP shape function of reproducing order 1 is the hat function. However, if the support of the associated window function is different from $[-1, 1]$, for example, $[-1.5, 1.5]$, then the argument of above theorem is not applicable. Hence the corresponding RKP function may not be the hat function.

Remark 3.3. In [3], it is stated that the regularity of a RKP shape function $\phi(x)$ is the same as that of the associated weight function $w(x)$. This statement is not applicable to the foregoing theorem because the support is not large enough to make for the moment matrix $M(x)$ to be non singular for all $x \in \mathbb{R}$.

3.2 Constructions of closed form \mathcal{C}^m -RKP shape functions ($m \geq 0$) that satisfy the Kronecker delta property

In parts [A] and [B] of this subsection, by solving the linear system (12), various one dimensional \mathcal{C}^m RKP shape functions of given reproducing orders are constructed.

In section 3.3, the effectiveness of these RKP shape functions are tested by applying them to the second order and the fourth order differential equations, respectively.

In Theorem 3.1, for a given symmetric support $[-K, K]$, we considered RKP shape functions with the reproducing property of odd order $2K - 1$. However, by applying similar arguments for non symmetric support $[-K, K + 1]$ (or $[-(K + 1), K]$), one can construct RKP shape functions satisfying the reproducing property of even order $2K$.

Lemma 3.1 makes it possible to construct the unique closed form \mathcal{C}^0 p.p. RKP shape function of reproducing order $2K - 1$ that satisfy the Kronecker delta property with respect to the given support $[-K, K]$.

However, by sacrificing one order from the maximal reproducing order $2K - 1$ (or extending the support to $[-(K + 1), (K + 1)]$ if we want to keep the reproducing order), for a given integer $m > 0$, we are able to construct a \mathcal{C}^m -p.p. RKP shape function on $[-K, K]$ (or $[-K, K + 1]$) that satisfies the Kronecker delta property.

For this purpose, we start with a piecewise smooth function $f(x)$ that satisfies

$$\sum_{k=-K+1}^K (x-k)^\alpha f(x-k) = \delta_0^\alpha, \alpha = 0, 1, 2, \dots, 2K-2, \text{ for } x \in (0, 1).$$

Then this system is under determined. In order to make for this system to be an exactly determined system, we add an additional equation to this system as follows: for $x \in (0, 1)$,

$$\begin{cases} \sum_{k=-K+1}^K (x-k)^\alpha f(x-k) & = \delta_0^\alpha, \alpha = 0, 1, 2, \dots, 2K-2, \\ \sum_{k=-K+1}^K (x-k)^{2K-1} f(x-k) & = G(x). \end{cases} \quad (13)$$

Then the coefficient matrix of this extended system becomes $2K \times 2K$ Vandermonde matrix. The right hand side becomes the $2K$ dimensional column vector

$$[1, 0, 0, \dots, 0, G(x)]^T.$$

If $G(0) = 0$ and $G(1) = 0$, then it follows from Lemma 3.1 that the solution vector

$$(f(x+K-1), f(x+(K-2)), \dots, f(x+1), f(x), f(x-1), \dots, f(x-K))^T, x \in (0, 1)$$

for $f(x)$ can be extended to be a continuous function $F(x)$ on $[-K, K]$ that satisfies the Kronecker delta property. Thus, we impose the following conditions on $G(x)$:

$$G(0) = 0 \text{ and } G(1) = 0. \quad (14)$$

Next, we are going to impose some additional condition on $G(x)$ so that $f(x)$ can be extended to a \mathcal{C}^2 shape function.

By differentiating the expanded system (13) twice, we have

$$\begin{aligned} & V(x) \cdot [f''(x+K-1), \dots, f''(x+1), f''(x), f''(x-1), \dots, f''(x-K)]^T \\ &= [0, 0, 2, 0, \dots, 0, 0, G'''(x)]^T. \end{aligned} \quad (15)$$

Since the right side of the last system has a non-zero entry on the odd number row, Lemma 3.1 implies that the solution vector of this system can be made to be a continuous function whenever $G''(0) = 0$ and $G''(1) = 0$. Thus, we impose the following condition:

$$G''(0) = 0 \text{ and } G''(1) = 0. \quad (16)$$

For this construction, we have to start with (13) when the freedom parameter function $G(x)$ has the form

$$G(x) = x(x-1)p(x),$$

where $p(x)$ is a polynomial.

In what follows, the indices of RKP shape function $\phi_{([a,b];m_2;m_3)}(x)$ indicate the following:

$$\begin{aligned} [a, b] &= \text{the support of } \phi(x), \\ m_2 &= \text{the order of the regularity (that is, } \phi(x) \in \mathcal{C}^{m_2}), \\ m_3 &= \text{the order of the reproducing property.} \end{aligned}$$

The graphs of these \mathcal{C}^0 -RKP shape functions are depicted in Fig. 1, from which one can observe that the RKP functions satisfy the Kronecker delta property.

In the following, we constructed the \mathcal{C}^m RKP shape function up to the regularity order $m = 2$. However, by using a similar argument, one can construct RKP functions that are regular up to any desired regularity order.

All RKP shape functions constructed in this section satisfy the Kronecker delta property.

A: The RKP shape functions with symmetric support $[-K, K]$.

[A.I] The support is $[-1, 1]$.

The unique RKP shape function of reproducing order 1 whose support is $[-1, 1]$ is the following hat function.

$$\phi_{([-1,1];0;1)}(x) = \begin{cases} x+1 & x \in [-1, 0] \\ -x+1 & x \in [0, 1] \\ 0 & x \in \mathbb{R} \setminus [-1, 1] \end{cases}$$

[A.II] The support is $[-2, 2]$.

We construct the RKP shape functions of various regularity order that have the reproducing property of order $k \geq 2$.

Consider the following system

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ (x-2) & (x-1) & x & (x+1) \\ (x-2)^2 & (x-1)^2 & x^2 & (x+1)^2 \\ (x-2)^3 & (x-1)^3 & x^3 & (x+1)^3 \end{bmatrix} \cdot \begin{bmatrix} \phi(x-2) \\ \phi(x-1) \\ \phi(x) \\ \phi(x+1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ G(x) \end{bmatrix} \quad (17)$$

We already know that $G(x)$ must have a factor $x(x-1)$. So it suffices to consider the following case:

$$G(x) = x(x-1)p(x)$$

Case II.1 $p(x) = 0$.

In this case, we obtain the unique \mathcal{C}^0 RKP shape function $\phi_{([-2,2];0;3)}(x)$ of reproducing order 3.

$$\phi_{([-2,2];0;3)}(x) = \begin{cases} \frac{1}{6}(x+1)(x+2)(x+3) & x \in [-2, -1] \\ -\frac{1}{2}(x-1)(x+1)(x+2) & x \in [-1, 0] \\ \frac{1}{2}(x-2)(x-1)(x+1) & x \in [0, 1] \\ -\frac{1}{6}(x-3)(x-2)(x-1) & x \in [1, 2] \\ 0 & x \in \mathbb{R} \setminus [-2, 2] \end{cases}$$

Case II.2 $p(x) = a_0 + a_1x$.

In this case, we can construct a \mathcal{C}^1 RKP function of reproducing order 2. In order for the solution of the extended system (17) to be extended to a \mathcal{C}^1 -function, $p(x)$ is uniquely determined as

$$a_0 = 1, \quad a_1 = -2.$$

$$\phi_{([-2,2];1;2)}(x) = \begin{cases} \frac{1}{2}(x+2)^2(x+1) & \in [-2, -1], \\ -\frac{1}{2}(x+1)(3x^2+2x-2) & \in [-1, 0], \\ \frac{1}{2}(x-1)(3x^2-2x-2) & \in [0, 1], \\ -\frac{1}{2}(x-2)^2(x-1) & \in [1, 2], \\ 0 & \in \mathbb{R} \setminus [-2, 2] \end{cases}$$

Case II.3 $p = a_0 + a_1x + a_2x^2 + a_3x^3$.

In this case, we can construct a \mathcal{C}^2 RKP function of reproducing order 2. In order for the solution of the extended system (17) to be extended to a \mathcal{C}^2 -function, $p(x)$ should be

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = -9, \quad a_3 = 6.$$

$$\phi_{([-2,2];2;2)}(x) = \begin{cases} -\frac{1}{2}(x+2)^3(x+1)(2x+1) & x \in [-2, -1], \\ \frac{1}{2}(x+1)(6x^4 + 9x^3 - 2x + 2) & x \in [-1, 0], \\ -\frac{1}{2}(x-1)(6x^4 - 9x^3 + 2x + 2) & x \in [0, 1], \\ \frac{1}{2}(x-2)^3(x-1)(2x-1) & x \in [1, 2], \\ 0 & x \in \mathbb{R} \setminus [-2, 2] \end{cases}$$

Case II.4 $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$.

In this case, we can construct a C^3 RKP function of reproducing order 2. In order for the solution of the extended system (17) to be extended to a C^1 -function, $p(x)$ should be

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = -30, \quad a_4 = 45, \quad a_5 = -18.$$

$$\phi_{([-2,2];3;2)}(x) = \begin{cases} \frac{1}{2}(x+2)^4(x+1)(6x^2 + 9x + 4) & x \in [-2, -1], \\ -\frac{1}{2}(x+1)(18x^6 + 45x^5 + 30x^4 + 2x - 2) & x \in [-1, 0], \\ \frac{1}{2}(x-1)(18x^6 - 45x^5 + 30x^4 - 2x - 2) & x \in [0, 1], \\ -\frac{1}{2}(x-2)^4(x-1)(6x^2 - 9x + 4) & x \in [1, 2], \\ 0 & x \in \mathbb{R} \setminus [-2, 2] \end{cases}$$

Case II.5 $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$.

In this case, we can construct a C^4 RKP function of reproducing order 2. The only choice of $p(x)$ for the solution of the extended system (17) to have a C^4 extension is

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = -105, \quad a_5 = 252, \quad a_6 = -210, \quad a_7 = 60.$$

$$\phi_{([-2,2];4;2)}(x) = \begin{cases} -\frac{1}{2}(x+2)^5(x+1)(20x^3 + 50x^2 + 44x + 13) & x \in [-2, -1], \\ \frac{1}{2}(x+1)(60x^8 + 210x^7 + 252x^6 + 105x^5 - 2x + 2) & x \in [-1, 0], \\ -\frac{1}{2}(x-1)(60x^8 - 210x^7 + 252x^6 - 105x^5 + 2x + 2) & x \in [0, 1], \\ \frac{1}{2}(x-2)^5(x-1)(20x^3 - 50x^2 + 44x - 13) & x \in [1, 2], \\ 0 & x \in \mathbb{R} \setminus [-2, 2] \end{cases}$$

[A.III] The support is $[-3, 3]$.

We construct the RKP shape functions of various regularity order that have the reproducing property of order $k \geq 4$.

In this case, we have the following equation

$$V_3(x) \cdot \begin{bmatrix} \phi(x-3) \\ \phi(x-2) \\ \phi(x-1) \\ \phi(x) \\ \phi(x+1) \\ \phi(x+2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ G(x) \end{bmatrix}, \quad (18)$$

where

$$V_3(x) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ (x-3) & (x-2) & (x-1) & x & (x+1) & (x+2) \\ (x-3)^2 & (x-2)^2 & (x-1)^2 & x^2 & (x+1)^2 & (x+2)^2 \\ (x-3)^3 & (x-2)^3 & (x-1)^3 & x^3 & (x+1)^3 & (x+2)^3 \\ (x-3)^4 & (x-2)^4 & (x-1)^4 & x^4 & (x+1)^4 & (x+2)^4 \\ (x-3)^5 & (x-2)^5 & (x-1)^5 & x^5 & (x+1)^5 & (x+2)^5 \end{bmatrix}$$

As before, it suffices to consider the following case:

$$G(x) = x(x-1)p(x)$$

Case III.1 $p(x) = 0$.

In this case, we obtain the unique C^0 RKP shape function $\phi_{([-3,3];0;5)}$ of reproducing order 5.

$$\phi_{([-3,3];0;5)}(x) = \begin{cases} \frac{1}{120}(x+1)(x+2)(x+3)(x+4)(x+5) & x \in [-3, -2] \\ -\frac{1}{24}(x-1)(x+1)(x+2)(x+3)(x+4) & x \in [-2, -1] \\ \frac{1}{12}(x-2)(x-1)(x+1)(x+2)(x+3) & x \in [-1, 0] \\ -\frac{1}{12}(x-3)(x-2)(x-1)(x+1)(x+2) & x \in [0, 1] \\ \frac{1}{24}(x-4)(x-3)(x-2)(x-1)(x+1) & x \in [1, 2] \\ -\frac{1}{120}(x-5)(x-4)(x-3)(x-2)(x-1) & x \in [2, 3] \\ 0 & x \in \mathbb{R} \setminus [-3, 3] \end{cases}$$

Case III.2 $p(x) = a_0 + a_1x$.

In this case, we can construct a C^1 RKP function of reproducing order 4. The only choice of $p(x)$ for the solution of the extended system (18) to have a \mathcal{C}^1 extension is

$$a_0 = 4, \quad a_1 = -8.$$

$$\phi_{([-3,3];1;4)}(x) = \begin{cases} \frac{1}{120}x(2+x)(3+x)^2(7+x) & x \in [-3, -2], \\ -\frac{1}{24}(1+x)(2+x)(-24-3x+6x^2+x^3) & x \in [-2, -1], \\ \frac{1}{12}(1+x)(12-12x-15x^2+2x^3+x^4) & x \in [-1, 0], \\ -\frac{1}{12}(-1+x)(12+12x-15x^2-2x^3+x^4) & x \in [0, 1], \\ \frac{1}{24}(-2+x)(-1+x)(24-3x-6x^2+x^3) & x \in [1, 2], \\ -\frac{1}{120}(-7+x)(-3+x)^2(-2+x)x & x \in [2, 3], \\ 0 & x \in \mathbb{R} \setminus [-3, 3] \end{cases}$$

Case III.3 $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

In this case, we can construct a C^2 RKP function of reproducing order 4. The only choice of $p(x)$ for the solution of the extended system (18) to have a C^2 extension is

$$a_0 = 4, \quad a_1 = 4, \quad a_2 = -36, \quad a_3 = 24.$$

$$\phi_{([-3,3];2;4)}(x) = \begin{cases} \frac{1}{24}(2+x)(3+x)^3(8+5x) & x \in [-3, -2], \\ -\frac{1}{24}(1+x)(2+x)(48+153x+114x^2+25x^3) & x \in [-2, -1], \\ \frac{1}{12}(1+x)(12-12x-3x^2+38x^3+25x^4) & x \in [-1, 0], \\ -\frac{1}{12}(-1+x)(12+12x-3x^2-38x^3+25x^4) & x \in [0, 1], \\ \frac{1}{24}(-2+x)(-1+x)(-48+153x-114x^2+25x^3) & x \in [1, 2], \\ -\frac{1}{24}(-3+x)^3(-2+x)(-8+5x) & x \in [2, 3], \\ 0 & x \in \mathbb{R} \setminus [-3, 3] \end{cases}$$

[A.IV] The support is $[-4, 4]$.

We construct the RKP shape functions of various regularity order that satisfy the reproducing property of order $k \geq 6$.

We consider the following equation:

$$V_4(x) \cdot \begin{bmatrix} \phi(x-4) \\ \phi(x-3) \\ \phi(x-2) \\ \phi(x-1) \\ \phi(x) \\ \phi(x+1) \\ \phi(x+2) \\ \phi(x+3) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ G(x) \end{bmatrix}, \quad (19)$$

where $V_4(x)$ is the 8×8 Vandermonde matrix.

It suffices to consider the following case:

$$G(x) = x(x-1)p(x)$$

Case IV.1 $p(x) = 0$.

In this case, we obtain the unique C^0 RKP function $\phi_{([-4,4];0;7)}$ of reproducing order 7.

$$\phi_{([-4,4];0;7)}(x) = \begin{cases} \frac{1}{5040}(1+x)(2+x)(3+x)(4+x)(5+x)(6+x)(7+x) & x \in [-4, -3], \\ -\frac{1}{720}(-1+x)(1+x)(2+x)(3+x)(4+x)(5+x)(6+x) & x \in [-3, -2], \\ \frac{1}{240}(-2+x)(-1+x)(1+x)(2+x)(3+x)(4+x)(5+x) & x \in [-2, -1], \\ -\frac{1}{144}(-3+x)(-2+x)(-1+x)(1+x)(2+x)(3+x)(4+x) & x \in [-1, 0], \\ \frac{1}{144}(-4+x)(-3+x)(-2+x)(-1+x)(1+x)(2+x)(3+x) & x \in [0, 1], \\ -\frac{1}{240}(-5+x)(-4+x)(-3+x)(-2+x)(-1+x)(1+x)(2+x) & x \in [1, 2], \\ \frac{1}{720}(-6+x)(-5+x)(-4+x)(-3+x)(-2+x)(-1+x)(1+x) & x \in [2, 3], \\ -\frac{1}{5040}(-7+x)(-6+x)(-5+x)(-4+x)(-3+x)(-2+x)(-1+x) & x \in [3, 4], \\ 0 & x \in \mathbb{R} \setminus [-4, 4] \end{cases}$$

Case IV.2 $p(x) = a_0 + a_1x$.

In this case, we can construct C^1 RKP function of reproducing order 6. The only choice of $p(x)$ for the solution of the extended system (19) to have a C^1 extension is

$$a_0 = -36, \quad a_1 = 72.$$

$$\phi_{([-4,4];1;6)}(x) = \begin{cases} \frac{1}{5040}(3+x)(4+x)^2(168 + 187x + 95x^2 + 17x^3 + x^4) & x \in [-4, -3], \\ -\frac{1}{720}(2+x)(3+x)(60 - 2x + 105x^2 + 73x^3 + 15x^4 + x^5) & x \in [-3, -2], \\ \frac{1}{240}(1+x)(2+x)(228 - 14x - 57x^2 + 13x^3 + 9x^4 + x^5) & x \in [-2, -1], \\ -\frac{1}{144}(1+x)(-144 + 144x + 160x^2 - 39x^3 - 17x^4 + 3x^5 + x^6) & x \in [-1, 0], \\ \frac{1}{144}(-1+x)(-144 - 144x + 160x^2 + 39x^3 - 17x^4 - 3x^5 + x^6) & x \in [0, 1], \\ -\frac{1}{240}(-2+x)(-1+x)(-228 - 14x + 57x^2 + 13x^3 - 9x^4 + x^5) & x \in [1, 2], \\ \frac{1}{720}(-3+x)(-2+x)(-60 - 2x - 105x^2 + 73x^3 - 15x^4 + x^5) & x \in [2, 3], \\ -\frac{1}{5040}(-4+x)^2(-3+x)(168 - 187x + 95x^2 - 17x^3 + x^4) & x \in [3, 4], \\ 0 & x \in \mathbb{R} \setminus [-4, 4] \end{cases}$$

Case IV.3 $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

In this case, we can construct C^2 RKP functions of reproducing order 6. The only choice of $p(x)$ for the solution of the extended system (19) to have a C^2 extension is

$$a_0 = -36, \quad a_1 = -36, \quad a_2 = 324, \quad a_3 = -216.$$

$$\phi_{([-4,4];2;6)}(x) = \begin{cases} \frac{1}{5040}(4+x)^3(3+x)(-525-173x+13x^2+x^3) & x \in [-4, -3], \\ -\frac{1}{720}(3+x)(2+x)(-3180-3998x-1515x^2-143x^3+15x^4+x^5) & x \in [-3, -2], \\ \frac{1}{240}(2+x)(1+x)(-420-1418x-1029x^2-203x^3+9x^4+x^5) & x \in [-2, -1], \\ -\frac{1}{144}(1+x)(-144+144x+52x^2-363x^3-233x^4+3x^5+x^6) & x \in [-1, 0], \\ \frac{1}{144}(-1+x)(-144-144x+52x^2+363x^3-233x^4-3x^5+x^6) & x \in [0, 1], \\ -\frac{1}{240}(-2+x)(-1+x)(420-1418x+1029x^2-203x^3-9x^4+x^5) & x \in [1, 2], \\ \frac{1}{720}(-3+x)(-2+x)(3180-3998x+1515x^2-143x^3-15x^4+x^5) & x \in [2, 3], \\ -\frac{1}{5040}(-4+x)^3(-3+x)(525-173x-13x^2+x^3) & x \in [3, 4], \\ 0 & x \in \mathbb{R} \setminus [-4, 4] \end{cases}$$

B: The RKP shape functions with non symmetric support $[-K, K+1], [-K-1, K]$.

Since a shape function with support $[-K-1, K]$ can be obtained from that on $[-K, K+1]$ by flipping it over with respect to the y -axis, we only construct the RKP shape functions with support $[-K, K+1]$.

All RKP shape functions constructed in the following has the Kronecker delta property.

[B.V] The support is $[-1, 2]$.

When the support is $[-1, 2]$, we also construct RKP shape functions that have the reproducing property of order $k \geq 1$.

In this case, we have the following equation

$$\begin{bmatrix} 1 & 1 & 1 \\ (x-1) & x & (x+1) \\ (x-1)^2 & x^2 & (x+1)^2 \end{bmatrix} \cdot \begin{bmatrix} \phi(x-1) \\ \phi(x) \\ \phi(x+1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ G(x) \end{bmatrix} \quad (20)$$

We already know that $G(x)$ must have a factor $x(x-1)$. So it suffices to consider the following case:

$$G(x) = x(x-1)p(x)$$

Case V.1 $p(x) = 0$.

In this case, we obtain the unique C^0 RKP function of the reproducing property of maximal order 2.

$$\phi_{([-1,2];0;2)}(x) = \begin{cases} \frac{1}{2}(1+x)(2+x) & x \in [-1, 0], \\ -(-1+x)(1+x) & x \in [0, 1], \\ \frac{1}{2}(-2+x)(-1+x) & x \in [1, 2], \\ 0 & x \in \mathbb{R} \setminus [-1, 2] \end{cases}$$

Case V.2 $p(x) = a_0 + a_1x$.

In this case, we can construct a C^1 RKP function of reproducing order 1. The only choice of $p(x)$ for the solution of the extended system (20) to have a C^1 extension is

$$a_0 = -1, a_1 = 2.$$

$$\phi_{([-1,2];1;1)}(x) = \begin{cases} -(-1+x)(1+x)^2 & x \in [-1, 0], \\ (-1+x)(-1-2x+2x^2) & x \in [0, 1], \\ -(-2+x)^2(-1+x) & x \in [1, 2], \\ 0 & x \in \mathbb{R} \setminus [-1, 2] \end{cases}$$

Case V.3 $p = a_0 + a_1x + a_2x^2 + a_3x^3$.

In this case, we can construct a C^2 RKP function of reproducing order 1. The only choice of $p(x)$ for the solution of the extended system (20) to have C^2 extension is

$$a_0 = -1, a_1 = -2, a_2 = 10, a_3 = -6.$$

$$\phi_{([-1,2];2;1)}(x) = \begin{cases} \frac{1}{2}(1+x)(2+8x^3+6x^4) & x \in [-1, 0], \\ -(-1+x)(1+2x+2x^2-10x^3+6x^4) & x \in [0, 1], \\ \frac{1}{2}(-2+x)(-1+x)(-16+40x-28x^2+6x^3) & x \in [1, 2], \\ 0 & x \in \mathbb{R} \setminus [-1, 2] \end{cases}$$

[B.VI] The support is $[-2, 3]$.

In this case, we construct the RKP shape functions of reproducing order $k \geq 3$. We consider the following equation

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ (x-2) & (x-1) & x & (x+1) & (x+2) \\ (x-2)^2 & (x-1)^2 & x^2 & (x+1)^2 & (x+2)^2 \\ (x-2)^3 & (x-1)^3 & x^3 & (x+1)^3 & (x+2)^3 \\ (x-2)^4 & (x-1)^4 & x^4 & (x+1)^4 & (x+2)^4 \end{bmatrix} \cdot \begin{bmatrix} \phi(x-2) \\ \phi(x-1) \\ \phi(x) \\ \phi(x+1) \\ \phi(x+2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ G(x) \end{bmatrix} \quad (21)$$

We already know that $G(x)$ must have a factor $x(x-1)$. So it suffices to consider the following case:

$$G(x) = x(x-1)p(x)$$

Case VI.1 $p(x) = 0$.

In this case, we obtain the unique C^0 RKP function of maximal reproducing order 4.

$$\phi_{([-2,3];0;4)}(x) = \begin{cases} \frac{1}{24}(1+x)(2+x)(3+x)(4+x) & x \in [-2, -1], \\ -\frac{1}{6}(-1+x)(1+x)(2+x)(3+x) & x \in [-1, 0], \\ \frac{1}{4}(-2+x)(-1+x)(1+x)(2+x) & x \in [0, 1], \\ -\frac{1}{6}(-3+x)(-2+x)(-1+x)(1+x) & x \in [1, 2], \\ \frac{1}{24}(-4+x)(-3+x)(-2+x)(-1+x) & x \in [2, 3], \\ 0 & x \in \mathbb{R} \setminus [-2, 3] \end{cases}$$

Case VI.2 $p(x) = a_0 + a_1x$.

In this case, we can construct a C^1 RKP function of reproducing order 3. The only choice of $p(x)$ for the solution of the extended system (21) to have C^1 extension is

$$a_0 = 2, \quad a_1 = -4.$$

$$\phi_{([-2,3];1;3)}(x) = \begin{cases} \frac{1}{24}(1+x)(2+x)(18+11x+x^2) & x \in [-2, -1], \\ -\frac{1}{6}(1+x)(-6+3x+8x^2+x^3) & x \in [-1, 0], \\ \frac{1}{4}(-1+x)(-4-6x+5x^2+x^3) & x \in [0, 1], \\ -\frac{1}{6}(-2+x)(-1+x)(-9+2x+x^2) & x \in [1, 2], \\ \frac{1}{24}(-3+x)(-2+x)(-6-x+x^2) & x \in [2, 3], \\ 0 & x \in \mathbb{R} \setminus [-2, 3] \end{cases}$$

Case VI.3 $p = a_0 + a_1x + a_2x^2 + a_3x^3$.

In this case, we can construct a C^2 RKP function of reproducing order 3. The only choice of $p(x)$ for the solution of the extended system (21) to have a C^2 extension is

$$a_0 = 2, \quad a_1 = 3, \quad a_2 = -19, \quad a_3 = 12.$$

$$\phi_{([-2,3];2;3)}(x) = \begin{cases} -\frac{1}{24}(1+x)(2+x)(16+64x+52x^2+12x^3) & x \in [-2, -1], \\ \frac{1}{6}(1+x)(6-3x-3x^2+16x^3+12x^4) & x \in [-1, 0], \\ -\frac{1}{4}(-1+x)(4+6x+2x^2-20x^3+12x^4) & x \in [0, 1], \\ \frac{1}{6}(-2+x)(-1+x)(-29+79x-56x^2+12x^3) & x \in [1, 2], \\ -\frac{1}{24}(-3+x)(-2+x)(-180+228x-92x^2+12x^3) & x \in [2, 3], \\ 0 & x \in \mathbb{R} \setminus [-2, 3] \end{cases}$$

[B.VII] The support is $[-3, 4]$.

In this case, we construct the RKP shape functions of reproducing order $k \geq 5$. we consider the following system of equations

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ (x-3) & (x-2) & (x-1) & x & (x+1) & (x+2) & (x+3) \\ (x-3)^2 & (x-2)^2 & (x-1)^2 & x^2 & (x+1)^2 & (x+2)^2 & (x+3)^2 \\ (x-3)^3 & (x-2)^3 & (x-1)^3 & x^3 & (x+1)^3 & (x+2)^3 & (x+3)^3 \\ (x-3)^4 & (x-2)^4 & (x-1)^4 & x^4 & (x+1)^4 & (x+2)^4 & (x+3)^4 \\ (x-3)^5 & (x-2)^5 & (x-1)^5 & x^5 & (x+1)^5 & (x+2)^5 & (x+3)^5 \\ (x-3)^6 & (x-2)^6 & (x-1)^6 & x^6 & (x+1)^6 & (x+2)^6 & (x+3)^6 \end{bmatrix} \cdot \begin{bmatrix} \phi(x-3) \\ \phi(x-2) \\ \phi(x-1) \\ \phi(x) \\ \phi(x+1) \\ \phi(x+2) \\ \phi(x+3) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ G(x) \end{bmatrix}. \quad (22)$$

We already know that $G(x)$ must have a factor $x(x-1)$. So it suffices to consider the following case:

$$G(x) = x(x-1)p(x)$$

Case VII.1 $p(x) = 0$.

In this case, we obtain the unique C^0 RKP shape function of of maximal reproducing order 6.

$$\phi_{([-3,4];0;6)}(x) =$$

$$\begin{cases} \frac{1}{720}(1+x)(2+x)(3+x)(4+x)(5+x)(6+x) & x \in [-3, -2], \\ -\frac{1}{120}(-1+x)(1+x)(2+x)(3+x)(4+x)(5+x) & x \in [-2, -1], \\ \frac{1}{48}(-2+x)(-1+x)(1+x)(2+x)(3+x)(4+x) & x \in [-1, 0], \\ -\frac{1}{36}(-3+x)(-2+x)(-1+x)(1+x)(2+x)(3+x) & x \in [0, 1], \\ \frac{1}{48}(-4+x)(-3+x)(-2+x)(-1+x)(1+x)(2+x) & x \in [1, 2], \\ -\frac{1}{120}(-5+x)(-4+x)(-3+x)(-2+x)(-1+x)(1+x) & x \in [2, 3], \\ \frac{1}{720}(-6+x)(-5+x)(-4+x)(-3+x)(-2+x)(-1+x) & x \in [3, 4], \\ 0 & x \in \mathbb{R} \setminus [-3, 4] \end{cases}$$

Case VII.2 $p(x) = a_0 + a_1x$.

In this case, we can construct a C^1 RKP function of reproducing order 5. The only choice of $p(x)$ for the solution of the extended system (22) to have a C^1 extension is

$$a_0 = -12, \quad a_1 = 24.$$

$$\phi_{([-3,4];1;5)}(x) =$$

$$\begin{cases} \frac{1}{720}(2+x)(3+x)(60+170x+89x^2+16x^3+x^4) & x \in [-3, -2], \\ -\frac{1}{120}(1+x)(2+x)(-96-11x+35x^2+11x^3+x^4) & x \in [-2, -1], \\ \frac{1}{48}(1+x)(48-32x-60x^2+x^3+6x^4+x^5) & x \in [-1, 0], \\ -\frac{1}{36}(-1+x)(36+48x-37x^2-13x^3+x^4+x^5) & x \in [0, 1], \\ \frac{1}{48}(-2+x)(-1+x)(60-2x-7x^2-4x^3+x^4) & x \in [1, 2], \\ -\frac{1}{120}(-3+x)(-2+x)(40-15x+19x^2-9x^3+x^4) & x \in [2, 3], \\ \frac{1}{720}(-4+x)(-3+x)(144-136x+65x^2-14x^3+x^4) & x \in [3, 4], \\ 0 & x \in \mathbb{R} \setminus [-3, 4] \end{cases}$$

Case VII.3 $p = a_0 + a_1x + a_2x^2 + a_3x^3$.

In this case, we can construct a C^2 RKP function of reproducing order 5. The only choice of $p(x)$ for the solution for the extended system (22) to have a C^2 extension is

$$a_0 = -12, \quad a_1 = -16, \quad a_2 = 112, \quad a_3 = -72.$$

$$\phi_{([-3,4];2;5)}(x) =$$

$$\begin{cases} \frac{1}{720}(2+x)(3+x)(1116+1482x+625x^2+88x^3+x^4) & x \in [-3, -2], \\ -\frac{1}{120}(1+x)(2+x)(112+445x+355x^2+83x^3+x^4) & x \in [-2, -1], \\ \frac{1}{48}(1+x)(48-32x-28x^2+105x^3+78x^4+x^5) & x \in [-1, 0], \\ -\frac{1}{36}(-1+x)(36+48x+3x^2-125x^3+73x^4+x^5) & x \in [0, 1], \\ \frac{1}{48}(-2+x)(-1+x)(-164+478x-335x^2+68x^3+x^4) & x \in [1, 2], \\ -\frac{1}{120}(-3+x)(-2+x)(-1064+1337x-525x^2+63x^3+x^4) & x \in [2, 3], \\ \frac{1}{720}(-4+x)(-3+x)(-2928+2520x-695x^2+58x^3+x^4) & x \in [3, 4], \\ 0 & x \in \mathbb{R} \setminus [-3, 4] \end{cases}$$

The C^2 -RKP shape functions, $\phi_{([-2,2];2;2)}(x)$, $\phi_{([-3,3];2;4)}(x)$, $\phi_{([-4,4];2;6)}(x)$, of reproducing order 2, 4, 6 respectively, and their first and second derivatives are plotted in Fig. 2. These C^2 -RKP shape functions are tested by applying to a fourth order equation in the following subsection.

[C] Another approach for the constructions of closed form RKP shape functions that may not satisfy the Kronecker delta property.

It follows from Theorem 3.1 that for a given support $[-K, K]$, the C^0 -RKP shape function exists and has the reproducing property of order $2K - 1$.

In the previous constructions, one order of reproducing property was sacrificed in order to get smoother RKP shape functions.

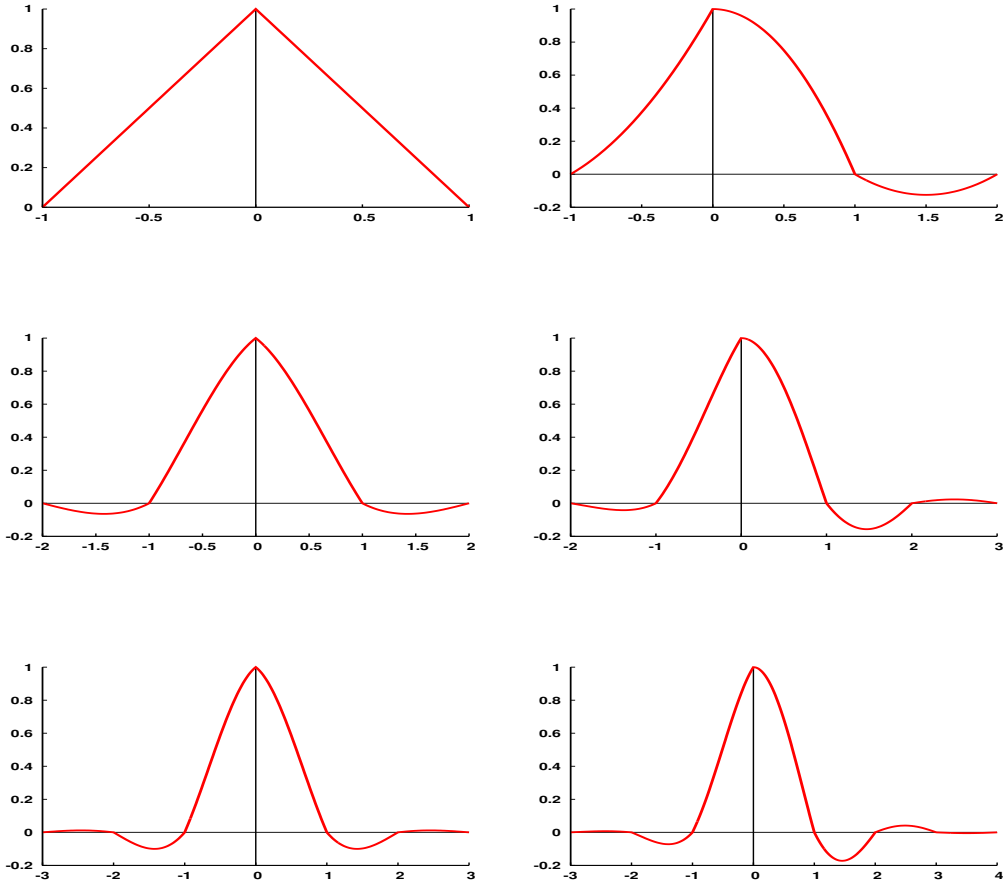


Figure 1: The continuous piecewise polynomial RKP shape functions of order $k = 1, 2, 3, 4, 5, 6$ (Left to Right and Top to Bottom).

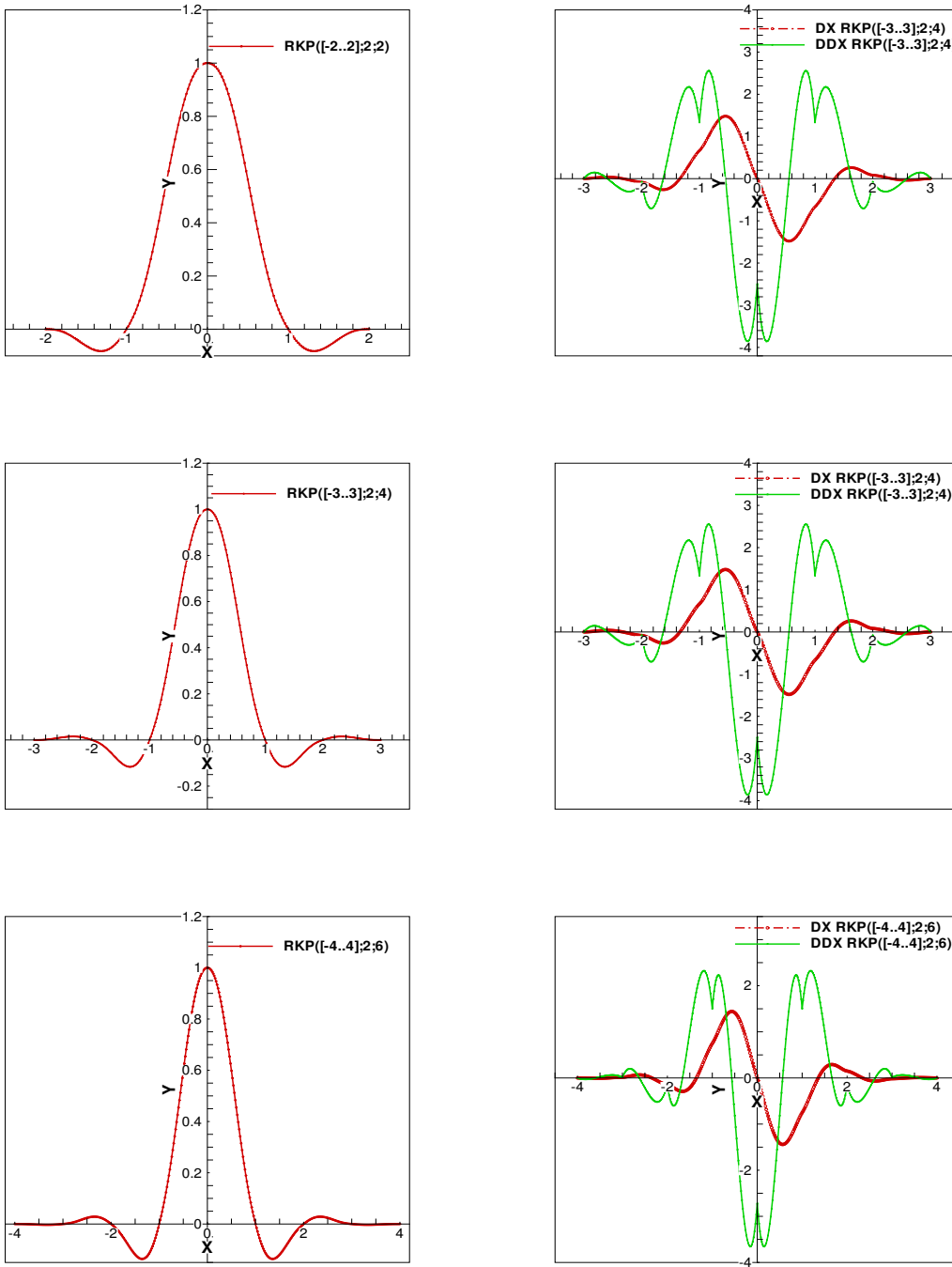


Figure 2: $\phi_{([-2..2];2;2)}(x)$, $\phi_{([-3..3];2;4)}(x)$, $\phi_{([-4..4];2;6)}(x)$, and their first and second derivatives.

Now in order to improve the regularity of the \mathcal{C}^0 - RKP shape function of reproducing order $2K - 1$, we extend the support to be $[-(K + 1), (K + 1)]$ so that it can contains enough particles.

Then one can surely have a smooth RKP shape function of order $2K - 1$ by using the standard method (5) and (6) because the support of the chosen weight function contains enough number of particles for the moment matrix $M(x)$ to be non-singular. However, a closed form RKP shape function is not guaranteed.

Suppose $\phi(x)$ is a RKP shape function of reproducing order $l < 2K + 1$ and its support is $[-(K + 1), (K + 1)]$.

Then by taking derivative of both sides of the definition (10), one can have the following systems of linear equations for $x \in (0, 1)$:

$$S_l : \sum_{j=-(K-1)}^K (x-j)^\beta F_l(x-j) = (-1)^l (l!) \delta_\beta^l. \quad (23)$$

The system is under determined if $(l + 1) < 2(K + 1)$. In other words, the corresponding coefficient matrices are a $(l + 1) \times 2(K + 1)$ Vandermonde matrices as follows:

$$V(x) \cdot \begin{bmatrix} F_l(x + K) \\ \vdots \\ F_l(x) \\ \vdots \\ F_l(x - (K + 1)) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ (-1)^l l! \end{bmatrix}, \quad (24)$$

where $V(x)$ is a rectangular Vandemode matrix

$$\begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ (x + K) & \cdots & (x + 1) & x & \cdots & (x - K) & (x - (K + 1)) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (x + K)^l & \cdots & (x + 1)^l & x^l & \cdots & (x - K)^l & (x - (K + 1))^l \end{bmatrix}.$$

There are many ways to expand $V(x)$ to be an invertible square matrix. However, not all expansions yield RKP shape functions. The following expansions are able to produce optimally regular RKP shape functions.

In order to make the under determined system (23) to be exactly determined,

1. add $2(K + 1) - (l + 1)$ additional equations to the bottom of the system;
2. add $2(K + 1) - (l + 1)$ additional equations to the top of the system;
3. add some additional equations to the top and some additional equations to the bottom of the system

For example, it follows from Theorem 3.1 that if the support of a shape function ϕ is $[-1, 1]$, the following system

$$\begin{bmatrix} 1 & 1 \\ x & (x-1) \end{bmatrix} \begin{bmatrix} \phi(x) \\ \phi(x-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

yields an \mathcal{C}^0 RKP function of reproducing order 1.

If we extend the support of ϕ to $[-2, 2]$ and differentiate the system, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ (x+1) & x & (x-1) & (x-2) \end{bmatrix} \begin{bmatrix} \phi'(x) \\ \phi'(x-1) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

(I) By adding additional two equations, this under determined system can be extended an exactly determined system as follows:

$$\begin{bmatrix} (x+1)^{-1} & x^{-1} & (x-1)^{-1} & (x-2)^{-1} \\ 1 & 1 & 1 & 1 \\ (x+1) & x & (x-1) & (x-2) \\ (x+1)^2 & x^2 & (x-1)^2 & (x-2)^2 \end{bmatrix} \begin{bmatrix} \phi'(x+1) \\ \phi'(x) \\ \phi'(x-1) \\ \phi'(x-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ p(x) \end{bmatrix}.$$

By Lemma 3.1, for a proper choice of the parameter $p(x)$, the solution of this extended system can be made to be a continuous shape function. Integrating $\phi'(x)$, we obtain a \mathcal{C}^1 -RKP shape function $\phi^*(x)$ of reproducing order 1.

(II) If we extend the support of ϕ to $[-3, 3]$ and differentiate the system twice, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ (x+2) & (x+1) & x & (x-1) & (x-2) & (x-3) \\ (x+2)^2 & (x+1)^2 & x^2 & (x-1)^2 & (x-2)^2 & (x-3)^2 \end{bmatrix} \begin{bmatrix} \phi''(x+2) \\ \phi''(x+1) \\ \phi''(x) \\ \phi''(x-1) \\ \phi''(x-2) \\ \phi''(x-3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ p_1(x) \\ p_2(x) \\ p_3(x) \end{bmatrix}.$$

By adding additional three equations, this under determined system can be extended an exactly determined system as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ (x+2) & (x+1) & x & (x-1) & (x-2) & (x-3) \\ (x+2)^2 & (x+1)^2 & x^2 & (x-1)^2 & (x-2)^2 & (x-3)^2 \\ (x+2)^3 & (x+1)^3 & x^3 & (x-1)^3 & (x-2)^3 & (x-3)^3 \\ (x+2)^4 & (x+1)^4 & x^4 & (x-1)^4 & (x-2)^4 & (x-3)^4 \\ (x+2)^5 & (x+1)^5 & x^5 & (x-1)^5 & (x-2)^5 & (x-3)^5 \end{bmatrix} \begin{bmatrix} \phi''(x+2) \\ \phi''(x+1) \\ \phi''(x) \\ \phi''(x-1) \\ \phi''(x-2) \\ \phi''(x-3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ p_1(x) \\ p_2(x) \\ p_3(x) \end{bmatrix}.$$

In the same manner, for a proper choice of $p_i(x)$, the solution of this extended system can be made to be a continuous shape function. Successively integrating $\phi''(x)$, we have a \mathcal{C}^2 -RKP shape function $\phi^*(x)$ of reproducing order 1.

Thus far, we have constructed several different closed form RKP shape functions. However, we have not mentioned which one has the most optimal approximability. For the partial answers, we have the following remarks.

- (1) The RKP shape functions obtained by the third construction method do not satisfy the Kronecker delta property. However, we cannot claim the approximability of the RKP shape function by the third method is less effective than those closed form RKP shape functions that satisfy the Kronecker delta property.
- (2) Moreover, those closed form RKP shape functions are much easier in implementing RKPM. However, we do not claim that our closed form p.p. RKP shape functions have better approximability than those implicit form RKP shape functions constructed by the standard method (5).
- (3) In ([2], [4]), a tool for checking the approximability of RKP shape functions is given. The largest eigenvalue of the matrix related to gradients of the amount failed to have the reproducing property of order $k + 1$ is one indicator (Theorem 4.4 of [2]).
- (4) Since, in two dimensional case, there are more possible closed form RKP shape functions, we check the quality (approximability) of our closed form RKP shape functions constructed this section along with the two dimensional extension in Part II of this paper.

3.3 Numerical Tests

In order to demonstrate the effectiveness of the RKP shape functions constructed in the previous section, some of those RKP basic shape functions are applied for the numerical solutions of the second order and the fourth order differential equations.

(A) [Second order equation:]

The RKPM with the unique C^0 -RKP shape functions of reproducing order $k = 1, 3, 5$ are applied to the second order equation.

Let $u(x) = e^x(1 - x^2)^6$, then $u(x)$ solves the following model problem

$$-\frac{d^2}{dx^2}u(x) = f(x) \text{ in } (-1, 1) \quad (25)$$

$$u(\pm 1) = 0, \quad (26)$$

where $f(x) = -e^x((1 - x^2)^6 - 24x(1 - x^2)^5 - 12(1 - x^2)^5 + 120x^2(1 - x^2)^4)$ (One can see this model problem in section 4).

Suppose h is the given mesh size such that $N = 2/h \in \mathbb{Z}$, the degree of freedom, and $\phi(x)$ is a basic RKP shape function of reproducing order $k := 2K - 1, K = 1, 2, 3, \dots$. Then the uniformly distributed particles x_1, \dots, x_{N+2K-2} are

$$[-1 - h(K - 1)], \dots, [-1 - h], \uparrow [-1], [-1 + h], \dots, [1 - h], [1], \uparrow, [1 + h], \dots, [1 + h(K - 1)].$$

Table 1: Relative Error in Energy Norm (%) obtained by applying the RKP shape functions $\phi_{([-1,1];0;1)}(x)$, $\phi_{([-2,2];0;3)}(x)$, $\phi_{([-3,3];0;5)}(x)$ of reproducing order $k = 1, 3, 5$, respectively to the model problem $-\Delta u = f$. The true solution of the model problem is $u(x) = e^x(1 - x^2)^6$.

h	$k = 1$	$k = 3$	$k = 5$
0.2e-0	25.1e-0	5.4e-0	1.8e-0
0.1e-0	12.7e-0	8.1e-1	8.9e-2
0.5e-1	6.4e-0	1.1e-1	3.6e-3
2.5e-2	3.2e-0	1.3e-2	1.2e-4
1.25e-2	1.6e-0	1.7e-3	5.4e-5

Let us note that $2(K - 1)$ particles are in the outside of the domain $[-1, 1]$ and two end points are particles.

Let $\psi_j(x) : [-K, K] \rightarrow \mathbb{R}$ be a shift function defined by $\psi_j(x) = hx + x_j$, where x_j is the j th particle. Then the RKP shape functions used for the solution of the variational equation of the model problem are

$$\{\phi(\psi_j(x)) : j = 1, \dots, (N + 2K - 2)\}.$$

For various sizes of h and various reproducing orders, relative errors in energy norm(%) of the resulting numerical solutions are computed in Table 1.

Remark 3.4. *It was proved in ([3], [8]) that the interpolation error in energy norm related to RKP shape functions of reproducing order k is $\mathcal{O}(h^k)$. Thus, we plot the relative error in energy norm versus the mesh sizes in log – log scales in Fig. 2. The slopes of those lines in this figure are close to 1, 3, 5, respectively.*

For the purpose of comparing PUFEM and RKPM, PUFEM with respect to the \mathcal{C}^2 - p.p. PU weight function (3) (which is denoted by ϕ_{g_3} in [15]) is applied to this model problem for various mesh sizes and polynomial degrees so that degrees of freedom are comparable to those of RKPM. The relative errors in energy norm of those solutions obtained by PUFEM ("the p -version Generalized FEM") and RKPM ("the h -version Generalized FEM") are depicted in Fig. 3, from which one can see that the results by the RKP shape function with the reproducing property of order 5 are almost as good as optimal results by PUFEM.

(B) [Fourth order equation:]

We apply the closed form \mathcal{C}^2 RKP basic shape functions of reproducing order $k = 2, 4, 6$,

$$\phi_{([-2,2];2;2)}(x), \phi_{([-3,3];2;4)}(x), \phi_{([-4,4];2;6)}(x),$$

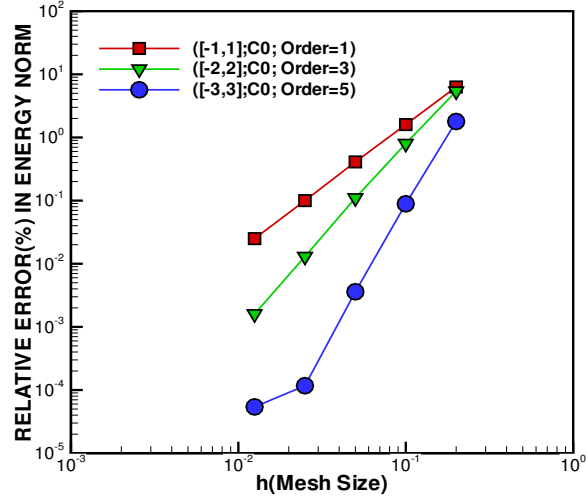


Figure 3: The relative errors(%) in the energy norm by RKPM of order 1, RKPM of order 3, RKPM of order 5 are plotted with respect to the mesh size h .

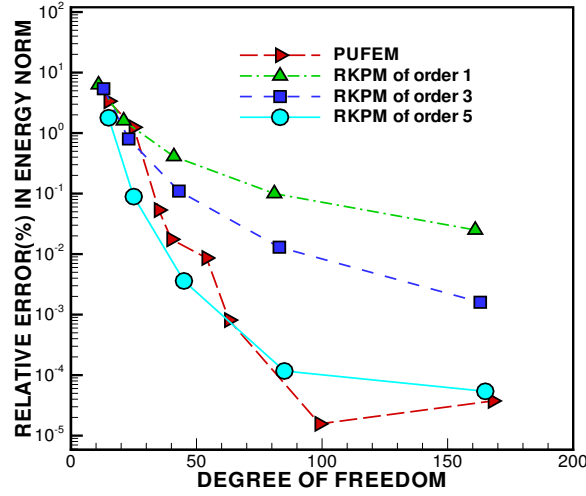


Figure 4: The relative errors(%) in the energy norm by RKPM of order 1, RKPM of order 3, RKPM of order 5, and Partition of Unity FEM with smooth piecewise polynomial PU shape function $\phi_{g3}^{(pp)}$ defined by (3).

Table 2: Relative Error in Energy Norm (%) of the fourth order differential equation, $\Delta^2 u = f$, obtained by using \mathcal{C}^2 RKP shape functions of reproducing order 2, 4, 6. The true solution is $u(x) = e^x(1 - x^2)^6$.

h	$\phi_{([-2,2];2;2)}(x)$	$\phi_{([-2,2];2;4)}(x)$	$\phi_{([-2,2];2;6)}(x)$
0.2e-0	52.79e-0	16.57e-0	5.94e-0
0.1e-0	33.86e-0	3.22e-0	4.61e-1
0.5e-1	18.60e-0	4.52e-1	1.94e-2
2.5e-2	9.56e-0	5.95e-2	9.26e-4
1.25e-2	4.82e-0	7.95e-3	5.75e-4

respectively, to the following fourth order differential equation.

$$\frac{d^4}{dx^4}u(x) = f(x) \text{ in } (-1, 1) \quad (27)$$

$$u(\pm 1) = \frac{d^2 u}{dx^2}(\pm 1) = 0, \quad (28)$$

where

$$f(x) = e^x [(1 - x^2)^6 - 48(1 - x^2)^5 x + 720(1 - x^2)^4 x^2 - 72(1 - x^2)^5 - 3840(1 - x^2)^3 x^3 + 1440(1 - x^2)^4 x + 5760(1 - x^2)^2 x^4 - 5760(1 - x^2)^3 x^2 + 360(1 - x^2)^4].$$

Then the true solution is $u(x) = e^x(1 - x^2)^6$.

Percentage relative errors in energy norm for each case are depicted in Fig. 5. Numeric data of these results are also shown in Table 2.

3.4 Closed Form RKP shape functions in Dimension Two

One dimensional RKP shape functions can be naturally extended to higher dimensional RKP functions through their tensor products.

There exist several other higher dimensional closed form RKP shape functions that are not tensor products of one dimensional RKP functions. We have to know which shape function has a better approximability over the tensor product RKP shape functions. This will be investigated in Part II of this paper.

By taking tensor product of two one dimensional RKP shape functions, we have the following two dimensional version of Theorem 3.1.

Theorem 3.2. *There exists a unique translation invariant \mathcal{C}^0 -RKP shape function $\phi(x, y)$ which satisfies the following conditions:*

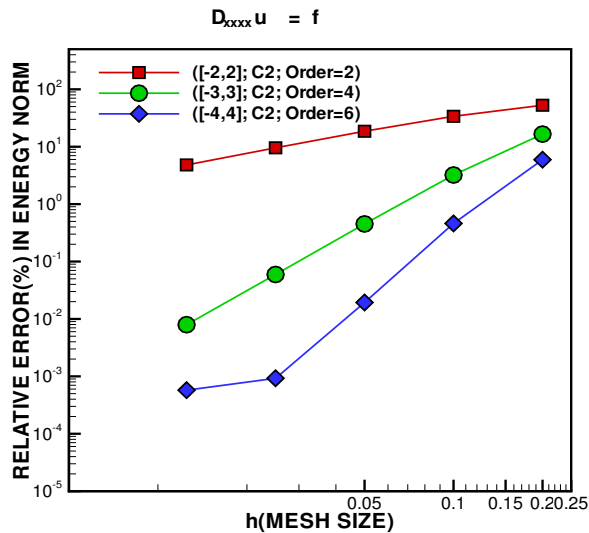


Figure 5: The Relative Error (%) in Energy Norm versus the mesh size for the fourth order differential equation

- (a) The support of $\phi(x, y)$ is $[-K_x, K_x] \times [-K_y, K_y]$ where K_x, K_y are positive integers.
- (b) The reproducing order of $\phi(x)$ is $\min\{2K_x - 1, 2K_y - 1\}$.
- (c) $\phi(x, y)$ is continuous.

This unique basic shape function satisfies the Kronecker delta property:

$$\phi(x - x_{j_1}, y - y_{j_2}) = \delta_{j_1}^{i_1} \delta_{j_2}^{i_2}$$

and it is a piecewise polynomial of degree $2K_x - 1$ in x and $2K_y - 1$ in y .

4 PU functions for non-uniformly distributed particles

Thus far, we considered translation invariant RKP shape functions when the particles are uniformly distributed. However, it may not be a severe restriction. We are able to apply the translation invariant RKP shape functions constructed in previous sections for a locally uniform distributed particles as shown in Fig. 10.

For this purpose, we construct a flexible p.p. partition of unity shape functions in this section.

4.1 Construction of PU shape functions by the convolution of the window functions with the characteristic functions

[A] Construction.

For a positive real number δ , let I_j^δ be the δ -cell centered at x_j , defined by

$$I_j^\delta = \{x \in \mathbb{R}^n : \|x - x_j\|_\infty \equiv \max_{i=1,2,\dots,n} |x^i - x_j^i| \leq \delta\}.$$

Let us consider the following tensor product of scaled conical weight functions

$$\beta_\delta(x) = \begin{cases} A \prod_{i=1}^n (1 - [|x^i|/\delta]^2)^l & \text{if } \|x\|_\infty \leq \delta \\ 0 & \text{if } \|x\|_\infty > \delta, \end{cases} \quad (29)$$

where

$$A = 1 / \int_{I_0^\delta} \prod_{i=1}^n (1 - [|x^i|/\delta]^2)^l dx.$$

By an almost everywhere (a.e.) δ -covering of a domain Ω , we define a family of mutually disjoint simply connected open subsets $E_j, j = 1, \dots, N$ of \mathbb{R}^n such that

$$\begin{aligned} \bigcup_{j=1}^N \overline{E_j} &\supset \{x : \text{dist}(x, \Omega) \leq \delta\} \equiv \Omega^\delta, \\ \bigcup_{j=1}^N E_j &\supset \partial\Omega, \text{ except finitely many points.} \end{aligned}$$

For a subset $E_k \subset \mathbb{R}^n$, let χ_{E_k} be the characteristic function, defined by

$$\chi_{E_k}(x) = \begin{cases} 1 & \text{if } x \in E_k \\ 0 & \text{if } x \in \mathbb{R}^n \setminus E_k. \end{cases} \quad (30)$$

Then the convolution of $\chi_{E_k}(x)$ and $\beta_\delta(x)$ becomes a piecewise polynomial as follow.

$$\psi_k^\delta(x) := (\chi_{E_k} * \beta_\delta)(x) = \begin{cases} 1 & \text{if } E_k \cap I_x^\delta = I_x^\delta, \\ \int_{E_k \cap I_x^\delta} \beta_\delta(t-x) dt & \text{if } E_k \cap I_x^\delta \neq I_x^\delta. \end{cases} \quad (31)$$

Suppose $\{E_j : j = 1, \dots, N\}$ be an a.e. δ -covering of $\Omega \subset \mathbb{R}^n$. Then

$$\sum_{j=1}^N \chi_{E_j} = 1 \text{ a. e. on } \Omega.$$

Thus we have

$$\sum_{j=1}^N \psi_j^\delta(x) = \left(\sum_{j=1}^N \chi_{E_j} \right) * (\beta_\delta)(x) = (1 * \beta_\delta)(x) = 1, \quad (32)$$

$$\text{supp } \psi_j^\delta(x) = \{x \in \mathbb{R}^n : \text{dist}(x, E_j) \leq \delta\} \equiv E_j^\delta. \quad (33)$$

Hence we have the following theorem

Theorem 4.1. (1) *If $\{E_j : j = 1, \dots, N\}$ be an a.e. δ -covering of Ω , then $\{\psi_j^\delta(x)\}$ are PU shape functions subordinated to the covering $\{E_j^\delta : j = 1, \dots, N\}$.*

(2) *If for a positive integer q , $\beta_\delta(x) \in C^q$, then $\psi_j^\delta(x) \in C^q$.*

Remark 4.1. *In above construction of PU shape functions, the choice of a.e. mutually disjoint coverings are very flexible. Actually, the construction allows to include non convex subsets in the coverings. Thus, it effectively handles domains with complex geometry. Moreover, the resulting PU shape functions are smooth piecewise polynomials and hence numerical integrations become simple and accurate. We noticed that our PUFEM with respect to this special construction of PU function is similar to those in RKEM (Reproducing Kernel Element Methods), developed in (Chapter 6 of [11], [12], [13]).*

[B] 1-D Example.

Let us consider one dimensional examples. Suppose $E_k = (a_k, b_k)$, then the support of ϕ_k^δ is $\eta_k := [a_k - \delta, b_k + \delta]$ and

$$\psi_k^\delta(x) = \begin{cases} \int_{x-b_k}^{\delta} \beta_\delta(t) dt & \text{if } x \in [b_k - \delta, b_k + \delta] \\ 1 & \text{if } x \in [a_k + \delta, b_k - \delta] \\ \int_{-\delta}^{x-a_k} \beta_\delta(t) dt & \text{if } x \in [a_k - \delta, a_k + \delta] \\ 0 & \text{if } x \in \mathbb{R} \setminus [a_k - \delta, b_k + \delta] \end{cases} \quad (34)$$

Example 4.1. *If $\delta = 0.1, l = 2$ in the conical weight function, $E_k = (2, 3)$, then we have the following.*

$$\beta_\delta(x) = \begin{cases} 9.375(1 - 100x^2)^2 & \text{if } |x| < 0.1 \\ 0 & \text{if } |x| \geq 0.1, \end{cases} \quad (35)$$

$$\psi_k^\delta(x) = \begin{cases} \int_{x-3}^{0.1} \beta_\delta(t) dt & \text{if } x \in [2.9, 3.1] \\ 1 & \text{if } x \in [2.1, 2.9] \\ \int_{-0.1}^{x-2} \beta_\delta(t) dt & \text{if } x \in [1.9, 2.1] \end{cases} \quad (36)$$

For various E_j , the PU functions $\psi_j^\delta, \delta = 0.1$, are plotted in Fig. 6

[C] 2-D Example. Next, let us consider two dimensional examples. Let $E_{kl} = (a_k, b_k) \times (c_l, d_l)$. Then the convolution $\psi_{kl}^\delta(x) = \chi_{E_{kl}} * [\beta_\delta \times \beta_\delta](x)$ is $\psi_k^\delta \times \psi_l^\delta$, which is given by the

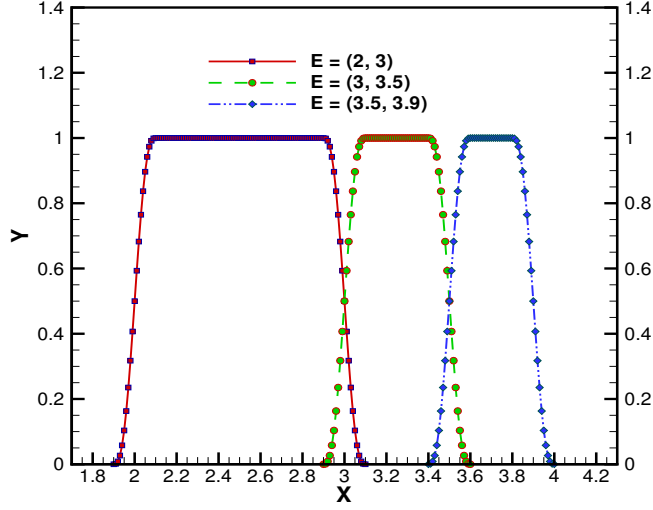


Figure 6: $\psi_1 = \chi_{E_1} * \beta_\delta$, $E_1 = (2, 3)$; $\psi_2 = \chi_{E_2} * \beta_\delta$, $E_2 = (3, 3.5)$; $\psi_3 = \chi_{E_3} * \beta_\delta$, $E_3 = (3.5, 3.9)$; when $\delta = 0.1$.

following.

$$\psi_{kl}(x, y) = \begin{cases} \int_{x-b_k}^{\delta} \beta_\delta(s) ds \int_{y-d_l}^{\delta} \beta_\delta(t) dt & \text{if } x \in [b_k - \delta, b_k + \delta] \times [d_l - \delta, d_l + \delta] \\ \int_{x-b_k}^{\delta} \beta_\delta(s) ds \cdot 1 & \text{if } x \in [b_k - \delta, b_k + \delta] \times [c_l + \delta, d_l - \delta] \\ \int_{x-b_k}^{\delta} \beta_\delta(s) ds \int_{-\delta}^{y-c_l} \beta_\delta(t) dt & \text{if } x \in [b_k - \delta, b_k + \delta] \times [c_l - \delta, c_l + \delta] \\ 1 \cdot \int_{y-d_l}^{\delta} \beta_\delta(t) dt & \text{if } x \in [a_k + \delta, b_k - \delta] \times [d_l - \delta, d_l + \delta] \\ 1 & \text{if } x \in [a_k + \delta, b_k - \delta] \times [c_l + \delta, d_l - \delta] \\ 1 \cdot \int_{-\delta}^{y-c_l} \beta_\delta(t) dt & \text{if } x \in [a_k + \delta, b_k - \delta] \times [c_l - \delta, c_l + \delta] \\ \int_{-\delta}^{x-a_k} \beta_\delta(s) ds \int_{y-d_l}^{\delta} \beta_\delta(t) dt & \text{if } x \in [a_k - \delta, a_k + \delta] \times [d_l - \delta, d_l + \delta] \\ \int_{-\delta}^{x-a_k} \beta_\delta(s) ds \cdot 1 & \text{if } x \in [a_k - \delta, a_k + \delta] \times [c_l + \delta, d_l - \delta] \\ \int_{-\delta}^{x-a_k} \beta_\delta(s) ds \int_{-\delta}^{y-c_l} \beta_\delta(t) dt & \text{if } x \in [a_k - \delta, a_k + \delta] \times [c_l - \delta, c_l + \delta] \end{cases} \quad (37)$$

The support of $\psi_{kl}^\delta(x)$ is

$$\eta_{kl} = [a_k - \delta, b_k + \delta] \times [c_l - \delta, d_l + \delta].$$

This two dimensional PU function is depicted in Fig. 7 when $E_{kl} = (2, 3) \times (2, 3)$ and $\delta = 0.1$.

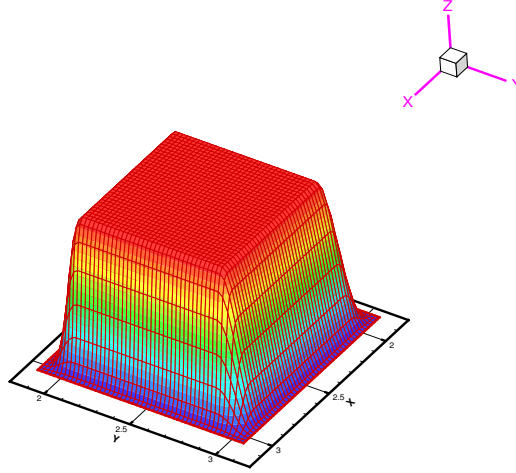


Figure 7: Two dimensional PU function $(\chi_E) * (\beta_\delta \times \beta_\delta)$ when $E = (2, 3) \times (2, 3)$ and $\delta = 0.1$.

4.2 PUFEM with respect to the convolution PU shape functions

Now, we apply the p.p. PU shape functions $\psi_j^\delta(x)$ to a model second order differential equation and a model fourth order differential equation.

Let $\mathcal{U}(w) = \frac{1}{2}\mathcal{B}(w, w)$ be the strain energy of w , where $\mathcal{B}(\cdot, \cdot)$ denote the bilinear forms defined in section 2. Then the relative error in energy norm (%) is defined as

$$\|e\|_{E,r} \text{ in } \% = \left[\frac{|\mathcal{U}(u_{ex}) - \mathcal{U}(u_{app})|}{\mathcal{U}(u_{ex})} \right]^{1/2} \times 100. \quad (38)$$

In the following examples, numerical solutions obtained by applying GFEM with respect to ψ_j^δ shape functions related to the following data

$$\begin{aligned} \Omega &= (-1, 1), \\ \delta &\in [0.01, 0.2], \\ \text{a.e. } \delta\text{-covering} &= \{(-1 - 4\delta, -0.6), (-0.6, -0.2), (-0.2, 0.2), (0.2, 0.6), (0.6, 1 + 4\delta)\}. \end{aligned}$$

Example 4.2. [Second Order Equation:] $u(x) = e^x(1 - x^2)^6$ solves the following model problem

$$-\frac{d^2}{dx^2}u(x) = f(x) \text{ in } (-1, 1), \quad (39)$$

$$u(\pm 1) = 0, \quad (40)$$

where $f(x) = -e^x((1-x^2)^6 - 24x(1-x^2)^5 - 12(1-x^2)^5 + 120x^2(1-x^2)^4)$. Then the strain energy is 1.8568504251271325.

[**Fourth Order Equation:**] $u(x) = e^x(1-x^2)^6$ solves the following model problem

$$\frac{d^4}{dx^4}u(x) = f(x) \text{ in } (-1, 1) \quad (41)$$

$$u(\pm 1) = \frac{d^2u}{dx^2}(\pm 1) = 0, \quad (42)$$

where

$$f(x) = e^x [(1-x^2)^6 - 48(1-x^2)^5x + 720(1-x^2)^4x^2 - 72(1-x^2)^5 - 3840(1-x^2)^3x^3 + 1440(1-x^2)^4x + 5760(1-x^2)^2x^4 - 5760(1-x^2)^3x^2 + 360(1-x^2)^4].$$

Then the strain energy is $\frac{1}{2} \int_{\Omega} (\frac{d^2u}{dx^2})^2 dx = 36.5967697946804$.

In order to find an optimal δ that gives the best result, we plot the relative errors in energy (%) verses the size of δ , $0.01 \leq \delta \leq 0.2$. The results for the second order equation and the fourth order equation, respectively, are plotted in Fig. 8 and Fig. 9.

Figures 8 and 9 show that the accuracy of PUFEM solutions with respect to the convolution PU shape functions, ψ_j^δ , are not sensitive on the size of δ .

Of course, the accuracy may depend on the choice of the window function $\beta_\delta(x)$ as well as δ .

Finally, in order to apply the translation invariant RKP basic shape functions to non-uniformly distributed particles, we make use of a similar idea to RKEM (Reproducing Kernel Element Method), developed in (Chapter 6 of [11], [12], [13]).

In other words, by multiplying the convolution PU shape functions to the closed form RKP basic shape functions, shifted to locally uniformly distributed particles, we can construct global basis functions that keep the reproducing property of the same order and the Kronecker delta property.

Numerical examples and implementing this approach in higher dimension will be shown in Part II of this paper.

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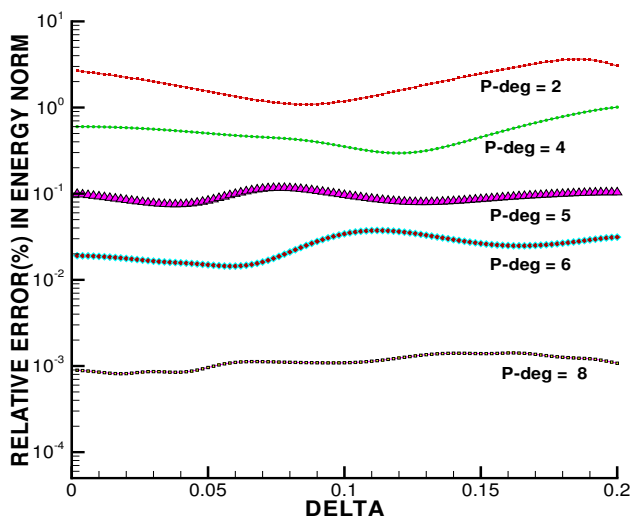


Figure 8: Relative Errors in energy norm (%) of GFEM solutions of the second order differential equation for various sizes of δ .

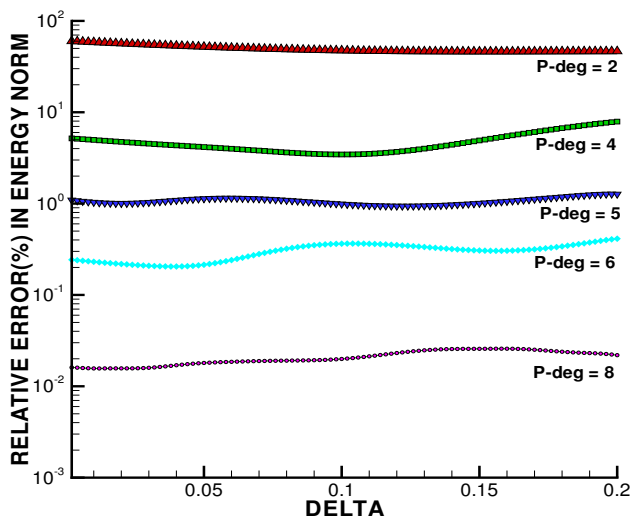


Figure 9: Relative Errors in energy norm (%) of GFEM solutions of the fourth order differential equation for various sizes of δ .

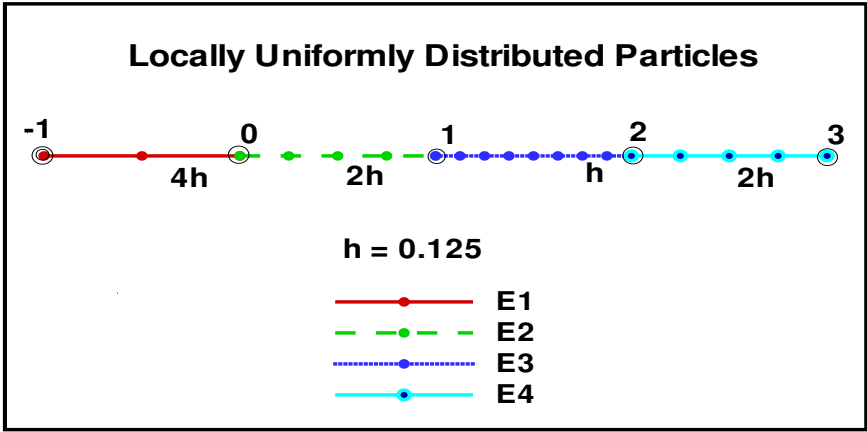


Figure 10: Locally uniform distributed Particles.

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