The Closed Form Reproducing Kernel Particle Shape Functions: Part 2. Non-Uniformly Distributed Particles

by

June G. Kim *†
Department of Mathematics,
Kangwon National University, Chunchon, 200-701, Korea

Hae-Soo Oh‡§
Department of Mathematics,
University of North Carolina at Charlotte, Charlotte, NC 28223

Jae Woo Jeong
Department of Mathematics,
University of North Carolina at Charlotte, Charlotte, NC 28223

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Abstract

In part 1 of this paper ([22]), for uniformly distributed particles, we construct highly regular piecewise polynomial RKP shape functions that have the polynomial reproducing property of order k for any given integer $k \geq 0$ and satisfy the Kronecker Delta Property. This discovery of closed form shape functions not only ensures high accuracy of RKPM, but also alleviates difficulties arising in implementing RKPM such as imposing Dirichlet boundary conditions and numerical integrations. However, uniformly distributed particles can be impractical, especially when the problems contain singularities or the solution domains are irregular. Thus, in this report, we generalize the construction of piecewise polynomial RKP shape functions described in part 1 to the case when the particles are non-uniformly distributed in $\mathbb{R}$ and to the case when the particles are non-uniformly distributed in a bounded closed interval. Furthermore, we present a more direct proof of an error estimate of the interpolation associated with these closed form RKP shape functions.

*supported in part by the Research Grant of the Kangwon National University
†Visiting Professor of the University of North Carolina at Charlotte
‡Corresponding author. Tel.: +1-704-687-4930; fax: +1-704-687-6415; E-mail: hso@uncc.edu
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1 Introduction

Recently several generalized finite element methods (GEFM) that circumvent the obstacles in conventional FEM such as mesh refinement and constructing smooth global basis functions were introduced. Among many GFEMs that use meshes minimally or do not use meshes at all ([1],[2],[3],[4]), those methods related to this paper are Element Free Galerkin Method (EFGM) ([1],[13],[14],[15]), Reproducing Kernel Particle Method (RKPM)([10],[15]), h-p Cloud Method([7],[8]), Partition of Unity Finite Element Method (PUFEM)([18],[24],[25]), and Reproducing Kernel Element Method (RKEM) ([15],[16],[17]).

The reproducing kernel particle method(RKPM) is a mesh free method that yields highly accurate approximation by using the reproducing kernel shape functions that can exactly interpolate the polynomials of a fixed degree. The RKP(reproducing kernel particle) shape functions can be constructed to be smooth up to any desired order by selecting smooth window functions.

However, the RKP shape functions constructed by using a specific window function are fractional functions with complicate denominators that are solutions of the system of algebraic equations. Thus, these RKP shape functions have the following difficulties:

(1) They do not satisfy the Kronecker delta property, and hence it has difficulties in dealing with Dirichlet boundary conditions.

(2) Accuracy is compromised in numerical integrations for these complex fractional shape functions.

In order to alleviate these obstacles, in part 1, we constructed piecewise polynomial $\mathcal{C}^r$-RKP shape functions associated with uniformly distributed particles, that satisfy the Kronecker delta property, for any integer $r \geq 0$. However, the RKP shape functions associated with uniformly distributed particles are not practical, especially when the problems contain singularities.

In this paper, we apply our methods to the RKP shape functions associated with the particles that have any desired forms of distributions in a given domain.

This paper is organized as follows: in section 2, definitions and terminologies are explained. For the construction of RKP shape functions corresponding to non-uniformly distributed particles, we reproduce representative piecewise polynomial RKP shape functions associated with uniformly distributed particles that were constructed in part 1.

In section 3, we construct smooth piecewise polynomial RKP shape functions satisfying the Kronecker delta property associated with non-uniformly distributed particles in $[0, \infty)$ as well as those associated with non-uniformly distributed particles in a closed bounded interval $[a, b]$. Next, we prove an error estimate of the interpolation associated with RKP shape functions of reproducing order $2K-1$ in $\mathbb{R}^d$. Furthermore, we also give a numerical example that demonstrates the effectiveness of these shape functions in solving elliptic differential equations.

For the case where particles are non-uniformly distributed in $(-\infty, \infty)$, we construct the corresponding smooth piecewise polynomial RKP shape functions that have the polynomial reproducing property of any given order in section 4.

Finally, piecewise polynomial RKP shape functions with polynomial reproducing property of high order, associated the particles distributed in $[0, \infty)$, are described in appendix.

These piecewise polynomial RKP shape functions are naturally extended to higher dimensional RKP shape functions satisfying the Kronecker delta property by taking tensor product of these single valued shape functions. However, the constructions of general higher dimensional closed form RKP shape functions that are not product of single valued functions are discussed elsewhere.

2 Preliminary

Throughout this paper, $\alpha, \beta \in \mathbb{Z}^d$ are multi indices and $x = ({}^1x, {}^2x, .., {}^dx), x_j = ({}^1x_j, {}^2x_j, .., {}^dx_j)$ denote points in $\mathbb{R}^d$. However, if there are no confusions, we also use the conventional notation for the points in $\mathbb{R}^d$ or $\mathbb{Z}^d$ as

$$x = (x_1, x_2, \cdots, x_n) \text{ and } \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d).$$

We also use the following notations:

$$\begin{aligned}
(x - x_j)^\alpha &:= ({}^1x - {}^1x_j)^{\alpha_1} ... ({}^dx - {}^dx_j)^{\alpha_d}, \\
|\alpha| &:= \alpha_1 + \alpha_2 + \cdots + \alpha_d, \\
\alpha! &:= \alpha_1!\alpha_2! \cdots \alpha_d!, \\
\partial_x^\alpha u &:= \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}
\end{aligned}$$

Let Ω be a domain in $\mathbb{R}^d$. For any nonnegative integer m, $\mathcal{C}^m(\Omega)$ denotes the space of all functions ϕ such that ϕ together with all their derivatives $D^\alpha \phi$ of orders $|\alpha| \leq m$, are continuous on Ω. The support of ϕ is defined by

$$\text{supp } \phi = \overline{\{x \in \Omega : \phi(x) \neq 0\}}.$$

In the following, a function $\phi \in \mathcal{C}^m(\Omega)$ is said to be a $\mathcal{C}^m$- function.

We also use the usual Sobolev space denoted by $H^k(\Omega)$. For $u \in H^k(\Omega)$, the norm is

$$\|u\|_{k,\Omega}^2 = \sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha u|^2 dx,$$

and the semi-norm is

$$|u|_{k,\Omega}^2 = \sum_{|\alpha| = k} \int_\Omega |\partial^\alpha u|^2 dx.$$

A weight function(or window function) is a non negative continuous function with compact support and is denoted by $w(x)$. For example, the widely used window functions include the following: For $x \in \mathbb{R}$,

(a) Conical:

$$w(x) = \begin{cases} (1-x^2)^l, & |x| \le 1, \\ 0, & |x| > 1, \end{cases} \tag{1}$$

which is a $\mathcal{C}^{l-1}$-function.

(b) Gaussian:

$$w(x) = \begin{cases} (e^{-1/1-x^2}) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1, \end{cases} \tag{2}$$

which is an infinitely smooth function.

(c) Partition of unity ([21]):

$$w(x) = \begin{cases} (1+x)^3 g(x) & \text{if } -1 \le x \le 0 \\ (1-x)^3 g(-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{if } |x| > 1, \end{cases} \tag{3}$$

where $g(x) = (1 - 3x + 6x^2)$.

In $\mathbb{R}^d$, the weight function $w(x)$ can be constructed from a one-dimensional weight function either as $w(x) = w(\|x\|)$ or as $w(x) = \prod_{i=1}^d w(x_i)$, where $x = (x_1, \cdots, x_d)$ and $\|x\|^2 = x_1^2 + \cdots + x_d^2$.

Let Λ be a finite index set and Ω denotes a bounded domain. Let $\{x_j : j \in \Lambda\}$ be a set of a finite number of points in $\mathbb{R}^d$, that are called particles.

Adopting those terminologies and notations of ([3]), we have the following: For $j = (j_1, j_2, \cdots, j_d) \in \mathbb{Z}^d$, and the mesh size $0 < h \le 1$, let

$$x_j^h = (j_1 h, \cdots, j_d h) = hj.$$

Then the points x_j^h are called uniformly distributed particles. Let ϕ be a continuous function with compact support that contains the origin 0. Then the particle shape functions associated to the uniformly distributed particles is defined by

$$\phi_j^h(x) = \phi(\frac{x - jh}{h}) = \phi(\frac{x_1 - j_1 h}{h}, \cdots, \frac{x_d - j_d h}{h}),$$

for $j \in \mathbb{Z}^d$ and $0 < h \le 1$. Then these particle shape functions are **translation invariant** in the sense that

$$x_{i+j}^h = x_i^h + x_j^h, \phi_j^h(x - ih) = \phi_{i+j}^h(x).$$

In part 1 of this paper, we have considered the particles that are uniformly distributed and the particles are allowed to go outside of the domain. Here, we consider the case when the particles are partially non-uniformly distributed and also discuss the case when the particles are only inside or on the boundary of Ω, that is, $\{x_j : j \in \Lambda\} \subset \overline{\Omega}$.

Definition 2.1. *Let k be a non negative integer. Then $\{\phi_j(x) : j \in \Lambda\}$ are called RKP shape functions with the polynomial reproducing property of order k (or simply, "of reproducing order k") if and only if it satisfies the following condition:*

$$\sum_{j\in\Lambda}(x_j)^\alpha \phi_j(x) = x^\alpha, \text{ for } x \in \Omega \subset \mathbb{R}^d \text{ and for } 0 \le |\alpha| \le k. \tag{4}$$

The RKP shape function, associated with the particle x_j, is constructed by

$$\phi_j(x) = w(x - x_j) \sum_{0\le|\alpha|\le k} (x - x_j)^\alpha b_\alpha(x) \tag{5}$$

where $b_\alpha(x)$ are chosen so that (4) is satisfied and $w(x)$ is a window function. This gives rise to a linear system in $b_\alpha(x)$, namely

$$\sum_{0\le|\alpha|\le k} m_{\alpha+\beta}(x) b_\alpha(x) = \delta^0_{|\beta|} \text{ for } 0 \le |\beta| \le k, \tag{6}$$

where $\delta^0_{|\beta|}$ is the Kronecker delta, and

$$m_\alpha(x) = \sum_{j\in\mathbb{Z}} w(x - x_j)(x - x_j)^\alpha. \tag{7}$$

For one dimensional case, this system can be written as

$$M(x) \cdot [b_0(x), b_1(x), \cdots, b_k(x)]^T = [1, 0, \cdots, 0]^T,$$

where

$$M(x) = \sum_{j\in\Lambda} w(x - x_j) \begin{bmatrix} 1 \\ (x - x_j)^1 \\ (x - x_j)^2 \\ \vdots \\ (x - x_j)^k \end{bmatrix} [1, (x - x_j)^1, \cdots, (x - x_j)^k].$$

The coefficient matrix $M(x)$ of the linear system (6) is called the moment matrix.

By applying a similar argument to [3], one can show that (4) is equivalent to

$$\sum_{j\in\Lambda}(x - x_j)^\beta \phi_j(x) = \delta^0_{|\beta|}, \qquad \text{for } 0 \le |\beta| \le k \text{ and } x \in \mathbb{R}^d. \tag{8}$$

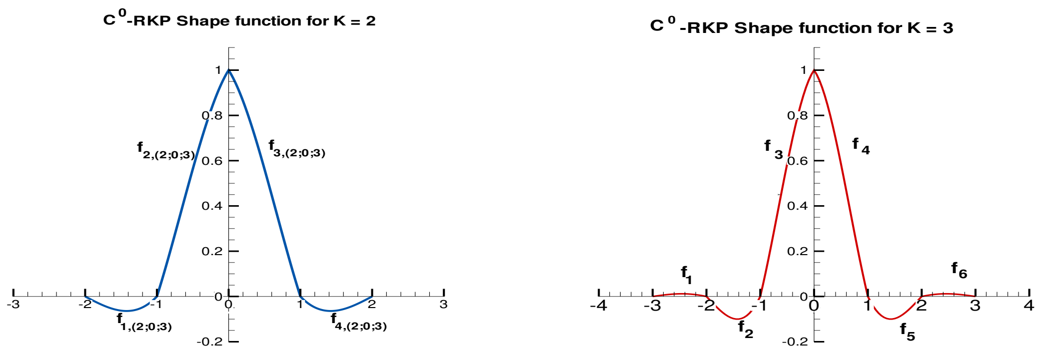


Figure 1: (Left:) The graph of the $\mathcal{C}^0$ RKP shape function of order 3, $\phi_{(2;0;3)}(x)$. (Right:) The graph of the $\mathcal{C}^0$ RKP shape function of order 5, $\phi_{(3;0;5)}(x)$.

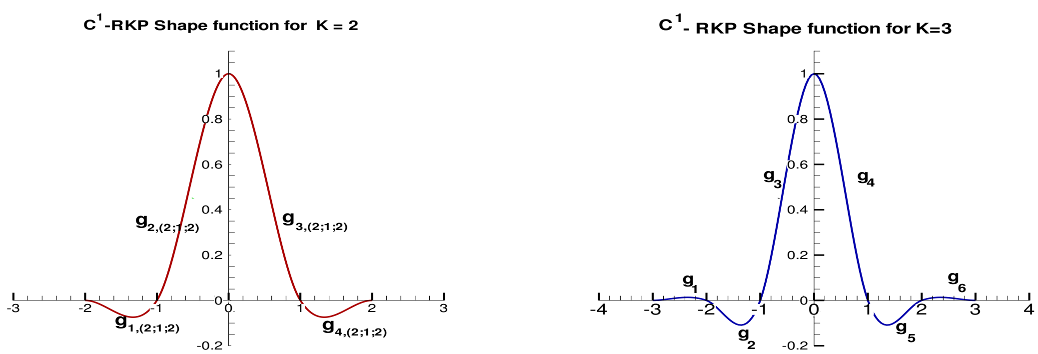


Figure 2: (Left:) The graph of the $\mathcal{C}^1$ RKP shape function of order 2, $\phi_{(2;1;2)}(x)$. (Right:) The graph of the $\mathcal{C}^1$ RKP shape function of order 5, $\phi_{(3;1;4)}(x)$.

2.1 Piecewise polynomial RKP shape functions for uniformly distributed particles

In this paper, the indices of polynomial reproducing shape function $\phi_{([a,b];m_2;m_3)}(x)$ indicate the following:

$$\begin{aligned} [a,b] &= \text{the support of } \phi(x), \\ m_2 &= \text{the order of the regularity(that is, } \phi(x) \in \mathcal{C}^{m_2}), \\ m_3 &= \text{the order of the reproducing property.} \end{aligned}$$

In particular, $\phi_{([-K,K];m_2;m_3)}(x)$ is also denoted by $\phi_{(K;m_2;m_3)}(x)$. Throughout this paper, K is a positive integer that is the radius of the support of a basic RKP shape function whose support is $[-K,K]$.

In this section, for the construction of RKP shape functions associated with non-uniformly distributed particles, we briefly describe those piecewise polynomial RKP shape functions corresponding to the uniformly distributed particles that can be found in the first part of this paper ([22]).

2.1.1 Translation invariant $\mathcal{C}^0$-RKP shape functions

Suppose the particles are uniformly distributed. From the equivalent definition (8), the $\mathcal{C}^0$-RKP basic shape function, $\phi_{([-K,K];0;2K-1)}(x)$, of reproducing order $2K-1$ is obtained by solving the following system of $2K$ equations:

$$\sum_{j=-K+1}^{K} (x-j)^{\alpha} f(x-j) = \delta_0^{\alpha}, \alpha = 0,1,2,\cdots,2K-1, \text{ for } x \in (0,1). \tag{9}$$

The coefficient matrix is the Vandermonde matrix. By Theorem 2.1 of ([22]), this system has a unique solution that can be extended to the unique continuous piecewise polynomial.

When $K=3$: For example, if $K=3$, the shape function obtained by the solution of the system (9) has the reproducing property of order 5, that satisfies the Kronecker delta property. It is defined as follows:

$$\phi_{([-3,3];0;5)}(x) = \begin{cases} f_1(x) := \frac{1}{120}(x+1)(x+2)(x+3)(x+4)(x+5) & x \in [-3,-2] \\ f_2(x) := -\frac{1}{24}(x-1)(x+1)(x+2)(x+3)(x+4) & x \in [-2,-1] \\ f_3(x) := \frac{1}{12}(x-2)(x-1)(x+1)(x+2)(x+3) & x \in [-1,0] \\ f_4(x) := -\frac{1}{12}(x-3)(x-2)(x-1)(x+1)(x+2) & x \in [0,1] \\ f_5(x) := \frac{1}{24}(x-4)(x-3)(x-2)(x-1)(x+1) & x \in [1,2] \\ f_6(x) := -\frac{1}{120}(x-5)(x-4)(x-3)(x-2)(x-1) & x \in [2,3] \\ 0 & x \notin [-3,3] \end{cases} \tag{10}$$

Here $f_1(x) = f(x-3), f_2(x) = f(x-2), \cdots, f_5(x) = f(x+1), f_6(x) = f(x+2)$ are the solutions of the system (9) and their graphs are depicted in Fig. 1.

When $K = 2$: Similarly, if $K = 2$, then we have the following $\mathcal{C}^0$-RKP shape function of reproducing order 3 that satisfy the Kronecker delta property:

$$\phi_{[-2,2];0;3}(x) = \begin{cases} f_{1,(2;0;3)}(x) := \frac{1}{6}(x+1)(x+2)(x+3) & x \in [-2,-1] \\ f_{2,(2;0;3)}(x) := -\frac{1}{2}(x-1)(x+1)(x+2) & x \in [-1,0] \\ f_{3,(2;0;3)}(x) := \frac{1}{2}(x-2)(x-1)(x+1) & x \in [0,1] \\ f_{4,(2;0;3)}(x) := -\frac{1}{6}(x-3)(x-2)(x-1) & x \in [1,2] \\ 0 & x \notin [-2,2]. \end{cases} \tag{11}$$

The graph of this shape function is shown in Fig. 1.

2.1.2 Translation invariant $\mathcal{C}^1$-RKP shape functions

Next, by sacrificing the order of polynomial reproducing property by one, one can construct the $\mathcal{C}^1$-RKP shape function, $\phi_{([-K,K];1;2K-2)}(x)$, with the reproducing property of order $2K-2$. For this end, we impose an additional condition to the last equation of the system (9) of $2K$ equations.

$$\begin{cases} \sum_{k=-K+1}^{K}(x-k)^{\alpha}g(x-k) & = \delta_0^{\alpha}, \alpha = 0,1,2,\cdots,2K-2, \\ \sum_{k=-K+1}^{K}(x-k)^{2K-1}g(x-k) & = G(x). \end{cases} \tag{12}$$

where $x \in (0,1)$. Then, the coefficient matrix of this constrained system becomes a $2K \times 2K$ Vandermonde matrix. The right hand side becomes the $2K$ dimensional column vector

$$[1,0,0,\cdots,0,G(x)]^T.$$

If $G(0) = 0$ and $G(1) = 0$, then by Lemma 2.1 of ([22]), the solution $g(x-j), j = -K+1,\cdots,K$, can be extended to a continuous function on $[-K,K]$ that satisfies the Kronecker delta property. Thus, we impose the following conditions on $G(x)$:

$$G(0) = 0 \text{ and } G(1) = 0. \tag{13}$$

That is, $G(x) = x(x-1)p(x)$. The $\mathcal{C}^1$-, $\mathcal{C}^2$- RKP shape functions are constructed in ([22]) by properly choosing $p(x)$.

When $K = 3$**:** Suppose $K = 3$ and $p(x) = 4-8x$. Then we obtain the unique $\mathcal{C}^1$- RKP shape function $\phi_{([-3,3];1;4)}(x)$ with the reproducing property of order 4 that satisfies the Kronecker delta property.

$$\phi_{([-3,3];1;4)}(x) = \begin{cases} g_1(x) := \frac{1}{120}x(x+2)(x+3)^2(x+7) & x \in [-3,-2], \\ g_2(x) := -\frac{1}{24}(x+1)(x+2)(x^3+6x^2-3x-24) & x \in [-2,-1], \\ g_3(x) := \frac{1}{12}(x+1)(x^4+2x^3-15x^2-12x+12) & x \in [-1,0], \\ g_4(x) := -\frac{1}{12}(x-1)(x^4-2x^3-15x^2+12x+12) & x \in [0,1], \\ g_5(x) := \frac{1}{24}(x-2)(x-1)(x^3-6x^2-3x+24) & x \in [1,2], \\ g_6(x) := -\frac{1}{120}(x-7)(x-3)^2(x-2)x & x \in [2,3], \\ 0 & x \notin [-3,3] \end{cases} \tag{14}$$

Here $g_1(x) = g(x-3), g_2(x) = g(x-2), \cdots, g_5(x) = g(x+1), g_6(x) = g(x+2)$, are the solutions of the system (12), whose graphs are depicted in Fig. 2.

When $K = 2$: With $G(x) = x(x-1)(1-2x)$ and $K = 2$, solving the system (12), we have the following $\mathcal{C}^1$-RKP shape function of reproducing order 2:

$$\phi_{([-2,2];1;3)}(x) = \begin{cases} g_{1,(2;1;2)}(x) := \frac{1}{2}(x+1)(x+2)^2 & x \in [-2,-1] \\ g_{2,(2;1;2)}(x) := -\frac{1}{2}(x+1)(3x^2+2x-2) & x \in [-1,0] \\ g_{3,(2;1;2)}(x) := \frac{1}{2}(x-1)(3x^2-2x-2) & x \in [0,1] \\ g_{4,(2;1;2)}(x) := -\frac{1}{2}(x-2)^2(x-1) & x \in [1,2] \\ 0 & x \notin [-2,2] \end{cases} \tag{15}$$

The graphs $g_{j,(2;1;2)}(x), j = 1,2,3,4$, are depicted in Fig. 2.

2.1.3 Translation invariant $\mathcal{C}^2$-RKP shape functions

With a proper choice for $p(x)$, solving the following system

$$\begin{cases} \sum_{k=-K+1}^{K}(x-k)^{\alpha}h(x-k) & = \delta_0^{\alpha}, \alpha = 0,1,2,\cdots,2K-2, \\ \sum_{k=-K+1}^{K}(x-k)^{2K-1}h(x-k) & = x(x-1)p(x), \end{cases} \tag{16}$$

we obtain a $\mathcal{C}^2$- RKP shape function $\phi_{([-K,K];2;2K-2)}$ with the reproducing property of order $2K-2$ that satisfy the Kronecker delta property.

For example, if $K = 3$ and $p(x) = 4 + 4x - 36x^2 + 24x^3$, the solution of this system yields the following piecewise polynomial $\mathcal{C}^2$-shape function of reproducing order 4.

$$\phi_{([-3,3];2;4)}(x) = \begin{cases} h_1(x) := \frac{1}{24}(x+2)(x+3)^3(5x+8) & x \in [-3,-2], \\ h_2(x) := -\frac{1}{24}(x+1)(x+2)(25x^3+114x^2+153x+48) & x \in [-2,-1], \\ h_3(x) := \frac{1}{12}(x+1)(25x^4+38x^3-3x^2-12x+12) & x \in [-1,0], \\ h_4(x) := -\frac{1}{12}(x-1)(25x^4-38x^3-3x^2+12x+12) & x \in [0,1], \\ h_5(x) := \frac{1}{24}(x-2)(x-1)(25x^3-114x^2+153x-48) & x \in [1,2], \\ h_6(x) := -\frac{1}{24}(x-3)^3(x-2)(5x-8) & x \in [2,3], \\ 0 & x \notin [-3,3] \end{cases} \tag{17}$$

Similarly, $h_1(x) = h(x-3), h_2(x) = h(x-2), \cdots, h_5(x) = h(x+1), h_6(x) = h(x+2)$, are the solutions of the system (16).

3 Piecewise polynomial RKP shape functions for non uniformly distributed particles that are in $[0, \infty)$

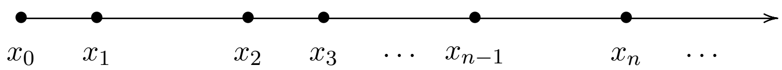
Suppose the domain is $[0, \infty)$, a_1, a_2, $\cdots$, a_n ($a_n \neq a_{n-1}$) are positive real numbers, and the particles are distributed as follows:

$$x_0, x_1, x_2, \cdots, x_n, x_{n+1}, \cdots$$

where

$$
\begin{aligned}
|x_j - x_{j-1}| &= a_j, \quad (j = 1, 2, \cdots, n-1), \\
|x_j - x_{j-1}| &= a_n, \quad (j \geq n), \quad \text{and} \quad x_0 = 0.
\end{aligned}
$$

The actual coordinates of particles are $x_0 = 0$, $x_1 = a_1$, $x_2 = a_1 + a_2$, $\cdots$, $x_n = \sum_{j=1}^{n} a_j$, $x_{n+1} = a_n + \sum_{j=1}^{n} a_j$, $x_{n+2} = 2a_n + \sum_{j=1}^{n} a_j$, and so on.



For each particle x_j, we will construct $\mathcal{C}^0$-piecewise polynomial RKP shape function $\phi_{(x_j)}(x)$ of reproducing order $2K - 1$, and we also construct that for $r \geq 1$, $\mathcal{C}^r$- piecewise polynomial RKP shape functions of reproducing order $2K - 2$.

As shown in Figs 1 and 2, it was proven in ([22]) that the unique translation invariant $\mathcal{C}^0$-piecewise polynomial basic RKP shape function with reproducing order $2K - 1$ must have support as large as $[-K, K]$, where K is a positive integer.

Thus, in order to construct piecewise polynomial RKP shape functions associated with non-uniformly distributed particles that are in $[0, \infty)$, we first give the conditions on an individual shape function and the set of particles that should be in the support of this shape function.

In the next section, we construct piecewise polynomial RKP shape functions for non-uniformly distributed particles that are allowed to go outside of the domain. In that case, like the shape functions associated with uniformly distributed particles, RKP shape functions have supports which consist of $2K$ consecutive intervals. However, when the particles are restricted to be in the interior of the domain or on the boundary of the domain, the supports of RKP shape functions corresponding to particles near boundary have various lengths. For example, the support of the shape function corresponding to the boundary particle consists of K subintervals. In this section, we will use the following notations for RKP shape functions.

$\phi_{(x_j)}(x) :=$ the shape function associated with the particle x_j (the shape function in the space coordinate system).

$\phi_j(x) :=$ the shape function centered at 0, obtained by translating $\phi_{(x_j)}(x)$ (the shape function in the reference coordinate system).

3.1 $\mathcal{C}^0$ piecewise polynomial RKP shape functions for arbitrary distributed particles in $[0, \infty)$

For a clear description of constructing $\mathcal{C}^0$-RKP shape functions for the particles that are non uniformly distributed on $[0, \infty)$, we start with a specific example such that $K = 2$ is the radius of the support of those shape functions corresponding to the uniformly distributed particles. In this example, we also assume that $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{2}$ and $a_4 = 1$ are the step sizes for

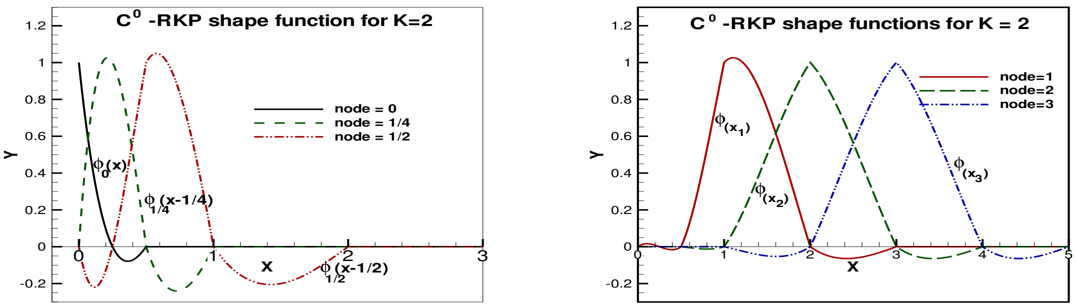


Figure 3: (Left:) The graph of the $\mathcal{C}^0$ RKP shape function of order 3, $\phi_{(x_0)}(x), \phi_{(x_1=1/4)}(x)\phi_{(x_2=1/2)}(x)$ (Right:) The graph of the $\mathcal{C}^0$ RKP shape function of order 3, $\phi_{(x_3=1)}(x), \phi_{(x_4=2)}(x), \phi_{(x_5=3)}(x)$. Let us note that these shape functions satisfy the Kronecker delta property, however parts of graphs are higher than 1. Actually, the maximum vale of these shape functions do not occur at nodal points.

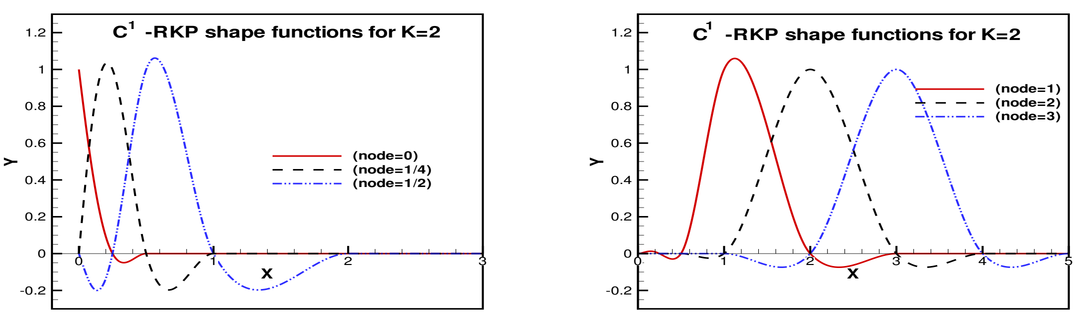


Figure 4: (Left:) The graph of the $\mathcal{C}^1$ RKP shape function of order 2, $\phi_{(x_0)}(x), \phi_{(x_1=1/4)}(x)\phi_{(x_2=1/2)}(x)$ (Right:) The graph of the $\mathcal{C}^1$ RKP shape function of order 2, $\phi_{(x_3=1)}(x), \phi_{(x_4=2)}(x), \phi_{(x_5=3)}(x)$.

non uniformly distributed particles. In this case, the coordinates of the related non uniformly distributed particles on $[0,\infty)$ are

$$x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = 1, \quad x_4 = 2, \quad x_5 = 3, \quad \cdots$$

Let us note that the $\mathcal{C}^0$-RKP shape functions $\phi_{(x_j)}$ corresponding to the above particles have the reproducing order $2K-1=3$ if and only if, for all $x \in [0,\infty)$,

$$\sum_{j=0}^{\infty}(x-x_j)^\alpha \phi_{(x_j)}(x) = \delta_0^\alpha, \quad \text{for } \alpha = 0,1,2,3. \tag{18}$$

In order for the above system of four equations to be solvable, we consider this system separately on each of subintervals $[0,1/2],[1/2,1],[1,2],[2,3],[3,4],\cdots$, with four shape functions corresponding to particles in the the following index sets:

$$\begin{aligned}
\Lambda_0 &= \{x_0,x_1,x_2,x_3\},\\
\Lambda_1 &= \{x_1,x_2,x_3,x_4\},\\
\Lambda_2 &= \{x_2,x_3,x_4,x_5\},\\
\Lambda_3 &= \{x_3,x_4,x_5,x_6\},\\
\Lambda_4 &= \{x_4,x_5,x_6,x_7\},\\
&\vdots
\end{aligned}$$

(I-C^0:) Let us start with the index set $\Lambda_4 = \{x_4 = 2, x_5 = 3, x_6 = 4, x_7 = 5\}$. Since the particle in this set is uniformly distributed, and the radius of the support is $K=2$, supp $\phi_{(x_j)}(x) = [x_j-2, x_j+2], j = 4,5,6,7$. Hence, the intersection of supports of $\phi_{(x_j)}, x_j \in \Lambda_4$ is $[3,4]$. If $x \in (3,4)$, then the system (18) becomes the following system for 4 unknowns:

$$\sum_{x_j\in\Lambda_4}(x-x_j)^\alpha \phi_{(x_j)}\mid_{[3,4]}(x) = \delta_0^\alpha, \quad \text{for } \alpha = 0,1,2,3.$$

Solving this equation, we have

$$\begin{cases}
\phi_{(x_4)}\mid_{[3,4]}(x) = -\dfrac{1}{6}(x-5)(x-4)(x-3),\\
\phi_{(x_5)}\mid_{[3,4]}(x) = \dfrac{1}{2}(x-5)(x-4)(x-2),\\
\phi_{(x_6)}\mid_{[3,4]}(x) = -\dfrac{1}{2}(x-5)(x-3)(x-2),\\
\phi_{(x_7)}\mid_{[3,4]}(x) = \dfrac{1}{6}(x-4)(x-3)(x-2).
\end{cases} \tag{19}$$

Using the unique $\mathcal{C}^0$-RKP shape function of reproducing order 3, defined by (11), for uniformly distributed particles, one can easily verify that

$$
\begin{aligned}
\phi_{(x_4)}\mid_{[3,4]}(x) &= f_{4,(2;0;3)}(x-2),\\
\phi_{(x_5)}\mid_{[3,4]}(x) &= f_{3,(2;0;3)}(x-3),\\
\phi_{(x_6)}\mid_{[3,4]}(x) &= f_{2,(2;0;3)}(x-4),\\
\phi_{(x_7)}\mid_{[3,4]}(x) &= f_{1,(2;0;3)}(x-5).
\end{aligned}
$$

By a similar manner, the solution functions of (18) for $x \in [k, k+1], k \geq 3$, with respect to an appropriate index set of four particles are translations of $\phi_{([-2,2];0;3)}(x)$ defined by (11). Thus, we assign the translated RKP shape function $\phi_{([-2,2];0;3)}(x - x_j)$ to each particle x_j if $j \geq 6$.

(II-C^0:) Next, if $x \in (2,3)$, then we use the set $\Lambda_3 = \{x_3 = 1, x_4 = 2, x_5 = 3, x_6 = 4\}$ so that the system (18) becomes the following system for four unknowns:

$$
\sum_{x_j \in \Lambda_3} (x - x_j)^\alpha \phi_{(x_j)}\mid_{[2,3]}(x) \quad = \quad \delta_0^\alpha, \quad \alpha = 0, 1, 2, 3.
$$

The solutions of this system are

$$
\begin{cases}
\phi_{(x_3)}\mid_{(2,3)}(x) = -\dfrac{1}{6}(x-4)(x-3)(x-2),\\
\phi_{(x_4)}\mid_{(2,3)}(x) = \dfrac{1}{2}(x-4)(x-3)(x-1),\\
\phi_{(x_5)}\mid_{(2,3)}(x) = -\dfrac{1}{2}(x-4)(x-2)(x-1),\\
\phi_{(x_6)}\mid_{(2,3)}(x) = \dfrac{1}{6}(x-3)(x-2)(x-1).
\end{cases} \tag{20}
$$

(III-C^0:) If $x \in (1,2)$, then we use the index set $\Lambda_2 = \left\{x_2 = \frac{1}{2}, x_3 = 1, x_4 = 2, x_5 = 3\right\}$. Then the system (18) becomes the following system for four unknowns:

$$
\sum_{x_j \in \Lambda_2} (x - x_j)^\alpha \phi_{(x_j)}\mid_{(1,2)}(x) \quad = \quad \delta_0^\alpha, \quad \alpha = 0, 1, 2, 3.
$$

The solutions of this system are

$$
\begin{cases}
\phi_{(x_2)}\mid_{[1,2]}(x) = -\dfrac{8}{15}(x-3)(x-2)(x-1),\\
\phi_{(x_3)}\mid_{[1,2]}(x) = \dfrac{1}{2}(x-3)(x-2)(2x-1),\\
\phi_{(x_4)}\mid_{[1,2]}(x) = -\dfrac{1}{3}(x-3)(x-1)(2x-1),\\
\phi_{(x_5)}\mid_{[1,2]}(x) = \dfrac{1}{10}(x-2)(x-1)(2x-1).
\end{cases} \tag{21}
$$

(IV-C^0:) If $x \in (1/2, 1)$, then we use the index set $\Lambda_1 = \left\{x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = 1, x_4 = 2\right\}$. Then the system (18) becomes the following system for four unknowns:

$$\sum_{x_j \in \Lambda_1} (x - x_j)^\alpha \phi_{(x_j)} \,|_{(1/2,1)}\, (x) \quad = \quad \delta_0^\alpha, \quad \alpha = 0, 1, 2, 3.$$

The solutions of this system are

$$\begin{cases} \phi_{(x_1)} \,|_{[\frac{1}{2},1]}\, (x) = -\dfrac{32}{21}(x-2)(x-1)(2x-1) \\ \phi_{(x_2)} \,|_{[\frac{1}{2},1]}\, (x) = \dfrac{4}{3}(x-2)(x-1)(4x-1) \\ \phi_{(x_3)} \,|_{[\frac{1}{2},1]}\, (x) = -\dfrac{1}{3}(x-2)(2x-1)(4x-1) \\ \phi_{(x_4)} \,|_{[\frac{1}{2},1]}\, (x) = \dfrac{1}{21}(x-1)(2x-1)(4x-1). \end{cases} \tag{22}$$

(V-C^0:) Thus far, for $x \in (1/2, \infty)$, we have solved

$$\sum_{j=0}^{\infty} (x - x_j)^\alpha \phi_{(x_j)}(x) \quad = \quad \delta_0^\alpha, \quad \alpha = 0, 1, 2, 3.$$

In order to get the RKP shape functions that satisfy the above system for all $x \in [0, \infty)$, we finally use the index set $\Lambda_0 = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$ as follows: for $x \in [0, 1/2)$, solve the following system

$$\sum_{x_j \in \Lambda_0} (x - x_j)^\alpha \phi_{(x_j)} \,|_{[0,\frac{1}{2}]}\, (x) \quad = \quad \delta_0^\alpha, \quad \alpha = 0, 1, 2, 3.$$

Then the solutions of this system for the four unknown functions, are

$$\begin{cases} \phi_{(x_0)} \,|_{[0,\frac{1}{2}]}\, (x) = -(x-1)(2x-1)(4x-1) \\ \phi_{(x_1)} \,|_{[0,\frac{1}{2}]}\, (x) = \dfrac{32}{3}(x-1)x(2x-1) \\ \phi_{(x_2)} \,|_{[0,\frac{1}{2}]}\, (x) = -4(x-1)x(4x-1) \\ \phi_{(x_3)} \,|_{[0,\frac{1}{2}]}\, (x) = \dfrac{1}{3}x(2x-1)(4x-1). \end{cases} \tag{23}$$

Now, assembling the solutions listed in Eqns (19), (20), (21), (22), and (23), we define the RKP shape functions of reproducing order 3 corresponding to the particles, $x_0, x_1, x_2, x_3, x_5, \cdots$, that have the following supports, respectively,

$$\begin{aligned} \text{supp} \quad \phi_{(x_0)} &= [0, 1/2], \\ \text{supp} \quad \phi_{(x_1)} &= [0, 1/2] \cup [1/2, 1], \\ \text{supp} \quad \phi_{(x_2)} &= [0, 1/2] \cup [1/2, 1] \cup [1, 2], \\ \text{supp} \quad \phi_{(x_3)} &= [0, 1/2] \cup [1/2, 1] \cup [1, 2] \cup [2, 3], \\ \text{supp} \quad \phi_{(x_4)} &= [1/2, 1] \cup [1, 2] \cup [2, 3] \cup [3, 4], \\ \text{supp} \quad \phi_{(x_5)} &= [1, 2] \cup [2, 3] \cup [3, 4] \cup [4, 5], \\ \text{supp} \quad \phi_{(x_j)} &= [x_j - 2, x_j + 2], \text{ if } j \geq 6. \end{aligned} \tag{24}$$

Since all partially generated RKP shape functions satisfy the Kronecker delta property, i.e. $\phi_{(x_j)}(x_i) = \delta^i_j$ for all $i, j \geq 0$. The assembled shape functions are continuous piecewise polynomials that satisfy the Kronecker delta property.

The assembled $\mathcal{C}^0$-RKP shape functions of reproducing order 3 are as follows:

1. $\phi_{(x_0)}(x) = \phi_0(x)$ and $\phi_0(x)$ is as follows:

$$\phi_0(x) = \begin{cases} -(x-1)(2x-1)(4x-1) & x \in [0, \frac{1}{2}] \\ 0 & x \notin [0, \frac{1}{2}] \end{cases}$$

2. $\phi_{(x_1)}(x) = \phi_1(x - 1/4)$ and $\phi_1(x)$ is as follows:

$$\phi_1(x) = \begin{cases} \frac{1}{3}(4x-3)(4x-1)(4x+1) & x \in [-\frac{1}{4}, \frac{1}{4}] \\ -\frac{1}{21}(4x-7)(4x-3)(4x-1) & x \in [\frac{1}{4}, \frac{3}{4}] \\ 0 & x \notin [-\frac{1}{4}, \frac{3}{4}] \end{cases}$$

3. $\phi_{(x_2)}(x) = \phi_2(x - 1/2)$ and $\phi_2(x)$ is as follows:

$$\phi_2(x) = \begin{cases} -(2x-1)(2x+1)(4x+1) & x \in [-\frac{1}{2}, 0] \\ \frac{1}{3}(2x-3)(2x-1)(4x+1) & x \in [0, \frac{1}{2}] \\ -\frac{1}{15}(2x-5)(2x-3)(2x-1) & x \in [\frac{1}{2}, \frac{3}{2}] \\ 0 & x \notin [-\frac{1}{2}, \frac{3}{2}] \end{cases}$$

4. $\phi_{(x_3)}(x) = \phi_3(x - 1)$ and $\phi_3(x)$ is as follows:

$$\phi_3(x) = \begin{cases} \frac{1}{3}(x+1)(2x+1)(4x+3) & x \in [-1, -\frac{1}{2}] \\ -\frac{1}{3}(x-1)(2x+1)(4x+3) & x \in [-\frac{1}{2}, 0] \\ \frac{1}{2}(x-2)(x-1)(2x+1) & x \in [0, 1] \\ f_{4,(2;0;3)}(x) & x \in [1, 2] \\ 0 & x \notin [-1, 2] \end{cases}$$

5. $\phi_{(x_4)}(x) = \phi_4(x - 2)$ and $\phi_4(x)$ is as follows:

$$\phi_4(x) = \begin{cases} \frac{1}{21}(x+1)(2x+3)(4x+7) & x \in [-\frac{3}{2}, -1] \\ -\frac{1}{3}(x-1)(x+1)(2x+3) & x \in [-1, 0] \\ f_{3,(2;0;3)}(x) & x \in [0, 1] \\ f_{4,(2;0;3)}(x) & x \in [1, 2] \\ 0 & x \notin [-\frac{3}{2}, 2] \end{cases}$$

6. $\phi_{(x_5)}(x) = \phi_5(x-3)$ and $\phi_5(x)$ is as follows:

$$\phi_5(x) = \begin{cases} \frac{1}{10}(x+1)(x+2)(2x+5) & x \in [-2,-1] \\ f_{2,(2;0;3)}(x) & x \in [-1,0] \\ f_{3,(2;0;3)}(x) & x \in [0,1] \\ f_{4,(2;0;3)}(x) & x \in [1,2] \\ 0 & x \notin [-2,2] \end{cases}$$

7. The shape function corresponding to the particle x_j $(j \geq 6)$ is

$$\phi_{(x_j)}(x) = \phi_{([-2,2];0;3)}(x - x_j)$$

The graph of these shape functions $\phi_{(x_j)}(x), j = 0,1,2,3,4,5,6$, are depicted in Fig. 3.

The construction of RKP shape functions for the general case are similar to the above example. Let us briefly describe the general procedure.

The $\mathcal{C}^0$-RKP shape functions $\phi_{(x_j)}$ corresponding to particles that are non uniformly distributed on $[0,\infty)$ have the polynomial reproducing order $2K-1$ if and only if, for all $x \in [0,\infty)$,

$$\sum_{j=0}^{\infty}(x-x_j)^{\alpha}\phi_{(x_j)}(x) = \delta_0^{\alpha} \quad \text{for } \alpha = 0,1,2,3,\cdots,2K-1. \tag{25}$$

In order for the above system of $2K$ equations to be solvable, we consider this system on the subintervals, as in the above example, with $2K$ shape functions corresponding to particles in the the following index sets

$$\begin{aligned} \Lambda_0 &= \{x_0, x_1, \cdots, x_{2K-1}\}, \\ \Lambda_1 &= \{x_1, x_2, \cdots, x_{2K}\}, \\ \Lambda_2 &= \{x_2, x_3, \cdots, x_{2K+1}\}, \\ &\vdots \\ \Lambda_n &= \{x_n, x_{n+1}, \cdots, x_{n+2K-1}\}, \\ &\vdots \end{aligned}$$

(nG) : Suppose the particles in Λ_n are uniformly distributed and are not affected by the non uniformly distributed particles. The intersection of supports of the RKP shape functions for the particles in Λ_n is $[x_{n+K-1}, x_{n+K}]$. For $x \in [x_{n+K-1}, x_{n+K}]$, we solve the system of $2K$ equations for $2K$ unknowns:

$$\sum_{x_j \in \Lambda_n} (x-x_j)^{\alpha}\phi_{(x_j)} |_{[x_{n+K-1},x_{n+K}]} (x) = \delta_0^{\alpha}, \quad \alpha = 0,1,2,3,\cdots,2K-1.$$

$((n-1)G)$: For $x \in [x_{n+K-2}, x_{n+K-1}]$, we solve the system of $2K$ equations for $2K$ unknowns:

$$\sum_{x_j \in \Lambda_{n-1}} (x - x_j)^\alpha \phi_{(x_j)} \mid_{[x_{n+K-2}, x_{n+K-1}]} (x) \quad = \quad \delta_0^\alpha, \quad \alpha = 0, 1, 2, 3, \cdots, 2K-1.$$

$\vdots$

$(1G)$: For $x \in [x_K, x_{K+1}]$, we solve the system of $2K$ equations for $2K$ unknowns:

$$\sum_{x_j \in \Lambda_1} (x - x_j)^\alpha \phi_{(x_j)} \mid_{[x_K, x_{K+1}]} (x) \quad = \quad \delta_0^\alpha, \quad \alpha = 0, 1, 2, 3, \cdots, 2K-1.$$

$(0G)$: Finally, for $x \in [0, x_K]$, we solve the system of $2K$ equations for $2K$ unknowns:

$$\sum_{x_j \in \Lambda_0} (x - x_j)^\alpha \phi_{(x_j)} \mid_{[0, x_K]} (x) \quad = \quad \delta_0^\alpha, \quad \alpha = 0, 1, 2, 3, \cdots, 2K-1.$$

By a similar method to the above example, assembling those partially defined RKP shape functions in the steps $(nG), ((n-1)G), \cdots, (1G)$, and $(0G)$ together, we have the required $\mathcal{C}^0$-RKP shape functions associated with particles x_j $(j \geq 0)$.

3.2 $\mathcal{C}^1$-piecewise polynomial RKP shape functions for arbitrary distributed particles that are in $[0, \infty)$

In the construction of $\mathcal{C}^0$-RKP shape functions in the previous section, we solved the system of $2K$ equations for $2K$ unknowns on each interval $[0, x_K]$, $[x_K, x_{K+1}]$, $\cdots$, $[x_{n+K-1}, x_{n+K}]$, $\cdots$. Without any restrictions, the solution functions are assembled to be $\mathcal{C}^0$-RKP shape functions of reproducing order $2K-1$.

However, as we discussed in part 1, in order for the assembled shape functions to be $\mathcal{C}^r, r > 0$ functions, we have to impose some conditions to the system of $2K$ equations for $2K$ unknowns on each of the above intervals. More specifically, we need to impose some conditions so that the solution functions have the same derivative at the nodal points $x_j, j > 0$.

In order to impose such conditions for smoothness, it is necessary to sacrifice the reproducing order by 1, by imposing a polynomial $G(x)$ on the right hand side of the last equation as follows:

$$\begin{cases} \sum_{j=0}^{\infty} (x - x_j)^\alpha \phi_{(x_j)}(x) & = \quad \delta_0^\alpha, \qquad \alpha = 0, 1, 2, \cdots, 2K-2, \\ \sum_{j=0}^{\infty} (x - x_j)^{2K-1} \phi_{(x_j)}(x) & = \quad G(x) \end{cases} \tag{26}$$

Then, obviously, the shape functions that are solutions of the system (26) of $2K$ equations have the reproducing property of order $2K-2$. Here $G(x)$ will be properly selected on each interval so that the solution of this system can be $\mathcal{C}^r, r > 0$, functions.

In this section, we sketch how to construct $\mathcal{C}^1$-RKP shape functions of order $2K-2$. In order to make the system (26) coupled with $2K$ unknowns on each interval, we use the same subsets of particles, $\Lambda_0, \Lambda_1, \cdots, \Lambda_n, \cdots$, as before.

For specific construction of $\mathcal{C}^1$-RKP shape functions, we describe it for the RKP shape functions of reproducing order $2K-2, K=2$, corresponding to the following non uniformly distributed particles:

$$x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = 1, x_4 = 2, x_5 = 3, \cdots$$

Following the same argument as the previous subsection, we consider the system (26) separately on each of subintervals

$$[0, 1/2], [1/2, 1], [1, 2], [2, 3], [3, 4], \cdots,$$

with four shape functions corresponding to particles in the the following index sets:

$$\Lambda_0 = \{x_0, x_1, x_2, x_3\}, \Lambda_1 = \{x_1, x_2, x_3, x_4\}, \Lambda_2 = \{x_2, x_3, x_4, x_5\}, \Lambda_3 = \{x_3, x_4, x_5, x_6\}, \Lambda_4 = \{x_4, x_5, x_6, x_7\}, \cdots.$$

If we consider only the particles corresponding to the index set Λ_k, the system (26) can be rewritten as follows:

$$\begin{cases} \sum_{x_j \in \Lambda_k} (x - x_j)^\alpha \phi_{(x_j)}(x) = \delta_0^\alpha, & \alpha = 0, 1, 2, \\ \sum_{x_j \in \Lambda_k} (x - x_j)^3 \phi_{(x_j)}(x) = G_k(x) \end{cases} \tag{27}$$

(I-C^1:) By the same reason as the previous section, if we start with the index set $\Lambda_4 = \{x_4 = 2, x_5 = 3, x_6 = 4, x_7 = 5\}$, then we only need to solve the system (27) for $x \in [3, 4]$. It was shown in ([22]) that in order for this system being uniquely solvable, 3 and 4 should be zeros of $G_k(x), k = 4$.

Solving the constrained system (27) for $G_4(x) = (x-4)(x-3)(7-2x), \Lambda_4$, and $x \in [3, 4]$, we have the following solutions:

$$\begin{cases} \phi_{(x_4)} \mid_{[3,4]} (x) = -\frac{1}{2}(x-4)^2(x-3) \\ \phi_{(x_5)} \mid_{[3,4]} (x) = \frac{1}{2}(x-4)(3x^2 - 20x + 31) \\ \phi_{(x_6)} \mid_{[3,4]} (x) = -\frac{1}{2}(x-3)(3x^2 - 22x + 38) \\ \phi_{(x_7)} \mid_{[3,4]} (x) = \frac{1}{2}(x-4)(x-3)^2 \end{cases} \tag{28}$$

Let us note that in this section, the choice of the constraint functions $G_k(x)$ are generally not unique.

By using the shape function (15) of reproducing order 3 for uniformly distributed particles, One can see that those functions in (28) can be written as translations of (15) as follows:

$$\begin{cases} \phi_{(x_4)} \mid_{[3,4]} (x) = g_{4,(2;1;2)}(x-2) \\ \phi_{(x_5)} \mid_{[3,4]} (x) = g_{3,(2;1;2)}(x-3) \\ \phi_{(x_6)} \mid_{[3,4]} (x) = g_{2,(2;1;2)}(x-4) \\ \phi_{(x_7)} \mid_{[3,4]} (x) = g_{1,(2;1;2)}(x-5) \\ 0 & x \notin [-2, 2] \end{cases} \tag{29}$$

(II-C^1:) Solving the constrained system (27) for $G_3(x) = (x-3)(x-2)(5-2x), \Lambda_3$, and $x \in [2,3]$, we have the following solutions:

$$\begin{cases} \phi_{(x_3)}\mid_{[2,3]}(x) = -\dfrac{1}{2}(x-3)^2(x-2) \\ \phi_{(x_4)}\mid_{[2,3]}(x) = \dfrac{1}{2}(x-3)(3x^2-14x+14) \\ \phi_{(x_5)}\mid_{[2,3]}(x) = -\dfrac{1}{2}(x-2)(3x^2-16x+19) \\ \phi_{(x_6)}\mid_{[2,3]}(x) = \dfrac{1}{2}(x-3)(x-2)^2 \end{cases} \tag{30}$$

(III-C^1:) Solving the constrained system (27) for $G_2(x) = (x-2)(x-1)(2-\frac{3}{2}x), \Lambda_2$, and $x \in [1,2]$, we have the following solutions:

$$\begin{cases} \phi_{(x_2)}\mid_{[1,2]}(x) = -\dfrac{4}{3}(x-2)^2(x-1) \\ \phi_{(x_3)}\mid_{[1,2]}(x) = \dfrac{1}{2}(x-2)(5x^2-14x+7) \\ \phi_{(x_4)}\mid_{[1,2]}(x) = -\dfrac{1}{3}(x-1)(5x^2-17x+11) \\ \phi_{(x_5)}\mid_{[1,2]}(x) = \dfrac{1}{2}(x-2)(x-1)^2 \end{cases} \tag{31}$$

(IV-C^1:) Solving the constrained system (27) for $G_1(x) = (x-1)(x-\frac{1}{2})(\frac{3}{2}-\frac{5}{2}x), \Lambda_1$, and $x \in [1/2,1]$, we have the following solutions:

$$\begin{cases} \phi_{(x_1)}\mid_{[\frac{1}{2},1]}(x) = -\dfrac{16}{3}(x-1)^2(2x-1) \\ \phi_{(x_2)}\mid_{[\frac{1}{2},1]}(x) = \dfrac{4}{3}(x-1)(14x^2-20x+5) \\ \phi_{(x_3)}\mid_{[\frac{1}{2},1]}(x) = -\dfrac{1}{3}(2x-1)(14x^2-25x+8) \\ \phi_{(x_4)}\mid_{[\frac{1}{2},1]}(x) = \dfrac{1}{3}(x-1)(2x-1)^2 \end{cases} \tag{32}$$

(V-C^1:) Finally, Solving the constrained system (27) for $G_0(x) = (x-1)(x-\frac{1}{2})(x-\frac{1}{4})x(6x-1), \Lambda_0$, and $x \in [0,1/2]$, we have the following solutions:

$$\begin{cases} \phi_{(x_0)} \mid_{[0,\frac{1}{2}]} (x) = (x-1)(2x-1)^2(3x+1)(4x-1) \\ \phi_{(x_1)} \mid_{[0,\frac{1}{2}]} (x) = -\dfrac{8}{3}(x-1)x(2x-1)(24x^2-10x-3) \\ \phi_{(x_2)} \mid_{[0,\frac{1}{2}]} (x) = 2(x-1)x(4x-1)(12x^2-8x-1) \\ \phi_{(x_3)} \mid_{[0,\frac{1}{2}]} (x) = -\dfrac{1}{3}x^2(2x-1)(4x-1)(6x-7) \end{cases} \tag{33}$$

By assembling those solutions of (28), (30), (31), (32), and (33), we obtain the $\mathcal{C}^1$-RKP shape functions of reproducing order 2 corresponding to the the following particles

$$x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = 1, x_4 = 2, x_5 = 3, \cdots$$

The assembled $\mathcal{C}^1$-RKP shape functions of reproducing order 2 that satisfy the Kronecker delta property are as follows:

1. $\phi_{(x_0)} = \phi_0(x-0)$, where $\phi_0(x)$ is as follows:

$$\phi_0(x) = \begin{cases} (x-1)(2x-1)^2(3x+1)(4x-1) & x \in [0, \frac{1}{2}] \\ 0 & x \notin [0, \frac{1}{2}] \end{cases}$$

2. $\phi_{(x_1)} = \phi_1(x-1/4)$, where $\phi_1(x)$ is as follows:

$$\phi_1(x) = \begin{cases} -\frac{1}{6}(4x-3)(4x-1)(4x+1)(12x^2+x-2) & x \in [-\frac{1}{4}, \frac{1}{4}] \\ -\frac{1}{6}(4x-3)^2(4x-1) & x \in [\frac{1}{4}, \frac{3}{4}] \\ 0 & x \notin [-\frac{1}{4}, \frac{3}{4}] \end{cases}$$

3. $\phi_{(x_2)} = \phi_2(x-1/2)$, where $\phi_2(x)$ is as follows: For the particle $x_2 = \frac{1}{2}$

$$\phi_2(x) = \begin{cases} (2x-1)(2x+1)(4x+1)(6x^2+2x-1) & x \in [-\frac{1}{2}, 0] \\ \frac{1}{3}(2x-1)(28x^2-12x-3) & x \in [0, \frac{1}{2}] \\ -\frac{1}{6}(2x-3)^2(2x-1) & x \in [\frac{1}{2}, \frac{3}{2}] \\ 0 & x \notin [-\frac{1}{2}, \frac{3}{2}] \end{cases}$$

4. $\phi_{(x_3)} = \phi_3(x-1)$, where $\phi_3(x)$ is as follows:

$$\phi_3(x) = \begin{cases} -\frac{1}{3}(x+1)^2(2x+1)(4x+3)(6x-1) & x \in [-1, -\frac{1}{2}] \\ -\frac{1}{3}(2x+1)(14x^2+3x-3) & x \in [-(\frac{1}{2}), 0] \\ \frac{1}{2}(x-1)(5x^2-4x-2) & x \in [0, 1] \\ g_{4,(2;1;2)}(x) & x \in [1, 2] \\ 0 & x \notin [-1, 2] \end{cases}$$

5. $\phi_{(x_4)} = \phi_4(x-2)$, where $\phi_4(x)$ is as follows:

$$\phi_4(x) = \begin{cases} \frac{1}{3}(x+1)(2x+3)^2 & x \in [-\frac{3}{2}, -1] \\ -\frac{1}{3}(x+1)(5x^2+3x-3) & x \in [-1, 0] \\ g_{3,(2;1;2)}(x) & x \in [0, 1] \\ g_{4,(2;1;2)}(x) & x \in [1, 2] \\ 0 & x \notin [-\frac{3}{2}, 2] \end{cases}$$

6. $\phi_{(x_5)} = \phi_5(x-3)$, where $\phi_5(x)$ is as follows:

$$\phi_5(x) = \begin{cases} \frac{1}{2}(x+1)(x+2)^2 & x \in [-2, -1] \\ g_{2,(2;1;2)}(x) & x \in [-1, 0] \\ g_{3,(2;1;2)}(x) & x \in [0, 1] \\ g_{4,(2;1;2)}(x) & x \in [1, 2] \\ 0 & x \notin [-2, 2] \end{cases}$$

7. For the particle x_j $(j \geq 6)$

$$\phi_{(x_j)}(x) = \phi_{([-2,2];1;3)}(x - x_j)$$

The graphs of the $\mathcal{C}^1$-RKP shape functions of reproducing order 2, associated with those particle $x_j, j = 0, 1, 2, 3, 4, 5$, are depicted in Fig. 4.

In general, to construct $\mathcal{C}^r, r \geq 1$, RKP shape function of reproducing order $2K-2$, we have to solve the following system of $2K$ equations with one constrained equation for index subsets Λ_k containing $2K$ particles:

$$\begin{cases} \sum_{x_j \in \Lambda_k} (x - x_j)^\alpha \phi_{(x_j)}(x) & = & \delta_0^\alpha & \alpha = 0, 1, 2, \cdots, 2K-2, \\ \sum_{x_j \in \Lambda_k} (x - x_j)^{2K-1} \phi_{(x_j)}(x) & = & G_i(x) \end{cases} \tag{34}$$

Here $G_k(x)$ is properly selected so that the resulting solutions have the same derivatives at the nodes and satisfy the Kronecker delta property.

Since the construction procedures of RKP shape functions associated with the particles in $[0, \infty)$ do not depend on non uniformity of particles, we put $\mathcal{C}^0$-RKP shape functions of reproducing order $5(K = 3)$ as well as $\mathcal{C}^1$-RKP shape functions of reproducing order $4(K = 3)$ associated with the uniformly distributed particles $x_j = j, j = 0, 1, 2, 3, \cdots$, in appendix.

3.3 $\mathcal{C}^r$-piecewise polynomial RKP shape functions for arbitrary distributed particles that are in a closed interval $[a, b]$

In the previous subsections, we constructed piecewise polynomial RKP shape functions when the particles are non-uniformly(or uniformly) distributed in $[0, \infty)$. By taking linear transformation of those shape functions, we can also have piecewise polynomial RKP shape functions of the same degree of reproducing order, associated with the particles distributed on $(-\infty, b]$. Thus, we can construct piecewise polynomial RKP shape functions associated with non-uniformly(or uniformly) distributed particles which are in a given bounded domain $[a, b]$.

Without loss of generality, we construct $\mathcal{C}^0$-RKP shape functions when the particles are in $[a, b], a < b$. Suppose $K = 3$ and the particles are uniformly distributed as follows:

$$x_0 = 0, x_1 = 1, x_2 = 2, \cdots, x_5 = 5, x_6 = 6, x_7 = 7, \cdots, x_{10} = 10, x_{11} = 10, x_{12} = 12, \tag{35}$$

on the closed bounded domain $[0, 12]$.

Let $\phi_{(x_j)}(x)$ be the $\mathcal{C}^0$-piecewise polynomial RKP shapes functions, that are constructed in Appendix [A-I], associated with the particles: $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6$. Let us note that

$$\phi_{(x_6)}(x) = \phi_{([-3,3];0;5)}(x - x_6)$$

which is symmetric about x_6.

Now, we assign piecewise polynomial $\mathcal{C}^0$-shape functions, ψ_{x_j}, associated with the uniformly distributed particles on the bounded domain $[0, 12]$, like those in (35), as follows:

1. $\psi_{x_j}(x) := \phi_{(x_j)}(x), \quad j = 0, 1, 2, 3, 4, 5,$
2. $\psi_{x_6}(x) := \phi_{([-3,3];0;5)}(x - x_6),$
3. $\psi_{x_j}(x) := \phi_{(12-x_j)}(-(x - 12)), \quad j = 7, 8, 9, 10, 11, 12.$

Then, these shape functions have the property of reproducing of order 5. Indeed,

$$\sum_{j=0}^{12}(x - x_j)^\alpha \psi_{x_j}(x) = \begin{cases} \displaystyle\sum_{j=0}^{8}(x - x_j)^\alpha \psi_{x_j}(x) & \text{for} \quad x \in [0, 6] \\ \displaystyle\sum_{j=4}^{12}(x - x_j)^\alpha \psi_{x_j}(x) & \text{for} \quad x \in [6, 12] \end{cases}$$

Since $\phi_{x_4}|_{[3,7]}(x) = \phi_{([-3,3];0;5)}|_{[-1,3]}(x - x_4)$, we have

$$\begin{aligned} \psi_{x_7}(x) &= \phi_{x_7}(-(x - 12)) \\ &= \phi_{([-3,3];0;5)}((-(x - 12)) - x_5) = \phi_{([-3,3];0;5)}((-(x - 12)) - 5) \\ &= \phi_{([-3,3];0;5)}(x - 7) \quad = \quad \phi_{x_7}(x), \text{ for all } x \in [4, 6]. \end{aligned}$$

Similarly, using the relation $\phi_{x_5}|_{[3,8]}(x) = \phi_{([-3,3];0;5)}|_{[-2,3]}(x - x_5)$, we can show

$$\psi_{x_8}(x) = \phi_{x_8}(x), \text{ for all } x \in [4, 6].$$

Thus, we have

$$\sum_{j=0}^{8}(x - x_j)^\alpha \psi_{x_j}(x) = \sum_{j=0}^{8}(x - x_j)^\alpha \phi_{x_j}(x) = x^\alpha, \quad \text{for all } x \in [0, 6], \quad \alpha = 0, 1, 2, 3, 4, 5.$$

On the other hand, the following Lemma leads us to

$$\sum_{j=4}^{12}(x-x_j)^\alpha\psi_{x_j}(x)=x^\alpha,\qquad \text{for all } x\in[6,12],\qquad \alpha=0,1,2,3,4,5.$$

Lemma 3.1. *Let ϕ_{x_j} be the RKP shape functions of reproducing order k associated with the particles $x_j\in[\alpha,\beta]$, $j=1,2,\cdots,n$. Let $T(x)=ax+b$, $a\neq 0$, be a linear transformation on $\mathbb{R}$. Then the functions defined by*

$$\psi_{\xi_j}(\xi):=(\phi_{(x_j)}\circ T^{-1})(\xi)\quad,\quad j=1,2,\cdots,n,$$

become polynomial reproducing shape functions associated with the particles $\xi_j=T(x_j)$ on $T([\alpha,\beta])$.

Proof.

$$\begin{aligned}\sum_{j=1}^{n}(\xi-\xi_j)^\alpha\psi_{\xi_j}(\xi) &= \sum_{j=1}^{n}\left(T(x)-T(x_j)\right)^\alpha\phi_{(x_j)}(T^{-1}(T(x)))\\ &= a^\alpha\sum_{j=1}^{n}\left(x-x_j\right)^\alpha\phi_{(x_j)}(x)\\ &= a^\alpha\delta_0^\alpha \;=\; \delta_0^\alpha,\ \text{for } \alpha=0,1,2,\cdots,k.\end{aligned}$$

The last equality follows from the fact that $a^0=1$. □

3.4 An Interpolation Error Estimate

For brevity of the proof of Lemma 3.1, we assume that the shape functions, defined by

$$\phi_j^h(x):=\phi(\frac{x-hj}{h}),x\in\mathbb{R}^d,$$

are the tensor product of those shape functions associated with uniformly distributed particles in $\mathbb{R}$ Then, we have the followings:

(1) supp $\phi_j^h(x)\subset\prod_{i=1}^d[hj_i-hK,hj_i+hK]$,

(2) Their polynomial reproducing order become $2K-1$ or $2K-2$ according as all $\phi_j^h(x)$ are $\mathcal{C}^0$-shape functions or $\mathcal{C}^r$-shape functions for $r>0$.

We will consider a smooth function $u(x)$ defined in Ω and study the interpolation error between u and $\mathcal{I}_hu$, where $\mathcal{I}_hu$ is the interpolant of u in terms of ϕ_j^h defined as follows:

$$\left(\mathcal{I}_hu\right)(x):=\sum_{j\in\Lambda}u(x_j^h)\phi_j^h(x).$$

It is immediate that this definition can be stated as

$$\big(\mathcal{I}_h u\big)(x) = \sum_{j\in A_x^h} u(x_j^h)\phi_j^h(x),$$

where

$$A_x^h := \{j \in \Lambda \mid x \in \text{supp } \phi_j^h\}$$

is the influence set for the point x. Note also that the number of elements of A_x^h is uniformly bounded.

An interpolation error estimate for RKP shape functions was proved in Han and Meng ([10]), and Babuska-Banerjee-Osborn([2]). However, we present a more direct proof of an interpolation error estimate in the following lemma.

Lemma 3.2. *Suppose $u \in H^{2K+l}(\Omega), l > d/2, 4K-2 > d$, and Ω is a convex domain in $\mathbb{R}^d$ with smooth boundary. Then we have*

$$\|u - \mathcal{I}_h u\|_{1,\Omega} \le Ch^{2K-1}|u|_{2K,\Omega}$$

where the constant C is independent of u and h, but depends on the support size K of the RKP shape functions.

Proof. If $l > d/2$, it follows from the Sobolev imbedding theorem([9]) that $H^{2K+l}(\Omega) \subset C_B^{2K}(\Omega) = \{u \in C^{2K}(\Omega) : D^\alpha u \in L_\infty(\Omega) \text{ for } |\alpha| \le 2K\}$. Hence, for $|\alpha| \le 2K$, $D^\alpha u(x)$ has point values in Ω. Thus, using the Cauchy-Taylor formula:

$$f(x+y) = \sum_{|\alpha|<k} \frac{\partial^\alpha f(x)}{\alpha!} y^\alpha + k\int_0^1 (1-t)^{k-1} \sum_{|\alpha|=k} \frac{\partial^\alpha f(x+ty)}{\alpha!} y^\alpha dt,$$

we have

$$u(x_j^h) = \sum_{|\alpha|<2K} \frac{\partial^\alpha u(x)}{\alpha!}(x_j^h - x)^\alpha + R_K u(x_j^h, x), \text{ for } x \in \Omega,$$

where

$$R_K u(x_j^h, x) = (2K) \sum_{|\alpha|=2K} \frac{1}{\alpha!}\Big(\int_0^1 (1-t)^{2K-1}\partial^\alpha u\big(x + t(x_j^h - x)\big)dt\Big)(x_j^h - x)^\alpha.$$

Therefore, according to (8), we have

$$\begin{aligned}
\mathcal{I}_h u(x) &= \sum_j u(x_j^h)\phi_j^h(x) \\
&= \sum_j \Big(\sum_{|\alpha|<2K} \frac{\partial^\alpha u(x)}{\alpha!}(x_j^h - x)^\alpha + R_K u(x_j^h, x)\Big)\phi_j^h(x) \\
&= u(x) + \sum_j R_K u(x_j^h, x)\phi_j^h(x)
\end{aligned}$$

and hence

$$u(x) - \mathcal{I}_h u(x) = -\sum_j R_K u(x_j^h, x)\phi_j^h(x). \tag{36}$$

(Step 1) We first prove the following L_2-estimate of the interpolation error:

$$\|u - \mathcal{I}_h u\|_{0,\Omega}^2 \le Ch^{2(2K)}|u|_{2K,\Omega}^2, \text{ if } 4K-2 \ge d. \tag{37}$$

Indeed, let us choose particles in $\overline{\Omega}$ such that

$$\Omega \subset \cup_j \eta_j^h, \text{ where } \eta_j^h = \text{supp}(\phi_j^h) = \{x \mid |x_i - x_{j_i}^h| \le Kh, i = 1, 2, \cdots, d\}.$$

Applying Cauchy inequality to the right hand side of (36) and letting $C = \sum_{|\alpha|=2K}[\frac{1}{\alpha!}]^2$, we have the following:

$$\begin{aligned} \|u - \mathcal{I}_h u\|^2 &= \int_\Omega |\sum_j R_K u(x_j^h, x)\phi_j^h(x)|^2 dx \\ &\le C(2K)^2 \int_0^1 (1-t)^{2(2K-1)} F(x,t)dt, \end{aligned} \tag{38}$$

where

$$F(x,t) = \sum_{|\alpha|=2K} \sum_j \int_\Omega \Big\{\partial^\alpha u\big((1-t)x + tx_j^h)\big)(x_j^h - x)^\alpha \phi_j^h(x)\Big\}^2 dx. \tag{39}$$

The support of the function $(\partial_x^\alpha u)((1-t)x + tx_j^h)(x_j^h - x)^\alpha \phi_j^h(x)$ is contained in η_j^h. Thus, we have $\|x - x_j^h\| \le \sqrt{d}Kh$, for all $x \in \eta_j^h$, and hence,

$$| (\partial_x^\alpha u)((1-t)x + tx_j^h)(x_j^h - x)^\alpha \phi_j^h(x) |^2 \le (\sqrt{d}Kh)^{2|\alpha|} | (\partial_x^\alpha u) \circ ((1-t)x + tx_j^h)\phi_j^h(x) |^2 .$$

Therefore, we have

$$\begin{aligned} &\sum_{|\alpha|=2K} \sum_j \int_\Omega \big\{\partial^\alpha u\big((1-t)x + tx_j^h\big)(x_j^h - x)^\alpha \phi_j^h(x)\big\}^2 dx \\ &\le \sum_{|\alpha|=2K} \sum_j \int_{\Omega\cap\eta_j^h} \big|(\partial_x^\alpha u) \circ \big((1-t)x + tx_j^h\big)(x_j^h - x)^\alpha \phi_j^h(x)\big|^2 dx \\ &\le (\sqrt{d}Kh)^{2(2K)} \sum_{|\alpha|=2K} \sum_j \int_{\Omega\cap\eta_j^h} \big|(\partial_x^\alpha u) \circ \big((1-t)x + tx_j^h\big)\phi_j^h(x)\big|^2 dx. \end{aligned} \tag{40}$$

Now, for each j, let $x = T_j(y) : \mathbb{R}^d \to \mathbb{R}^d$ be a transformation defined by

$$T_j(y) = \frac{1}{1-t}(y - tx_j^h),$$

and suppose $T_j^{-1}(\eta_j^h) = \tilde{\eta}_j^h$. Then the last integral of the inequality (40) can be written as follows:

$$(1-t)^{-d}(\sqrt{d}Kh)^{2(2K)} \sum_j \int_{\tilde{\eta}_j^h \cap \Omega} \left[|(\partial_y^\alpha u(y) \phi_j^h(T_j(y))|^2 \right] dy. \tag{41}$$

Since $0 < t < 1$, for $x \in \eta_j^h$,

$$|\frac{T_j^{-1}(x) - hj}{h}| \quad = \quad (1-t)|\frac{x-hj}{h}| \leq (1-t)K \leq K. \tag{42}$$

Therefore, we have

$$\tilde{\eta}_j^h \subset \eta_j^h. \tag{43}$$

From the construction of shape function $\phi(x)$, there exists a constant C such that

$$|\phi_j^h(T_j(y))| \leq C, \text{ for all } j, h, y. \tag{44}$$

According to (43) and (44), Eqt. (41) can be written as

$$\begin{aligned}
&(1-t)^{-d}(\sqrt{d}Kh)^{2(2K)} \sum_j \int_{\tilde{\eta}_j^h \cap \Omega} \left[|(\partial_y^\alpha u(y) \phi_j^h(T(y))|^2 \right] dy \\
&\leq C(1-t)^{-d}(\sqrt{d}Kh)^{2(2K)} \sum_j \int_{\eta_j^h \cap \Omega} |\ \partial_y^\alpha u(y)\ |^2\ dy \\
&\leq qC(1-t)^{-d}(\sqrt{d}Kh)^{2(2K)} \int_\Omega |\ \partial_y^\alpha u(y)\ |^2\ dy.
\end{aligned} \tag{45}$$

Here $q = \max_j[\text{card } \{k : \eta_j^h \cap \eta_k^h \neq 0\}]$. Thus, from (39), (40), and (45), we have

$$\begin{aligned}
F(x,t) \quad &\leq \quad qC(1-t)^{-d}(\sqrt{d}Kh)^{2(2K)} \sum_{|\alpha|=2K} \int_\Omega |\ \partial_y^\alpha u(y)\ |^2\ dy \\
&= \quad qC(1-t)^{-d}(\sqrt{d}Kh)^{2(2K)} |u|_{2K}^2.
\end{aligned} \tag{46}$$

Finally, according to (38) and (46), we have

$$\begin{aligned}
\|u - \mathcal{I}u\|^2 \quad &\leq \quad C(2K)^2 \hat{C} (\sqrt{d}Kh)^{2(2K)} |u|_{2K}^2 \int_0^1 (1-t)^{4K-2-d} dt \\
&\leq \quad C(Kh)^{2(2K)} |u|_{2K,\Omega}^2, \text{ if } 4K-2 \geq d.
\end{aligned} \tag{47}$$

(Step 2) Next, we estimate the L_2-norm of the derivative of the interpolation error. For any $\beta \in \mathbb{Z}^d, |\beta| = 1$, by taking β derivative of the following relation

$$\sum_j (x-j)^\alpha \phi(x-j) = \delta_\alpha^0, \text{ for } 0 \leq |\alpha| \leq 2K-1,$$

we get the following relations:

$$\begin{aligned}
\sum_j \phi^{(\beta)}(x-j) &= 0, \\
\sum_j (x-j)^\alpha \phi^{(\beta)}(x-j) &= -\delta_\alpha^\beta, |\alpha| = 1, \\
\sum_j (x-j)^\alpha \phi^{(\beta)}(x-j) &= 0, 1 < |\alpha| \le 2K-1.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(\mathcal{I}_h u)^{(\beta)}(x) &= \sum_j u(x_j^h) \frac{1}{h} \phi^{(\beta)}(\frac{x-hj}{h}) \\
&= \sum_j \Big(\sum_{|\alpha|<2K} \frac{\partial^\alpha u(x)}{\alpha!} (x_j^h - x)^\alpha + R_K u(x_j^h, x) \Big) \frac{1}{h} \phi^{(\beta)}(\frac{x-hj}{h}) \\
&= u^{(\beta)}(x) + \sum_j R_K u(x_j^h, x) \frac{1}{h} \phi^{(\beta)}(\frac{x-hj}{h}),
\end{aligned}$$

and hence,

$$u^{(\beta)}(x) - (\mathcal{I}_h u)^{(\beta)}(x) = -\sum_j R_K u(x_j^h, x) \frac{1}{h} \phi^{(\beta)}(\frac{x-hj}{h}), \ |\beta| = 1.$$

This is the same form as that in (step 1), except the additional factor $\frac{1}{h}$.

Therefore, we get the required estimate of the interpolation error:

$$\|u - \mathcal{I}_h u\|_{1,\Omega} = \Big(\|u - \mathcal{I}_h u\|_{0,\Omega}^2 + \sum_{|\beta|=1} \|u^{(\beta)} - (\mathcal{I}_h u)^{(\beta)}\|_{0,\Omega}^2 \Big)^{1/2} \le C h^{2K-1} |u|_{2K,\Omega}.$$

□

Now suppose $u \in \mathcal{C}^m(\Omega)$ or $u \in H^{m+[d/2]+1}$, where $[x] := \max\{n \in \mathbb{Z} : n \le x\}$, and $\Omega \subset \mathbb{R}^d$. Then, by Sobolev imbedding theorem, $u \in \mathcal{C}^0(\Omega)$ and hence $\mathcal{I}_h u$ is defined. Thus, by using Lemma 3.2 and Céa's theorem, we have the following error estimate of RKPM:

Theorem 3.1. *Suppose $u \in H^{m+[d/2]+1}(\Omega)$(or $u \in \mathcal{C}^m(\Omega)$), $4K-2 > d$, and u^R is an RKPM approximate solution of an elliptic boundary value problem on a convex domain $\Omega \subset \mathbb{R}^d$. Then, we have*

$$\|u - u^R\|_{1,\Omega} \le C h^{l-1} |u|_{l,\Omega}.$$

where

$$\begin{aligned}
l &= \min\{m, 2K\} \text{ if the RKP shape functions are } \mathcal{C}^0, \\
l &= \min\{m, 2K-1\} \text{ if the RKP shape functions are } \mathcal{C}^r, r > 0.
\end{aligned}$$

Here the constant C is independent of u and h and the diameters of supports of shape functions are $\leq \sqrt{d}Kh$.

3.5 Numerical Examples

In this section, we demonstrate the effectiveness of the constructed piecewise polynomial RKP shape functions. Without loss of generality, we test the effectiveness of the closed form shape functions associated with the particles that are uniformly distributed in $[0, \infty)$.

Example 3.1.

$$-\frac{d^2u}{dx^2} = f \text{ in } I = (0,1), \tag{48}$$

$$u(0) = 0 \text{ and } u(1) = 1. \tag{49}$$

RKPM associated with the $\mathcal{C}^1$-piecewise polynomial RKP shape functions of reproducing order 4, listed in Appendix, is applied for the approximate solutions of this problem. In this case, all particles are on the right hand side of the left boundary node $x_1 = 0$.

The relative errors in percent with respect to the energy norm are depicted in Fig. 5 when the true solutions are, respectively, $x^{0.7}, x^{1.7}, x^{2.7}$, and $x^{4.7}$. These true solutions are in $\mathcal{C}^1(I), \mathcal{C}^2(I), \mathcal{C}^3(I)$, and $\mathcal{C}^5(I)$, respectively.

The proof of Lemma 3.1 and Theorem 3.1 imply that the errors versus the mesh sizes in log-log scale is below the line $y = \alpha x + b$, where $\alpha = \min\{k+1, m\} - 1$, $k :=$ the polynomial reproducing order of shape function, and $m :=$ the regularity of the true solution.

In Fig. 5, this example strongly supports the theory. Indeed, since $k = 4$, the slope α becomes $0, 1, 2$, and 4, respectively, according as the true solution is $x^{0.7}, x^{1.7}, x^{2.7}$, and $x^{4.7}$. On the other hand, the slope of the computed relative errors are $0.2, 1.2, 2.2, 4.1$, respectively.

4 Piecewise polynomial RKP shape functions for non uniformly distributed particles that are in $(-\infty, \infty)$.

Suppose $a_1, a_2, a_3, \cdots, a_n$ are positive rational numbers and the particles are non uniformly distributed as follows:

$$\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots, x_{n-1}, x_n, x_{n+1}, x_{n+2}, \cdots.$$

where

$$\begin{aligned} \cdots = |x_{-2} - x_{-3}| &= |x_{-1} - x_{-2}| = |x_0 - x_{-1}| = a_1, \\ |x_j - x_{j-1}| &= a_j, j = 1, 2, \cdots, n \\ |x_{n+1} - x_n| &= |x_{n+2} - x_{n+1}| = \cdots = a_n \end{aligned}$$

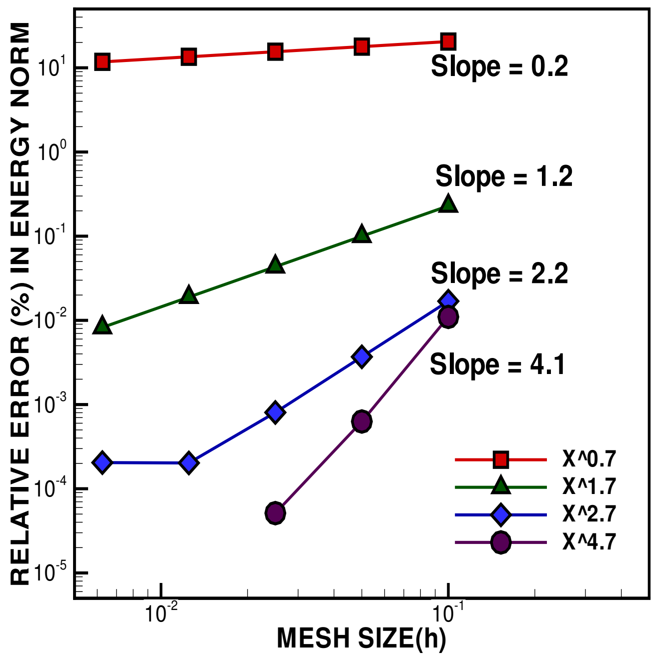


Figure 5: $\frac{\|u-u^R\|_1}{\|u\|_1} \times 100$ (Relative errors in percent) when the true solutions are $x^{0.7}, x^{1.7}, x^{2.7}$ and $x^{4.7}$, respectively.

For each particle x_j, we construct piecewise polynomial RKP shape function $\phi_{(x_j)}(x)$ of reproducing order $2K-1$ whose support consists of $2K$ consecutive intervals

$$\text{supp } \phi_{(x_j)}(x) = [x_{j-K}, x_{j-K+1}] \cup \cdots \cup [x_{j-1}, x_j] \cup [x_j, x_{j+1}] \cup \cdots \cup [x_{K-1}, x_K].$$

In this paper, we use the following notational convention.

(1) $\phi_{x_j}(x)$ is the shape function associated with the particle x_j.

(2) $\phi_j(x)$ is the shape function centered at 0, obtained by scaling and normalizing $\phi_{x_j}(x)$.

Without loss of generality, from now on, we assume $K = 3$. Then, to get the RKP basis function $\phi_{(x_j)}(x)$ for the particle x_j, we have to solve the systems on each subinterval of supp $\phi_{(x_j)}$.

(*i*) $\phi_{(x_j)}|_{[x_{j-3},x_{j-2}]}(x)$ is determined by

$$\sum_{k\in\{j-5,j-4,j-3,j-2,j-1,j\}} (x-x_k)^\alpha \phi_{(k)}(x) = \delta_0^\alpha, \alpha = 0,1,2,\cdots,5, x \in [x_{j-3}, x_{j-2}]$$

(*ii*) $\phi_{(x_j)}|_{[x_{j-2},x_{j-1}]}(x)$ is determined by

$$\sum_{k\in\{j-4,j-3,j-2,j-1,j,j+1\}} (x-x_k)^\alpha \phi_{(k)}(x) = \delta_0^\alpha, \alpha = 0,1,2,\cdots,5, x \in [x_{j-2}, x_{j-1}]$$

(*iii*) $\phi_{(x_j)}|_{[x_{j-1},x_j]}(x)$ is determined by

$$\sum_{k\in\{j-3,j-2,j-1,j,j+1,j+2\}} (x-x_k)^\alpha \phi_{(k)}(x) = \delta_0^\alpha, \alpha = 0,1,2,\cdots,5, x \in [x_{j-1}, x_j]$$

(*iv*) $\phi_{(x_j)}|_{[x_{j+1},x_j]}(x)$ is determined by

$$\sum_{k\in\{j-2,j-1,j,j+1,j+2,j+3\}} (x-x_k)^\alpha \phi_{(k)}(x) = \delta_0^\alpha, \alpha = 0,1,2,\cdots,5, x \in [x_j, x_{j+1}]$$

(*v*) $\phi_{(x_j)}|_{[x_{j+2},x_{j+1}]}(x)$ is determined by

$$\sum_{k\in\{j-1,j,j+1,j+2,j+3,j+4\}} (x-x_k)^\alpha \phi_{(k)}(x) = \delta_0^\alpha, \alpha = 0,1,2,\cdots,5, x \in [x_{j+1}, x_{j+2}]$$

(*vi*) $\phi_{(x_j)}|_{[x_{j+3},x_{j+2}]}(x)$ is determined by

$$\sum_{k\in\{j,j+1,j+2,j+3,j+4,j+5\}} (x-x_k)^\alpha \phi_{(k)}(x) = \delta_0^\alpha, \alpha = 0,1,2,\cdots,5, x \in [x_{j+2}, x_{j+3}]$$

Let

$$|x_{j+l} - x_{j+(l-1)}| = h_l, l = -4, -3, \cdots, 1, 2, 3, 4, 5.$$

Then we have

$$\phi_{(x_j)}|_{[x_{j-3},x_{j-2}]}(x) = \begin{cases} f_1(hx+x_j) & \text{if } h_{-4}=h_{-3}=\cdots=h_0=h, \\ c_1(h^*x+x_j:x_j) & \text{otherwise,} \end{cases}$$

$$\phi_{(x_j)}|_{[x_{j-2},x_{j-1}]}(x) = \begin{cases} f_2(hx+x_j) & \text{if } h_{-3}=h_{-2}=\cdots=h_1=h, \\ c_2(h^*x+x_j:x_j) & \text{otherwise,} \end{cases}$$

$$\phi_{(x_j)}|_{[x_{j-1},x_{j}]}(x) = \begin{cases} f_3(hx+x_j) & \text{if } h_{-2}=h_{-1}=\cdots=h_2=h, \\ c_3(h^*x+x_j:x_j) & \text{otherwise,} \end{cases}$$

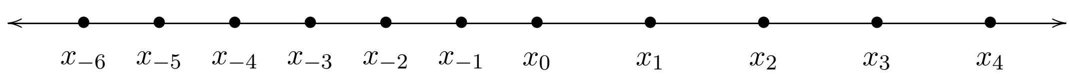
$$\phi_{(x_j)}|_{[x_{j},x_{j+1}]}(x) = \begin{cases} f_4(hx+x_j) & \text{if } h_{-1}=h_{0}=\cdots=h_3=h, \\ c_4(h^*x+x_j:x_j) & \text{otherwise,} \end{cases}$$

$$\phi_{(x_j)}|_{[x_{j+1},x_{j+2}]}(x) = \begin{cases} f_5(hx+x_j) & \text{if } h_{0}=h_{1}=\cdots=h_4=h, \\ c_5(h^*x+x_j:x_j) & \text{otherwise,} \end{cases}$$

$$\phi_{(x_j)}|_{[x_{j+2},x_{j+3}]}(x) = \begin{cases} f_6(hx+x_j) & \text{if } h_{1}=h_{2}=\cdots=h_5=h, \\ c_6(h^*x+x_j:x_j) & \text{otherwise,} \end{cases}$$

Here the corrected functions $c_k(x:x_j)$ are polynomials and h^* is a proper small positive real number.

In what follows, without loss of generality, we construct polynomial reproducing basic shape functions of reproducing order 5 for the following type of non uniform particle distribution.



Here, on the left of the particle x_0, the particles are uniformly distributed with unit 1, whereas on the right of the particle x_0, the particles are distributed with unit 2.

If $\{x_j : \text{supp } \phi_{(x_j)}(x) \subset (x_2,\infty)\} \subset [0,\infty)\cap 2\mathbb{Z}$, then the six particles in the above system are uniformly distributed. Thus, if $j \geq 5$, then $\phi_{(x_j)}(x) = \phi_{([-3,3];0;5)}(2x+x_j)$. By a similar reason, if $j \leq -5$, $\phi_{(x_j)}(x) = \phi_{([-3,3];0;5)}(x+x_j)$.

On the other hand, suppose

$$S_{(j)} = \text{ supp } \phi_{(j)}(x) \cap (x_{-2},x_2) \neq 0,$$

then $\phi_{(x_j)}(x)$ on $S_{(j)}$ should be determined by solving the above six systems.

In this section, we construct closed form RKP basic shape functions whose support is $[-K,K]=[-3,3]$ for non uniform particle distribution. However, a similar argument can be applied for an arbitrary integer K to get closed form RKP shape functions for desired reproducing order.

4.1 $\mathcal{C}^0$ Polynomial Reproducing Basic Function

Suppose the distance between non uniform particles are $\cdots,1,1,1,2,2,2,\cdots,$. For piecewise polynomial RKP basic shape functions for this form of locally uniform particles, we define the following 11 basic shape functions.

First, we consider the unique $\mathcal{C}^0$ RKP shape functions reproducing order $2K-1$ whose supports are $[-K,K]$ and $[-2K,2K]$, respectively.

1. $\phi_{(K;0;5)}(x)$ **with foot length 1:** This shape function is defined by (10). The RKP basic function corresponding to $x_j, j \le -5$ is obtained by translating this function to x_j.

2. $\phi_{(2K;0;5)}(x)$ **with foot length 2:** This RKP shape function is defined by

$$\phi_{(2K;0;5)}(x) = \begin{cases} f_1(x/2) & x \in [-6,-4] \\ f_2(x/2) & x \in [-4,-2] \\ f_3(x/2) & x \in [-2,0] \\ f_4(x/2) & x \in [0,2] \\ f_5(x/2) & x \in [2,4] \\ f_6(x/2) & x \in [4,6] \\ 0 & x \notin [-6,6] \end{cases} . \tag{50}$$

The polynomials $f_i, i = 1,2,3,4,5$ are those defined by (10). The RKP basic function corresponding to $x_j, j \ge 5$ is obtained by translating this shape function $\phi_{(2K;0;5)}(x)$ to x_j.

The supports of the nine shape functions corresponding to the particles located at

$$x_{-4} = -4, x_{-3} = -3, x_{-2} = -2, x_{-1} = -1, x_0 = 0, x_1 = 2, x_2 = 4, x_3 = 6, x_4 = 8,$$

have non void intersections with an interval of length one as well as an interval length two. Thus, the basic shape functions corresponding to these nodes should be corrected so that the reproducing property of order 5 can be retained.

In the following piecewise polynomial shape functions, the parts of basic shape functions to be corrected The correction functions are denoted by

$c_m(x:n)$ and $d_m(x:n)$:
$m = 1,2,3,4,5,6$ indicate the six parts of the support,
$n = -4,-3,-2,-1,0,2,4,6,8$ that indicate the coordinates of particles.

3: Nine basic shape functions with mixed foot length:

(1) Shape functions associated with the particles x = -4 & x = -3:

$$\phi_{(-4)}(x) = \begin{cases} f_1(x) & x \in [-3,-2] \\ f_2(x) & x \in [-2,-1] \\ f_3(x) & x \in [-1,0] \\ f_4(x) & x \in [0,1] \\ f_5(x) & x \in [1,2] \\ c_6(x:-4) & x \in [2,4] \\ 0 & x \notin [-3,4] \end{cases} ; \quad \phi_{(-3)}(x) = \begin{cases} f_1(x) & x \in [-3,-2] \\ f_2(x) & x \in [-2,-1] \\ f_3(x) & x \in [-1,0] \\ f_4(x) & x \in [0,1] \\ c_5(x:-3) & x \in [1,2] \\ c_6(x:-3) & x \in [2,3] \\ 0 & x \notin [-3,3] \end{cases} \tag{51}$$

(2) Shape functions associated with the particles x = -2 & x = -1:

$$\phi_{(-2)}(x) = \begin{cases} f_1(x) & x \in [-3,-2] \\ f_2(x) & x \in [-2,-1] \\ f_3(x) & x \in [-1,0] \\ c_4(x:-2) & x \in [0,1] \\ c_5(x:-2) & x \in [1,2] \\ c_6(x:-2) & x \in [2,4] \\ 0 & x \notin [-3,4] \end{cases} \quad ; \quad \phi_{(-1)}(x) = \begin{cases} f_1(x) & x \in [-3,-2] \\ f_2(x) & x \in [-2,-1] \\ c_3(x:-1) & x \in [-1,0] \\ c_4(x:-1) & x \in [0,1] \\ c_5(x:-1) & x \in [1,3] \\ c_6(x:-1) & x \in [3,5] \\ 0 & x \notin [-3,5] \end{cases} \tag{52}$$

(3) Shape functions associated with the particles x = 0 & x = 2:

$$\phi_{(0)}(x) = \begin{cases} f_1(x) & \in [-3,-2] \\ c_2(x:0) & x \in [-2,-1] \\ c_3(x:0) & x \in [-1,0] \\ c_4(x:0) & x \in [0,2] \\ c_5(x:0) & x \in [2,4] \\ f_6(x/2) & x \in [4,6] \\ 0 & x \notin [-3,6] \end{cases} \quad ; \quad \phi_{(2)}(x) = \begin{cases} c_1(x:2) & x \in [-4,-3] \\ c_2(x:2) & x \in [-3,-2] \\ c_3(x:2) & x \in [-2,0] \\ c_4(x:2) & x \in [0,2] \\ f_5(x/2) & x \in [2,4] \\ f_6(x/2) & x \in [4,6] \\ 0 & x \notin [-4,6] \end{cases} \quad ; \tag{53}$$

(4) Shape functions associated with the particles x = 4 & x = 6:

$$\phi_{(4)}(x) = \begin{cases} c_1(x:4) & x \in [-5,-4] \\ c_2(x:4) & x \in [-4,-2] \\ c_3(x:4) & x \in [-2,0] \\ f_4(x/2) & x \in [0,2] \\ f_5(x/2) & x \in [2,4] \\ f_6(x/2) & x \in [4,6] \\ 0 & x \notin [-5,6] \end{cases} \quad ; \quad \phi_{(6)}(x) = \begin{cases} c_1(x:6) & x \in [-6,-4] \\ c_2(x:6) & x \in [-4,-2] \\ f_3(x/2) & x \in [-2,0] \\ f_4(x/2) & x \in [0,2] \\ f_5(x/2) & x \in [2,4] \\ f_6(x/2) & x \in [4,6] \\ 0 & x \notin [-6,6] \end{cases} \tag{54}$$

(5) Shape function associated with the particle x = 8:

$$\phi_{(8)}(x) = \begin{cases} c_1(x:8) & x \in [-6,-4] \\ f_2(x/2) & x \in [-4,-2] \\ f_3(x/2) & x \in [-2,0] \\ f_4(x/2) & x \in [0,2] \\ f_5(x/2) & x \in [2,4] \\ f_6(x/2) & x \in [4,6] \\ 0 & x \notin [-6,6] \end{cases} \tag{55}$$

In order to determine the correction functions $c_m(x:n)$, the influenced domain $[x_{-2}, x_2]$ is divided into 4 sub-domains

$$[x_{-2}, x_{-1}] = [-2,-1], \ [x_{-1}, x_0] = [-1,0], \ [x_0, x_1] = [0,2], \ [x_1, x_2] = [2,4].$$

Let us note that our basic shape functions consist of six pieces of polynomials and their supports consist of six intervals to keep the reproducing property of order 5.

Let $\Lambda = \{-n, 2n; n \in \mathbb{Z}\}$ be an index set.

1. On the interval $[x_{-2}, x_{-1}] = [-2, -1]$, we have

$$\sum_{j\in\Lambda}(x-x_j)^k \phi_{(x_j)}(x-x_j) \quad = \sum_{j\in\{-4,-3,-2,-1,0,2\}} (x-j)^k \phi_{(j)}(x-j)$$

In other words, if those basic shape functions $\phi_{(-4)}, \phi_{(-3)}, \phi_{(-2)}, \phi_{(-1)}, \phi_{(0)}, \phi_{(2)}$, are translated to the particles $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, x_1$, respectively, then the interval $[x_{-2}, x_{-1}]$ is the intersection of the supports of all these six basic shape functions. Therefore, $\phi_{(j)}(x-j)|_{[-2,-1]}, j = -4, -3, -1, 0, 2$, must be modified, because $\{-4, -3, -2, -1, 0, 2\}$ are not uniformly distributed. This can be done simply by solving the following system of equations.

$$\sum_{j\in\{-4,-3,-2,-1,0,2\}} (x-j)^k \phi_{(j)}(x-j) = \delta_0^k, \; k = 0, 1, \cdots, 5. \tag{56}$$

which becomes

$$\begin{aligned} (x+4)^k c_6(x+4:-4) \quad &+ \quad (x+3)^k c_5(x+3:-3) + (x+2)^k c_4(x+2:-2) \\ &+ \quad (x+1)^k c_3(x+1:-1) + (x-0)^k c_2(x-0:0) + (x-2)^k c_1(x-2:2) \\ &= \quad \delta_0^k, \; k = 0, 1, \cdots, 5. \end{aligned}$$

By solving the above system of six equations, we can correct the basic shape function whose lags fall on the affected interval $[-2, -1]$ as follows:

$$\begin{aligned} c_6(x:-4) &= -\frac{1}{144}(x-6)(x-4)(x-3)(x-2)(x-1), \\ c_5(x:-3) &= \frac{1}{30}(x-5)(x-3)(x-2)(x-1)(x+1), \\ c_4(x:-2) &= -\frac{1}{16}(x-4)(x-2)(x-1)(x+1)(x+2), \\ c_3(x:-1) &= \frac{1}{18}(x-3)(x-1)(x+1)(x+2)(x+3), \\ c_2(x:\;\;0) &= -\frac{1}{48}(x-2)(x+1)(x+2)(x+3)(x+4), \\ c_1(x:\;\;2) &= \frac{1}{720}(x+2)(x+3)(x+4)(x+5)(x+6) \end{aligned}$$

These are the sixth part of the basic shape function $\phi_{(-4)}$, the fifth part of $\phi_{(-3)}$, the fourth part of $\phi_{(-2)}$, the third part of $\phi_{(-1)}$, the second part of $\phi_{(0)}$ and the first part of $\phi_{(2)}$.

2. Similarly, on the interval $[x_{-1}, x_0] = [-1, 0]$, we have

$$\sum_{j\in\Lambda}(x-x_j)^k\phi_{(j)}(x-x_j) \;=\; \sum_{j\in\{-3,-2,-1,0,2,4\}}(x-j)^k\phi_{(j)}(x-j).$$

Thus, by solving the following system

$$\sum_{j\in\{-3,-2,-1,0,2,4\}}(x-j)^k\phi_{(j)}(x-j) \;=\; \delta_0^k,\;\; k=0,1,\cdots,5,$$

we can determine the sixth part of the basic shape function $\phi_{(-3)}$, the fifth part of $\phi_{(-2)}$, the fourth part of $\phi_{(-1)}$, the third part of $\phi_{(0)}$, the second part of $\phi_{(2)}$ and the first part of $\phi_{(4)}$ as follows:

$$\begin{aligned}
c_6(x:-3) &= -\frac{1}{210}(x-7)(x-5)(x-3)(x-2)(x-1),\\
c_5(x:-2) &= \frac{1}{48}(x-6)(x-4)(x-2)(x-1)(x+1),\\
c_4(x:-1) &= -\frac{1}{30}(x-5)(x-3)(x-1)(x+1)(x+2),\\
c_3(x:\;\;0) &= \frac{1}{48}(x-4)(x-2)(x+1)(x+2)(x+3),\\
c_2(x:\;\;2) &= -\frac{1}{240}(x-2)(x+2)(x+3)(x+4)(x+5),\\
c_1(x:\;\;4) &= \frac{1}{1680}(x+2)(x+4)(x+5)(x+6)(x+7).
\end{aligned}$$

3. On the interval $[x_0, x_1] = [0, 2]$, we have

$$\sum_{j\in\Lambda}(x-x_j)^k\phi_{(j)}(x-x_j) \;=\; \sum_{j\in\{-2,-1,0,2,4,6\}}(x-j)^k\phi_{(j)}(x-j).$$

Similarly, by solving the following system

$$\sum_{j\in\{-2,-1,0,2,4,6\}}(x-j)^k\phi_{(j)}(x-j) \;=\; \delta_0^k,\;\; k=0,1,\cdots,5,$$

we can determine the sixth part of the basic shape function $\phi_{(-2)}$, the fifth part of $\phi_{(-1)}$, the fourth part of $\phi_{(0)}$, the third part of $\phi_{(2)}$, the second part of $\phi_{(4)}$ and the first part of

$\phi_{(6)}$ as follows:

$$\begin{aligned}
c_6(x:-2) &= -\frac{1}{384}(x-8)(x-6)(x-4)(x-2)(x-1), \\
c_5(x:-1) &= \frac{1}{105}(x-7)(x-5)(x-3)(x-1)(x+1), \\
c_4(x:\ \ 0) &= -\frac{1}{96}(x-6)(x-4)(x-2)(x+1)(x+2), \\
c_3(x:\ \ 2) &= \frac{1}{192}(x-4)(-2+x)(x+2)(x+3)(x+4), \\
c_2(x:\ \ 4) &= -\frac{1}{480}(x-2)(x+2)(x+4)(x+5)(x+6), \\
c_1(x:\ \ 6) &= \frac{1}{2688}(x+2)(x+4)(x+6)(x+7)(x+8).
\end{aligned}$$

4. On the interval $[x_1, x_2] = [2, 4]$, by solving the following system

$$\sum_{j\in\{-1,0,2,4,6,8\}} (x-j)^k \phi_{(j)}(x-j) = \delta_0^k, \ k = 0, 1, \cdots, 5,$$

the following correction functions are determined.

$$\begin{aligned}
c_6(x:-1) &= -\frac{1}{945}(x-9)(x-7)(x-5)(x-3)(x-1), \\
c_5(x:\ \ 0) &= \frac{1}{384}(x-8)(x-6)(x-4)(x-2)(x+1), \\
c_4(x:\ \ 2) &= -\frac{1}{288}(x-6)(x-4)(x-2)(x+2)(x+3), \\
c_3(x:\ \ 4) &= \frac{1}{320}(x-4)(x-2)(x+2)(x+4)(x+5), \\
c_2(x:\ \ 6) &= -\frac{1}{672}(x-2)(x+2)(x+4)(x+6)(x+7), \\
c_1(x:\ \ 8) &= \frac{1}{3456}(x+2)(x+4)(x+6)(x+8)(x+9).
\end{aligned}$$

4.2 $\mathcal{C}^1$ Polynomial Reproducing Basic shape Function

By sacrificing reproducing order by one, we can construct closed form $\mathcal{C}^1$ RKP basic shape functions for locally uniform particles, which are separated apart by $\cdots, 1, 1, 1, 2, 2, 2, \cdots$ and satisfy the reproducing property of order 4.

1. $\phi_{(K;1;4)}(x)$ with foot length 1 and $\phi_{(2K;1;4)}(x)$ with foot length 2:

The basic shape function associated with the particles that are on the left of $x = -4$ is $\phi_{(K;1;4)}(x)$. On the other hand, the basic shape function for the particles whose coordinates are on the left of $x = 8$ is $\phi_{(2K;1;4)}(x)$, that is obtained by linearly stretching $\phi_{(K;1;4)}(x)$ onto $[-6, 6]$.

$$\phi_{(K;1;4)}(x) = \begin{cases} g_1(x) & x \in [-3,-2] \\ g_2(x) & x \in [-2,-1] \\ g_3(x) & x \in [-1,0] \\ g_4(x) & x \in [0,1] \\ g_5(x) & x \in [1,2] \\ g_6(x) & x \in [2,4] \\ 0 & x \notin [-3,4] \end{cases} \quad ; \quad \phi_{(2K;1;4)}(x) = \begin{cases} g_1(x/2) & x \in [-6,-4] \\ g_2(x/2) & x \in [-4,-2] \\ g_3(x/2) & x \in [-2,0] \\ g_4(x/2) & x \in [0,2] \\ g_5(x/2) & x \in [2,4] \\ g_6(x/2) & x \in [4,6] \\ 0 & x \notin [-6,6] \end{cases} , \qquad (57)$$

where $g_1, g_2, \cdots, g_6$ are those defined by (14).

2: Nine basic shape functions with mixed foot length:

The other basic shape functions corresponding to all the other particles whose coordinates are in $[-4, 8]$ should be modified so that the reproducing property of order 4 can be retained.

(1) $\mathcal{C}^1$ shape functions for nodes x = -4 & x = -3:

$$\phi^1_{(-4)}(x) = \begin{cases} g_1(x) & x \in [-3,-2] \\ g_2(x) & x \in [-2,-1] \\ g_3(x) & x \in [-1,0] \\ g_4(x) & x \in [0,1] \\ g_5(x) & x \in [1,2] \\ d_6(x:-4) & x \in [2,4] \\ 0 & x \notin [-3,4] \end{cases} \quad ; \quad \phi^1_{(-3)}(x) = \begin{cases} g_1(x) & x \in [-3,-2] \\ g_2(x) & x \in [-2,-1] \\ g_3(x) & x \in [-1,0] \\ g_4(x) & x \in [0,1] \\ d_5(x:-3) & x \in [1,2] \\ d_6(x:-3) & x \in [2,3] \\ 0 & x \notin [-3,3] \end{cases} \qquad (58)$$

(2) $\mathcal{C}^1$ shape functions for nodes x = -2 & x = -1:

$$\phi^1_{-2}(x) = \begin{cases} g_1(x) & x \in [-3,-2] \\ g_2(x) & x \in [-2,-1] \\ g_3(x) & x \in [-1,0] \\ d_4(x:-2) & x \in [0,1] \\ d_5(x:-2) & x \in [1,2] \\ d_6(x:-2) & x \in [2,4] \\ 0 & x \notin [-3,4] \end{cases} \quad ; \quad \phi^1_{(-1)}(x) = \begin{cases} g_1(x) & x \in [-3,-2] \\ g_2(x) & x \in [-2,-1] \\ d_3(x:-1) & x \in [-1,0] \\ d_4(x:-1) & x \in [0,1] \\ d_5(x:-1) & x \in [1,3] \\ d_6(x:-1) & x \in [3,5] \\ 0 & x \notin [-3,5] \end{cases} \qquad (59)$$

(3) $\mathcal{C}^1$ shape functions for nodes x = 0 & x = 2:

$$\phi^1_{(0)}(x) = \begin{cases} g_1(x) & \in [-3,-2] \\ d_2(x:0) & x \in [-2,-1] \\ d_3(x:0) & x \in [-1,0] \\ d_4(x:0) & x \in [0,2] \\ d_5(x:0) & x \in [2,4] \\ g_6(x/2) & x \in [4,6] \\ 0 & x \notin [-3,6] \end{cases} \quad ; \quad \phi^1_{(2)}(x) = \begin{cases} d_1(x:2) & x \in [-4,-3] \\ d_2(x:2) & x \in [-3,-2] \\ d_3(x:2) & x \in [-2,0] \\ d_4(x:2) & x \in [0,2] \\ g_5(x/2) & x \in [2,4] \\ g_6(x/2) & x \in [4,6] \\ 0 & x \notin [-4,6] \end{cases} \quad ; \qquad (60)$$

(3) $\mathcal{C}^1$ shape functions for nodes x = 4 & x = 6:

$$\phi^1_{(4)}(x) = \begin{cases} d_1(x:4) & x \in [-5,-4] \\ d_2(x:4) & x \in [-4,-2] \\ d_3(x:4) & x \in [-2,0] \\ g_4(x/2) & x \in [0,2] \\ g_5(x/2) & x \in [2,4] \\ g_6(x/2) & x \in [4,6] \\ 0 & x \notin [-5,6] \end{cases} \quad ; \quad \phi^1_{(6)}(x) = \begin{cases} d_1(x:6) & x \in [-6,-4] \\ d_2(x:6) & x \in [-4,-2] \\ g_3(x/2) & x \in [-2,0] \\ g_4(x/2) & x \in [0,2] \\ g_5(x/2) & x \in [2,4] \\ g_6(x/2) & x \in [4,6] \\ 0 & x \notin [-6,6] \end{cases} \tag{61}$$

(3) $\mathcal{C}^1$ shape functions for node x = 8:

$$\phi^1_{(8)}(x) = \begin{cases} d_1(x:8) & x \in [-6,-4] \\ g_2(x/2) & x \in [-4,-2] \\ g_3(x/2) & x \in [-2,0] \\ g_4(x/2) & x \in [0,2] \\ g_5(x/2) & x \in [2,4] \\ g_6(x/2) & x \in [4,6] \\ 0 & x \notin [-6,6] \end{cases} \tag{62}$$

The influenced domain $[x_{-2}, x_2]$ consists of four intervals. That is,

$$[x_{-2}, x_2] = [-2,-1] \cup [-1,0] \cup [0,2] \cup [2,4].$$

Note that our basic function are supported by 6 intervals of length one or two.
Let $\Lambda = \{-n, 2n; n \in \mathbb{Z}\}$ be an index set.

1. On the interval $[x_{-2}, x_{-1}] = [-2,-1]$, we have

$$\sum_{j \in \Lambda} (x - x_j)^k \phi_{(j)}(x - x_j) = \sum_{j \in \{-4,-3,-2,-1,0,2\}} (x - j)^k \phi_{(j)}(x - j).$$

The intersection of the support of the translated basic shape functions to particle $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, x_1$ is the interval $[-2,-1]$. These shape function restricted to $[-2,-1]$ must be modified so that the reproducing property of order 4 can be retained. This can be done simply by solving the following system of algebraic equations.

$$\sum_{j \in \{-4,-3,-2,-1,0,2\}} (x - j)^k \phi_{(j)}(x - j) = \delta_0^k, \; k = 0, 1, \cdots, 4 \tag{63}$$

$$\sum_{j \in \{-4,-3,-2,-1,0,2\}} (x - j)^k \phi_{(j)}(x - j) = G(x) \tag{64}$$

which becomes

$$\begin{aligned}(x+4)^k d_6(x+4:-4) &+ (x+3)^k d_5(x+3:-3) + (x+2))^k d_4(x+2):-2)\\ &+ (x+1)^k d_3(x+1:-1) + (x-0)^k d_2(x-0:0)\\ &+ (x-2)^k d_1(x-2:2) = \delta_0^k,\ k=0,1,\cdots,4\\ (x+4)^5 d_6(x+4:-4) &+ (x+3)^5 d_5(x+3:-3) + (x+2)^5 d_4(x+2:-2)\\ &+ (x+1)^5 d_3(x+1:-1) + (x-0)^5 d_2(x-0:0)\\ &+ (x-2)^5 d_1(x-2:2) = G(x),\end{aligned}$$

where $G(x) = (x+2)(x+1)(-16-10x)$.

By solving the above equation, we can correct the basic shape function whose lags fall on the affected interval $[-2,-1]$ as follows:

$$\begin{aligned}d_6(x:-4) &= -\frac{1}{144}(x-8)(x-3)^2(x-2)x,\\ d_5(x:-3) &= \frac{1}{30}(x-2)(x-1)(x^3-7x^2-3x+29),\\ d_4(x:-2) &= -\frac{1}{16}(x-1)(x^4-3x^3-18x^2+16x+16),\\ d_3(x:-1) &= \frac{1}{18}(x+1)(x^2-3x-6)(x^2+4x-3),\\ d_2(x:\ \ 0) &= -\frac{1}{48}(x+1)(x+2)(x^3+5x^2-12x-40),\\ d_1(x:\ \ 2) &= \frac{1}{720}(x+3)(x+4)^2(x^2+9x+6),\end{aligned}$$

2. On the interval $[x_{-1}, x_0] = [-1, 0]$, we have

$$\begin{aligned}\sum_{j\in\{-3,-2,-1,0,2,4\}} (x-j)^k \phi_{(j)}(x-j) &= \delta_0^k,\ k=0,1,\cdots,4\\ \sum_{j\in\{-3,-2,-1,0,2,4\}} (x-j)^5 \phi_{(j)}(x-j) &= G(x),\end{aligned}$$

where $G(x) = x(x-1)(-16-22x)$.

By solving the above system of algebraic equations, we can determine the sixth part of the basic shape function $\phi^1_{(-3)}$, the fifth part of $\phi^1_{(-2)}$, the fourth part of $\phi^1_{(-1)}$, the third part of $\phi^1_{(0)}$, the second part of $\phi^1_{(2)}$ and the first part of $\phi^1_{(4)}$ as follows:

$$
\begin{aligned}
d_6(x:-3) &= -\frac{1}{210}(x-3)^2(x-2)(x^2-10x-5),\\
d_5(x:-2) &= \frac{1}{48}(x-2)(x-1)(x^3-9x^2-8x+52),\\
d_4(x:-1) &= -\frac{1}{30}(x-1)(x^4-5x^3-29x^2+35x+30),\\
d_3(x:\ \ 0) &= \frac{1}{48}(x+1)(x^4-x^3-38x^2-12x+48),\\
d_2(x:\ \ 2) &= -\frac{1}{240}(x+2)(x+3)(x^3+7x^2-20x-100),\\
d_1(x:\ \ 4) &= \frac{1}{1680}(x+4)(x+5)^2(x^2+10x-4)
\end{aligned}
$$

3. On the interval $[x_0, x_1] = [0, 2]$, we have the following system of algebraic equations:

$$
\begin{aligned}
\sum_{j\in\{-2,-1,0,2,4,6\}} (x-j)^k\phi_{(j)}(x-j) &= \delta_0^k,\ k=0,1,\cdots,4\\
\sum_{j\in\{-3,-2,-1,0,2,4\}} (x-j)^5\phi_{(j)}(x-j) &= G(x),
\end{aligned}
$$

where $G(x) = x(x-1)(8-16x)$.

The solutions of this system give rise to the following correction functions:

$$
\begin{aligned}
d_6(x:-2) &= -\frac{1}{384}(x-4)^2(x-2)(x^2-11x+2),\\
d_5(x:-1) &= \frac{1}{105}(x-3)(x-1)(x^3-11x^2+7x+59),\\
d_4(x:\ \ 0) &= -\frac{1}{96}(x-2)(x^4-7x^3-20x^2+60x+48),\\
d_3(x:\ \ 2) &= \frac{1}{192}(x-6)(x+2)(x^3+7x^2+4x-16),\\
d_2(x:\ \ 4) &= -\frac{1}{480}(x+2)(x+4)(x^3+9x^2-8x-116),\\
d_1(x:\ \ 6) &= \frac{1}{2688}(x+4)(x+6)^2(x^2+11x+4).
\end{aligned}
$$

4. Similarly, on the interval $[x_1, x_2] = [2, 4]$, the system for construction of RKP shape func-

tions becomes as follows:

$$
\begin{aligned}
\sum_{j\in\{-1,0,2,4,6,8\}} (x-j)^k \phi_{(j)}(x-j) &= \delta_0^k,\ k=0,1,\cdots,4\\
\sum_{j\in\{-3,-2,-1,0,2,4\}} (x-j)^5 \phi_{(j)}(x-j) &= G(x),
\end{aligned}
$$

where $G(x) = x(x-1)(80-28x)$.

By solving this system, we have the following correction functions:

$$
\begin{aligned}
d_6(x:-1) &= -\frac{1}{945}(x-5)^2(x-3)(x^2-12x-9),\\
d_5(x:\ \ 0) &= \frac{1}{384}(x-4)(x-2)(x^3-13x^2+6x+128),\\
d_4(x:\ \ 2) &= -\frac{1}{288}(x-2)(x^4-5x^3-48x^2+84x+144),\\
d_3(x:\ \ 4) &= \frac{1}{320}(x+2)(x^4+3x^3-54x^2-80x+160),\\
d_2(x:\ \ 6) &= -\frac{1}{672}(x+2)(x+4)(x^3+11x^2-12x-172),\\
d_1(x:\ \ 8) &= \frac{1}{3456}x(x+4)(x+6)^2(x+13).
\end{aligned}
$$

Appendix

.1 $\mathcal{C}^r$-Piecewise polynomial RKP shape functions associated with the particles distributed in $[0,\infty)$ when $K=3$.

Without loss of generality, it is sufficient to construct the piecewise polynomial RKP shape functions associate with uniformly distributed particles in $[0,\infty)$ such that

$$x_k = k, k = 0,1,2,3,\cdots,.$$

[A-I] $\mathcal{C}^0$ piecewise polynomial RKP shape functions of reproducing order $(2K-1), K=3$.

1. $\phi_{(x_0)}(x) = \phi_0(x)$ and $\phi_0(x)$ is as follows:

$$
\phi_0(x) = \begin{cases} -\frac{1}{120}(x-5)(x-4)(x-3)(x-2)(x-1) & x \in [0,3] \\ 0 & x \notin [0,3] \end{cases}
$$

2. $\phi_{(x_1)}(x) = \phi_1(x-1)$ and $\phi_0(x)$ is as follows:

$$\phi_1(x) = \begin{cases} \frac{1}{24}(x-4)(x-3)(x-2)(x-1)(x+1) & x \in [-1,2] \\ -\frac{1}{120}(x-5)(x-4)(x-3)(x-2)(x-1) & x \in [2,3] \\ 0 & x \notin [-1,3] \end{cases}$$

3. $\phi_{(x_2)}(x) = \phi_2(x-2)$ and $\phi_2(x)$ is as follows:

$$\phi_2(x) = \begin{cases} -\frac{1}{12}(x-3)(x-2)(x-1)(x+1)(x+2) & x \in [-2,1] \\ \frac{1}{24}(x-4)(x-3)(x-2)(x-1)(x+1) & x \in [1,2] \\ -\frac{1}{120}(x-5)(x-4)(x-3)(x-2)(x-1) & x \in [2,3] \\ 0 & x \notin [-2,3] \end{cases}$$

4. $\phi_{(x_3)}(x) = \phi_3(x-3)$ and $\phi_3(x)$ is as follows:

$$\phi_3(x) = \begin{cases} \frac{1}{12}(x-2)(x-1)(x+1)(x+2)(x+3) & x \in [-3,0] \\ -\frac{1}{12}(x-3)(x-2)(x-1)(x+1)(x+2) & x \in [0,1] \\ \frac{1}{24}(x-4)(x-3)(x-2)(x-1)(x+1) & x \in [1,2] \\ -\frac{1}{120}(x-5)(x-4)(x-3)(x-2)(x-1) & x \in [2,3] \\ 0 & x \notin [-3,3] \end{cases}$$

5. $\phi_{(x_4)}(x) = \phi_4(x-4)$ and $\phi_4(x)$ is as follows:

$$\phi_4(x) = \begin{cases} -\frac{1}{24}(x-1)(x+1)(x+2)(x+3)(x+4) & x \in [-4,-1] \\ \frac{1}{12}(x-2)(x-1)(x+1)(x+2)(x+3) & x \in [-1,0] \\ -\frac{1}{12}(x-3)(x-2)(x-1)(x+1)(x+2) & x \in [0,1] \\ \frac{1}{24}(x-4)(x-3)(x-2)(x-1)(x+1) & x \in [1,2] \\ -\frac{1}{120}(x-5)(x-4)(x-3)(x-2)(x-1) & x \in [2,3] \\ 0 & x \notin [-4,3] \end{cases}$$

6. $\phi_{(x_5)}(x) = \phi_5(x-5)$ and $\phi_5(x)$ is as follows:

$$\phi_5(x) = \begin{cases} \frac{1}{120}(x+1)(x+2)(x+3)(x+4)(x+5) & x \in [-5,-2] \\ -\frac{1}{24}(x-1)(x+1)(x+2)(x+3)(x+4) & x \in [-2,-1] \\ \frac{1}{12}(x-2)(x-1)(x+1)(x+2)(x+3) & x \in [-1,0] \\ -\frac{1}{12}(x-3)(x-2)(x-1)(x+1)(x+2) & x \in [0,1] \\ \frac{1}{24}(x-4)(x-3)(x-2)(x-1)(x+1) & x \in [1,2] \\ -\frac{1}{120}(x-5)(x-4)(x-3)(x-2)(x-1) & x \in [2,3] \\ 0 & x \notin [-5,3] \end{cases}$$

7. For $j \geq 6$, $\phi_{(x_j)}(x) = \phi_{([-3,3];0;5)}(x-6)$ and $\phi_{([-3,3];0;5)}(x)$ is defined by (10).

[A-II] Piecewise polynomial $\mathcal{C}^1$ RKP shape functions of reproducing order $(2K-2), K=3$.

1. $\phi_{(x_0)}(x) = \phi_0(x)$ and $\phi_0(x)$ is as follows:

$$\phi_0(x) = \begin{cases} \frac{1}{5400}(x-5)(x-4)(x-3)^2(x-2)(x-1)(8x+15) & x \in [0,3] \\ 0 & x \notin [0,3] \end{cases}$$

2. $\phi_{(x_1)}(x) = \phi_1(x-1)$ and $\phi_1(x)$ is as follows:

$$\phi_1(x) = \begin{cases} -\frac{1}{1080}(x-4)(x-3)(x-2)(x-1)(x+1)(8x^2-x-45) & x \in [-1,2] \\ -\frac{1}{120}(x-7)(x-3)^2(x-2)x & x \in [2,3] \\ 0 & x \notin [-1,3] \end{cases}$$

3. $\phi_{(x_2)}(x) = \phi_2(x-2)$ and $\phi_2(x)$ is as follows:

$$\phi_2(x) = \begin{cases} \frac{1}{540}(x-3)(x-2)(x-1)(x+1)(x+2)(8x^2+7x-45) & x \in [-2,1] \\ \frac{1}{24}(x-2)(x-1)(x^3-6x^2-3x+24) & x \in [1,2] \\ -\frac{1}{120}(x-7)(x-3)^2(x-2)x & x \in [2,3] \\ 0 & x \notin [-2,3] \end{cases}$$

4. $\phi_{(x_3)}(x) = \phi_3(x-3)$ and $\phi_3(x)$ is as follows:

$$\phi_3(x) = \begin{cases} -\frac{1}{540}(x-2)(x-1)(x+1)(x+2)(x+3)(8x^2+15x-45) & x \in [-3,0] \\ -\frac{1}{12}(x-1)(x^4-2x^3-15x^2+12x+12) & x \in [0,1] \\ \frac{1}{24}(x-2)(x-1)(x^3-6x^2-3x+24) & x \in [1,2] \\ -\frac{1}{120}(x-7)(x-3)^2(x-2)x & x \in [2,3] \\ 0 & x \notin [-3,3] \end{cases}$$

5. $\phi_{(x_4)}(x) = \phi_4(x-4)$ and $\phi_4(x)$ is as follows:

$$\phi_4(x) = \begin{cases} \frac{1}{1080}(x-1)(x+1)(x+2)(x+3)(x+4)(8x^2+23x-45) & x \in [-4,-1] \\ \frac{1}{12}(x+1)(x^4+2x^3-15x^2-12x+12) & x \in [-1,0] \\ -\frac{1}{12}(x-1)(x^4-2x^3-15x^2+12x+12) & x \in [0,1] \\ \frac{1}{24}(x-2)(x-1)(x^3-6x^2-3x+24) & x \in [1,2] \\ -\frac{1}{120}(x-7)(x-3)^2(x-2)x & x \in [2,3] \\ 0 & x \notin [-4,3] \end{cases}$$

6. $\phi_{(x_5)}(x) = \phi_5(x-5)$ and $\phi_5(x)$ is as follows:

$$\phi_5(x) = \begin{cases} -\frac{1}{5400}(x+1)(x+2)(x+3)(x+4)(x+5)^2(8x-9) & x \in [-5,-2] \\ -\frac{1}{24}(x+1)(x+2)(x^3+6x^2-3x-24) & x \in [-2,-1] \\ \frac{1}{12}(x+1)(x^4+2x^3-15x^2-12x+12) & x \in [-1,0] \\ -\frac{1}{12}(x-1)(x^4-2x^3-15x^2+12x+12) & x \in [0,1] \\ \frac{1}{24}(x-2)(x-1)(x^3-6x^2-3x+24) & x \in [1,2] \\ -\frac{1}{120}(x-7)(x-3)^2(x-2)x & x \in [2,3] \\ 0 & x \notin [-5,3] \end{cases}$$

7. For $j \geq 6$, $\phi_{(x_j)}(x) = \phi_{([-3,3];1;4)}(x-j)$ and $\phi_{([-3,3];1;4)}(x)$ is defined by (14).

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