The Smooth Piecewise Polynomial Particle Shape Functions corresponding to Patch-wise Non-Uniformly Spaced Particles for Meshfree Particle Methods

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Abstract

In the previous papers ([12],[24]), for uniformly or locally non-uniformly distributed particles, we constructed highly regular piecewise polynomial particle shape functions that have the polynomial reproducing property of order k for any given integer $k \geq 0$ and satisfy the Kronecker Delta Property. In this paper, in order to make these piecewise polynomial particle shape functions to be more useful in dealing with problems on complex geometries, we introduce highly regular piecewise polynomial particle shape functions corresponding to the particles that are mostly non-uniformly distributed within the domain, and their supports are proper subsets of the given domain. An error estimate of the interpolation associated with such flexible piecewise polynomial particle shape functions is proven.

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1 Introduction

For the last half century, the finite element method(FEM) has been widely used to solve many important science and engineering problems. However, the classical FEM has several obstacles, such as mesh refinement and constructing smooth global basis functions.

Recently several generalized finite element methods (GEFM) that circumvent the obstacles of the conventional FEM were introduced. Among many GFEMs that use meshes minimally or do not use meshes at all ([1],[2],[3],[4]), those methods related to this paper are Element Free Galerkin Method (EFGM) ([1],[10],[11],[13],[14], [15],[18],[19],[20]), h-p Cloud Method([8],[9]), Partition of Unity Finite Element Method (PUFEM)([21],[25],[26]), and Reproducing Kernel Element Method (RKEM) ([15],[16],[17]). This paper is closely related to those element free methods: RKPM and RKEM.

The Reproducing Kernel Particle Method (RKPM) ([10],[11],[15],[18],[19],[20]) is a mesh free method that yields highly accurate approximation to smooth functions by using the reproducing kernel shape functions that can exactly interpolate the polynomials of a fixed degree. The RKP(reproducing kernel particle) shape functions can be constructed to be smooth up to any desired order by selecting smooth window functions.

However, the RKP shape functions constructed by using specific window functions are generally fractional functions with complicate denominators that are solutions of the system of algebraic equations. Thus, these RKP shape functions have the following difficulties:

(1) They do not satisfy the Kronecker delta property; hence, it has difficulties in dealing with Dirichlet boundary conditions.

(2) Accuracy is compromised in numerical integrations for these complex fractional shape functions.

In order to alleviate these obstacles, in our previous papers([12],[24]), we constructed piecewise polynomial $\mathcal{C}^r$-reproducing polynomial particle shape functions associated with uniformly (or non-uniformly) distributed particles, that satisfy the Kronecker delta property, for any integer $r \geq 0$, and any desired reproducing order. However, the particle shape functions obtained by the tensor product of these piecewise polynomial one dimension particle shape functions are not practical in higher dimensional problems, especially when the solution domain is non-convex.

In this paper, we propose a flexible method to construct piecewise polynomial particle shape functions associated with patch-wise uniformly (or non-uniformly) distributed particles in a convex (or non-convex) polygonal domain. The proposed method is schematically described as follows:

(1) We construct a reference patch $\hat{\Omega}$ that satisfies the following:

- the particles are non-uniformly or uniformly spaced in $\hat{\Omega}$.
- the assigned particle shape functions $\phi_j(\xi)$ to these particles are smooth piecewise polynomial with reproducing property of any given order.
- the particle shape functions satisfy the Kronecker delta property.
- their supports are small proper subsets of $\hat{\Omega}$.

(2) For small number of particles near the boundary $\partial\hat{\Omega}$, we extend one polynomial section of each piecewise polynomial particle shape function $\phi_j(\xi)$ corresponding to these particles to global polynomials by allowing for their supports to go outside of $\hat{\Omega}$. Let $\phi_j^*(\xi)$ be the extended $\phi_j(\xi)$.

(3) Decompose the polygonal domain into non-overlapping patches Ω_k of any desired sizes.

(4) Construct affine mappings T_k from the reference patch to spacial patches Ω_k.

(5) Construct a smooth piecewise polynomial wide flat top partition of unity shape functions $\psi_k^\delta(x)$ subordinate to the covering Ω_k^δ, which denotes δ-neighborhood of Ω_k.

(6) Capping the extended particle shape function $\phi_j^* \circ T_k^{-1}$ by multiplying the flat-top partition of unity and the transformed extended particle shape functions, the resulting piecewise polynomial particle shape functions have compact supports and have polynomial reproducing property of reduced order.

Since the resulting closed form particle shape functions satisfy the Kronecker delta property except at few particles, they can easily handle the Dirichlet boundary conditions. Moreover, by adjusting the support sizes of particle shape functions, one can construct the piecewise polynomial particle shape functions with given regularity order as well as given reproducing order.

This paper is organized as follows: In section 2, notations and definitions used in this paper are stated. In section 3, we propose a new method to construct piecewise polynomial particle shape functions associated with patch-wise uniformly (or non-uniformly) spaced particles. In section 4, we prove an error estimate of the interpolation associated with those particle shape functions constructed in section 3. In section 5, we demonstrate the effectiveness of our proposed method and perform numerical tests that support the theoretical results in section 4. In section 6, the essential components of higher dimension cases of the proposed method are introduced. Finally, in the appendix, piecewise polynomial particle shape functions corresponding to particles in $[0, n]$ are constructed such that their supports are contained in $[0, n]$.

2 RKP shape functions and RPP shape functions

Throughout this paper, $\alpha, \beta \in \mathbb{Z}^d$ are multi-indices and $x = ({}^1x, {}^2x, .., {}^dx), x_j = ({}^1x_j, {}^2x_j, .., {}^dx_j)$ denote points in $\mathbb{R}^d$. However, if there is no confusion, we also use the conventional notation for the points in $\mathbb{R}^d$ or $\mathbb{Z}^d$ as

$$x = (x_1, x_2, \cdots, x_d) \text{ and } \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d).$$

We also use the following notations:

$$\begin{aligned}
(x - x_j)^\alpha &:= ({}^1x - {}^1x_j)^{\alpha_1} ... ({}^dx - {}^dx_j)^{\alpha_d}, \\
|\alpha| &:= \alpha_1 + \alpha_2 + \cdots + \alpha_d, \\
\alpha! &:= \alpha_1!\alpha_2! \cdots \alpha_d!, \\
\partial_x^\alpha u &:= \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}
\end{aligned}$$

Let Ω be a domain in $\mathbb{R}^d$. For any non-negative integer m, $\mathcal{C}^m(\Omega)$ denotes the space of all functions ϕ such that ϕ together with all their derivatives $D^\alpha \phi$ of orders $|\alpha| \le m$, are continuous on Ω. The support of ϕ is defined by

$$\text{supp } \phi = \overline{\{x \in \Omega : \phi(x) \neq 0\}}.$$

In the following, a function $\phi \in \mathcal{C}^m(\Omega)$ is said to be a $\mathcal{C}^m$- function.

We also use the usual Sobolev space denoted by $H^k(\Omega)$. For $u \in H^k(\Omega)$, the norm is

$$\|u\|_{k,\Omega}^2 = \sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u|^2 dx,$$

and the semi-norm is

$$|u|_{k,\Omega}^2 = \sum_{|\alpha| = k} \int_\Omega |\partial^\alpha u|^2 dx.$$

A weight function(or window function) is a non-negative continuous function with compact support and is denoted by $w(x)$. For example, the widely used window functions include the following: For $x \in \mathbb{R}$,

(a) Conical:

$$w(x) = \begin{cases} (1 - x^2)^l, & |x| \le 1, \\ 0, & |x| > 1, \end{cases} \tag{1}$$

which is a $\mathcal{C}^{l-1}$-function.

(b) Gaussian:

$$w(x) = \begin{cases} (e^{-1/1-x^2}) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1, \end{cases} \tag{2}$$

which is an infinitely smooth function.

(c) Partition of unity ([23]):

$$w(x) = \begin{cases} (1+x)^3 g(x) & \text{if } -1 \le x \le 0 \\ (1-x)^3 g(-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{if } |x| > 1, \end{cases} \tag{3}$$

where $g(x) = (1 - 3x + 6x^2)$.

In $\mathbb{R}^d$, the weight function $w(x)$ can be constructed from a one-dimensional weight function either as $w(x) = w(\|x\|)$ or as $w(x) = \prod_{i=1}^d w(x_i)$, where $x = (x_1, \cdots, x_d)$ and $\|x\|^2 = x_1^2 + \cdots + x_d^2$. In this paper, we use the later one for a higher dimensional window function.

Let Λ be a finite index set and Ω denotes a bounded domain in $\mathbb{R}^d$. Let $\{x_j : j \in \Lambda\}$ be a set of a finite number of uniformly or non-uniformly spaced points in $\mathbb{R}^d$, that are called particles.

Definition 2.1. *Let k be a non-negative integer. Then the functions $\phi_j(x)$ corresponding to the particles $x_j, j \in \Lambda$ are called the RPP(reproducing polynomial particle) shape functions with the reproducing property of order k (or simply, "of reproducing order k") if and only if it satisfies the following condition:*

$$\sum_{j\in\Lambda} (x_j)^\alpha \phi_j(x) = x^\alpha, \ \textit{for } x \in \Omega \subset \mathbb{R}^d \textit{ and for } 0 \le |\alpha| \le k. \tag{4}$$

The RKP (Reproducing Kernel Particle) shape function, associated with the particle $x_j, j \in \Lambda$, is constructed by

$$\phi_j(x) = w(x - x_j) \sum_{0\le|\alpha|\le k} (x - x_j)^\alpha b_\alpha(x) \tag{5}$$

where $b_\alpha(x)$ are chosen so that (4) is satisfied and $w(x)$ is a window function. This gives rise to a linear system in $b_\alpha(x)$, namely

$$\sum_{0\le|\alpha|\le k} m_{\alpha+\beta}(x) b_\alpha(x) = \delta^0_{|\beta|} \text{ for } 0 \le |\beta| \le k, \tag{6}$$

where $\delta^0_{|\beta|}$ is the Kronecker delta, and

$$m_\alpha(x) = \sum_{j\in\mathbb{Z}} w(x - x_j)(x - x_j)^\alpha. \tag{7}$$

For one dimensional case, this system can be written as

$$M(x) \cdot [b_0(x), b_1(x), \cdots, b_k(x)]^T = [1, 0, \cdots, 0]^T,$$

where

$$M(x) = \sum_{j\in\Lambda} w(x - x_j) \begin{bmatrix} 1 \\ (x - x_j)^1 \\ (x - x_j)^2 \\ \vdots \\ (x - x_j)^k \end{bmatrix} [1, (x - x_j)^1, \cdots, (x - x_j)^k].$$

The coefficient matrix $M(x)$ of the linear system (6) is called the moment matrix.

By applying a similar argument to ([2],[10]), one can easily show the following:

Lemma 2.1. *The condition (4) for the RPP shape functions is equivalent to*

$$\sum_{j\in\Lambda}(x-x_j)^\beta \phi_j(x) = \delta^0_{|\beta|}, \qquad \text{for } 0 \le |\beta| \le k \text{ and } x \in \mathbb{R}^d. \tag{8}$$

Adopting those terminologies and notations of ([3]), we have the following: For $j = (j_1, j_2, \cdots, j_d) \in \mathbb{Z}^d$, and the mesh size $0 < h \le 1$, let

$$x_j^h = (j_1 h, \cdots, j_d h) = hj.$$

Then the points x_j^h are called uniformly distributed particles. Let ϕ be a continuous function with compact support that contains the origin 0. Then the particle shape functions associated to the uniformly distributed particles is defined by

$$\phi_j^h(x) = \phi(\frac{x-jh}{h}) = \phi(\frac{x_1 - j_1 h}{h}, \cdots, \frac{x_d - j_d h}{h}),$$

for $j \in \mathbb{Z}^d$ and $0 < h \le 1$. Then these particle shape functions are **translation invariant** in the sense that

$$x_{i+j}^h \;\; = \;\; x_i^h + x_j^h, \phi_j^h(x - ih) = \phi_{i+j}^h(x).$$

If the RKP shape functions are translation invariant, in order to retain the polynomial reproducing property, some particles and their supports should go outside of the domain.

However, in this paper, we consider the case when the particles are mostly non-uniformly and partly uniformly distributed within the given domain, that is, $\{x_j : j \in \Lambda\} \subset \overline{\Omega}$, and their supports are proper subsets of the given domain.

Remark 2.1. *The piecewise polynomial (p.p.) RPP shape functions and the RKP shape functions are slightly different. Their differences are as follows:*

(1) The piecewise polynomial RPP shape functions are constructed by solving the system (8) without using window function. On the other hand, the RKP shape functions are constructed by solving the system (5) with respect to a specific window function. Thus, the RKP shape functions are not piecewise polynomials in general.

In other words, the RKP shape functions are constructed with respect to specific window functions, whereas what we call the RPP(Reproducing Polynomial Particle) shape functions in this paper have no relation with any specific window functions, but are concerned with the sizes of supports. However, both are constructed to have the polynomial reproducing property. Thus, we will not distinguish them, unless window functions should be specified.

(2) The supports of our p.p. RPP shape functions are bounded by the particles (for example, $[-3, 3]$ *in one dimensional integer particles case), whereas the boundaries of the supports of RKP*

shape function are allowed to be points between two particles (for example, $[-3.1, 3.1], [-3.8, 3.8], [-3, 3]$ *etc).*

(3) Even if the boundaries of the selected window function are particles (for example, the Gaussian window $e^{-1/(1-|x/3|^2)}$*), the resulting RKP shape functions are obviously not piecewise polynomial, and hence cannot be p.p. RPP shape functions.*

(4) We do not know for what kind of window functions RKP shape functions are equivalent to the RPP shape functions in general. However, if the window function is a $\mathcal{C}^0$*-function with support* $[-K, K]$ *for an integer* K*, for example* $w(x) = (1 - |x/K|^2)$ *(the conical window function), and the polynomial reproducing order is* $2K - 1$*, then the RKP shape function becomes the RPP shape function (Theorem 3.1 of [24]).*

3 Patch-wise Non-Uniformly Spaced Particles

In this paper, the indices of reproducing polynomial shape function $\phi_{([a,b];m_2;m_3)}(x)$ indicate the following:

$$\begin{aligned} [a,b] &= \text{the support of } \phi(x), \\ m_2 &= \text{the order of the regularity(that is, } \phi(x) \in \mathcal{C}^{m_2}), \\ m_3 &= \text{the order of the reproducing property.} \end{aligned}$$

In particular, $\phi_{([-K,K];m_2;m_3)}(x)$ is also denoted by $\phi_{(K;m_2;m_3)}(x)$. Throughout this paper, K is a positive integer that is the radius of the support of a basic particle shape function whose support is $[-K, K]$.

In [24], it was shown that the basic $\mathcal{C}^1$-particle shape function of reproducing order 2 with support $[-2, 2]$ for uniformly distributed particles is uniquely determined by

$$\phi_{([-2,2];1;2)}(x) = \begin{cases} \frac{1}{2}(x+1)(x+2)^2, & x \in [-2,-1], \\ -\frac{1}{2}(x+1)(3x^2+2x-2), & x \in [-1,0], \\ \frac{1}{2}(x-1)(3x^2-2x-2), & x \in [0,1], \\ -\frac{1}{2}(x-2)^2(x-1), & x \in [1,2]. \end{cases}$$

In [12], we constructed the $\mathcal{C}^1$-particle shape functions $\phi_{(x_j)}$ of reproducing order 2 defined on $[0, n]$ that are associated with the particles, $x_j = j, j = 0, 1, 2, \cdots, n$. In other words,

$$\sum_{j=0}^{n} x_j^\alpha \phi_{(x_j)}(x) = x^\alpha, \text{ for all } x \in [0,n], \alpha = 0,1,2.$$

Since $\phi_{(x_0)}|_{[0,2]}, \phi_{(x_1)}|_{[0,2]}, \phi_{(x_2)}|_{[0,2]}$, and $\phi_{(x_3)}|_{[0,2]}$ are polynomials, as shown in Figs. 1 and 2, the parts of these piecewise polynomial shape functions can be globally extended as follows:

(i) The extended particle shape function corresponding to the particle $x_0 = 0$ is the following:

$$\hat{\phi}_{(x_0)}(x) = \begin{cases} \frac{1}{72}(x-3)(x-2)^2(x-1)(5x+6), & x \in (-\infty, 2], \\ 0, & x \notin (-\infty, 2]. \end{cases}$$

(ii) The extended particle shape function corresponding to the particle $x_1 = 1$ is the following:

$$\hat{\phi}_{(x_1)}(x) = \begin{cases} -\frac{1}{24}(x-3)(x-2)x(5x^2-9x-8), & x \in (-\infty, 2], \\ \phi_{([-2,2];1;2)}(x-1), & x \in [2,3], \\ 0, & x \notin (-\infty, 3]. \end{cases}$$

(iii) The extended particle shape function corresponding to the particle $x_2 = 2$, is the following:

$$\hat{\phi}_{(x_2)}(x) = \begin{cases} \frac{1}{24}(x-3)(x-1)x(5x^2-14x-4), & x \in (-\infty, 2], \\ \phi_{([-2,2];1;2)}(x-2), & x \in [2,4], \\ 0, & x \notin (-\infty, 4]. \end{cases}$$

(iv) The extended particle shape function corresponding to the particle $x_3 = 3$, is the following:

$$\hat{\phi}_{(x_3)}(x) = \begin{cases} -\frac{1}{72}(x-2)(x-1)x^2(5x-19), & x \in (-\infty, 2], \\ \phi_{([-2,2];1;2)}(x-3), & x \in [2,5], \\ 0, & x \notin (-\infty, 5]. \end{cases}$$

(v) For the particles x_j $(j = 4, 5, \cdots, n-4)$, we assign the particle shape functions defined as follows:

$$\hat{\phi}_{(x_j)}(x) = \phi_{([-2,2];1;2)}(x-j).$$

(vi) The extended particle shape function corresponding to the particles $x_j, j = n-3, n-2, n-1, n$, are the following:

$$\hat{\phi}_{(x_j)}(x) = \phi_{(x_{n-j})}(-(x-n)).$$

For brevity, in this section, we assume that $n = 8$.

In what follows, we construct patch-wise equally spaced particles as well as their corresponding particle shape functions that are piecewise polynomials and satisfy the Kronecker delta property.

1: Consider the following partition of $\Omega = [\alpha, \beta]$:

$$\alpha = x_1 < x_2 < \cdots < x_{N+1} = \beta.$$

The N patches obtained by this partition are denoted as follows:

$$\Omega_k = (x_k, x_{k+1}), k = 1, 2, \cdots, N.$$

2: Let

$$\begin{aligned} \hat{\Omega} &= [0, 8] \\ \xi_j &= j, \text{ for } j = 0, 1, 2 \cdots, 8, \end{aligned}$$

be the reference patch and nodes. Let $\hat{\phi}_{(\xi_j)}(\xi), j = 0, 1, \cdots, 8$ be the extended $\mathcal{C}^1$-particle shape functions defined above(see, Figs 1 & 2).

Then we have

$$\sum_{j=0}^{8} \hat{\phi}_{(\xi_j)}(\xi)\xi_j^{\alpha} = \xi^{\alpha}, \text{ for all } \xi \in (-\infty, \infty), \alpha = 0, 1, 2. \tag{9}$$

Indeed, for $\xi \in (-\infty, 2]$,

$$\sum_{j=0}^{8} \hat{\phi}_{(\xi_j)}(\xi)\xi_j^{\alpha} = \sum_{j=0}^{3} \hat{\phi}_{(\xi_j)}(\xi)\xi_j^{\alpha}.$$

The particle shape functions, $\hat{\phi}_{(\xi_0)}, \hat{\phi}_{(\xi_1)}, \hat{\phi}_{(\xi_2)}, \hat{\phi}_{(\xi_3)}$ are global polynomials on $(-\infty, 2]$, and for all $\xi \in [0, 2]$ and $\alpha = 0, 1, 2$, we have

$$\hat{\phi}_{(\xi_0)}(\xi)\xi_0^{\alpha} + \hat{\phi}_{(\xi_1)}(\xi)\xi_1^{\alpha} + \hat{\phi}_{(\xi_2)}(\xi)\xi_2^{\alpha} + \hat{\phi}_{(\xi_3)}(\xi)\xi_3^{\alpha} = \xi^{\alpha}.$$

Therefore, for all $\xi \in (\infty, 0]$ and $\alpha = 0, 1, 2$, we have

$$\hat{\phi}^{Ext}_{(\xi_0)}(\xi)\xi_0^{\alpha} + \hat{\phi}^{Ext}_{(\xi_1)}(\xi)\xi_1^{\alpha} + \hat{\phi}^{Ext}_{(\xi_2)}(\xi)\xi_2^{\alpha} + \hat{\phi}^{Ext}_{(\xi_3)}(\xi)\xi_3^{\alpha} = \xi^{\alpha},$$

which implies

$$\sum_{j=0}^{8} \hat{\phi}_{(\xi_j)}(\xi)\xi_j^{\alpha} = \xi^{\alpha}, \text{ for all } \xi \in (-\infty, 2], \alpha = 0, 1, 2.$$

Similarly, we can show that

$$\sum_{j=0}^{8} \xi_j^{\alpha} \hat{\phi}_{(\xi_j)}(\xi) = \xi^{\alpha}, \text{ for all } \xi \in [6, \infty), \alpha = 0, 1, 2. \tag{10}$$

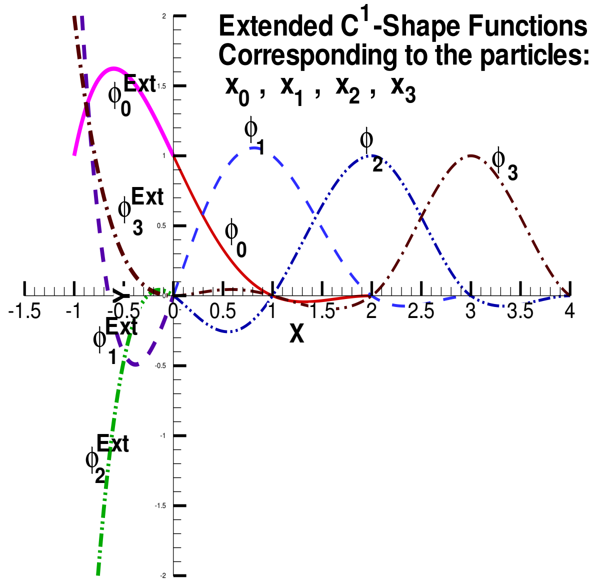
Of course, these particle shape functions satisfy the polynomial reproducing property of order 2 for $x \in [2, 6]$.

3: For each $k = 1, 2, \cdots, N$, we define an affine transformation $T_k : [0, 8] \to \overline{\Omega}_k = [x_k, x_{k+1}]$ as follows:

$$T_k(\xi) = h_k \xi + x_k, \quad \text{ where } h_k = \frac{x_{k+1} - x_k}{8}.$$

Then, each patch $\overline{\Omega}_k$ has exactly 9 equally spaced particles:

$$\begin{aligned} x_{(k,0)} &:= T_k(\xi_0), \\ x_{(k,1)} &:= T_k(\xi_1), \\ &\vdots \\ x_{(k,8)} &:= T_k(\xi_8). \end{aligned}$$



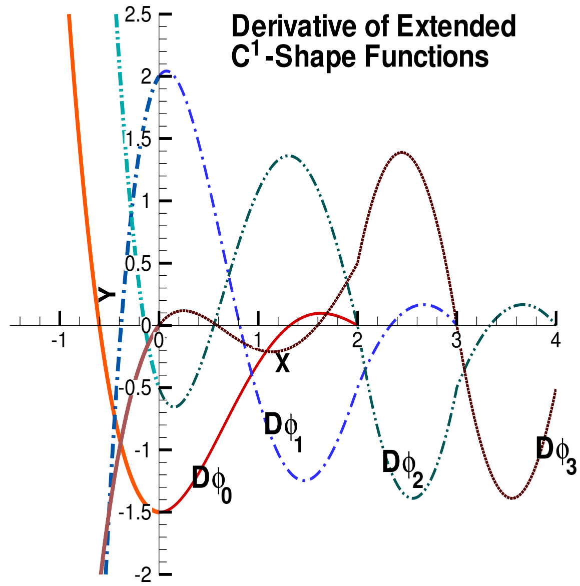


Figure 1: Extended Particle shape functions $\hat{\phi}_{(\xi_j)}(x) = \overline{\phi}_j(x)$(original P.W. polynomial) $+\phi_j^{Ext}(x)$(extended global polynomial), and their derivatives, $j = 0, 1, 2, 3$.

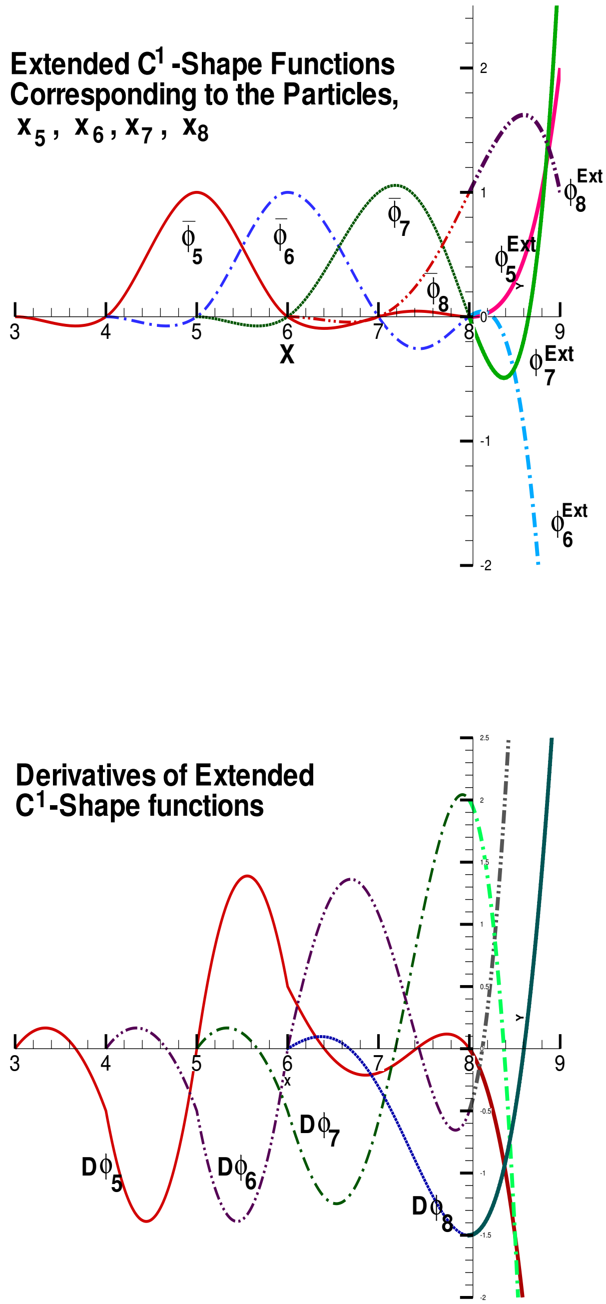


Figure 2: Extended Particle shape functions $\hat{\phi}_{(\xi_j)}(x) = \overline{\phi}_j(x)$(original P.W. polynomial) $+\phi_j^{Ext}(x)$(extended global polynomial), and their derivatives, $j = 5, 6, 7, 8$.

Now to these particles, we associate the particle shape functions

$$\phi^*_{(k,j)}(x) := (\hat{\phi}_j \circ T_k^{-1})(x), j = 0, 1, \cdots, 8; k = 1, 2, \cdots, N.$$

Thus, we have constructed patch-wise uniformly distributed particles such that in each patch $\overline{\Omega}_k$, particles are spaced by h_k. It follows from Lemma 3.1 that the constructed particle shape functions have the reproducing property of order 2. That is, for each $k = 1, 2, \cdots, N$, and $\alpha = 0, 1, 2$,

$$\sum_{j=0}^{8} x^{\alpha}_{(k,j)} \phi^*_{(k,j)}(x) = x^{\alpha}, \quad \text{for } x \in \mathbb{R}. \tag{11}$$

Lemma 3.1. *Let $\hat{\phi}_{(\xi_j)}(\xi)$ be the particle shape functions of reproducing order k associated with the particles $\xi_j \in \hat{\Omega} \subset \mathbb{R}$, $j \in \Lambda$. Let $T(\xi) = a\xi + b$, $a \neq 0$, be a affine transformation on $\mathbb{R}$. Then the functions defined by*

$$\phi_{x_j}(x) := (\hat{\phi}_{(\xi_j)} \circ T^{-1})(x) \quad , \quad j \in \Lambda,$$

become reproducing polynomial particle shape functions associated with the particles $x_j = T(\xi_j)$ on $T(\hat{\Omega})$.

Proof.

$$\begin{aligned} \sum_{j\in\Lambda} (x - x_j)^{\alpha} \phi_{x_j}(x) &= \sum_{j\in\Lambda} (T(\xi) - T(\xi_j))^{\alpha}\, \hat{\phi}_{(\xi_j)}(T^{-1}(T(\xi))) \\ &= a^{\alpha} \sum_{j\in\Lambda} (\xi - \xi_j)^{\alpha}\, \hat{\phi}_{(\xi_j)}(\xi) \\ &= a^{\alpha}\delta_0^{\alpha} = \delta_0^{\alpha}, \text{ for all } x \in \Omega. \end{aligned}$$

The last equality follows from the fact that $a^0 = 1$. □

In what follows, we assume

$$H = \min\{H_k = (x_{k+1} - x_k) : k = 1, 2, \cdots, N\}, \tag{12}$$

$$0 << \delta < \frac{H}{3}. \tag{13}$$

and we use the scaled window function defined by

$$\beta_{\delta} = Aw(\frac{x}{\delta}), \tag{14}$$

where $w(x)$ is the conical window function defined by (1) and

$$A^{-1} = \int_{-\delta}^{\delta} w(\frac{x}{\delta})dx.$$

Let

$$\psi_k^\delta(x) := \int_{\Omega_k} \beta_\delta(y-x)dy$$
$$= \begin{cases} \int_{x-x_{k+1}}^{\delta} \beta_\delta(t)dt = f_{k+1}(x), & \text{if } x \in [x_{k+1}-\delta, x_{k+1}+\delta], \\ 1, & \text{if } x \in [x_k+\delta, x_{k+1}-\delta], \\ \int_{-\delta}^{x-x_k} \beta_\delta(t)dt = f_k(x), & \text{if } x \in [x_{k+1}-\delta, x_k+\delta], \\ 0, & \text{if } x \in \mathbb{R}\backslash[x_k-\delta, x_{k+1}+\delta]. \end{cases} \tag{15}$$

Then $\psi_k^\delta(x)$ is the convolution $\chi_{\Omega_k} * \beta_\delta(x)$, where the characteristic function χ_{Ω_k} is defined by

$$\chi_{\Omega_k}(x) = \begin{cases} 1, & \text{if } x \in \Omega_k, \\ 0, & \text{if } \xi \notin \Omega_k. \end{cases} \tag{16}$$

Moreover, we have the following: since $\beta_\delta(x)$ is a polynomial,

- $\psi_k^\delta(x)$ is a piecewise polynomial function. Indeed, if $l = 5$ in the definition of the conical window function, then

$$f_{k+1}(x) = (\frac{512\delta}{693})\Big\{-(x-x_{k+1}) + \frac{(x-x_{k+1})^{11}}{11\delta^{10}} - \frac{5(x-x_{k+1})^9}{9\delta^8} + \frac{10(x-x_{k+1})^7}{7\delta^6} - \frac{2(x-x_{k+1})^5}{\delta^4} + \frac{5(x-x_{k+1})^3}{3\delta^2} + \frac{256\delta}{693}\Big\}, \tag{17}$$

$$f_k(x) = -(\frac{512\delta}{693})\Big\{-(x-x_k) + \frac{(x-x_k)^{11}}{11\delta^{10}} - \frac{5(x-x_k)^9}{9\delta^8} + \frac{10(x-x_k)^7}{7\delta^6} - \frac{2(x-x_k)^5}{\delta^4} + \frac{5(x-x_k)^3}{3\delta^2} + \frac{256\delta}{693}\Big\}, \tag{18}$$

$$\max|\frac{d}{dx}(\psi_k^\delta(x))| = \max|\beta_\delta| = \mathcal{O}(\delta^{-1}). \tag{19}$$

For example, if $l = 5$ in the definition of the conical window function, then

$$\max|\frac{d}{dx}(\chi_{\Omega_k} * \beta_\delta)| < (1.28) \times 10^p, \quad \text{if } \delta = 10^{-p}, p = 0, 1, 2, \ldots.$$

For $\delta = 0.1$, $\psi_k^\delta(x)$ is depicted in Fig. 3 when Ω_k is $(2,3), (3,3.5)$ and $(3.5,3.9)$, respectively.

Now, we show that the particle shape functions $\{\psi_k^\delta(x) : k = 1, \cdots, N\}$ are partition of unity subordinated to the non-uniform covering $\{(x_k-\delta, x_{k+1}+\delta) : k = 1, \cdots, N\}$.

For the purpose of higher dimensional extension of the proposed method, we prove it in the d dimensional space $\mathbb{R}^d$. Let us consider an almost everywhere (a.e.) δ-covering of a domain Ω,

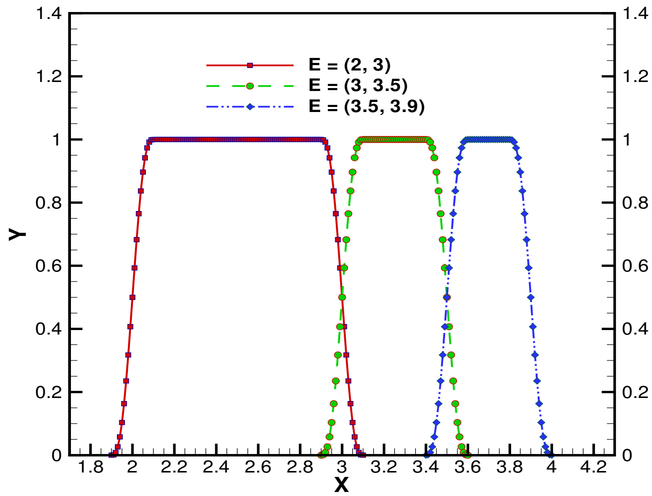


Figure 3: The graphs of the convolution function $\psi_k^\delta(x)$ for $\delta = 0.1$ and various Ω_k. $\psi_1^\delta = \chi_{\Omega_1} * \beta_\delta, \Omega_1 = (2,3)$; $\psi_2^\delta = \chi_{\Omega_2} * \beta_\delta, \Omega_2 = (3,3.5)$; $\psi_3^\delta = \chi_{\Omega_3} * \beta_\delta, \Omega_3 = (3.5,3.9)$.

which means a family of mutually disjoint simply connected open subsets $\Omega_k, k = 1, ..., N$ of $\mathbb{R}^d$ such that

$$\bigcup_{k=1}^{N} \overline{\Omega}_k \supset \{x : dist(x, \Omega) \le 2\delta\} \quad (\text{ Thus, } \quad \bigcup_{k=1}^{N} \overline{\Omega}_k \supset \partial\Omega).$$

Then we have

$$\sum_{k=1}^{N} \chi_{\Omega_k}(x) = 1, a.e. \text{ on } \Omega, \text{ and hence } \sum_{k=1}^{N} (\chi_{\Omega_k} * \beta_\delta)(x) = (1 * \beta_\delta)(x) = 1, \text{ for all } x \in \Omega.$$

Hence we have the following theorem

Lemma 3.2. *(i) If $\{\Omega_k : k = 1, \cdots, N\}$ be an a.e. δ-covering of Ω, then $\{\psi_k^\delta(x)\}$ are piecewise polynomial partition of unity shape functions subordinated to the covering $\{\Omega_k^\delta : k = 1, \cdots, N\}$, where $\Omega_k^\delta = \{x : dist(x, \Omega_k) \le \delta\}$.*

(ii) If for a positive integer q, $\beta_\delta(x) \in \mathcal{C}^q$, then $\psi_k^\delta(x) \in \mathcal{C}^q$.

Since $x_{(k,8)} = x_{(k+1,0)}$, for $k = 1, 2, \cdots, N-1$, we introduce the following global numbering for the $8N+1$ particles:

$$
\begin{aligned}
x_{(1,0)} &:= x_{(1,0)} \\
&\vdots \\
x_{(k,8)} &:= x_{(k,8)} = x_{(k+1,0)} \text{ for } k = 1, \cdots, N-1 \\
&\vdots \\
x_{(N,8)} &:= x_{(N,8)}.
\end{aligned}
$$

On the other hand, we define the particle shape function corresponding to these particles as follows:

$$
\begin{cases}
\phi_{(1,0)}(x) & := \phi^*_{(1,0)}(x), \\
\phi_{(1,1)}(x) & := \phi^*_{(1,1)}(x), \\
\phi_{(1,2)}(x) & := \phi^*_{(1,2)}(x),
\end{cases} \tag{20}
$$

$$
\begin{cases}
\phi_{(N,6)}(x) & := \phi^*_{(N,6)}(x), \\
\phi_{(N,7)}(x) & := \phi^*_{(N,7)}(x), \\
\phi_{(N,8)}(x) & := \phi^*_{(N,8)}(x),
\end{cases} \tag{21}
$$

$$
\begin{cases}
\phi_{(k,8)}(x) & := \left\{ \psi^\delta_k(x)\phi^*_{(k,8)}(x) + \psi^\delta_{k+1}(x)\phi^*_{(k+1,0)}(x) \right\}, k = 1, 2, \cdots, N-1, \\
\phi_{(k,j)}(x) & := \psi^\delta_k(x)\phi^*_{(k,j)}(x), k = 2, \cdots, N-1.
\end{cases} \tag{22}
$$

Then these functions are RPP shape functions of order 2 as follows:

Theorem 3.1. *1. For all $x \in \Omega$,*

$$
x^\alpha_{(1,0)}\phi_{(1,0)}(x) + \sum_{k=1}^{N}\sum_{j=1}^{8} x^\alpha_{(k,j)}\phi_{(k,j)}(x) = x^\alpha, \alpha = 0, 1, 2. \tag{23}
$$

2.

$$
\begin{cases}
\phi_{(1,i)}(x_{(1,j)}) & = \delta_{ij}, & \textit{if } 0 \le i, j \le 8, \\
\phi_{(N,i)}(x_{(N,j)}) & = \delta_{ij}, & \textit{if } 1 \le i, j \le 8, \\
\phi_{(k-1,i)}(x_{(k,j)}) & \ne \delta_{ij}, & \textit{if } i \in \{5,6,7,8\} \textit{ and } j \in \{1,2,3\}, \\
\phi_{(k,i)}(x_{(k-1,j)}) & \ne \delta_{ij}, & \textit{if } i \in \{1,2,3,4\} \textit{ and } j \in \{5,6,7,8\}, \\
\phi_{(k,i)}(x_{(l,j)}) & = \delta_{kl}\delta_{ij}, & \textit{for all other cases.}
\end{cases}
$$

Proof. (1) For notational convenience, we write the particle shape functions ϕ^* constructed above as two parts:

$$
\phi^* = \overline{\phi}^*(\text{original p.w. polynomial}) + \widehat{\phi}^*(\text{extended global polynomial}).
$$

It suffices to prove

$$\sum_{(k,j)\in\Lambda} x^{\alpha}_{(k,j)}\phi^*_{(k,j)}(x) := x^{\alpha}_{(1,0)}\phi_{(1,0)}(x) + \sum_{k=1}^{N}\sum_{j=1}^{8} x^{\alpha}_{(k,j)}\phi_{(k,j)}(x) = x^{\alpha}$$

for the following cases:

(i) For $x \in [x_k - \delta, x_k)$,

$$\begin{aligned}\sum_{(k,j)\in\Lambda} x^{\alpha}_{(k,j)}\phi^*_{(k,j)}(x) &= \psi^{\delta}_{k-1}(x)\sum_{j=6}^{8} x^{\alpha}_{(k-1,j)}\overline{\phi}^*_{(k-1,j)}(x) + \psi^{\delta}_{k}(x)\sum_{j=0}^{3} x^{\alpha}_{(k,j)}\widehat{\phi}^*_{(k,j)}(x) \\ &= \psi^{\delta}_{k}(x)x^{\alpha} + \psi^{\delta}_{k+1}(x)x^{\alpha} = x^{\alpha}.\end{aligned}$$

(ii) For $x = x_k = x_{(k,8)}$,

$$\sum_{(k,j)\in\Lambda} \phi^*_{(k,j)}(x_k) = \psi^{\delta}_{k}(x)x^{\alpha}_k + \psi^{\delta}_{k+1}(x)x^{\alpha}_k = x^{\alpha}_k.$$

(iii) For $x \in (x_k, x_k + \delta]$,

$$\begin{aligned}\sum_{(k,j)\in\Lambda} x^{\alpha}_{(k,j)}\phi^*_{(k,j)}(x) &= \psi^{\delta}_{k-1}(x)\sum_{j=6}^{8} x^{\alpha}_{(k-1,j)}\widehat{\phi}^*_{(k-1,j)}(x) + \psi^{\delta}_{k}(x)\sum_{j=0}^{3} x^{\alpha}_{(k,j)}\overline{\phi}^*_{(k,j)}(x) \\ &= \psi^{\delta}_{k}(x)x^{\alpha} + \psi^{\delta}_{k+1}(x)x^{\alpha} = x^{\alpha}.\end{aligned}$$

(iv) For $x \in [x_k + \delta, x_{k+1} - \delta)$, it is obvious.

(2) It is obvious that $\phi_{(k,0)}(x_{(l,0)}) = \delta_{kl}, \quad \phi_{(k,8)}(x_{(l,8)}) = \delta_{kl}, \quad 1 \le k, l \le N$. However, $\phi_{(k,j)}(x_{(l,j)}) = C\delta_{kl}$, where $C < 1$ if $j = 1$; $C = 1$ if $j \neq 1$.

□

In this section, we constructed p.p.(piecewise polynomial) RPP shape functions associated with patch-wise uniform particles by selecting a reference patch $[0, 8]$, in which nine particles are equally spaced. However, if we choose a reference patch $[0, n]$ in which $n + 1$ particles are non-uniformly spaced and particle shape functions are piecewise polynomials (constructed in [12]), we can obtain p.p. RPP shape functions corresponding to path-wise non-uniformly spaced particles.

4 Error Estimate

In this section, we assume that the reference domain $\hat{\Omega}$ is $[0, n]$ for a fixed integer n, in which there are equally spaced $(n+1)$-integer particles. Of the $(n+1)$ p.p. RPP shape functions associated with the particles in $[0, n]$, at least one of them is a translation of the shape function

$$\phi_{([-K,K];r;2K-2)}(\xi),$$

whose support is $[-K, K]$ for a positive integer K. Since this is the largest support for the reference particle shape functions, the supports of all other reference RPP shape functions have diameters $\leq 2K$.

Let

$$h = \max\{h_k = \frac{H_k}{n} : 1 \leq k \leq N\}. \tag{24}$$

We also assume that the distance between particles are quasi-uniform in the following sense:

$$0 < K\delta_1 < \frac{h_k}{h} < K\delta_2, \tag{25}$$

where δ_1 and δ_2 are fixed positive constants. Let

$$\Lambda = \{(1,0)\} \cup \{(k,j) : 1 \leq k \leq N, 1 \leq j \leq n\}.$$

Then the interpolation of $u(x)$ in terms of the particle shape functions constructed in previous section, $\phi_{(k,j)}(x)$, is defined by

$$\mathcal{I}_h u(x) := \sum_{(k,j)\in\Lambda} u(x_{(k,j)})\phi_{(k,j)}(x).$$

An interpolation error estimate for particle shape functions was proved in Han and Meng ([10]), and Babuska-Banerjee-Osborn([2]). However, with respect to piecewise polynomial particle shape functions constructed in the previous section, we present a more direct proof of an interpolation error estimate for the one dimensional case. For the higher dimensions, we will modify the proof in the forthcoming paper ([12]).

Lemma 4.1. *Suppose $u \in H^{m+1}(\Omega), (2m-3) \geq 0$, the particle shape functions $\phi_{(k,j)}(x)$ have polynomial reproducing property of order $(m-1)$, and Ω is a domain in $\mathbb{R}$. Then we have*

$$\|u - \mathcal{I}_h u\|_{0,\Omega} \leq C_0 h^m |u|_{m,\Omega}, \tag{26}$$

$$\|\frac{d}{dx}(u - \mathcal{I}_h u)\|_{0,\Omega} \leq C h^m \left\{C_1 h^{-1} + C_2 \delta^{-1}|\right\} |u|_{m,\Omega}, \tag{27}$$

where the constants, C_0, C_1, C_2, are independent of u and h. δ is a fixed constant with $h < \delta < H/3$ that is the radius of the scaled conical window function.

Thus, δ depends on the diameter H of the smallest patch. On the other hand, h does not depend on the patch sizes, but depends on n, the number of particles in the reference patch.

Proof. It follows from the Sobolev imbedding theorem that $H^{m+1}(\Omega) \subset \mathcal{C}^m(\Omega)$. Thus, $u^{(m)}(x)$ has point values in Ω.

Now, using the Cauchy integral remainder formula:

$$\begin{aligned} f(y) &= \sum_{\alpha=0}^{m-1} \frac{f^{(\alpha)}(x)}{\alpha!}(y-x)^\alpha + \frac{1}{(m-1)!}\int_x^y f^{(m)}(\theta)(y-\theta)^{m-1}d\theta \\ &= \sum_{\alpha=0}^{m-1} \frac{f^{(\alpha)}(x)}{\alpha!}(y-x)^\alpha + m\int_0^1 (1-t)^{m-1}\frac{f^{(m)}((1-t)x+ty))}{m!}(y-x)^m dt, \end{aligned} \tag{28}$$

where $t = (\theta - x)/(y-x)$, we have

$$u(x_{(k,j)}) = \sum_{\alpha=0}^{m-1} \frac{u^{(\alpha)}(x)}{\alpha!}(x_{(k,j)} - x)^\alpha + R_m u(x_{(k,j)}, x), \text{ for } x \in \Omega,$$

where

$$R_m u(x_{(k,j)}, x) = m\int_0^1 (1-t)^{m-1}[\frac{f^{(m)}((1-t)x + tx_{(k,j)})}{m!}(x_{(k,j)} - x)^m]dt.$$

Since we assumed that $\{\phi_{(k,j)}(x) : (k,j) \in \Lambda\}$ have the polynomial reproducing property of order $m-1$, according to (8), we have

$$\begin{aligned} \mathcal{I}_h u(x) &= \sum_{(k,j)\in\Lambda} u(x_{(k,j)})\phi_{(k,j)}(x) \\ &= \sum_{(k,j)\in\Lambda} \Big(\sum_{\alpha=0}^{m-1} \frac{u^{(\alpha)}(x)}{\alpha!}(x_{(k,j)} - x)^\alpha + R_m u(x_{(k,j)}, x)\Big)\phi_{(k,j)}(x) \\ &= u(x) + \sum_{(k,j)\in\Lambda} R_m u(x_{(k,j)}, x)\phi_{(k,j)}(x) \end{aligned}$$

and hence

$$u(x) - \mathcal{I}_h u(x) = -\sum_{(k,j)\in\Lambda} R_m u(x_{(k,j)}, x)\phi_{(k,j)}(x). \tag{29}$$

(Step 1:) First, we estimate the interpolation error in the $L_2(\Omega)$-norm. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\|u - \mathcal{I}_h u\|^2_{L_2(\Omega)} = \int_\Omega (u(x) - \mathcal{I}_h u(x))^2 dx \\ &= \big(\frac{1}{(m-1)!}\big)^2 \int_\Omega \sum_{(k,j)\in\Lambda} \left[\int_0^1 (1-t)^{m-1} u^{(m)}((1-t)x + tx_{(k,j)})(x_{(k,j)} - x)^m \phi_{(k,j)}(x)dt\right]^2 dx \\ &\le \big(\frac{1}{(m-1)!}\big)^2 \sum_{(k,j)\in\Lambda} \int_\Omega \left[\int_0^1 (1-t)^{2(m-1)} \left\{u^{(m)}((1-t)x + tx_{(k,j)})(x_{(k,j)} - x)^m \phi_{(k,j)}(x)\right\}^2 dt\right] dx \\ &= \big(\frac{1}{(m-1)!}\big)^2 \int_0^1 (1-t)^{2(m-1)} F(x,t)dt, \end{aligned}$$

where

$$F(x,t) = \sum_{(k,j)\in\Lambda} \int_\Omega \left[u^{(m)}((1-t)x + tx_{(k,j)})(x_{(k,j)} - x)^m \phi_{(k,j)}(x)\right]^2 dx. \tag{30}$$

Since

$$\eta_k^j := \text{ supp } \phi_{(k,j)} \subset [x_{(k,j)} - \delta_2 hK, x_{(k,j)} + \delta_2 hK],$$

$|x_{(k,j)} - x| \le \delta_2 Kh$ for all $x \in \eta_k^j$. Thus, we have

$$|F(x,t)| \le (\delta_2 K)^{2m} M^2 h^{2m} \sum_{(k,j)\in\Lambda} \int_{\Omega\cap\eta_k^j} [u^{(m)}((1-t)x + tx_{(k,j)})]^2 dx,$$

where

$$M = \max\{|\phi_{(k,j)}(x)| : (k,j) \in \Lambda, x \in \Omega\}.$$

Let $y = (1-t)x + tx_{(k,j)} := L_k^j(x)$ be a affine transformation on $\mathbb{R}$ and let

$$\tilde{\eta}_k^j = L_k^j(\eta_k^j).$$

Then, according to the change of variable, we have

$$|F(x,t)| \le M^2(\delta_2 K)^{2m} h^{2m} \sum_{(k,j)\in\Lambda} \int_{\Omega\cap\tilde{\eta}_k^j} [u^{(m)}(y)]^2 (1-t)^{-1} dy. \tag{31}$$

If $y \in \tilde{\eta}_k^j$, then $y = (1-t)x + tx_{(k,j)}$ for some $x \in \eta_k^j$, and hence

$$|y - x_{(k,j)}| = |(1-t)(x - x_{(k,j)}| \le |x - x_{(k,j)}| \le \delta_2 Kh, \text{ since } 0 \le t \le 1.$$

Therefore, $\tilde{\eta}_k^j \subset \eta_k^j$ and hence (31) can be rewritten as follows:

$$\begin{aligned} |F(x,t)| &\le M^2(\delta_2 K)^{2m} h^{2m} (1-t)^{-1} \sum_{(k,j)\in\Lambda} \int_{\Omega\cap\eta_k^j} [u^{(m)}(y)]^2 dy \\ &\le qM^2(\delta_2 K)^{2m} h^{2m} (1-t)^{-1} \int_\Omega [u^{(m)}(y)]^2 dy \\ &= qM^2(\delta_2 K)^{2m} h^{2m} (1-t)^{-1} |u|_m^2, \end{aligned} \tag{32}$$

where $q = max\{card\{(k',j') : \eta_k^j \cap \eta_{k'}^{j'} \neq \emptyset : (k,j) \in \Lambda\}$. Thus, from (30) and (31), we have the following estimate:

$$\begin{aligned} \|u - \mathcal{I}_h u\|_{L_2(\Omega)}^2 &\le \frac{qM^2(\delta_2 K)^{2m} h^{2m} |u|_m^2}{[(m-1)!]^2} \int_0^1 (1-t)^{2(m-1)-1} dt \\ &\le \left[\frac{qM^2(\delta_2 K)^{2m}}{[(m-1)!]^2}\right] h^{2m} |u|_m^2, \text{ if } (2m-3) \ge 0. \end{aligned} \tag{33}$$

(Step 2:) Next, we estimate the derivative of the interpolation error in the $L_2(\Omega)$-norm. From the property of reproducing polynomial of order $m-1$:

$$\begin{aligned}\sum_{(k,j)\in\Lambda} \phi_{(k,j)}(x) &= 1,\\ \sum_{(k,j)\in\Lambda} (x-x_{(k,j)})^\alpha \phi_{(k,j)}(x) &= 0, \quad 1\le \alpha \le m-1,\end{aligned}$$

we have

$$\left\{\begin{aligned}\sum_{(k,j)\in\Lambda} \frac{d}{dx}\phi_{(k,j)}(x) &= 0,\\ \sum_{(k,j)\in\Lambda} (x-x_{(k,j)})\frac{d}{dx}\phi_{(k,j)}(x) &= -1,\\ \sum_{(k,j)\in\Lambda} (x-x_{(k,j)})^\alpha\frac{d}{dx}\phi_{(k,j)}(x) &= 0, \quad 1<\alpha\le m-1.\end{aligned}\right. \tag{34}$$

Since $K\delta_2 h < h_k$ for all k, we have the following relation:

$$\begin{aligned}\frac{d}{dx}\phi_{(k,j)}(x) &= \frac{d}{dx}\left[\hat{\phi}_j(\frac{x-x_k}{h_k})\psi_k^\delta(x)\right]\\ &= \frac{d}{dx}\left[\hat{\phi}_j(\frac{x-x_k}{h_k})\right]\psi_k^\delta(x) + \hat{\phi}_j(\frac{x-x_k}{h_k})\frac{d}{dx}\left[\psi_k^\delta(x)\right]\\ &= \frac{1}{h_k}\left\{\left[\frac{d}{d\xi}\hat{\phi}_j\right](\frac{x-x_k}{h_k})\psi_k^\delta(x)\right\} + \left\{\hat{\phi}_j(\frac{x-x_k}{h_k})\frac{d}{dx}\left[\psi_k^\delta(x)\right]\right\},\end{aligned}$$

which implies

$$|\frac{d}{dx}\phi_{(k,j)}(x)| \le \frac{C}{h}\left[\frac{d}{d\xi}\hat{\phi}_j(\frac{x-x_k}{h_k})\right] + \left\{\hat{\phi}_j(\frac{x-x_k}{h_k})\max|\beta_\delta|\right\}. \tag{35}$$

Now, applying the relation (34), we have

$$\begin{aligned}\frac{d}{dx}\mathcal{I}_h u(x) &= \sum_{(k,j)\in\Lambda} u(x_{(k,j)})\frac{d}{dx}\left[\phi_{(k,j)}(x)\right]\\ &= \sum_{(k,j)\in\Lambda}\Big(\sum_{\alpha=0}^{m-1}\frac{u^{(\alpha)}(x)}{\alpha!}(x_{(k,j)}-x)^\alpha + R_m u(x_{(k,j)},x)\Big)\frac{d}{dx}\left[\phi_{(k,j)}(x)\right]\\ &= \frac{d}{dx}\left[u(x)\right] + \sum_{(k,j)\in\Lambda} R_m u(x_{(k,j)},x)\frac{d}{dx}\left[\phi_{(k,j)}(x)\right],\end{aligned}$$

and hence,

$$\frac{d}{dx}\mathcal{I}_h u(x) - \frac{d}{dx}u(x) = \sum_{(k,j)\in\Lambda} R_m u(x_{(k,j)},x)\frac{d}{dx}[\phi_{(k,j)}(x)]. \tag{36}$$

Using Eqt. (35), $\max|\beta_\delta| = \mathcal{O}(\delta^{-1})$, and applying the same arguments as Step 1, we can get

$$\|\frac{d}{dx}\mathcal{I}_h u(x) - \frac{d}{dx}u(x)\|^2_{L_2(\Omega)} \leq h^{(2m)}\left\{C_1 h^{-2} + C_2\delta^{-2}\right\}|u|^2_m, \quad \text{if } (2m-3) \geq 0. \tag{37}$$

Finally, the desired interpolation error estimates follow from (33) and (37). □

Remark 4.1. *(1) The derivative of the interpolation error in $L_2(\Omega)$-norm depends on the size of δ determined by (13). If δ is getting smaller,* $\max|\beta_\delta|$ *becomes larger. Hence it is recommended to chose $\delta \in (0, H/3]$ that is not too small.*

(2) On the other hand, if δ/h is a large number and x_k is boundaries of patches, $(x_k-\delta, x_k+\delta)$ contains many particles at which the particle shape functions do not satisfy the Kronecker delta property. Moreover, $\phi_j(x)^{Ext}$ may take large values on $(x_k-\delta, x_k+\delta)$ In this case, in order for $\phi_j(x)^{Ext}\psi_k(x)$ to take small values, the damping effect of partition of unity function $\psi_k(x)$ can be increased by taking the degree of the conical window function as high as the degree of the extended polynomial $\phi_j(x)^{Ext}$.

Suppose $u \in H^{m+1}(\Omega)$, and $\Omega \subset \mathbb{R}$. Then $u \in \mathcal{C}^0(\Omega)$ and hence $\mathcal{I}u$ is defined. Thus, by using Lemma 4.1 and Céa's theorem, we have the following error estimate of Reproducing Polynomial Particle Method (RPPM) whose global basis functions are constructed in section 3:

Theorem 4.1. *Suppose $u \in H^{m+1}(\Omega), 2m-3 > 0$, and u^R is an ERR approximate solution of an elliptic boundary value problem on the domain $\Omega \subset \mathbb{R}$. Then, we have*

$$\|u - u^R\|_{1,\Omega} \leq Ch^l\left[h^{-1} + \delta^{-1}\right]|u|_{l,\Omega}.$$

where

$l \;=\; \min\{m, 2K\}$ *if the RPP shape functions have reproducing order* $2K-1$ *($\mathcal{C}^0$-functions),*

$l \;=\; \min\{m, 2K-1\}$ *if the RPP shape functions have reproducing order* $2K-2$ *($\mathcal{C}^r$-functions, $r > 0$).*

Here the constant C is independent of u and h.

Now, let us have some remarks on the relation between our piecewise polynomial particle shape functions and the spline functions.

1. The maximum norm estimate (1.3) of [7] is similar to the $L_2(\Omega)$-norm estimate (26) of Lemma 3.1.

2. Existence and uniqueness for the particle shape function $\phi_{([-K,K];0;2K-1)}(x)$ was proved in Theorem 3.1 of [24]. We noticed that Theorem 4.4.1 of [7] is virtually the same as our uniqueness theorem. However, the two proofs are different. Furthermore, our theorem did not use a sequence of double knots and the unique piecewise polynomial is specifically constructed from the Lagrange interpolation polynomials associated with the particles in $\mathrm{supp}\phi_{([-K,K];0;2K-1)}(x)$. Thus, the $\mathcal{C}^0$-spline functions associated with nodes $-K, \cdots K$, is the same as $\phi_{([-K,K];0;K)}$.

3. Let r be a positive integer. Then our $\mathcal{C}^r$-piecewise polynomial particle shape functions $\phi_{([-k,k];r;2K-2)}$ of reproducing order $2K-2$ could not be found in the literatures on the spline functions.

5 Numerical Examples

Let $\mathcal{U}(w) = \frac{1}{2}\mathcal{B}(w,w)$ be the strain energy of w, where $\mathcal{B}(\cdot,\cdot)$ denote the bilinear form of the related differential equation. Then the relative error in energy norm (%) is defined as

$$\|e\|_{E,r} \text{ in } \% = \left[\frac{|\mathcal{U}(u_{ex}) - \mathcal{U}(u_{app})|}{\mathcal{U}(u_{ex})}\right]^{1/2} \times 100. \tag{38}$$

In the following two examples, numerical solutions are obtained by applying the local approximation space defined as follows:

(1) The solution domains Ω of two examples are $(-1,1)$ and $(0,2)$, respectively. The reference patch $[0,n]$ is

$$\hat{\Omega} = [-2,2].$$

(2) The reference RPP shape functions of reproducing order 4 are the Lagrange interpolating global polynomials:

$$\begin{aligned}
\phi_{(-2)}(\xi) &= (\xi-2)(\xi-1)\xi(\xi+1)/24,\\
\phi_{(-1)}(\xi) &= -(\xi-2)(\xi-1)\xi(\xi+2)/6,\\
\phi_{(0)}(\xi) &= (\xi-2)(\xi-1)(\xi+2)/4,\\
\phi_{(1)}(\xi) &= -(\xi-2)(\xi+1)\xi(\xi+2)/6,\\
\phi_{(2)}(\xi) &= (\xi-1)(\xi+1)\xi(\xi+2)/24
\end{aligned}$$

(3) $\Omega = \cup_{k=1}^N \Omega_k$ is the subdivision of the domain Ω into uniform patches. The flat-top partition of unity shape functions subordinate to the covering $\{\Omega_k^\delta = (x_k-\delta, x_{k+1}+\delta) : k = 1,\cdots,N\}$ are defined by

$$\psi_\delta(x) = Aw(\frac{x}{\delta}) * \chi_{\Omega_k}(\text{convolution}), \tag{39}$$

where $w(x)$ is the conical window function with $w(x) = (1-|x|^2)^4$ for $|x| \le 1$, and

$$\delta = \text{diam}(\Omega_k)/3.$$

(4) The Finite Element approximation space is the vector space spanned by the following particle shape function of reproducing order 4:

$$\phi_{(k,j)}(x) := \psi_k^\delta(x)\cdot\phi_{(j)}(T_k^{-1}(x)), \quad j=-2,-1,0,1,2; \quad k=1,\cdots,N,$$

where $T_k : \hat{\Omega} \to \Omega_k$ is an affine mapping from the reference patch onto the spacial patch Ω_k.

For brevity, in this section, we use uniform patches and use the Lagrangian interpolators for the reference particle shape functions (NOTE: in this case, our reference particle shape functions are similar to the global partition polynomials of RKEM(Chapter 6 of Li-Liu[15])).

In these particular examples, we have the following features:

1. The mesh size h is $diam(\Omega_k)$, because the support of these particular reference particle shape functions are the diameter of uniform patch (NOTE: in the proof of Lemma 4.1, $h = K \times (diam(\Omega_k)/n$, where $(n+1)$ is the number of particles in the reference patch, $2K$ is the maximum of the diameters of supports of the reference particle shape functions). In other words, to get a smaller h, $diam(\Omega_k)$ must be small when the reference particle shape functions are global polynomials. However, in our proposed method, reference particle shape functions are piecewise polynomials with small support. Hence, a smaller h is obtained by taking a large number n, while the patch sizes and δ are fixed.

2. Thus, we have to choose δ with $0 << \delta < h/2$. However, if δ is too small, the condition number of the related stiffness matrix becomes large. In Fig. 3, the number of uniform patches are $5, 10, 20, 40$ and hence $h = 0.4, 0.2, 0.1, 0.05$, respectively. For these cases, the corresponding δ are selected as $0.1, 0.08, 0.05, 0.02$, respectively.

On the other hand, our proposed method has the following features:

- The constant δ for the flat-top partition of unity shape function is fixed and not too small (one third of the diameter of the smallest patch). One can control patch sizes so that the smallest patch is not too small. That is, δ is independent of h.

- The mesh size $h_k = diam(\Omega_k)/n$ on the patch Ω_k can be small as n goes to ∞, where $(n+1)$ is the number of particles in the reference patch $[0, n]$ and the associated reference particle shape functions are constructed in the appendix for uniformly spaced particles.

[**Second Order Equation-I:**] $u(x) = e^x(1-x^2)^6$ solves the following model problem

$$-\frac{d^2}{dx^2}u(x) = f(x) \text{ in } (-1,1), \tag{40}$$

$$u(\pm 1) = 0, \tag{41}$$

where $f(x) = -e^x((1-x^2)^6 - 24x(1-x^2)^5 - 12(1-x^2)^5 + 120x^2(1-x^2)^4)$. Then the strain energy is 1.8568504251271325.

Since the $u(x)$ is smooth and particle shape functions have the reproducing property of order 4, by theorem 4.1, if the error in $H^1(\Omega)$-norm versus the mesh size is plotted in log-log scale, the slope of the convergence line is better than or equal to 3. Fig. 3 shows that the relative error makes a line of slope ≈ 3.8.

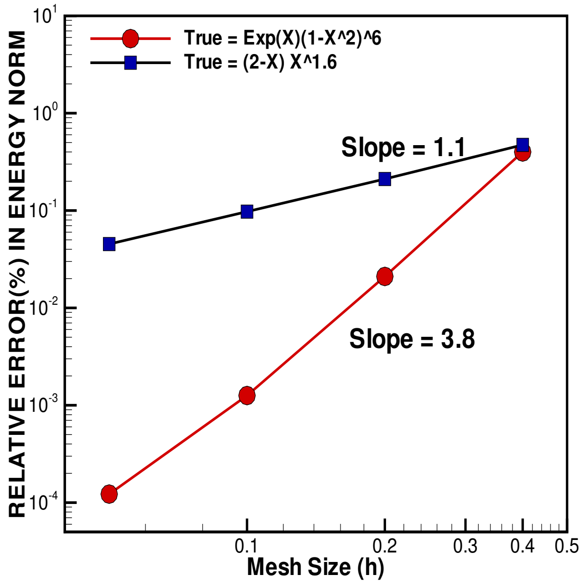


Figure 4: (Relative Errors) ×100 are plotted in log-log scale which support the theory for convergence of the proposed method.

[**Second Order Equation-II:**] $u(x) = (2-x)x^{1.6}$ solves the following model problem

$$
\begin{aligned}
-\frac{d^2}{dx^2}u(x) &= f(x) \text{ in } (0,2), && (42)\\
u(0) &= u(2) = 0, && (43)
\end{aligned}
$$

where $f(x) = -3.2x^{0.6} + 0.96(2-x)/(x^{0.4})$. Then the strain energy is 1.59127044848075.

In the second example, the true solution is in $H^2(\Omega)$. Thus, by theorem 4.1, if the error in $H^1(\Omega)$-norm versus the mesh size is plotted in log-log scale, the slope of the line of convergence is better than or equal to 1. Fig. 3 shows that the relative error makes a line of slope ≈ 1.1.

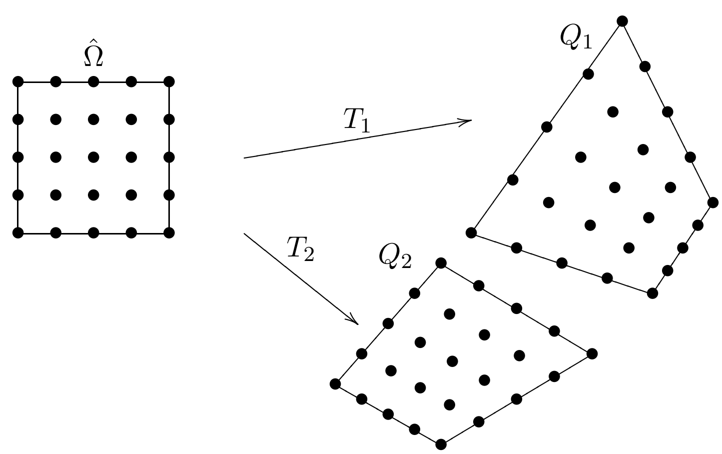
6 Remarks for Higher Dimensional Extension

The arguments and results presented for one dimensional case can be smoothly extended to higher dimensional cases without involving much complexity. In other words, the given polygonal domain $\Omega \subset \mathbb{R}^d$ is subdivided into non-overlapping rectangles (or hexahedrons), just as a subdivision for the conventional finite element method. The tensor product of the one dimensional reference domain, along with the tensor product of one dimensional particle shape functions, can be used for the reference domain and the reference piecewise polynomial particle shape functions, respectively.

The essential components of two dimensional case for our proposed method are as follows:

1. Suppose n is a fixed positive integer and the reference particle shape functions have the polynomial reproducing property of order m:
$$\sum_{j=0}^{n} \xi_j^{\alpha}\phi_j(\xi) = \xi^{\alpha}, \text{ for } 0 \le \alpha \le m.$$
Then the reference domain is
$$\hat{\Omega}_{(q)} = [0, n] \times [0, n]$$
and the reference particle shape functions have the extended polynomial reproducing property as follows:
$$\sum_{(j_1,j_2)\in\Lambda} \xi_{j_1}^{\alpha_1}\eta_{j_2}^{\alpha_2}\phi_{j_1}(\xi)\phi_{j_2}(\eta) = \xi^{\alpha_1}\eta^{\alpha_2}, \text{ for } 0 \le \alpha_1 \le m, 0 \le \alpha_2 \le m. \tag{44}$$
Note that the tensor product of RPP shape functions has serendipity reproducing orders, compared with the definition (4).

2. Let Q be a rectangle whose four vertices are
$$(x_i, y_i), i = 1, 2, 3, 4.$$
Then a bijective mapping $T : \hat{\Omega}_{(q)} \longrightarrow Q$ is defined by
$$(x, y) = T(\xi, \eta) = (T_x(\xi, \eta), T_y(\xi, \eta)),$$
where
$$\begin{aligned} x &= \frac{x_1}{n^2}(n-\xi)(n-\eta) + \frac{x_2}{n^2}(\xi)(n-\eta) + \frac{x_3}{n^2}(\xi)(\eta) + \frac{x_4}{n^2}(n-\xi)(\eta), \\ y &= \frac{y_1}{n^2}(n-\xi)(n-\eta) + \frac{y_2}{n^2}(\xi)(n-\eta) + \frac{y_3}{n^2}(\xi)(\eta) + \frac{y_4}{n^2}(n-\xi)(\eta). \end{aligned}$$



Let

$$\phi^*_{(j_1,j_2)}(x,y) = \phi_{(j_1,j_2)}(T^{-1}(x,y)),$$

where

$$\phi_{(j_1,j_2)}(\xi,\eta) = \phi_{j_1}(\xi) \times \phi_{j_2}(\eta).$$

Then the transformed particle shape functions have the polynomial reproducing property as follows:

Lemma 6.1. *Suppose the reproducing property (44) holds for*

$$0 \leq (\alpha_1 + \alpha_2) \leq m.$$

Then the transformed particle shape functions have the following reproducing order

$$\sum_{(j_1,j_2)\in\Lambda} x_{j_1}^{\beta_1} y_{j_2}^{\beta_2} \phi^*_{(j_1,j_2)}(x,y) = x^{\beta_1} y^{\beta_2},\ \textit{for}\ 0 \leq (\beta_1+\beta_2) \leq m/2.$$

Proof. Suppose $(\beta_1+\beta_2) \leq m/2$, then we have

$$\begin{aligned}
\sum_{(j_1,j_2)} x_{j_1}^{\beta_1} y_{j_2}^{\beta_2} \phi^*_{(j_1,j_2)}(x,y) &= \sum_{(j_1,j_2)} [T_x(\xi_{j_1},\eta_{j_2})]^{\beta_1} [T_y(\xi_{j_1},\eta_{j_2})]^{\beta_2} [\phi_{(j_1,j_2)} \circ T^{-1}](x,y) \\
&= \sum_{(j_1,j_2)} [T_x(\xi_{j_1},\eta_{j_2})]^{\beta_1} [T_y(\xi_{j_1},\eta_{j_2})]^{\beta_2} \phi_{j_1}(\xi)\phi_{j_2}(\eta) \\
&= \sum_{(j_1,j_2)} \left[\sum_{0\leq s,t\leq(\beta_1+\beta_2)} C_{st} \cdot (\xi_{j_1})^s (\eta_{j_2})^t \phi_{j_1}(\xi)\phi_{j_2}(\eta) \right] \text{(by expansion)} \\
&= \sum_{0\leq s,t\leq(\beta_1+\beta_2)} C_{st} \left[\sum_{(j_1,j_2)} (\xi_{j_1})^s (\eta_{j_2})^t \phi_{j_1}(\xi)\phi_{j_2}(\eta) \right] \\
&= \sum_{0\leq s,t\leq(\beta_1+\beta_2)} C_{st} \cdot (\xi)^s (\eta)^t (\text{ since } (s+t) \leq 2(\beta_1+\beta_2) \leq m) \\
&= T_x(\xi,\eta)^{\beta_1} T_y(\xi,\eta)^{\beta_2} (\text{ by factoring}) \\
&= x^{\beta_1} y^{\beta_2}.
\end{aligned}$$

□

3. Let $\hat{\Omega}_{(t)}$ be a right-angle triangle whose vertices are $(0,0), (3,0), (0,3)$ and consider 10 particles in $\hat{\Omega}_{(t)}$ as follows:

$(0,3)$			
$(0,2)$	$(1,2)$		
$(0,1)$	$(1,1)$	$(2,1)$	
$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$

Suppose the Lagrange interpolating polynomials associated with these ten particles are the reference particle shape functions. Then the reference particle shape functions are global polynomials, and satisfies the Kronecker delta property and the reproducing property (44) of order 3. However, this is not the best choice for the reference particle shape function for the triangular reference patch $\hat{\Omega}_{(t)}$. The piecewise polynomial reference particle shape functions with much smaller support are desirable.

4. The convolution of the characteristic function χ_{Q_k} of a patch set Q_k and the conical window function $A(1-(\frac{x}{\delta})^2)^l(1-(\frac{x}{\delta})^2)^l$ is piecewise polynomial, even though it becomes more complex than the one dimensional case. Indeed, the partition of unity shape functions

$$\chi_{Q_k} * [A(1-(\frac{x}{\delta})^2)^l(1-(\frac{y}{\delta})^2)^l]$$

are the double integrals of the polynomial $A(1-(\frac{x}{\delta})^2)^l(1-(\frac{y}{\delta})^2)^l$ over the small regions $[x-\delta, x+\delta] \times [y-\delta, y+\delta] \cap Q_k$ which are either a triangle or a rectangle. Here, as before,

$$A^{-1} = \int_{-\delta}^{\delta}(1-(\frac{x}{\delta})^2)^l dx \int_{-\delta}^{\delta}(1-(\frac{y}{\delta})^2)^l dy,$$

and l is an integer with $2 \leq l < \infty$.

We elaborate these arguments in the forthcoming papers.

Appendix

In this appendix, we construct the reference particle shape functions for patch-wise equally spaced particles in the following cases:

- $\mathcal{C}^0$ particle shape functions with reproducing property of order 3, corresponding to the particles in $[0, n] (n \geq 8)$.
- $\mathcal{C}^1$ particle shape functions with reproducing property of order 2, corresponding to the particles in $[0, n] (n \geq 8)$, which are shown in section 3.
- $\mathcal{C}^0$ particle shape functions with reproducing property of order 5, corresponding to the particles in $[0, n] (n \geq 12)$.
- $\mathcal{C}^1$ particle shape functions with reproducing property of order 4, corresponding to the particles in $[0, n] (n \geq 12)$.

A Piecewise polynomial $\mathcal{C}^0$ particle shape functions of reproducing order 3 whose supports are subsets of $[0, n]$

It is shown in [24] that the basic $\mathcal{C}^0$-particle shape function with support $[-2, 2]$ and reproducing property of order 3 for uniformly distributed particles is uniquely determined as follows:

$$\phi_{([-2,2];0;3)}(x) = \begin{cases} \frac{1}{6}(x+1)(x+2)(x+3) & x \in [-2,-1] \\ -\frac{1}{2}(x-1)(x+1)(x+2) & x \in [-1,0] \\ \frac{1}{2}(x-2)(x-1)(x+1) & x \in [0,1] \\ -\frac{1}{6}(x-3)(x-2)(x-1) & x \in [1,2] \end{cases}$$

As before, since $\phi_{(x_0)}|[0,2], \phi_{(x_1)}|[0,2], \phi_{(x_2)}|[0,2]$, and $\phi_{(x_3)}|[0,2]$ are polynomials, that part of these piecewise polynomial particle shape functions can be globally extended. Note that n must be greater than or equal to 8.

1. The extended particle shape function corresponding to the particle $x_0 = 0$ is the following:

$$\phi_{(x_0)}(x) = \begin{cases} -\frac{1}{6}(x-3)(x-2)(x-1) & x \in [0,2] \cup (-\infty, 0] \\ 0 & x \notin (-\infty, 2] \end{cases}$$

2. The extended particle shape function corresponding to the particle $x_1 = 1$ is the following:

$$\phi_{(x_1)}(x) = \begin{cases} \frac{1}{2}(x-3)(x-2)x & x \in [0,2] \cup (-\infty, 0] \\ \phi_{([-2,2];0;3)}(x-1) & x \in [2,3] \\ 0 & x \notin (-\infty, 3] \end{cases}$$

3. The extended particle shape function corresponding to the particle $x_2 = 2$ is the following:

$$\phi_{(x_2)}(x) = \begin{cases} -\frac{1}{2}(x-3)(x-1)x & x \in [0,2] \cup (-\infty, 0] \\ \phi_{([-2,2];0;3)}(x-2) & x \in [2,4] \\ 0 & x \notin (-\infty, 4] \end{cases}$$

4. The extended particle shape function corresponding to the particle $x_3 = 3$ is the following:

$$\phi_{(x_3)}(x) = \begin{cases} \frac{1}{6}(x-2)(x-1)x & x \in [0,2] \cup (-\infty, 0] \\ \phi_{([-2,2];0;3)}(x-3) & x \in [2,5] \\ 0 & x \notin (-\infty, 5] \end{cases}$$

5. For particle x_j $(j = 4, 5, \cdots, n-4)$

$$\phi_{(x_j)}(x) = \phi_{([-2,2];0;3)}(x-j)$$

6. For particle x_j $(j = n-3, n-2, n-1, n)$

$$\phi_{(x_j)}(x) = \phi_{(x_{n-j})}(-(x-n))$$

B $\mathcal{C}^0$ particle shape functions of reproducing order 5 whose supports are subsets of $[0, n]$

The basic $\mathcal{C}^0$-particle shape function with support $[-3, 3]$ and reproducing property of order 5 for uniformly distributed particles is uniquely determined as follows ([24]):

$$\phi_{([-3,3];0;5)}(x) = \begin{cases} \frac{1}{120}(x+1)(x+2)(x+3)(x+4)(x+5) & x \in [-3,-2], \\ -\frac{1}{24}(x-1)(x+1)(x+2)(x+3)(x+4) & x \in [-2,-1], \\ \frac{1}{12}(x-2)(x-1)(x+1)(x+2)(x+3) & x \in [-1,0], \\ -\frac{1}{12}(x-3)(x-2)(x-1)(x+1)(x+2) & x \in [0,1], \\ \frac{1}{24}(x-4)(x-3)(x-2)(x-1)(x+1) & x \in [1,2], \\ -\frac{1}{120}(x-5)(x-4)(x-3)(x-2)(x-1) & x \in [2,3]. \end{cases}$$

Since $\phi_{(x_0)}|_{[0,3]}, \phi_{(x_1)}|_{[0,3]}, \phi_{(x_2)}|_{[0,3]}, \phi_{(x_3)}|_{[0,3]}, \phi_{(x_4)}|_{[0,3]}$, and $\phi_{(x_5)}|_{[0,3]}$ are polynomials, that part of these piecewise polynomial particle shape functions can be globally extended. Note that n must be greater than or equal to 12.

1. The extended particle shape function corresponding to the particle $x_0 = 0$ is the following:

$$\phi_{(x_0)}(x) = \begin{cases} -\frac{1}{120}(x-5)(x-4)(x-3)(x-2)(x-1) & x \in [0,3] \cup (-\infty, 0] \\ 0 & x \notin [-\delta, 3] \end{cases}$$

2. The extended particle shape function corresponding to the particle $x_1 = 1$ is the following:

$$\phi_{(x_1)}(x) = \begin{cases} \frac{1}{24}(x-5)(x-4)(x-3)(x-2)x & x \in [0,3] \cup (-\infty, 0] \\ \phi_{([-3,3];0;5)}(x-1) & x \in [3,4] \\ 0 & x \notin (-\infty, 4] \end{cases}$$

3. The extended particle shape function corresponding to the particle $x_2 = 2$ is the following:

$$\phi_{(x_2)}(x) = \begin{cases} -\frac{1}{12}(x-5)(x-4)(x-3)(x-1)x & x \in [0,3] \cup (-\infty, 0] \\ \phi_{([-3,3];0;5)}(x-2) & x \in [3,5] \\ 0 & x \notin (-\infty, 5] \end{cases}$$

4. The extended particle shape function corresponding to the particle $x_3 = 3$ is the following:

$$\phi_{(x_3)}(x) = \begin{cases} \frac{1}{12}(x-5)(x-4)(x-2)(x-1)x & x \in [0,3] \cup (-\infty, 0] \\ \phi_{([-3,3];0;5)}(x-3) & x \in [3,6] \\ 0 & x \notin (-\infty, 6] \end{cases}$$

5. The extended particle shape function corresponding to the particle $x_4 = 4$ is the following:

$$\phi_{(x_4)}(x) = \begin{cases} -\frac{1}{24}(x-5)(x-3)(x-2)(x-1)x & x \in [0,3] \cup (-\infty, 0] \\ \phi_{([-3,3];0;5)}(x-4) & x \in [3,7] \\ 0 & x \notin (-\infty, 7] \end{cases}$$

6. The extended particle shape function corresponding to the particle $x_5 = 5$ is the following:

$$\phi_{(x_5)}(x) = \begin{cases} \frac{1}{120}(x-4)(x-3)(x-2)(x-1)x & x \in [0,3] \cup (-\infty, 0] \\ \phi_{([-3,3];0;5)}(x-5) & x \in [3,8] \\ 0 & x \notin (-\infty, 8] \end{cases}$$

7. For particle x_j $(j = 6, 7, \cdots, n-6)$

$$\phi_{(x_j)}(x) = \phi_{([-3,3];0;5)}(x-j)$$

8. For particle x_j $(j = n-5, n-4, n-3, n-2, n-1, n)$

$$\phi_{(x_j)}(x) = \phi_{(x_{n-j})}(-(x-n))$$

C $\mathcal{C}^1$ particle shape functions of reproducing order 4 whose supports are subsets of $[0, n]$

The basic $\mathcal{C}^1$-particle shape function with support $[-3, 3]$ and reproducing property of order 4 for uniformly distributed particles is uniquely determined as follows ([24]):

$$\phi_{([-3,3];1;4)}(x) = \begin{cases} \frac{1}{120}x(x+2)(x+3)^2(x+7) & x \in [-3,-2] \\ -\frac{1}{24}(x+1)(x+2)(x^3+6x^2-3x-24) & x \in [-2,-1] \\ \frac{1}{12}(x+1)(x^4+2x^3-15x^2-12x+12) & x \in [-1,0] \\ -\frac{1}{12}(x-1)(x^4-2x^3-15x^2+12x+12) & x \in [0,1] \\ \frac{1}{24}(x-2)(x-1)(x^3-6x^2-3x+24) & x \in [1,2] \\ -\frac{1}{120}(x-7)(x-3)^2(x-2)x & x \in [2,3] \end{cases}$$

As mentioned above, since $\phi_{(x_0)}|_{[0,3]}, \phi_{(x_1)}|_{[0,3]}, \phi_{(x_2)}|_{[0,3]}, \phi_{(x_3)}|_{[0,3]}, \phi_{(x_4)}|_{[0,3]}$, and $\phi_{(x_5)}|_{[0,3]}$ are polynomials, that part of these piecewise polynomial particle shape functions can be globally extended. Note that n should be greater than or equal to 12.

1. The extended particle shape function corresponding to the particle $x_0 = 0$ is the following:

$$\phi_{(x_0)}(x) = \begin{cases} \frac{1}{5400}(x-5)(x-4)(x-3)^2(x-2)(x-1)(8x+15) & x \in [0,3] \cup (-\infty, 0] \\ 0 & x \notin (-\infty, 3] \end{cases}$$

2. The extended particle shape function corresponding to the particle $x_1 = 1$ is the following:

$$\phi_{(x_1)}(x) = \begin{cases} -\frac{1}{1080}(x-5)(x-4)(x-3)(x-2)x(8x^2-17x-36) & x \in [0,3] \cup (-\infty, 0] \\ \phi_{([-3,3];1;4)}(x-1) & x \in [3,4] \\ 0 & x \notin (-\infty, 4] \end{cases}$$

3. The extended particle shape function corresponding to the particle $x_2 = 2$ is the following:

$$\phi_{(x_2)}(x) = \begin{cases} \frac{1}{540}(x-5)(x-4)(x-3)(x-1)x(8x^2-25x-27) & x \in [0,3] \cup (-\infty, 0] \\ \phi_{([-3,3];1;4)}(x-2) & x \in [3,5] \\ 0 & x \notin (-\infty, 5] \end{cases}$$

4. The extended particle shape function corresponding to the particle $x_3 = 3$ is the following:

$$\phi_{(x_3)}(x) = \begin{cases} -\frac{1}{540}(x-5)(x-4)(x-2)(x-1)x(8x^2-33x-18) & x \in [0,3] \cup (-\infty, 0] \\ \phi_{([-3,3];1;4)}(x-3) & x \in [3,6] \\ 0 & x \notin (-\infty, 6] \end{cases}$$

5. The extended particle shape function corresponding to the particle $x_4 = 4$ is the following:

$$\phi_{(x_4)}(x) = \begin{cases} \frac{1}{1080}(x-5)(x-3)(x-2)(x-1)x(8x^2-41x-9) & x \in [0,3] \cup (-\infty, 0] \\ \phi_{([-3,3];1;4)}(x-4) & x \in [3,7] \\ 0 & x \notin (-\infty, 7] \end{cases}$$

6. The extended particle shape function corresponding to the particle $x_5 = 5$ is the following:

$$\phi_{(x_5)}(x) = \begin{cases} -\frac{1}{5400}(x-4)(x-3)(x-2)(x-1)x^2(8x-49) & x \in [0,3] \cup (-\infty, 0] \\ \phi_{([-3,3];1;4)}(x-5) & x \in [3,8] \\ 0 & x \notin (-\infty, 8] \end{cases}$$

7. For particle x_j $(j = 6, 7, \cdots, n-6)$

$$\phi_{(x_j)}(x) = \phi_{([-3,3];1;4)}(x-j)$$

8. For particle x_j $(j = n-5, n-4, n-3, n-2, n-1, n)$

$$\phi_{(x_j)}(x) = \phi_{(x_{n-j})}(-(x-n))$$

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