TCHEBYSHEV TRIANGULATIONS OF STABLE SIMPLICIAL COMPLEXES

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ABSTRACT. We generalize the notion of the Tchebyshev transform of a graded poset to a triangulation of an arbitrary simplicial complex in such a way that, at the level of the associated F-polynomials $\sum_j f_{j-1}((x-1)/2)^j$, the triangulation induces taking the Tchebyshev transform of the first kind. We also present a related multiset of simplicial complexes whose association induces taking the Tchebyshev transform of the second kind. Using the reverse implication of a theorem by Schelin we observe that the Tchebyshev transforms of Schur stable polynomials with real coefficients have interlaced real roots in the interval $(-1,1)$, and present ways to construct simplicial complexes with Schur stable F-polynomials. We show that the order complex of a Boolean algebra is Schur stable. Using and expanding the recently discovered relation between the derivative polynomials for tangent and secant and the Tchebyshev polynomials we prove that the roots of the corresponding pairs of derivative polynomials are all pure imaginary, of modulus at most one, and interlaced.

INTRODUCTION

As Michael Hoffman noted in his paper on derivative polynomials [8]: "Sometimes problems naturally occur in pairs, and it is best to tackle both at the same time." He then studied the derivative polynomials for tangent and secant, and some generalizations thereof, faithfully to this principle. Another pair of polynomials that naturally occurs in pairs is the pair of Tchebyshev polynomials of the first and second kind. They may be thought of as the pair of bases naturally arising when we substitute a unit complex number into a polynomial and take the real and imaginary part.

In this paper we show that there is a connection between these pairs of polynomials, and use the connection to prove that the zeros of the derivative polynomials for tangent and secant are not only all pure imaginary and have multiplicity one, but they are also interlaced the same way the zeroes of the n-th Tchebyshev polynomial of the first kind and the zeros of the $(n-1)$-th Tchebyshev polynomial of the second kind are interlaced. The statement in itself is perhaps less interesting than its proof, using a mix of combinatorics and complex analysis, that has many actual and potential further uses.

After the Preliminaries, in Sections 2, 3 and 4 we introduce the Tchebyshev triangulation of a simplicial complex and a collection of subcomplexes in it that we associate to the Tchebyshev polynomials of the second kind. These new complexes, associated to the original have the property that their face enumerating polynomial $F(x) := \sum_{j\geq 0} f_{j-1}((x-1)/2)^j$ arises from the F-polynomial of the original complex by the linear map that sends the monomial x^n into the Tchebyshev polynomials $T_n(x)$ (first kind) or $U_{n-1}(x)$ (second kind) respectively. This construction generalizes the Tchebyshev transform for graded posets introduced and studied by the present author [5], [7], and by Ehrenborg and Readdy [2]. The connection to this earlier theory is explained along the way and in Section 5.

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In Section 6 we introduce a new notion: the stability of a simplicial complex. In control theory a polynomial is called Schur stable resp. Hurwitz stable if all its zeros are inside the unit disk $|x| < 1$ resp. in the open left t-plane. There is a Möbius transform connecting the two notions without establishing strict equivalence. It is remarkable that this transform connects the Hurwitz stability h-polynomial of a simplicial complex (widely studied in the literature) with the Schur stability of the F-polynomial $\sum_{j\geq 0} f_{j-1}((x-1)/2)^j$ rarely used by anyone besides the present author. (One notable earlier application of this invariant is in [4] where it helped explain a connection between central Delannoy numbers and Jacobi polynomials, discovered decades ago, but branded as a "coincidence".) We focus on Schur-stability because of a result of Schelin [13] providing a Cauchy-index formula for the number of zeros of a polynomial with real coefficients within the unit disk $|x| < 1$, in terms of the Tchebyshev transforms of a polynomial. Schelin's theorem is usually applied to verify Schur stability, here we observe that it may also be used the opposite way: if we know that a polynomial with real coefficients is Schur stable then, using Schelin's theorem we can show that the zeros of its Tchebyshev transforms are real, inside $(-1, 1)$, and interlaced. The question naturally arises: are there simplicial complexes and graded posets whose F-polynomial is Schur stable by some geometric or combinatorial reason? It is easy to observe that the class of stable simplicial complexes is closed under taking the join, and we suspect that the class of graded posets whose order complex is stable is closed under taking the direct product. We are only able to prove this in the case when at least one of the two stable posets is the graded poset of rank 1, but this already suffices to conclude that all Boolean algebras are stable. The proof involves combinatorial enumeration and an application of Lucas' theorem stating that the zeros of the derivative of a polynomial belong to the convex hull of the zeros of the same polynomial.

In Section 8 we show that the derivative polynomials for hyperbolic tangent and secant are closely related to the F-polynomials of the Tchebyshev transforms of the Boolean algebras. (Switching to hyperbolic tangent and secant from the classical setup results only in rotating all zeros involved around the origin by 90^o which thus become all real, in closer analogy to the Tchebyshev polynomials.) The connection between the Boolean analogues of the Tchebyshev polynomials of the first kind and the derivative polynomials for secant was already observed in [5]. Here we complete the picture by establishing the connection between derivative polynomials for (hyperbolic) tangent and the Boolean analogues of the Tchebyshev polynomials of the second kind. As a consequence of the relations established we may use the stability of the Boolean algebra and the reverse implication of Schelin's theorem to prove the interlacing property for the zeros of the derivative polynomials for tangent and secant.

The concluding Section 9 contains a few observations that may be useful in proving our conjecture that taking the direct product preserves the Schur stability of graded posets.

1. Preliminaries

1.1. **Stable polynomials, Tchebyshev transforms, and the Cauchy index.** In control theory, a polynomial is called *Hurwitz stable*, if all its roots have negative real part. It is called *Schur stable* if all its roots lie inside the unit disk $|x| < 1$. The two conditions may *almost* be reduced to each other by mapping the interior of the unit circle $|x| < 1$ upon the open left t-plane by means of the fractional linear transformation $x \mapsto t = (x-1)/(x+1)$ whose inverse is $t \mapsto x = (1+t)/(1-t)$. The number of zeros of the polynomial $F(x)$ of degree n inside the unit disk $|x| < 1$ is the same as the number of zeros of the transformed polynomial

$$h(t) := (1-t)^n F\left((1+t)/(1-t)\right) \tag{1}$$

in the open left t-plane. See, e.g., Marden [12, Chapter 10]. (The Schur stability of $F(x)$ is not equivalent to the Hurwitz stability of $h(t)$, since their degrees might differ. See more on this issue in Section 6.) As Marden notes, this observation may be used to count the zeros of $F(x)$ inside the unit disk $|x| < 1$ by counting the zeros of $h(t)$ in the left t-plane, which is a widely discussed issue, but he also proposes a direct approach to counting the zeros of $F(x)$ inside the unit disk $|x| < 1$.

In the case when $F(x)$ has real coefficients, a result of Schelin [13] may also be used instead. We phrase Schelin's theorem in terms of the *Tchebyshev transforms* of a polynomial.

Definition 1.1. *Given a polynomial $F(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$, the Tchebyshev transforms $T(F)(x)$ and $U(F)(x)$ of the first and second kind of $F(x)$ are given by*

$$T(F)(x) = a_n T_n(x) + \cdots + a_1 T_1(x) + a_0 \quad \textit{and} \quad U(F)(x) = a_n U_{n-1}(x) + \cdots + a_1 U_0(x).$$

Here, for all $m \geq 0$, $T_m(x)$ is the Tchebyshev polynomial of the first kind, determined by $T_m(\cos(\alpha)) = \cos(m \cdot \alpha)$ and $U_m(x)$ is the Tchebyshev polynomial of the second kind, determined by $U_m(\cos(\alpha)) = \sin((m+1)\cdot\alpha)/\sin(\alpha)$.

It follows from the definition immediately that the real part of $F(\exp(\mathbf{i}\cdot\alpha))$ is $T(F)(\cos(\alpha))$ and the imaginary part of $F(\exp(\mathbf{i}\cdot\alpha))$ is $U(\cos(\alpha))\cdot\sin(\alpha)$. (Throughout this paper, $\mathbf{i}$ stands for the complex square root of -1.) According to Schelin [13], the number of zeros within the unit disk of $F(x) \in \mathbb{R}[x]$ may be expressed using a *Cauchy index* associated to $U(F)/T(F)$.

Definition 1.2. *Suppose g is a real valued function of a real valuable whose only singularities are poles. The* Cauchy index *$I_g(\xi)$ of g at the pole ξ is*

$$I_g(\xi) = \frac{1}{2} \lim_{\varepsilon \to 0^+} (\operatorname{sign}(g(\xi+\varepsilon)) - \operatorname{sign}(g(\xi-\varepsilon))).$$

The Cauchy index $I_\alpha^\beta(g)$ of g over an interval (α, β) whose endpoints are not poles is the sum of the Cauchy indices $I_g(\xi)$ taken over all poles $\xi \in (\alpha, \beta)$.

If $g(x)$ is a rational function $p(x)/q(x)$, where $p, q \in \mathbb{R}[x]$ have no common zeros, then the poles of g are the zeros of q. If ξ is a zero of q of multiplicity m then $I_g(\xi) = 0$ if m is even, and it is equal to the sign of $p(\xi)q^{(m)}(\xi)$ if m is odd. Schelin's theorem [13, Theorem 2.3] is the following.

Theorem 1.3 (Schelin)**.** *If $p(x) \in \mathbb{R}[x]$ has no zeros on the unit circle then the number of zeros of p inside the unit circle, including multiplicities, is $I_{-1}^1 U(p)/T(p)$.*

Remark 1.4. It is important to assume that $F(x)$ has no zeros on the unit circle. Consider for example $F(x) = x^2 + 1$. This has two zeros on the unit circle, and none inside. However, since $T(F)(x) = 2x^2$ and $U(F)(x) = 2x$, we have $I_{-1}^1 U(p)/T(p) = 1$.

1.2. **Simplicial complexes.** We use the terminology of Stanley [14]. An abstract simplicial complex $\triangle$ is a family of subsets of a finite vertex set V, closed under inclusion, containing all singletons. The elements of $\triangle$ are *faces*, the dimension of a face $\sigma \in \triangle$ is $|\sigma| - 1$. The number of j-dimensional faces is denoted by f_j and the vector $(f_{-1}, f_0, \ldots, f_{d-1})$ is the *f-vector* of $\triangle$. Here $d-1$ is the dimension of $\triangle$, i.e., d is the maximum cardinality of its faces. An equivalent way to encode the f-vector is by the h-vector $(h_0, \ldots, h_d)$ which may be defined by $\sum_{i=0}^d h_i t^i = \sum_{j=0}^d f_{j-1} t^j (1-t)^{d-j}$. Here $h(t) := \sum_{i=0}^d h_i t^i$ is the *h-polynomial* of $\triangle$. Sometimes we will use the notation $h[\triangle](t)$. The *reduced Euler-characteristic* of a simplicial complex is $\tilde{\chi}(\triangle)=\sum_{j=0}^{d}(-1)^{j-1}f_{j-1}=(-1)^{d-1}h_d$. A simplicial complex is *Eulerian* if the reduced characteristic of each of its links is (-1) raised to the dimension of the link. Here the link $\operatorname{lk}(\sigma,\triangle)$ of a face $\sigma\in\triangle$ is the simplicial complex $\{\tau\in\triangle\ :\ \tau\cup\sigma\in\triangle,\tau\cap\sigma=\emptyset\}$. A special type of simplicial complexes that is important in this paper is the *order complex* $\triangle(P)$ of some partially ordered set P whose vertex set is P, and whose faces are the increasing chains in P. An operation we will use is the *join* $\triangle_1 * \triangle_2$ of two simplicial complexes $\triangle_1$ and $\triangle_2$: it is the simplicial complex $\triangle_1 * \triangle_2=\{\sigma\cup\tau\ :\ \sigma\in\triangle_1,\tau\in\triangle_2\}$. A special instance of the join operation is the *suspension* operation: the suspension $\triangle * \partial(\triangle^1)$ of a simplicial complex $\triangle$ is the join of $\triangle$ with the boundary complex of the one dimensional simplex. (A $(d-1)$-dimensional simplex is the family of all subsets of a d-element set, its boundary is obtained by removing its only facet from the list of faces.)

1.3. **The Tchebyshev transform of a poset.** The idea of a Tchebyshev transform was introduced by the present author [7]. Given any poset Q, we may define another poset $T(Q)$ whose elements are the open intervals $(x,y)\subset Q$, and the partial order is given by $(x_1,y_1)\leq(x_2,y_2)$ if either $x_1=x_2$ and $y_1<y_2$, or $y_2\leq x_1$ holds. Using the operation we may define the Tchebyshev transform of a graded poset (this notation and terminology was introduced by Ehrenborg and Readdy [2]), as follows. Given a poset P with minimum element $\widehat{0}$ and maximum element $\widehat{1}$, we introduce a new minimum element $\widehat{-1}<\widehat{0}$ and a new maximum element $\widehat{2}$. The graded Tchebyshev transform of the graded poset P is then the interval $[(\widehat{-1},\widehat{0}),(\widehat{1},\widehat{2})]$ in $T(P\cup\{\widehat{-1},\widehat{2}\})$. By abuse of notation, for a graded poset P we will denote its graded Tchebyshev transform by $T(P)$. Many interesting results on the Tchebyshev transform may be found in the papers cited above. These include showing that the Tchebyshev transform preserves grading [7], the Eulerian property [7], the positivity of the *cd*-index [2], and shellability [2]. The result motivating our present paper was shown in [5, Theorem 1.10], although not stated in this generality. (The same proof is applicable.)

Theorem 1.5. *Let P be a graded poset and $T(P)$ its graded Tchebyshev transform. Then the order complex $\triangle(T(P)\setminus\{(\widehat{-1},\widehat{0}),(\widehat{1},\widehat{2})\})$ triangulates the suspension of $\triangle(P\setminus\{\widehat{0},\widehat{1}\})$.*

1.4. **Derivative polynomials for tangent and secant.** The derivative polynomials $P_n(x)$ and $Q_n(x)$ for tangent and secant may be defined by the formulas

$$\frac{d^n}{dx^n}\tan(x)=P_n(\tan x)\quad\text{and}\quad\frac{d^n}{dx^n}\sec(x)=Q_n(\tan x)\cdot\sec(x).$$

They were studied most recently by Hoffman [8] and [9] but their study goes back to Krichnamachary and Rao [11], Haigh [3] and to Knuth and Buckholtz [10]. The table of the coefficients of $Q_n(x)$ is sequence A008294 in the "The Online Encyclopedia of Integer Sequences" [1]. As it was observed by the present author [5, Corollary 9.3], the derivative polynomial $Q_n(x)$ for secant is related to the Tchebyshev transform of the Boolean algebra B_n of rank n by the formula

$$\sum_{j=0}^{n} f_{j-1}\left(\triangle\left(T(B_n)\setminus\{(\widehat{-1},\widehat{0}),(\widehat{1},\widehat{2})\}\right)\right)\cdot\left(\frac{x-1}{2}\right)^j=\mathbf{i}^{-n}Q_n(x\cdot\mathbf{i}). \tag{2}$$

2. The Tchebyshev triangulation

In this section we define the Tchebyshev triangulation of a simplicial complex in such a way that it generalizes the already existing definitions for order complexes of posets. Given a simplicial complex $\triangle$ with vertex set V, let us fix a linear order $<$ on V and introduce a new element $\widehat{1}$ that is larger than all elements of V.

Definition 2.1. *The Tchebyshev triangulation $T_<(\triangle)$ is the simplicial complex on the vertex set*

$$\{(u,v)\ :\ u<v, \{u,v\}\in\triangle\}\cup\{(u,\widehat{1})\ :\ u\in V\}$$

whose faces are all sets $\{(u_1,v_1),\ldots,(u_k,v_k)\}$ satisfying $v_1<\cdots<v_k$ and the following conditions.

(i) *The set $\{u_1,u_2,\ldots,u_k,v_k\}\setminus\{\widehat{1}\}$ is a face of $\triangle$.*
(ii) *For all $i\le k-1$ either $u_i=u_{i+1}$ or $v_i\le u_{i+1}$ holds.*

Figure 1 represents a simplicial complex with four vertices, and its Tchebyshev triangulation with respect to the order $v_1<v_2<v_3<v_4$.

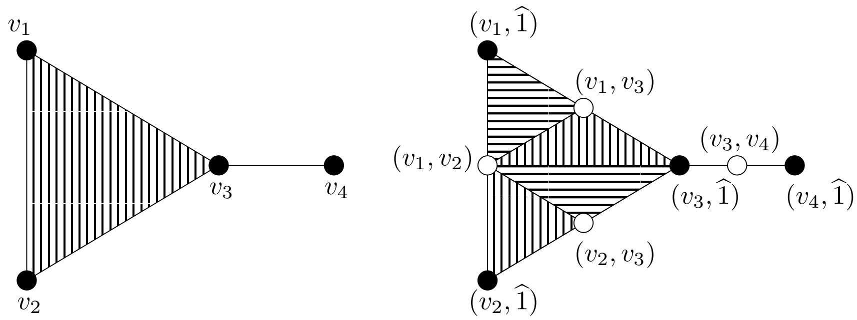

FIGURE 1. The Tchebyshev triangulation induced by $v_1<v_2<v_3<v_4$

Our definition of a Tchebyshev triangulation depends on the linear order chosen on V. However, the face numbers of the resulting complex are always the same. This will be shown in Section 3. In anticipation of this result we drop the $<$ sign and use $T(\triangle)$ as a shorthand for $T_<(\triangle)$. First we justify the use of the word triangulation. From our definition only that is obvious that $T(\triangle)$ is a simplicial complex. Recall that every simplicial complex has a *standard geometric representation* where we represent each vertex v with a basis vector e_v in some $|V|$-dimensional Euclidean space and we represent each face $\{v_1,\ldots,v_k\}$ as the convex hull of $e_{v_1},\ldots,e_{v_k}$. More generally we may make the following definition.

Definition 2.2. *Given a simplicial complex $\triangle$ with vertex set V and a Euclidean space E we say that the map $\eta: V\longrightarrow E$* induces a non-degenerate geometric realization of $\triangle$ *if for every $\sigma\in\triangle$ and $u\in\sigma$, the point $\eta(v)$ does not belong to the convex hull of $\{\eta(v)\ :\ v\in\sigma\setminus\{u\}\}$.*

The condition given in our definition is necessary and sufficient to insure that when we represent each face σ with the convex hull $\eta(\sigma)$ of $\{\eta(v)\ :\ v\in\sigma\}$, the set of represented faces $\eta(\sigma)$ ordered by inclusion is isomorphic to the faces of $\triangle$ ordered by inclusion. Using this more general notion of geometric realization we may define triangulations as follows.

Definition 2.3. *Assume that the maps $\eta: V(\triangle)\longrightarrow E$ and $\eta': V(\triangle')\longrightarrow E$ induce non-degenerate geometric realizations of the simplicial complexes $\triangle$ and $\triangle'$ respectively. We say that $\triangle'$, represented by η'* triangulates *$\triangle$, represented by η if*

(i) *For all $\sigma'\in\triangle'$ there is a $\sigma\in\triangle$ such that $\eta'(\sigma')\subseteq\eta(\sigma)$;*
(ii) *for all $\sigma\in\triangle$, every point of $\eta(\sigma)$ is contained in some $\eta'(\sigma')$ where $\sigma'\in\triangle'$.*

Using this (fairly narrow) definition of triangulation we may show the following.

Theorem 2.4. *Assume that a geometric representation of the simplicial complex $\triangle$ is induced by the map $\eta : V(\triangle) \longrightarrow E$. Then the map $\eta' : V(T(\triangle)) \longrightarrow E$ given by*

$$\eta'(u,v) = \begin{cases} \frac{1}{2}\cdot(\eta(u)+\eta(v)) & \text{if } v \in V(\triangle) \\ \eta(u) & \text{if } v = \widehat{1} \end{cases}$$

induces a non-degenerate geometric realization of $T(\triangle)$. Moreover, the geometric realization of $T(\triangle)$ triangulates the geometric realization of $\triangle$.

Proof. Consider first any face $\{(u_1,v_1),\ldots,(u_k,v_k)\}$ of $T(\triangle)$ where we assume $v_1 < \cdots < v_k$. Then, by the definition of $T(\triangle)$, the set $\sigma := \{u_1,v_1,\ldots,u_k,v_k\} \setminus \{\widehat{1}\}$ belongs to $\triangle$. By the definition of η' each (u_i,v_i) is either the midpoint of an edge of the (non-degenerate) simplex $\eta(\sigma)$ or a vertex of this simplex. (Only $\eta'(u_k,v_k)$ may be a vertex, exactly when $v_k = \widehat{1}$.) Since $v_1 < \cdots < v_k$ and since $u_i < v_i$ holds for all i, these edge-midpoints (and vertex) are pairwise different. Obviously they are also all vertices of their convex hull, thus η' induces a non-degenerate geometric realization. From our reasoning it also follows that condition (i) of Definition 2.3 is satisfied.

We are left to prove condition (ii) of Definition 2.3. Recall that the *k-skeleton* $\mathrm{Skel}_k(\triangle)$ of $\triangle$ is the subcomplex of all faces of dimension at most k. We prove by induction that for each $k \geq 1$ the complex $T(\mathrm{Skel}_k(\triangle))$, represented by the appropriate restriction of η', triangulates $\mathrm{Skel}_k(\triangle)$, represented by the appropriate restriction of η. For $k = 1$ this observation is trivial since the restriction of η' to $T(\mathrm{Skel}_1(\triangle))$ induces subdividing each edge $[\eta(u),\eta(v)]$ (where we may assume $u < v$) into two edges:

$$\left[\eta(u), \frac{1}{2}\cdot(\eta(u)+\eta(v))\right] = \eta'\left(\{(u,v),(u,\widehat{1})\}\right) \text{ and } \left[\frac{1}{2}\cdot(\eta(u)+\eta(v)), \eta(v)\right] = \eta'\left(\{(u,v),(v,\widehat{1})\}\right).$$

The verification of the details is left to the reader. Consider now the k-skeletons of $\triangle$ and $T(\triangle)$. Given a any k-face $\sigma \in \triangle$ we need to show that every interior point of $\eta(\sigma)$ is contained in some $\eta'(\sigma')$ where $\sigma' \in T(\triangle)$. (Boundary points belong to the $(k-1)$-skeleton, so we know the claim for them by induction.) Assume that $\sigma = \{v_1,\ldots,v_k\}$ where $v_1 < \cdots < v_k$. Then the realization of the following vertices of $T(\triangle)$ belong to $\eta(\sigma)$:

- all vertices (v_i,v_j) where $1 \leq i < j \leq k$ (represented by $\frac{1}{2}\cdot(\eta(v_i)+\eta(v_j))$;
- all vertices $(v_j,\widehat{1})$ where $1 \leq j \leq k$ (represented by $\eta(v_j)$).

It is easy to see that any face σ' of $T(\triangle)$ formed by the above vertices remains a face if we add the vertex (v_1,v_2). In fact, the only vertex whose second coordinate is v_2 is (v_1,v_2) so if it was not already an element of σ', condition (ii) in Definition 2.1 remains valid. On the other hand, any $(u,v) \neq (v_1,v_2)$ satisfies either $u = v_1$ or $u \geq v_2$. Therefore, the faces $\eta'(\sigma')$ contained in $\eta(\sigma)$ form a cone over $\eta'(v_1,v_2)$. Since coning the boundary of a simplex over the midpoint of one of its edges yields the entire simplex, it follows from our induction hypothesis that every interior point of $\eta(\sigma)$ is contained in some $\eta'(\sigma')$. □

We conclude this section by explaining the connection between the Tchebyshev triangulation and the Tchebyshev transform of a graded poset.

Proposition 2.5. *Let P be a graded poset and $T(P)$ its graded Tchebyshev transform. Then $\triangle(T(P)) \setminus \{(\widehat{-1},\widehat{0}),(\widehat{1},\widehat{2})\}$ is isomorphic to a Tchebyshev triangulation of the suspension of $\triangle(P \setminus \{\widehat{0},\widehat{1}\})$. This triangulation is induced by any linear extension of $P \cup \{\widehat{-1}\} \setminus \{\widehat{1}\}$.*

Proof. By definition, the graded Tchebyshev transform $T(P)$ has the following elements besides its minimum element $(\widehat{-1},\widehat{0})$ and its maximum element $(\widehat{1},\widehat{2})$:

(i) all pairs (u,v) where $u<v$ in the partial order of P and $u,v\in P\setminus\{\widehat{0},\widehat{1}\}$;
(ii) all pairs $(u,\widehat{1})$ where $u\in P\setminus\{\widehat{0},\widehat{1}\}$;
(iii) all pairs $(\widehat{-1},u)$ where $u\in P\setminus\{\widehat{1}\}$;
(iv) all pairs $(\widehat{0},u)$ where $u\in P\setminus\{\widehat{0}\}$.

Condition (i) above is equivalent to requiring $u<v$ in the linear extension and $\{u,v\}\in\triangle(P\setminus\{\widehat{0},\widehat{1}\})$. The elements listed in (iii) and (iv) motivate adding the suspending vertices $\widehat{-1}$ and $\widehat{0}$ to $V(\triangle(P\setminus\{\widehat{0},\widehat{1}\}))$ and setting $\widehat{-1}<\widehat{0}<u$ for all $u\in P\setminus\{\widehat{0},\widehat{1}\}$. Given this suspension of $\triangle(P\setminus\{\widehat{0},\widehat{1}\})$ with this linear order, the vertices of the Tchebyshev triangulation become exactly the elements of $T(P)\setminus\{(\widehat{-1},\widehat{0}),(\widehat{1},\widehat{2})\}$ listed above. It is easy to verify that $\{(u_1,v_1),\ldots,(u_k,v_k)\}$ (where $v_1<\cdots<v_k$ in the linear order) is a face of the Tchebyshev triangulation, if and only if $(u_1,v_1)<\cdots<(u_k,v_k)$ is an increasing chain in $T(P)\setminus\{(\widehat{-1},\widehat{0}),(\widehat{1},\widehat{2})\}$. In fact, the condition of being a face implies that the elements $\{u_1,v_1,\ldots,u_k,v_k\}$ must form an increasing chain in some order, and the assumption that the linear order defining the Tchebyshev triangulation is a linear extension of the partial order on P forces $v_1<\cdots<v_k$ and $\forall i(u_{i+1}=u_i\vee v_i\leq u_i)$ in the partial order of P. The converse is also easily verified. □

It is worth noting that Theorem 1.5 is a consequence of Theorem 2.4 and Proposition 2.5.

3. The face numbers of the Tchebyshev triangulation

Theorem 3.1. *For $k\geq 1$ the number $(k-1)$-faces of $T(\triangle)$ is given by*

$$f_{k-1}(T(\triangle))=\sum_{j=k}^{2k} f_{j-1}(\triangle)\cdot 2^{2k-j-1}\left(\binom{k}{2k-j}+\binom{k-1}{2k-j}\right).$$

Proof. We select any $(k-1)$-face $\sigma'=\{(u_1,v_1),\ldots,(u_k,v_k)\}\in T(\triangle)$ in two steps, as follows. First we choose the face $\sigma:=\{u_1,v_1,\ldots,u_k,v_k\}\setminus\{\widehat{1}\}\in\triangle$ then we select the elements $u_1,v_1,\ldots,u_k,v_k$ in $\sigma\cup\{\widehat{1}\}$. Since $u_1<v_1<v_2<\cdots<v_k$ and only v_k may be equal $\widehat{1}$, the face σ has at least k elements. On the other hand, σ has at most $2k$ elements (the worst case occurs when $u_1,v_1,\ldots,u_k,v_k$ are pairwise different and $v_k\neq\widehat{1}$). Hence, assuming $|\sigma|=j$, we must sum over $j=k,\ldots,2k$ and in each case we have $f_{j-1}(\triangle)$ choices to fix σ.

Assume now we have fixed a $(j-1)$-face $\sigma\in\triangle$, and let us count the number of ways to select $u_1,v_1,\ldots,u_k,v_k$. Since $v_1<\ldots<v_k$, exactly k elements of $\sigma\cup\{\widehat{1}\}$ are v's. Let us mark these with a "+" sign and the other elements of $\sigma\cup\{\widehat{1}\}$ with a "−" sign, and write down the sequence of $(j+1)$ signs in increasing order of the underlying elements. For example, when $k=3$, $j=5$, and $u_1<v_1<u_2<v_2<u_3<v_3=\widehat{1}$, we record $-+-+-+$. On the other hand, for $k=3$, $j=5$, and $u_1<v_1<u_2=u_3<v_2<v_3<\widehat{1}$ we record $-+-++-$. This encoding with signs is not unique, since for $u_1<v_1<u_2<v_2=u_3<v_3<\widehat{1}$ we also mark $-+-++-$. Fortunately we can exactly describe, how many sets σ' correspond to each code.

Each valid code starts with a "$-$" sign, associated to u_1, and has exactly k "+" signs. We can not have two consecutive "$-$" signs because of the following observations:

Every element of $\sigma\cup\{\widehat{1}\}$ that is strictly between v_i and v_{i+1} is equal to u_{i+1}. Every element of $\sigma\cup\{\widehat{1}\}$ preceding v_1 is equal to u_1. Every element of $\sigma\cup\{\widehat{1}\}$ succeeding v_k is equal to $\widehat{1}$.

In fact, if u_t is strictly between v_i and v_{i+1} then we must have $t\geq i$ (otherwise $u_t<v_t\leq v_i$) and if $t>i+1$ then, by condition (ii) of Definition 2.1, the only way to avoid $u_t>v_{t-1}\geq v_{i+1}$ is by setting $u_t=u_{t-1}$. Repeated use of this argument leads to $u_t=u_{t-1}=\cdots=u_{i+1}$. The proof of the second observation is similar, the third is obvious.

Conversely, assume we are given a string of "$-$" and "+" signs of length $j+1$, such that the first sign is "$-$", there are exactly k "+" signs, and there are no two consecutive "$-$" signs. We claim that the number of faces σ' corresponding to this string is 2 to the power of the number of consecutive "++" patterns in the string. In fact, the "+" signs unambiguously mark the elements $v_1<\cdots<v_k$. If there is a "$-$" sign between the "+" signs associated to v_i and v_{i+1} then, by the observations highlighted above, that "$-$" sign marks u_{i+1}. Similarly, the first "$-$" sign marks u_1. We are left to place those elements u_{i+1} for which v_i and v_{i+1} mark consecutive "+" signs. Let us determine them in increasing order. By condition (ii) of Definition 2.1, such a u_{i+1} is either equal to u_i (and is given recursively) or equal to v_i. Thus we have exactly two options to choose from, each time we encounter a "++" pattern. These choices may be made independently of each other.

We have found that, to obtain $f_{k-1}(T(\triangle))$, each $f_{j-1}(\triangle)$ needs to be multiplied by the total weight of all strings of length $(j+1)$ made of exactly k "+" signs and $(j+1-k)$ "$-$" signs, not containing consecutive "$-$" signs, and starting with a "$-$" sign. The weight of each such string is 2 to the power of the number of "++" patterns contained in the string as a substring. Let us distinguish two cases depending on the whether the last sign is "$-$" or "+". In the first case, we need to place $(j-k)$ "$-$" signs in between the "+" signs. Since consecutive "$-$" signs are not allowed, this amounts to choosing $(j-k)$ "gaps" from the $(k-1)$ "gaps" between consecutive "+" signs. This may be done in $\binom{k-1}{j-k}$ ways, and we are left with $k-1-(j-k)=2k-j-1$ substrings "++". Thus each string enumerated in the first case has weight 2^{2k-j-1}. The second case is similar, except that now we are only placing $(j-k-1)$ "$-$" signs in between the "+" signs, thus we have $\binom{k-1}{j-k-1}$ such strings and each has weight 2^{2k-j}. Therefore, $f_{j-1}(\triangle)$ needs to be multiplied with

$$2^{2k-j-1}\binom{k-1}{j-k}+2^{2k-j}\binom{k-1}{j-k-1}.$$

Note that this coefficient is also correct for the "extreme cases" $j=k$ and $j=2k$ if we use the conventions $\binom{k-1}{k}=0$ and $\binom{k-1}{-1}=0$. For $j=k$ then he coefficient of $f_{k-1}(\triangle)$ is 2^{k-1} (the only valid string of length $j+1$ is"$-+\cdots+$), while for $j=2k$ the coefficient of $f_{2k-1}(\triangle)$ is 1 (the only valid string of length $j+1$ is"$-+-+\cdots-+-$). We may avoid the use of "degenerate" binomial coefficients if we rewrite the coefficient of $f_{j-1}(\triangle)$ as

$$2^{2k-j-1}\left(\binom{k-1}{j-k}+\binom{k-1}{j-k-1}+\binom{k-1}{j-k-1}\right)$$

and use the identity $\binom{n+1}{m+1}=\binom{n}{m}+\binom{n}{m+1}$ (which, assuming $\binom{n}{-1}=\binom{n}{n+1}=0$, is true even when $m=-1$ or $m=n$) to get

$$\binom{k-1}{j-k}+\binom{k-1}{j-k-1}=\binom{k}{j-k}=\binom{k}{2k-j}\text{ and }\binom{k-1}{j-k-1}=\binom{k-1}{2k-j}.$$

□

We conclude this section with a result that justifies the word "Tchebyshev" in the term "Tchebyshev triangulation". For that purpose we need a somewhat unusual definition of the f-polynomial of a simplicial complex. (We use capital F to stress the difference between our definition and the usual notion of f-polynomial in the literature)

Definition 3.2. *We define the F-polynomial $F_{\triangle}(x)$ of a $(d-1)$ dimensional simplicial complex $\triangle$ as*

$$F_{\triangle}(x) = \sum_{j=0}^{d} f_{j-1}(\triangle) \cdot \left(\frac{x-1}{2}\right)^{j}.$$

As a consequence of Theorem 3.1 we have the following result.

Proposition 3.3. *The Tchebyshev transform of the F-polynomial of a simplicial complex is the F-polynomial of its Tchebyshev triangulation:*

$$T(F_{\triangle})(x) = F_{T(\triangle)}(x).$$

Proof. The polynomial $T(F_{\triangle})(x)$ is completely determined by its restriction to $[-1,1]$. Each $x \in [-1,1]$ may be written as $x = \cos(\alpha)$ for some $\alpha \in [0,2\pi]$. As observed in the Preliminaries, $T(F_{\triangle})(\cos(\alpha))$ is the real part of

$$F_{\triangle}(\exp(\mathbf{i}\cdot\alpha)) = \sum_{j=0}^{d} f_{j-1}(\triangle) \cdot \left(\frac{(\cos(\alpha)-1)+\mathbf{i}\cdot\sin(\alpha)}{2}\right)^{j},$$

thus we have

$$T(F_{\triangle})(x) = \sum_{j=0}^{d} \frac{f_{j-1}(\triangle)}{2^{j}} \cdot \sum_{k=0}^{\lfloor j/2\rfloor} \binom{j}{2k} (x^2-1)^{k}(x-1)^{j-2k}.$$

On the other hand

$$\begin{aligned} F_{T(\triangle)}(x) &= \sum_{k=0}^{d} \left(\frac{x-1}{2}\right)^{k} \sum_{j=k}^{2k} f_{j-1}(\triangle) 2^{2k-j-1} \left(\binom{k}{2k-j} + \binom{k-1}{2k-j}\right) \\ &= \sum_{j=0}^{d} f_{j-1}(\triangle) \sum_{k=\lceil j/2\rceil}^{j} 2^{2k-j-1} \left(\binom{k}{2k-j} + \binom{k-1}{2k-j}\right) \left(\frac{x-1}{2}\right)^{k}, \end{aligned}$$

hence it is sufficient to show that the contribution of $f_{j-1}(\triangle)$ is the same in both sums, i.e.,

$$\frac{1}{2^{j}} \sum_{k=0}^{\lfloor j/2\rfloor} \binom{j}{2k} (x^2-1)^{k}(x-1)^{j-2k} = \sum_{k=\lceil j/2\rceil}^{j} 2^{2k-j-1} \left(\binom{k}{2k-j} + \binom{k-1}{2k-j}\right) \left(\frac{x-1}{2}\right)^{k}$$

holds for all $j \geq 0$. Let us denote the left hand side of by $\mathcal{L}_j(x)$, and the right hand side by $\mathcal{R}_j(x)$. By the binomial theorem,

$$\mathcal{L}_j(x) = \frac{1}{2}\left(\left(\frac{x-1}{2} + \sqrt{\frac{x^2-1}{4}}\right)^{j} + \left(\frac{x-1}{2} - \sqrt{\frac{x^2-1}{4}}\right)^{j}\right).$$

The polynomials $\mathcal{L}_j(x)$ form a Fibonacci-type sequence, determined by $\mathcal{L}_0(x) = 1$, $\mathcal{L}_1(x) = (x-1)/2$, and the recursion formula $\mathcal{L}_{j+2}(x) - (x-1)\mathcal{L}_{j+1}(x) - (x-1)\mathcal{L}_j(x)/2 = 0$. It is easy to verify that the polynomials $\mathcal{R}_j(x)$ verify the same initial conditions and recursion formula. □

4. The Tchebyshev triangulation of the second kind

In analogy to the definition for posets in [2], we define the Tchebyshev triangulation of the second kind as follows.

Definition 4.1. *The* Tchebyshev triangulation of the second kind $U(\triangle)$ *of a simplicial complex* $\triangle$ *on the vertex set* V *is the multiset of simplicial complexes* $\{\mathrm{lk}((v,\widehat{1}), T(\triangle)) : v \in V\}$.

Remark 4.2. If $\triangle$ is the one dimensional simplex on the vertex set $\{v_1, v_2\}$ then $\mathrm{lk}((v_1,\widehat{1}) = \mathrm{lk}((v_2,\widehat{1})$ in any Tchebyshev triangulation. In such cases we want to list each simplicial complex as many time as it equals to the link of a vertex of the form $(v,\widehat{1})$.

As before, we tacitly assume that a linear order has been fixed on V, on which $U(\triangle)$ depends, but its face numbers do not, as we will see below. We may extend the notions of face numbers and F-polynomial to multisets of simplicial complexes, the most obvious difference being that, now $f_{-1} \geq 1$ is the number of simplicial complexes listed. In particular, by definition we have $f_{-1}(U(\triangle)) = |V|$.

In analogy to Theorem 3.1 we have the following.

Theorem 4.3. *For* $k \geq 1$ *the number* $(k-1)$*-faces in* $U(\triangle)$ *is given by*

$$f_{k-1}(U(\triangle)) = \sum_{j=k+1}^{2k+1} f_{j-1}(\triangle) \cdot 2^{2k+1-j} \binom{k}{2k+1-j}.$$

Proof. Analogous to the proof of Theorem 3.1. A $(k-1)$-face in $U(\triangle)$ is a set $\{(u_1, v_1), \ldots, (u_k, v_k), (u_{k+1}, v_{k+1})\} \in T(\triangle)$ such that $v_1 < \cdots < v_k$ and $v_{k+1} = \widehat{1}$. (The vertex $(u_{k+1},\widehat{1})$ marks the link from which the face is taken and is not counted towards the dimension.) Let j be the cardinality of $\{u_1, v_1, u_2, v_2, \ldots, v_k, u_{k+1}\} \in \triangle$. Clearly $k+1 \leq j \leq 2k+1$ must hold. Let us associate to each $(k-1)$-face a string of $+$ and $-$ signs the same way as in the proof of Theorem 3.1. Since $v_{k+1} = \widehat{1}$, each valid code must end with a "+" sign. Since $(u_{k+1},\widehat{1})$ is not counted towards the dimension, we have to consider valid strings with $(k+1)$ "+" signs. We obtain that $f_{j-1}(\triangle)$ is multiplied with the total weight of all strings of length $j+1$, consisting of exactly $(k+1)$ "+" signs and $(j-k)$ "−" signs, starting with a "−" sign and ending with a "+", containing no consecutive "−" signs. The weight of each valid string is 2 to the power of the number of "++" patterns contained in the string as a substring. There are k gaps between consecutive "+" signs of which $j-k-1$ needs to be selected as the position of a (non-first) "−". There are $\binom{k}{j-k-1} = \binom{k}{2k+1-j}$ ways to do so, and each such string contains "++" as a substring $(2k+1-j)$ times. □

In analogy to Proposition 3.3 we have:

Proposition 4.4. *The Tchebyshev transform of the second kind of the* F*-polynomial of a simplicial complex is the half of the* F*-polynomial of its Tchebyshev triangulation:*

$$U(F_\triangle)(x) = \frac{1}{2} F_{U(\triangle)}(x).$$

Proof. Analogous to the proof of Proposition 3.3. Restricting $U(F_\triangle)(x)$ to $[-1,1]$ and considering the imaginary part of $F_\triangle(\exp(\mathbf{i}\cdot\alpha))$ yields the formula

$$U(F_\triangle)(x)=\sum_{j=0}^{d}\frac{f_{j-1}(\triangle)}{2^j}\cdot\sum_{k=0}^{\lfloor (j-1)/2\rfloor}\binom{j}{2k+1}(x^2-1)^k(x-1)^{j-2k-1}.$$

As a consequence of Theorem 4.3 we have

$$\frac{1}{2}F_{U(\triangle)}(x)=\sum_{j=1}^{d}f_{j-1}(\triangle)\cdot\sum_{k=\lceil (j-1)/2\rceil}^{j-1}2^{2k-j}\binom{k}{2k+1-j}\left(\frac{x-1}{2}\right)^k.$$

Comparing the contribution $f_{j-1}(\triangle)$, it is sufficient to show that

$$\sum_{k=0}^{\lfloor (j-1)/2\rfloor}\binom{j}{2k+1}\left(\frac{x^2-1}{4}\right)^k\left(\frac{x-1}{2}\right)^{j-2k-1}=\sum_{k=\lceil (j-1)/2\rceil}^{j-1}2^{2k+1-j}\binom{k}{2k+1-j}\left(\frac{x-1}{2}\right)^k$$

holds for all $j\geq 1$. It may be shown using induction that both sides are equal to

$$\frac{1}{\sqrt{x^2-1}}\left(\left(\frac{x-1}{2}+\sqrt{\frac{x^2-1}{4}}\right)^j-\left(\frac{x-1}{2}-\sqrt{\frac{x^2-1}{4}}\right)^j\right),$$

a Fibonacci-type sequence, satisfying the same recursion as in the proof of Proposition 3.3, but subject to different initial conditions. □

We conclude this section by extending the notion of the Tchebyshev transform of the second kind to graded posets.

Definition 4.5. *Given a graded poset P, we define its Tchebyshev transform of the second kind $U(P)$ as the multiset of intervals*

$$\left\{[(\widehat{-1},\widehat{0}),(w,\widehat{1})]\subset T(P)\ :\ x\in P\cup\{\widehat{-1}\}\setminus\{\widehat{1}\}\right\}.$$

It should be noted that the coatoms in $T(P)$ are exactly all elements of the form $(w,\widehat{1})$ where $x\in P\cup\{\widehat{-1}\}\setminus\{\widehat{1}\}$. The "second type analogue" of Proposition 2.5 is also its consequence.

Corollary 4.6. *Let P be a graded poset and $T(P)$ its graded Tchebyshev transform. Then $\triangle(U(P))$, defined as the multiset of simplicial complexes*

$$\left\{\triangle(((\widehat{-1},\widehat{0}),(w,\widehat{1})))\ :\ [(\widehat{-1},\widehat{0}),(w,\widehat{1})]\in U(P)\right\},$$

is isomorphic to a Tchebyshev triangulation of the second kind of the suspension of $\triangle(P\setminus\{\widehat{0},\widehat{1}\})$. This triangulation is induced by any linear extension of $P\cup\{\widehat{-1}\}\setminus\{\widehat{1}\}$. (Note that the intervals $((\widehat{-1},\widehat{0}),(w,\widehat{1}))$ are "open", they do not contain $(\widehat{-1},\widehat{0})$ nor $(w,\widehat{1})$.)

Remark 4.7. It should be noted that the Tchebyshev transform of the second kind, introduced by Ehrenborg and Readdy [2] takes the *ab*-index of a graded poset P into 1/2 times the sum of the *ab*-indices of all graded posets $[(\widehat{-1},\widehat{0}),(w,\widehat{1})]$ listed in the multiset of graded posets $U(P)$.

5. F-polynomials of graded posets and their Tchebyshev transforms

Motivated by Proposition 2.5 and Corollary 4.6 we define the *F-polynomial of a graded poset* as follows.

Definition 5.1. *The F*-polynomial *$F_P(x)$ of a graded poset P is the F-polynomial of the suspension of $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$.*

As a consequence we have

$$F_P(x) = x \cdot F_{\triangle(P\setminus\{\widehat{0},\widehat{1}\})}(x). \tag{3}$$

We extend the notion of the F polynomial to multisets of posets by linearity. Propositions 3.3 and 4.4 may then be rephrased as follows.

Corollary 5.2. *The following relations hold between the F-polynomials of a graded poset P and its Tchebyshev transforms $T(P)$ and $U(P)$:*

$$F_{T(P)}(x) = T(F_P(x)) \quad \textit{and} \quad U(F_P(x)) = \frac{1}{2} F_{U(P)}(x).$$

The following rephrasing is also very useful.

Corollary 5.3. *$U(F_P)(x)$ may be calculated as the total weight of all chains in the graded poset $T(P)$ that contain one coatom $(w, \widehat{1}) \in T(P)$, avoid $\widehat{0}_{T(P)}$ and $\widehat{1}_{T(P)}$. Each coatom here contributes a factor of $1/2$, and all remaining elements in the chain (if any) contribute a factor of $(x-1)/2$.*

6. H-stable and S-stable simplicial complexes

Given a simplicial complex $\triangle$ of dimension $(d-1)$, Schelin's theorem provides a criterion for the Schur stability of $F_\triangle(x)$, and in the previous sections we have seen that the Tchebyshev transforms $T(F_\triangle)(x)$ and $U(F_\triangle)(x)$ used are also F-polynomials of (collections of) simplicial complexes. This in itself should make the F-polynomial of a simplicial complex an interesting invariant. When we try to use the transformation $x \mapsto t = (x-1)/(x+1)$ to relate the Schur stability of $F_\triangle(x)$ to the Hurwitz stability of the corresponding polynomial $h(t)$, this turns out to be the well-known h-polynomial:

$$(1-t)^d \cdot F_\triangle\left(\frac{1+t}{1-t}\right) = (1-t)^d \sum_{j=0}^{d} f_j \left(\frac{t}{1-t}\right)^j = h_\triangle(t).$$

Corollary 6.1. *The number of zeros of $F_\triangle(x)$ inside the unit disk $|x| < 1$ equals the number of zeros of $h_\triangle(t)$ inside the left t-halfplane.*

The only reason we might not conclude that $F_\triangle(x)$ is Schur stable if and only if $h_\triangle(t)$ is Hurwitz stable is that the degrees of these polynomials might differ. For example, the h-polynomial of a simplex of any dimension is 1, hence it is Hurwitz stable, whereas the degree of the F-polynomial of any simplicial complex of dimension $(d-1)$ is d.

Definition 6.2. *We call a $(d-1)$-dimensional simplicial complex complex $\triangle$* S-stable, *if its F-polynomial is Schur stable, and H-stable, if its h-polynomial is Hurwitz stable.*

Corollary 6.3. *Every S-stable simplicial complex $\triangle$ of dimension $d-1$ is also H-stable, and satisfies $\deg h_\triangle(t) = d$.*

In the opposite direction we may state the following.

Proposition 6.4. *An H-stable simplicial complex $\triangle$ is S-stable if and only if its reduced Euler characteristic $\tilde{\chi}(\triangle) = -F_{\triangle}(-1) = (-1)^{d-1}h_d$ is not zero.*

Proof. If the complex $\triangle$ is S-stable, of dimension $(d-1)$ then $h_{\triangle}(t)$ has degree d, since this is the number of zeros of $F_{\triangle}(x)$ inside the unit disk $|x| < 1$ and this is a lower bound for the number of zeros of $h_{\triangle}(t)$. Thus we must have $h_d \neq 0$, and the reduced Euler characteristic is not zero.

Conversely, if $h_d \neq 0$ then the degree of $h_{\triangle}(t)$ is d and, by the H-stability, $h_{\triangle}(t)$ has d roots in the left t-halfplane. This is also the number of zeros of $F_{\triangle}(x)$ inside the unit disk $|x| < 1$ and d is also the degree of $F_{\triangle}(x)$. ☐

Schelin's theorem is always useful to decide the Schur stability of a polynomial. Here we propose to focus on a consequence of the opposite implication of Theorem 1.3.

Proposition 6.5. *Assume that all zeros of a polynomial $p(x) \in \mathbb{R}[x]$ of degree d have modulus less than 1. Then all roots of the Tchebyshev transforms $T(p)$ and $U(p)$ are real, have multiplicity 1, and lie in the open interval $(-1,1)$. Moreover, the roots $t_1 < \cdots < t_d$ of $T(p)$ and the roots $u_1 < \cdots < u_{d-1}$ of $U(p)$ are interlaced, i.e., $t_1 < u_1 < t_2 < u_2 < \cdots < u_{d-1} < t_d$ holds.*

Proof. No root of $p(x)$ is on the unit circle, Theorem 1.3 is thus applicable. Since the contribution of each pole of $U(p)/T(p)$ to $I_{-1}^{1}U(p)/T(p)$ is at most 1, and p has degree d, the rational function $U(p)/T(p)$ must have at least d distinct poles in $(-1,1)$. This is only possible, if all d roots of $T(p)$ are distinct and belong to $(-1,1)$. Since each root t_i has multiplicity 1, the graph of $T(p)$ crosses the horizontal axis at each of its roots. Thus, if $T(p)$ is locally increasing at t_i, it must be locally decreasing at t_{i+1} and vice versa. In other words, $\operatorname{sign}(T(p)'(t_i)) = -\operatorname{sign}(T(p)'(t_{i+1}))$ must hold for $i = 1,\ldots,d-1$. On the other hand $I_{-1}^{1}U(p)/T(p) = d$ can only be achieved if for each we have $I_{t_i}U(p)/T(p) = 1$, i.e., the sign of $U(p)(t_i)$ must be equal to the sign of $T(p)'(t_i)$. Therefore $\operatorname{sign}(U(p)(t_i)) = -\operatorname{sign}(U(p)(t_{i+1}))$ must also hold for $i = 1,\ldots,d-1$. By the Intermediate Value Theorem, $U(p)$ has at least one root in each of (t_1,t_2), (t_2,t_3), $\ldots$, (t_{d-1},t_{d-2}). Since the degree of $U(p)$ is $d-1$, it has exactly one root in each of these open intervals. ☐

Thus, if we know by some structural (geometric or combinatorial) reason that a simplicial complex is S-stable then the roots of the Tchebyshev transforms of its F-polynomial will be real, distinct, and interlaced, as described in Proposition 6.5. We will see an important infinite sequence of such related pairs of polynomials in Section 8. This motivates the question to describe the S-stable simplicial complexes geometrically or combinatorially. We conclude this section with a discouraging example and an encouraging observation.

Example 6.6. The boundary complex $\partial(\triangle^d)$ of a d-dimensional simplex $\triangle^d$ is H-stable (S-stability is equivalent since $\tilde{\chi}(\partial(\triangle^d)) \neq 0$) only for $d \leq 2$. In fact, $h[\partial(\triangle^d)] = (1-t^{d+1})/(1-t)$, its roots are all $(d+1)$-th roots of unity, except 1. For $d \geq 3$, the real part of the root $\exp(\mathbf{i}\cdot 2\pi/(d+1))$ is nonnegative. For $d \leq 2$ the polynomials 1, $1+t$, and $1+t+t^2$ are Hurwitz stable.

Proposition 6.7. *The join $\triangle_1 * \triangle_2$ is S-stable (H-stable) if and only if both $\triangle_1$ and $\triangle_2$ are S-stable (H-stable).*

Proof. It is easy to verify that

$$F_{\triangle_1 * \triangle_2}(x) = F_{\triangle_1}(x) \cdot F_{\triangle_2}(x). \tag{4}$$

The statement on S-stability follows immediately. To prove the statement on H-stability one may either note the analogous equation

$$h_{\triangle_1 * \triangle_2}(t) = h_{\triangle_1}(t) \cdot h_{\triangle_2}(t),$$

or note that $F_{\triangle_1 * \triangle_2}(-1) \neq 0$ iff. $F_{\triangle_1}(-1)$ and $F_{\triangle_2}(-1)$ are nonzero, and so we may also use Proposition 6.4. □

Corollary 6.8. *A simplicial complex $\triangle$ is S-stable (H-stable) if and only if its suspension $\triangle * \partial(\triangle^1)$ has the same property.*

7. Stable graded posets

We may extend the notions of S-stability and H-stability from simplicial complexes to graded posets as follows.

Definition 7.1. *We call a graded poset P* S-stable *resp.* H-stable *if the same holds for the order complex $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$.*

As a consequence of Corollary 5.2, a graded poset is S-stable resp. H-stable if and only if the same holds for the suspension $\triangle(P \setminus \{\widehat{0}, \widehat{1}\}) * \partial(\triangle^1)$. Recall that the F-polynomial F-polynomial of this suspension.

Corollary 7.2. *A graded poset P is S-stable iff. $F_P(x)$ is Schur stable.*

It is important to note the following. As a consequence of Corollary 6.3 and Proposition 6.4, S-stability and H-stability is equivalent for the graded posets P satisfying $\widetilde{\chi}(\triangle(P \setminus \{\widehat{0}, \widehat{1}\})) \neq 0$. The reduced Euler characteristic of $\triangle(P \setminus \{\widehat{0}, \widehat{1}\}))$ equals the Möbius function of P (see, e.g. Stanley [15, Proposition 3.8.6]. A graded poset is *Eulerian* if the Möbius function of each interval $[u, v]$ is $(-1)^{\rho(u,v)}$ where $\rho(u, v)$ is the rank of the interval.

Corollary 7.3. *For each interval $[u, v]$ in an Eulerian poset P, the interval $[u, v]$ is S-stable iff. it is H-stable.*

It is usual in the literature of graded posets to exclude the unique minimum and maximum from the set of vertices of the order complex. One reason to do so is topological. The order complex $\triangle(P)$ is obtained from $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$ by joining the singleton $\widehat{0}$ and then the singleton $\widehat{1}$, so their presence always makes $\triangle(P)$ contractible even if $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$ is far from being so. As a consequence of Proposition 6.7 we may observe that the distinction makes a big difference for S-stability and no difference for H-stability.

Corollary 7.4. *Given a graded poset P, the simplicial complex $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$ is H-stable if and only if the same holds for $\triangle(P)$, whereas no nontrivial graded P has an S-stable $\triangle(P)$.*

We should also note that equation (4) implies

$$F_{\triangle(P)}(x) = F_{\triangle(P \setminus \{\widehat{0}, \widehat{1}\})}(x) \cdot \left(\frac{x+1}{2}\right)^2. \tag{5}$$

As a consequence of Proposition 6.7, the join of graded posets introduced by Stanley [16] preserves both S-stability and H-stability. Recall that the *join* $P*Q$ of two graded posets is the set $P\setminus\{\widehat{1}_P\}\cup Q\setminus\{\widehat{0}_Q\}$ in which $x\leq y$ holds if either $x\leq y$ in $P\setminus\{\widehat{1}_P\}$ or $x\leq y$ in $Q\setminus\{\widehat{0}_Q\}$, or $x\in P\setminus\{\widehat{1}_P\}$ and $y\in Q\setminus\{\widehat{0}_Q\}$.

Corollary 7.5. *$P*Q$ is S-stable (H-stable) if and only if the same holds for P and Q.*

In fact $\triangle(P*Q\setminus\{\widehat{0},\widehat{1}\})$ is the join of the simplicial complexes $\triangle(P\setminus\{\widehat{0},\widehat{1}\})$ and $\triangle(Q\setminus\{\widehat{0},\widehat{1}\})$.

We conclude this section with a key result that leads to an important conjecture. Let us denote the unique graded poset of rank 1 by I.

Theorem 7.6. *If P is an S-stable graded poset then the same holds for the direct product $P\times I$.*

Corollary 7.7. *All Boolean algebras $B_n=I\times I\times\cdots\times I$ are S-stable.*

Theorem 7.6 inspires the following.

Conjecture 7.8. *The direct product of S-stable graded posets is S-stable.*

The conjecture might seem bold but we are not aware of any counterexample to it.

We prove Theorem 7.6 by showing a sequence of Lemmas which are sometimes more general than needed, in the hope that some of these Lemmas may be useful in proving Conjecture 7.8.

Lemma 7.9. *For any pair of graded posets P and for all $j>0$ we have*

$$f_{j-1}(\triangle(P\times Q))=\sum_{k,l\leq jk+l\geq j+1}\binom{j-1}{j-k,j-l,k+l-j-1}f_{k-1}(\triangle(P))f_{l-1}(\triangle(Q)).$$

(Here all order complexes include the minimum and maximum elements.)

Proof. Consider a j-element chain $(u_1,v_1)<\cdots<(u_j,v_j)$ in $P\times Q$. Assume that the "underlying sets" $U:=\{u_1,\ldots,u_j\}\in\triangle(P)$ and $V:=\{v_1,\ldots,v_j\}\in\triangle(Q)$ have k resp. l elements. To reconstruct the chain in $P\times Q$ we need to select a j-element sequence from of $U\times V$ such that the first element is (u_1,v_1), the last element is (u_j,v_j) and, while reading the sequence left to right, in each step we may move only to the next largest u_s resp. v_t. This is equivalent to selecting a lattice path of length $j-1$ from $(1,1)$ to (k,l), using the steps $(1,0)$, $(0,1)$ and $(1,1)$ only. The number of such lattice paths is $\binom{j}{j-k,j-l,k+l-j-1}$ the number of sets U resp. V is $f_{k-1}(\triangle(P))$ resp. $f_{l-1}(\triangle(Q))$. All choices (including the choice of k and l may be made independently. ☐

Lemma 7.10.

$$f_{j-1}(\triangle(P\times I))=(j+1)f_{j-1}(\triangle(P))+(j-1)f_{j-2}(\triangle(P)).$$

holds for $j>0$.

Proof. Using Lemma 7.9 for $Q=I$, only $l=1$ and $l=2$ are viable options with $f_0(\triangle(I))=2$ and $f_1(\triangle(I))=1$, so $k+l\geq j+1$ forces $k\geq j-1$ while $k\leq j$ is also necessary. For $k=j$ either $l=1$ or $l=2$ is possible, and $f_{j-1}(\triangle(P))$ is multiplied with

$$\binom{j-1}{0,j-1,0}\cdot 2+\binom{j-1}{0,j-2,1}\cdot 1=j+1,$$

whereas $k = j-1$ forces $l = 2$ and $f_{j-2}(\triangle(P))$ is multiplied with

$$\binom{j-1}{1, j-2, 0} = j-1.$$

Note that the formula is also valid for $j = 1$. □

Lemma 7.11.

$$F_{\triangle(P\times I)}(x) = F_{\triangle(P)}(x) + \frac{x^2-1}{2}\frac{d}{dx}F_{\triangle(P)}(x).$$

Proof. Using Lemma 7.10 we may write

$$F_{\triangle(P\times I)}(x) = 1 + \sum_{j>0}((j+1)f_{j-1}(\triangle(P)) + (j-1)f_{j-2}(\triangle(P)))\cdot\left(\frac{x-1}{2}\right)^j$$

Here

$$1 + \sum_{j>0}(j+1)f_{j-1}(\triangle(P))\left(\frac{x-1}{2}\right)^j = \frac{d}{dx}(x-1)F_{\triangle(P)}(x) = F_{\triangle(P)}(x) + (x-1)\frac{d}{dx}F_{\triangle(P)}(x), \quad \text{and}$$

$$\sum_{j>0}(j-1)f_{j-2}(\triangle(P))\left(\frac{x-1}{2}\right)^j = \frac{(x-1)^2}{2}\frac{d}{dx}F_{\triangle(P)}(x).$$

□

Lemma 7.12.

$$F_{\triangle(P\times I\setminus\{\widehat{0},\widehat{1}\})}(x) = \frac{d}{dx}\frac{x^2-1}{2}F_{\triangle(P\setminus\{\widehat{0},\widehat{1}\})}(x).$$

Proof. Using Equation (5) we may rewrite Lemma 7.11 as

$$\left(\frac{x+1}{2}\right)^2 F_{\triangle(P\times I\setminus\{\widehat{0},\widehat{1}\})}(x) = \left(\frac{x+1}{2}\right)^2 F_{\triangle(P\setminus\{\widehat{0},\widehat{1}\})}(x) + \frac{x^2-1}{2}\frac{d}{dx}\left(\frac{x+1}{2}\right)^2 F_{\triangle(P\setminus\{\widehat{0},\widehat{1}\})}(x).$$

Using the product rule, and dividing both sides by $((x+1)/2)^2$ yields

$$F_{\triangle(P\times I\setminus\{\widehat{0},\widehat{1}\})}(x) = x\cdot F_{\triangle(P\setminus\{\widehat{0},\widehat{1}\})}(x) + \frac{x^2-1}{2}\frac{d}{dx}F_{\triangle(P\setminus\{\widehat{0},\widehat{1}\})}(x),$$

which is equivalent to the stated equation by the product rule. □

Theorem 7.6 is now a consequence of Lucas' Theorem, which is the following. (See Marden [12, Theorem 6.1], we state it in a weaker and simpler form.)

Theorem 7.13 (Lucas)**.** *All zeros of the derivative of a polynomial f lie in the convex hull H of the set of zeros of f.*

If all zeros of $F_{\triangle(P\setminus\{\widehat{0},\widehat{1}\})}(x)$ lie strictly inside the unit circle then the convex hull of the zeros of $\frac{x^2-1}{2}F_{\triangle(P\setminus\{\widehat{0},\widehat{1}\})}(x)$ intersects the boundary of the unit circle in the set $\{-1, 1\}$. By Lucas' Theorem and Lemma 7.12 all zeros of $F_{\triangle(P\times I\setminus\{\widehat{0},\widehat{1}\})}(x)$ lie inside the unit circle, except possibly for -1 and 1. But -1 and 1 are single roots of $\frac{x^2-1}{2}F_{\triangle(P\setminus\{\widehat{0},\widehat{1}\})}(x)$ so they can not be roots of its derivative.

8. Derivative polynomials for hyperbolic tangent and secant

Equation (2) suggests having a closer look at the hyperbolic analogues of the derivative polynomials for tangent and secant.

Definition 8.1. *We call the polynomials $\widetilde{P}_n(x)$ and $\widetilde{Q}_n(x)$, defined by*

$$\frac{d^n}{dx^n}\tanh(x)=\widetilde{P}_n(\tanh(x)) \quad \text{and} \quad \frac{d^n}{dx^n}\operatorname{sech}(x)=\widetilde{Q}_n(\tanh(x))\cdot\operatorname{sech}(x).$$

the derivative polynomials for hyperbolic tangent and secant.

It is easy to verify that $\tanh(x)=\mathbf{i}\tan(x/\mathbf{i})$ hold for all complex numbers that are in the domain of $\tanh(x)$. Using this relation we obtain

$$\left(\frac{d}{dx}\right)^n\tanh(x)=\mathbf{i}^{-n}\left(\frac{d}{d(\mathbf{i}x)}\right)^n\mathbf{i}\tan(x/\mathbf{i})=\mathbf{i}^{-n+1}P_n(\tan(x/\mathbf{i}))=\mathbf{i}^{-n+1}P_n(\tanh(x)/\mathbf{i}),$$

implying

$$\widetilde{P}_n(x)=\mathbf{i}^{-n+1}P_n(x/\mathbf{i}). \tag{6}$$

It should be noted that the degree of $P_n(x)$ and $\widetilde{P}_n(x)$ is $n+1$, and the degrees of all of their terms have the same parity. The linear map induced by $x^{n+1-2k}\mapsto\mathbf{i}^{-n+1}(x/\mathbf{i})^{n+1-2k}=(-1)^{n-k}x^{n+1-2k}$ is an involution on the vector space whose basis is $\{x^{n+1-2k}\ :\ 0\le k\le\lfloor(n+1)/2\rfloor\}$. The same linear map may also described as $x^{n+1-2k}\mapsto\mathbf{i}^{n-1}(x\cdot\mathbf{i})^{n+1-2k}$, so we also have

$$\widetilde{P}_n(x)=\mathbf{i}^{n-1}P_n(x\cdot\mathbf{i}). \tag{7}$$

Similarly, $\operatorname{sech}(x)=\sec(x/\mathbf{i})$ implies

$$\widetilde{Q}_n(x)=\mathbf{i}^{-n}Q_n(x/\mathbf{i}), \tag{8}$$

the polynomials $Q_n(x)$ and $\widetilde{Q}_n(x)$ have degree n, and the degrees of all of their terms have the same parity. The linear map induced by $x^{n-2k}\mapsto\mathbf{i}^{-n}(x/\mathbf{i})^{n-2k}=(-1)^{n-k}x^{n-2k}$ is an involution on the vector space whose basis is $\{x^{n-2k}\ :\ 0\le k\le\lfloor n/2\rfloor\}$. The same linear map may also described as $x^{n-2k}\mapsto\mathbf{i}^{n}(x\cdot\mathbf{i})^{n-2k}$, so we also have

$$\widetilde{Q}_n(x)=\mathbf{i}^{n}Q_n(x\cdot\mathbf{i}), \tag{9}$$

Using this last equation we may rephrase equation (2) as follows.

Proposition 8.2. *The n-th derivative polynomial for hyperbolic secant equals up to a sign to the F-polynomial of the Tchebyshev transform of the Boolean algebra of rank n:*

$$F_{T(B_n)}(x)=T(F_{B_n})(x)=(-1)^n\widetilde{Q}_n(x).$$

Form now on, we will use $T_n^B(x)$ as a shorthand for $F_{T(B_n)}(x)=T(F_{B_n})(x)$, and call the polynomials $T_n^B(x)$ the *Boolean analogues of the Tchebyshev polynomials of the first kind*. Similarly, we introduce $U_{n-1}^B(x)$ as a shorthand for $U(F_{B_n})(x)=1/2\cdot F_{U(B_n)}(x)$ and call the polynomials $U_{n-1}^B(x)$ the *Boolean analogues of the Tchebyshev polynomials of the second kind*. Our next goal is to establish a relation between the polynomials $U_{n-1}^B(x)$ and the derivative polynomials $\widetilde{P}_n(x)$.

Lemma 8.3. *The polynomials $U_{n-1}^B(x)$ satisfy the recursion formula*

$$U_{n-1}^B(x)=2^{n-1}+\frac{x-1}{2}\cdot\sum_{m=1}^{n-1}\binom{n}{m}2^{n-m}U_{m-1}^B(x).$$

Proof. The argument is analogous to the calculation indicated in [5, Section 9]. As stated in Corollary 5.3, $U^B_{n-1}(x)$ is total weight of all chains in $T(B_n)$ that contain at least one coatom $(w,\widehat{1})$. Here $w \in B_n \setminus \{\widehat{1}\} \cup \{\widehat{-1}\}$, the 2^n singletons (containing a coatom only) contribute 2^{n-1}. For any other chain, consider its least element (u,v). Let us denote the rank of v by $n-m$. There are $\binom{n}{m}$ ways to choose v and 2^{n-m} ways to choose $u \in [\widehat{0}, v) \cup \{\widehat{-1}\}$. All chains whose least element is (u,v) are identifiable with all chains contributing to $U(F_{[v,\widehat{1}]})$. (The role of $\widehat{-1}$ is played by u). Since $[v,\widehat{1}] \cong B_m$, the total contribution of all chains with least element (u,v) is $U^B_{m-1}(x)$. □

Lemma 8.4. *The polynomials $U^B_{n-1}(x)$ have the generating function*

$$\sum_{n=1}^{\infty} U^B_{n-1}(x)\frac{t^n}{n!} = \frac{\sinh(t)}{\cosh(t) - x\cdot\sinh(t)}$$

Proof. The statement is equivalent to

$$\sum_{n=1}^{\infty} U^B_{n-1}(x)\frac{t^n}{n!} = \frac{\frac{1}{2}(e^{2t}-1)}{\frac{x+1}{2} - \frac{x-1}{2}e^{2t}},$$

We may rewrite this as

$$\frac{x+1}{2}\sum_{n=1}^{\infty} U^B_{n-1}(q)\cdot\frac{t^n}{n!} = \frac{1}{2}\left(e^{2t}-1\right) + \frac{x-1}{2}e^{2t}\sum_{n=1}^{\infty} U^B_{n-1}(x)\cdot\frac{t^n}{n!}.$$

Comparing the coefficients of t^n on both sides (and subtracting $(x-1)U^B_{n-1}/(2n!)$ from both sides) we obtain that the statement is equivalent to Lemma 8.3. □

Proposition 8.5. *The following relation exists between the derivative polynomials for hyperbolic tangent and the Boolean analogues of the Tchebyshev polynomials of the second kind for $n \geq 1$:*

$$(-1)^n\widetilde{P}_n(x) = (x^2-1)U^B_{n-1}(x).$$

Proof. In analogy to Hoffman's formula [8]

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = \frac{x+\tan(t)}{1-x\tan(t)}$$

we have

$$\sum_{n\geq 0} \widetilde{P}_n(x)\frac{t^n}{n!} = \frac{x+\tanh(t)}{1+x\tanh(t)}$$

This may be shown, for example, using Hoffman' formula, Equation 6 and the relation $\tanh(t) = \mathbf{i}\tan(t/\mathbf{i})$. Replacing t with $-t$ we obtain

$$\sum_{n\geq 0}(-1)^n\widetilde{P}_n(x)\frac{t^n}{n!} = \frac{x-\tanh(t)}{1-x\tanh(t)} = x + \frac{(x^2-1)\tanh(t)}{1-x\tanh(t)} = x + \frac{(x^2-1)\sinh(t)}{\cosh(t)-x\sinh(t)}.$$

The statement follows from Lemma 8.4 by comparing the coefficient of t^n. □

Theorem 8.6. *For $n \geq 1$, the zeroes $-1 < u_1 < \cdots < u_{n-1} < 1$ of $\widetilde{P}_n(x)$ and the zeros $t_1 < \cdots < t_n$ of $\widetilde{Q}_n(x)$ are all real, belong to the interval $[-1,1]$, and are interlaced $-1 < t_1 < u_1 < t_2 < \cdots < u_{n-1} < t_n < 1$.*

Proof. By Proposition 8.2 the zeros of $\widetilde{Q}_n(x)$ are exactly the same as the zeros of $T_n^B(x)$ (with the same multiplicities). By Proposition 8.5, the zeros of $\widetilde{P}_n(x)$ consist of -1, 1 (both with multiplicity 1), and the zeros of $U_n^B(x)$ (with the same multiplicities as in $U_n^B(x)$.) By Corollary 7.7 the Boolean algebra B_n is S-stable. By Proposition 6.5 the zeros of $T_n^B(x)$ and $U_{n-1}^B(x)$ are real, have multiplicity one, belong to the open interval $(-1,1)$ and are interlaced as stated. □

Corollary 8.7. *The zeros of $P_n(x)$ and $Q_n(x)$ are pure imaginary, have multiplicity 1, belong to the line segment $[-\mathbf{i},\mathbf{i}]$ and are interlaced with $-\mathbf{i}$ and $\mathbf{i}$ being zeros of $P_n(x)$.*

This is a consequence of equations (6) and (8).

9. The basis $U_{n-1}^B(x)$ and concluding remarks

The polynomials $U_{n-1}^B(x)$ have the following remarkable property:

Proposition 9.1.

$$U_{n-1}^B(x) = 2^{n-1} F_{\triangle(B_n \setminus \{\widehat{0},\widehat{1}\})}(x)$$

holds for $n \geq 1$.

Proof. The two sides are clearly equal for $n = 1$, and we proceed by induction. By Lemma 7.12, the polynomials $F_{\triangle(B_n \setminus \{\widehat{0},\widehat{1}\})}(x)$ satisfy the recursion formula

$$F_{\triangle(B_{n+1} \setminus \{\widehat{0},\widehat{1}\})}(x) = \frac{d}{dx}\frac{x^2-1}{2} F_{\triangle(B_n \setminus \{\widehat{0},\widehat{1}\})}(x), \tag{10}$$

hence it is sufficient to show that the polynomials $U_{n-1}^B(x)$ satisfy the recursion formula

$$U_n^B(x) = \frac{d}{dx}\left((x^2-1)U_{n-1}^B(x)\right). \tag{11}$$

One way to derive this is using Hoffman's formula [8, (1)]:

$$P_{n+1}(x) = P_n'(x)(x^2+1),$$

Equation (6), and Proposition 8.5. Another way is to rewrite (10) as a recursion for the polynomials $F_{B_n}(x)$ and deduce a recursion for the Tchebyshev transform of the second kind using a calculation similar to the ones in the proofs of Propositions 3.3 and 4.4. We mention this second way because it makes showing (11) independent of Hoffman's formula cited above. Thus we may generate a second proof of Proposition 8.5 by using Hoffman's formula above and (11) in an induction. □

Corollary 9.2. *The polynomials $U_{n-1}^B(x)$ are eigenvectors of the linear operator $p(x) \mapsto U(x \cdot p(x))$ defined on the vectorspace $\mathbb{R}[x]$.*

A generalization of this Corollary appears in the work of Ehrenborg and Readdy [2, Theorem 10.10] where they provide all eigenvectors for a generalization of the operator $p(x) \mapsto U(x \cdot p(x))$ to a vector space of polynomials in noncommuting variables, associated to flag enumeration. In our case this result suggest that the polynomials $U_{n-1}^B(x)$ form a very suitable basis for Tchebyshev transform calculations. Proposition 9.1 yields the explicit formula

$$U_{n-1}^B(x) = 2^{n-1}\sum_{j=0}^{n-1} S(n,j+1)(j+1)!\left(\frac{x-1}{2}\right)^j \tag{12}$$

where the numbers $S(n, j+1)$ are the Stirling numbers of the second kind. In fact, $f_{j-1}(\triangle(B_n \setminus \{\widehat{0}, \widehat{1}\}))$ equals the number of j-element increasing chains $\emptyset \neq \sigma_1 \subset \cdots \subset \sigma_j \subset \{1, \ldots, n\}$ of subsets in an n-element set which is equal to the number of ordered set partitions of $\{1, \ldots, n\}$ into $(j+1)$ parts. A similar explicit formula for the polynomials $P_n(x)$ follows immediately. Using the well known relations between the Stirling numbers of first and second kind, one easily obtains the formula

$$\sum_{m=0}^{n} s(n+1, m+1) \frac{U_m^B(x)}{2^m} = \left(\frac{x-1}{2}\right)^n (m+1)!$$

It should be noted that $s(n+1, m+1)/(m+1)!$ is the coefficient of x^{m+1} in $\binom{x}{j+1}$. Hence for any simplicial complex $\triangle$ we may write

$$F_\triangle(x) = \sum_{j \geq 0} f_{j-1}(\triangle) \left(\frac{x-1}{2}\right)^j = \sum_{m \geq 0} \frac{U_m^B(x)}{2^m} [x^{m+1}] \sum_{j \geq 0} f_{j-1}(\triangle) \binom{x}{j+1}. \tag{13}$$

The polynomial

$$s_\triangle(x) := \sum_{j \geq 0} f_{j-1}(\triangle) \binom{x}{j+1}$$

is a variant of the Stirling polynomial of a simplicial complex studied in [6]. For a graded poset we introduce

$$s_P(x) := \sum_{j \geq 0} f_{j-1}(\triangle(P \setminus \{\widehat{0}, \widehat{1}\})) \binom{x}{j+1}$$

and then we get from (13)

$$F_P(x) = x \cdot \sum_{j \geq 0} f_{j-1}(\triangle(P \setminus \{\widehat{0}, \widehat{1}\})) \left(\frac{x-1}{2}\right)^j = \sum_{m \geq 0} \frac{x U_m^B(x)}{2^m} [x^{m+1}] s_P(x). \tag{14}$$

Using the definition of $\widetilde{P}_n(x)$ and Proposition 8.5 we have

$$\left(\frac{d}{dz}\right)^{m+1} \tanh(z) = (-1)^{m+1} U_m^B(\tanh(z))(\tanh^2(x) - 1)$$

hence, substituting $x = \tanh(z)$ and multiplying both sides with $(\tanh^2(z) - 1)$ in (14) yields

$$(\tanh^2(z) - 1) F_P(\tanh(z)) = \sum_{m \geq 0} \frac{\tanh(z)(-1)^{m+1} \left(\frac{d}{dz}\right)^{m+1} \tanh(z)}{2^m} [x^{m+1}] s_P(x),$$

that is,

$$(\tanh^2(z) - 1) F_P(\tanh(z)) = (-2) \tanh(z) s_P\left(\frac{-d}{2dz}\right) \tanh(z). \tag{15}$$

This formula could perhaps be useful in proving Conjecture 7.8 some day, since the polynomials $s_P(x)$ have the following multiplicative property.

Proposition 9.3. *We have $s_{P \times Q}(x) = s_P(x) \cdot s_Q(x)$ for all pairs of graded posets P, Q.*

Proof. First let us note that $s_P(x)$ may be rewritten in terms of the face numbers of $\triangle(P)$ as follows:

$$s_P(x) = \sum_{j \geq 0} f_{j-1}(\triangle(P)) \binom{x-2}{j}. \tag{16}$$

In fact,

$$f_{j-1}(\triangle(P)) = f_{j-1}(\triangle(P \setminus \{\widehat{0},\widehat{1}\})) + 2f_{j-2}(\triangle(P \setminus \{\widehat{0},\widehat{1}\})) + f_{j-3}(\triangle(P \setminus \{\widehat{0},\widehat{1}\}))$$

holds since each j-chain in P contains up to two elements of $\{0,\widehat{1}\}$ and the rest is a j-chain, $(j-1)$-chain or $(j-2)$-chain in $P \setminus \{\widehat{0},\widehat{1}\}$. This equality and

$$\binom{x}{j+1} = \binom{x-2}{j+1} + 2\binom{x-2}{j} + \binom{x-2}{j-1}$$

imply (16). Now the statement follows from Lemma 7.9 and an appropriately shifted variant of [6, Lemma 3.2]. □

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References

[1] N. J. A. Sloane, "The On-Line Encyclopedia of Integer Sequences," published electronically at www.research.att.com/~njas/sequences/.
[2] R. Ehrenborg and M. Readdy, The Tchebyshev Transforms of the First and Second Kind, preprint 2004, arXiv:math.CO/0412124.
[3] G. Haigh, A "natural" approach to Pick's theorem, *Math. Gaz.* **64** (1980), 173–180.
[4] G. Hetyei, Central Delannoy numbers and balanced Cohen-Macaulay complexes, preprint 2005, `http://www.math.uncc.edu/preprint/2005/2005_02.pdf`, to appear in *Annals of Combinatorics*.
[5] G. Hetyei, Matrices of formal power series associated to binomial posets, *J. Algebraic Combin.* **22** (2005), 65–104.
[6] G. Hetyei, The Stirling polynomial of a simplicial complex, *Discrete & Comput. Geom.* **35** (2006), 437-455.
[7] G. Hetyei, Tchebyshev posets, *Discrete & Comput. Geom.* **32** (2004), 493–520.
[8] M. E. Hoffman, Derivative Polynomials for Tangent and Secant, *Amer. Math. Monthly* **102** (1995), 23–30.
[9] M. E. Hoffman, Derivative Polynomials, Euler Polynomials, and Associated Integer Sequences, *Electron. J. Comb.* **6** (1999), #R21.
[10] D. E. Knuth and T. J. Buckholtz, Computation of tangent, Euler and Bernoulli numbers, *Math. Comp.* **21** (1967) 663–688.
[11] C. Krichnamachary and Rao M. Bhimasena, On a table for calculating Eulerian numbers based on a new method, *Proc. London Math. Soc.* (2) **22** (1923) 73–80.
[12] M. Marden, "Geometry of polynomials," second ed., American Mathematical Society, Providence, Rhode Island, 1966.
[13] Charles W. Schelin, Counting zeros of real polynomials within the unit disk, *SIAM J. Numer. Anal.* **20** (1983), 1023–1031.
[14] R. P. Stanley, "Combinatorics and Commutative Algebra," second ed., Birkhäuser Boston, 1996.
[15] R. P. Stanley, "Enumerative Combinatorics, Volume I," Cambridge University Press, Cambridge, 1997.
[16] R. P. Stanley, Flag f-vectors and the *cd*-index, *Math. Z.* **216** (1994), 483–499.

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