SHIFTED JACOBI POLYNOMIALS AND DELANNOY NUMBERS

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À la mémoire de Pierre Leroux

Abstract. We express a weigthed generalization of the Delannoy numbers in terms of shifted Jacobi polynomials. A specialization of our formulas extends a relation between the central Delannoy numbers and Legendre polynomials, observed over 50 years ago [8], [13], [14], to all Delannoy numbers and certain Jacobi polynomials. Another specialization provides a weighted lattice path enumeration model for shifted Jacobi polynomials and allows the presentation of a combinatorial, non-inductive proof of the orthogonality of Jacobi polynomials with natural number parameters. The proof relates the orthogonality of these polynomials to the orthogonality of (generalized) Laguerre polynomials, as they arise in the theory of rook polynomials. We also find finite orthogonal polynomial sequences of Jacobi polynomials with negative integer parameters and expressions for a weighted generalization of the Schröder numbers in terms of the Jacobi polynomials.

Introduction

It has been noted more than fifty years ago [8], [13], [14] that the diagonal entries of the Delannoy array $(d_{m,n})$, introduced by Henri Delannoy [5], and the Legendre polynomials $P_n(x)$ satisfy the equality

$$d_{n,n} = P_n(3), \tag{1}$$

but this relation was mostly considered a "coincidence". An important observation of our present work is that (1) can be extended to

$$d_{n+\alpha,n} = P_n^{(\alpha,0)}(3) \quad \text{for all } \alpha \in \mathbb{Z} \text{ such that } \alpha \geq -n, \tag{2}$$

where $P_n^{(\alpha,0)}(x)$ is the Jacobi polynomial with parameters $(\alpha, 0)$. This observation in itself is a strong indication that the interaction between Jacobi polynomials (generalizing Legendre polynomials) and the Delannoy numbers is more than a mere coincidence. In this paper we develop formulas for weighted Delannoy numbers, obtain lattice path enumeration models for Jacobi polynomials with integer parameters, and provide a combinatorial proof for the orthogonality of several sequences of such Jacobi polynomials.

In Section 2 we introduce weighted Delannoy paths associating the same weight u to each east step, the same weight v to each north step and the same weight w to each

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northeast step. We show that the total weight of all Delannoy paths in a rectangle may be easily expressed by substitution into a shifted Jacobi polynomial with the appropriate parameters. The shifting is achieved by replacing x with $2x-1$ in the definition of the Jacobi polynomial.

In particular, a specific substitution into the parameters u, v, w yields the shifted Jacobi polynomials themselves. Using this observation, in Section 3 we provide a lattice path enumeration proof for the orthogonality of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ when $\alpha, \beta \in \mathbb{N}$. The proof reduces the weight enumeration to a formula that is known in the classical theory of rook polynomials, and is used there to show the orthogonality of the (generalized) Laguerre polynomials. This connection is explored a little further in Section 4, but a lot of work remains to be done in this area.

Using our weighted lattice path enumeration model, we can show a very simple formula expressing each Jacobi polynomial $P_n^{(0,-\beta)}(x)$ with negative integer second parameter and satisfying $n \geq \beta$, in terms of a Jacobi polynomial with positive integer second parameter. The remaining sequence $\{P_n^{(0,-\beta)}(x) : n \leq \beta - 1\}$ is symmetric, and its first half forms a finite orthogonal polynomial sequence. A combinatorial proof of this result may be found in Section 5.

Finally, in Section 6 we generalize Schröder numbers in a way that is completely analogous to our definition of weighted Delannoy numbers. We show that ordinary Schröder numbers too may be obtained by substituting 3 into an appropriate Jacobi polynomial, and prove a formula for iterated antiderivatives of the shifted Legendre polynomials.

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1. Preliminaries

1.1. **Delannoy numbers.** The *Delannoy array* $(d_{i,j} : i, j \in \mathbb{Z})$ was introduced by Henri Delannoy [5]. It may be defined by the recursion formula

$$d_{i,j} = d_{i-1,j} + d_{i,j-1} + d_{i-1,j-1} \tag{3}$$

with the conditions $d_{0,0} = 1$ and $d_{i,j} = 0$ if $i < 0$ or $j < 0$. The historic significance of these numbers is explained in the paper "Why Delannoy numbers?" [2] by Banderier and Schwer. The diagonal elements $(d_{n,n} : n \geq 0)$ in this array are the *central Delannoy numbers* (A001850 of Sloane [16]). These numbers are known through the books of Comtet [4] and Stanley [18], but it is Sulanke's paper [20] that gives the most complete enumeration of all known uses of the central Delannoy numbers.

1.2. **Jacobi and Legendre polynomials.** The n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of type (α,β) is defined as

$$P_n^{(\alpha,\beta)}(x) = (-2)^{-n}(n!)^{-1}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right).$$

If α and β are real numbers satisfying $\alpha,\beta > -1$, then the polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ form an orthogonal basis with respect to the inner product

$$\langle f,g\rangle := \int_{-1}^{1} f(x)\cdot g(x)\cdot (1-x)^{\alpha}(1+x)^{\beta}\, dx.$$

The following formula, stated only slightly differently in [19, (4.21.2)], may be used to extend the definition of $P_n^{(\alpha,\beta)}(x)$ to arbitrary complex values of α and β:

$$P_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n}\binom{n+\alpha+\beta+j}{j}\binom{n+\alpha}{n-j}\left(\frac{x-1}{2}\right)^j. \tag{4}$$

The following important relation provides a way to "swap" the parameters α and β (see [3, Chapter V, (2.8)]).

$$(-1)^n P_n^{(\alpha,\beta)}(-x) = P_n^{(\beta,\alpha)}(x). \tag{5}$$

The Legendre polynomials $\{P_n(x)\}_{n\geq 0}$ are the Jacobi polynomials $\{P_n^{(0,0)}(x)\}_{n\geq 0}$. They form an orthogonal basis with respect to the inner product

$$\langle f,g\rangle := \int_{-1}^{1} f(x)\cdot g(x)\, dx.$$

Substituting $\alpha=\beta=0$ into (4) yields

$$P_n(x) = \sum_{j=0}^{n}\binom{n+j}{j}\binom{n}{n-j}\left(\frac{x-1}{2}\right)^j. \tag{6}$$

As a specialization of (5) we obtain

$$(-1)^n P_n(-x) = P_n(x). \tag{7}$$

The shifted Legendre polynomials $\widetilde{P}_n(x)$ are defined by the linear substitution

$$\widetilde{P}_n(x) := P_n(2x-1).$$

They form an orthogonal basis with respect to the inner product

$$\langle f,g\rangle := \int_{0}^{1} f(x)\cdot g(x)\, dx.$$

They may be calculated using the following formula:

$$\widetilde{P}_n(x) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}x^k = \sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}\binom{2k}{k}x^k. \tag{8}$$

A generalization of (8) is stated and shown in Proposition 2.4. Shifted Legendre polynomials are widely used, they even have a table entry in the venerable opus of Abramowitz and

Stegun [1, 22.2.11]. Their obvious generalization, the *shifted Jacobi polynomials* $\widetilde{P}_n^{(\alpha,\beta)}(x)$, defined by the formula

$$\widetilde{P}_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(2x-1), \tag{9}$$

seem to occur much less frequently in the literature, a sample reference is the work of Gatteschi [6].

1.3. **Favard's theorem.** A good reference for the fundamental facts on orthogonal polynomials is Chihara's book [3]. A *moment functional* $\mathcal{L}$ is a linear map $\mathbb{C}[x] \to \mathbb{C}$. A sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ is an *orthogonal polynomial sequence* with respect to $\mathcal{L}$ if $p_n(x)$ has degree n, $\mathcal{L}(p_m(x)p_n(x)) = 0$ for $m \neq n$, and $\mathcal{L}(p_n^2(x)) \neq 0$ for all n. Such a sequence exists if and only if $\mathcal{L}$ is *quasi-definite* (see [3, Ch. I, Theorem 3.1], the term quasi-definite is introduced in [3, Ch. I, Definition 3.2]). Whenever an orthogonal polynomial sequence exists, each of its elements is determined up to a non-zero constant factor (see [3, Ch. I, Corollary of Theorem 2.2]).

A way to verify whether a sequence of polynomials is orthogonal is Favard's theorem [3, Chapter I, Theorem 4.4]. This states that a sequence of monic polynomials $\{p_n(x)\}_{n\geq 0}$ is an orthogonal polynomial sequence, if and only if it satisfies the recurrence formula

$$p_n(x) = (x - c_n)p_{n-1}(x) - \lambda_n p_{n-2}(x) \quad n = 1, 2, 3, \ldots \tag{10}$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$, the numbers c_n and λ_n are constants, $\lambda_n \neq 0$ for $n \geq 2$, and λ_1 is arbitrary (see [3, Ch. I, Theorem 4.1]). Conversely, for every sequence of monic polynomials defined in the above way there is a unique quasi-definite moment functional $\mathcal{L}$ such that $\mathcal{L}(1) = \lambda_1$ and $\{P_n(x)\}_{n=0}^{\infty}$ is the monic orthogonal polynomial sequence with respect to $\mathcal{L}$. The original proof provides only a recursive description of $\mathcal{L}$ for a given $\{p_n(x)\}_{n\geq 0}$ satisfying (10). Viennot [21] gave a combinatorial proof of Favard's theorem, upon which he has built a general combinatorial theory of orthogonal polynomials. In his theory, the values $\mathcal{L}(x^n)$ are explicitly given as sums of weighted *Motzkin paths.*

1.4. **Central Delannoy numbers and Legendre polynomials.** Equation (1) linking the central Delannoy numbers to the Legendre polynomials has been known for over 50 years [8], [13], [14]. Until recently there was a consensus that this link is not very relevant. Banderier and Schwer [2] note that there is no "natural" correspondence between Legendre polynomials and the original lattice path enumeration problem associated to the Delannoy array, while Sulanke [20] states that "the definition of Legendre polynomials does not appear to foster any combinatorial interpretation leading to enumeration".

The present author made two attempts to provide a combinatorial explanation. In [9] we find a generalization (1) to substitutions of 3 into Jacobi polynomials and an asymmetric variant of the Delannoy array, having the same diagonal elements. In [10] we find a construction of a polytope whose face numbers are the coefficients of the powers of $(x-1)/2$ in (6) and the Delannoy numbers enumerate faces having nonempty intersections with certain generalized orthants. Both attempts are "imperfect" in some sense: the first does not relate the original Delannoy array to substitutions into Jacobi polynomials, the second does not involve lattice path enumeration. The present work overcomes both of these "imperfections".

1.5. **Rook polynomials and orthogonal polynomials.** An excellent short summary of the classical theory of rook polynomials is given by Gessel in [7, Section 2]. For more information we follow his suggestion and refer the reader to Kaplansky and Riordan [12] and Riordan [15, pp. 163–277]. Let B be a subset of $[n] \times [n]$, i.e., a *board*. Here $[n]$ is a shorthand for $\{1, \ldots, n\}$. A subset S of B is called *compatible* if no two elements of S agree in either coordinate. The *rook polynomial of* B is defined as

$$r_B(x) := \sum_{k=0}^{n} (-1)^k r_k x^{n-k}$$

where r_k is the number of compatible k-subsets of B. Let $\mathcal{L}$ be the linear functional on polynomials in x defined by $\mathcal{L}(x^n) := n!$. Then

$$\mathcal{L}(p(x)) = \int_0^\infty e^{-x} p(x)\, dx$$

and the number of permutations π of $[n] \times [n]$ such that no $(i, \pi(i))$ belongs to B is $\mathcal{L}(r_B(x))$.

The rook polynomial of $[n] \times [n]$ is the *Laguerre polynomial*

$$l_n(x) := \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 k! x^{n-k} \tag{11}$$

In terms of the simple Laguerre polynomial as usually normalized,

$$l_n(x) = (-1)^n n! L_n(x).$$

It can be shown combinatorially that the Laguerre polynomials form an orthogonal basis with respect to the inner product induced by $\mathcal{L}$. We have

$$\mathcal{L}(l_m(x) l_n(x)) = \delta_{m,n} n!$$

Here $\delta_{m,n}$ is the Kronecker delta.

2. Weighted Delannoy numbers and shifted Legendre polynomials

A *Delannoy path* is a lattice path using only three kinds of steps: $(0,1)$ (east), $(1,0)$ (north), and $(1,1)$ (northeast). One of the most plausible generalizations of the Delannoy array is the following.

Definition 2.1. *Let u, v, w be commuting variables. We define the* weighted Delannoy numbers $d_{m,n}^{u,v,w}$ *as the total weight of all Delannoy paths from $(0,0)$ to (m,n), where each east step $(0,1)$ has weight u, each north step has weight v, and each northeast step has weight w. The weight of a lattice path is the product of the weights of its steps.*

The usual Delannoy numbers $d_{m,n}$ are obtained by substituting $u = v = w = 1$. Equation (3) generalizes to

$$d_{i,j}^{u,v,w} = u \cdot d_{i-1,j}^{u,v,w} + v \cdot d_{i,j-1}^{u,v,w} + w \cdot d_{i-1,j-1}^{u,v,w}, \tag{12}$$

and we have

$$d_{n,n}^{u,v,w} = \sum_{k=0}^{n} \binom{2n-k}{k} \binom{2n-2k}{n-k} u^{n-k} v^{n-k} w^k. \tag{13}$$

In fact, a Delannoy path from $(0,0)$ to (n,n) having k northeast steps has $(n-k)$ east and $(n-k)$ north steps, and there are

$$\binom{2n-k}{k, n-k, n-k} = \binom{2n-k}{k} \binom{2n-2k}{n-k}$$

ways to arrange these steps in some order. Using (8) and (13) we may show the following formula.

Lemma 2.2. *The weighted central Delannoy numbers are linked to the shifted Legendre polynomials by*

$$d_{n,n}^{u,v,w} = (-w)^n \widetilde{P}_n \left(-\frac{uv}{w} \right).$$

Proof. In a Delannoy path from $(0,0)$ to (n,n) the total number of north and northeast steps is n. Thus we have

$$d_{n,n}^{u,v,w} = (-w)^n d_{n,n}^{u,-v/w,-1}$$

which is equal to $(-w)^n d_{n,n}^{1,-uv/w,-1}$ since the number of east steps is the same as the number of north steps. By (13) we have

$$d_{n,n}^{1,-uv/w,-1} = \sum_{k=0}^{n} \binom{2n-k}{k} \binom{2n-2k}{n-k} \left(-\frac{uv}{w} \right)^{n-k} (-1)^k,$$

and the statement follows from (8). □

Substituting $u = v = w = 1$ into Lemma 2.2 yields

$$d_{n,n} = d_{n,n}^{1,1,1} = (-1)^n \widetilde{P}_n(-1) = (-1)^n \cdot P_n(-3).$$

Equation (1) now follows from the "swapping rule" (7). We may use the same rule to rewrite Lemma 2.2 as follows.

Proposition 2.3. *The weighted central Delannoy numbers and the shifted Legendre polynomials satisfy*

$$d_{n,n}^{u,v,w} = w^n \widetilde{P}_n \left(\frac{uv}{w} + 1 \right).$$

Proof. By definition, we have $(-1)^n \widetilde{P}_n(-x) = (-1)^n P_n(-2x-1)$. Using (7) we obtain

$$(-1)^n P_n(-2x-1) = P_n(2x+1) = \widetilde{P}_n(x+1).$$

Thus we may rewrite (7) for shifted Legendre polynomials as

$$(-1)^n \widetilde{P}_n(-x) = \widetilde{P}_n(x+1). \tag{14}$$

The statement now follows immediately from Lemma 2.2. □

Using Lemma 2.2 and Proposition 2.3 we obtain two infinite sets of weightings yielding the central Delannoy numbers $d_{n,n}$ as the total weight of all Delannoy paths from $(0,0)$ to (n,n). In fact, Lemma 2.2 implies

$$d_{n,n} = d_{n,n}^{r,1/r,1} \quad \text{for all } r \in \mathbb{R} \setminus \{0\},$$

whereas Proposition 2.3 yields

$$d_{n,n} = d_{n,n}^{r,2/r,-1} \quad \text{for all } r \in \mathbb{R} \setminus \{0\}.$$

In order to extend the validity of our formulas to non-central (weighted) Delannoy numbers, we prove the following generalization of (8) for the shifted Jacobi polynomials.

Proposition 2.4. *For all $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{C}$ we have*

$$(x-1)^\alpha \widetilde{P}_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n+\alpha} (-1)^{n+\alpha-k} x^k \binom{n+\alpha}{k} \binom{n+\beta+k}{n}.$$

Proof. Using (4) we may write

$$\begin{aligned}(x-1)^\alpha \widetilde{P}_n^{(\alpha,\beta)}(x) &= \sum_{j=0}^{n} \binom{n+\alpha+\beta+j}{j} \binom{n+\alpha}{n-j} (x-1)^{\alpha+j} \\ &= \sum_{j=0}^{n} \binom{n+\alpha+\beta+j}{j} \binom{n+\alpha}{\alpha+j} \sum_{k=0}^{\alpha+j} (-1)^{\alpha+j-k} \binom{\alpha+j}{k} x^k.\end{aligned}$$

Since

$$\binom{n+\alpha}{\alpha+j} \cdot \binom{\alpha+j}{k} = \binom{n+\alpha}{k} \cdot \binom{n+\alpha-k}{\alpha+j-k} = \binom{n+\alpha}{k} \cdot \binom{n+\alpha-k}{n-j},$$

changing the order of summation in the previous equation yields

$$(x-1)^\alpha \widetilde{P}_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n+\alpha} x^k \binom{n+\alpha}{k} \sum_{j=0}^{n} (-1)^{\alpha+j-k} \binom{n+\alpha+\beta+j}{j} \binom{n+\alpha-k}{n-j}.$$

It should be noted that $\binom{n+\alpha-k}{n-j} = 0$ if $\alpha < k$ and $j < k-\alpha$. Since

$$(-1)^j \binom{n+\alpha+\beta+j}{j} = \binom{-n-\alpha-\beta-1}{j},$$

the last equation may be rewritten as

$$(x-1)^\alpha \widetilde{P}_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n+\alpha} (-1)^{\alpha-k} x^k \binom{n+\alpha}{k} \sum_{j=0}^{n} \binom{-n-\alpha-\beta-1}{j} \binom{n+\alpha-k}{n-j}.$$

Using the well-known polynomial identity

$$\sum_{j=0}^{n}\binom{X}{j}\binom{Y}{n-j}=\binom{X+Y}{n}$$

we obtain

$$\sum_{j=0}^{n}\binom{-n-\alpha-\beta-1}{j}\binom{n+\alpha-k}{n-j}=\binom{-\beta-k-1}{n}.$$

The statement now follows from the last equation for $(x-1)^{\alpha}\widetilde{P}_n^{(\alpha,\beta)}(x)$ and from

$$\binom{-\beta-k-1}{n}=(-1)^n\binom{\beta+k+n}{n}.$$

□

Corollary 2.5. *In the case when $\alpha=0$ we obtain*

$$\widetilde{P}_n^{(0,\beta)}(x)=\sum_{k=0}^{n}(-1)^{n-k}x^k\binom{n}{k}\binom{n+\beta+k}{n}.$$

Equation (13) may be generalized to the following statement.

Proposition 2.6. *The weighted Delannoy numbers are given by the formula*

$$d_{m,n}^{u,v,w}=\sum_{k=0}^{n}\binom{m+n-k}{k}\binom{m+n-2k}{n-k}u^{m-k}v^{n-k}w^k.$$

In fact, a Delannoy path from $(0,0)$ to (m,n) containing k northeast steps must contain $(m-k)$ east and $(n-k)$ north steps, and these steps may be listed in

$$\binom{m+n-k}{k,m-k,n-k}=\binom{m+n-k}{k}\binom{m+n-2k}{n-k}\quad\text{ways.}$$

Using Corollary 2.5 and Proposition 2.6 we may generalize Lemma 2.2 to the main result of this section.

Theorem 2.7. *The weighted Delannoy numbers and the shifted Jacobi polynomials are linked by the formula*

$$d_{n+\beta,n}^{u,v,w}=u^{\beta}(-w)^n\widetilde{P}_n^{(0,\beta)}\left(-\frac{uv}{w}\right).$$

Here $\beta\in\mathbb{Z}$ is any integer satisfying $\beta\geq -n$.

Proof. By Proposition 2.6 we have

$$d_{n+\beta,n}^{u,v,w}=\sum_{k=0}^{n}\binom{2n+\beta-k}{k}\binom{2n+\beta-2k}{n-k}u^{n+\beta-k}v^{n-k}w^k.$$

Replacing k with $(n-k)$ yields

$$d^{u,v,w}_{n+\beta,n} = \sum_{k=0}^{n} \binom{n+\beta+k}{n-k}\binom{\beta+2k}{k} u^{\beta+k} v^k w^{n-k}.$$

Here

$$\binom{n+\beta+k}{n-k}\binom{\beta+2k}{k} = \binom{n+\beta+k}{n-k, \beta+k, k} = \binom{n+\beta+k}{n}\binom{n}{k},$$

thus we also have

$$d^{u,v,w}_{n+\beta,n} = \sum_{k=0}^{n} \binom{n+\beta+k}{n}\binom{n}{k} u^{\beta+k} v^k w^{n-k}.$$

On the other hand, by Corollary 2.5 we have

$$(-w)^n \widetilde{P}^{(0,\beta)}_n\left(-\frac{uv}{w}\right) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+\beta+k}{n} u^k v^k w^{n-k},$$

which differs from $d^{u,v,w}_{n+\beta,n}$ only by a factor of u^β. □

Direct substitution of $u = v = w = 1$ into Theorem 2.7 yields

$$d_{n+\beta,n} = (-1)^n \widetilde{P}^{(0,\beta)}_n(-1) = (-1)^n P^{(0,\beta)}_n(-3), \tag{15}$$

a simpler formula may be obtained by using the "swapping rule" (5), which, after replacing the letter β with α, yields (2). We conclude this section with the generalization of Proposition 2.3 to all weighted Delannoy numbers.

Proposition 2.8. *The weighted Delannoy numbers and the shifted Jacobi polynomials satisfy*

$$d^{u,v,w}_{n+\beta,n} = u^\beta w^n \widetilde{P}^{(\beta,0)}_n\left(\frac{uv}{w}+1\right).$$

Proof. In analogy to the argument seen in the proof of Proposition 2.3, we may rewrite (5) for shifted Jacobi polynomials as

$$(-1)^n \widetilde{P}^{(\alpha,\beta)}_n(-x) = \widetilde{P}^{(\beta,\alpha)}_n(x+1). \tag{16}$$

The statement is an immediate consequence of this swapping rule and Theorem 2.7. □

3. Combinatorial proofs of certain orthogonality relations

As a consequence of Theorem 2.7 we obtain the following lattice path representations of the shifted Legendre polynomials $\widetilde{P}_n(x) = \widetilde{P}^{(0,0)}_n(x)$ and the shifted Jacobi polynomials $\widetilde{P}^{(0,\beta)}_n(x)$.

Corollary 3.1. *For all $\beta \in \mathbb{Z}$ satisfying $\beta \geq -n$, the shifted Jacobi polynomial $\widetilde{P}^{(0,\beta)}_n(x)$ is $d^{1,x,-1}_{n+\beta,n}$, i.e., the total weight of all Delannoy paths from $(0,0)$ to $(n+\beta,n)$, where each east step has weight 1, each north step has weight x, and each northeast step has weight -1.*

In this section we use this Proposition to provide a combinatorial proof for all $\alpha, \beta \in \mathbb{N}$, of the fact that the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ form an orthogonal basis with respect to the inner product

$$\langle f, g\rangle := \int_{-1}^{1} f(x)\cdot g(x)\cdot (1-x)^{\alpha}(1+x)^{\beta}\ dx.$$

Using the substitution $u := (x+1)/2$ yields

$$\int_{-1}^{1} f(x)\cdot g(x)\cdot (1-x)^{\alpha}(1+x)^{\beta}\ dx = 2^{\alpha+\beta+1}\int_{0}^{1} f(2u-1)\cdot g(2u-1)\cdot (1-u)^{\alpha}u^{\beta}\ du$$

thus the orthogonality relations we want to prove may be restated for shifted Jacobi (and Legendre) polynomials as follows.

Theorem 3.2. *For all $\alpha, \beta \in \mathbb{N}$, the shifted Jacobi polynomials $\widetilde{P}_n^{(\alpha,\beta)}(x)$ form an orthogonal basis with respect to the inner product*

$$\langle f, g\rangle := \int_{0}^{1} f(x)\cdot g(x)\cdot (1-x)^{\alpha}x^{\beta}\ dx.$$

First we prove Theorem 3.2 for the case $\alpha = 0$ only. Since an inner product of any polynomial with itself is nonzero, it is sufficient to show the following orthogonality relation.

Proposition 3.3. *For all $m, n, \beta \in \mathbb{N}$ satisfying $m < n$ we have*

$$\int_{0}^{1} x^{m+\beta}\cdot \widetilde{P}_n^{(0,\beta)}(x)\ dx = 0.$$

Proof. Using the lattice path representation stated in Corollary 3.1, each Delannoy path from $(0,0)$ to $(n+\beta, n)$ containing k north steps contributes a term $(-1)^{n-k}x^k$ to $\widetilde{P}_n^{(0,\beta)}(x)$, and a term $(-1)^{n-k}x^{k+m+\beta}$ to $x^{m+\beta}\cdot \widetilde{P}_n^{(0,\beta)}(x)$. Since

$$\int_{0}^{1} x^{k+m+\beta}\ dx = \frac{1}{k+m+\beta+1},$$

we obtain that each Delannoy path from $(0,0)$ to $(n+\beta, n)$ containing k north steps contributes a term

$$\frac{(-1)^{n-k}(n+m+\beta+1)!}{k+m+\beta+1} \quad \text{to} \quad (n+m+\beta+1)!\cdot\int_{0}^{1} x^{m+\beta}\cdot \widetilde{P}_n^{(0,\beta)}(x)\ dx.$$

Therefore

$$(n+m+\beta+1)!\cdot\int_{0}^{1} x^{m+\beta}\cdot \widetilde{P}_n^{(0,\beta)}(x)\ dx$$

is the total weight of all pairs (L, σ) where L is a Delannoy path from $(0,0)$ to $(n+\beta, n)$ and σ is a bijection $\{r, a_1, \ldots, a_{n+\beta}, b_1, \ldots, b_m\} \to \{1, \ldots, m+n+\beta+1\}$, subject to the following rules:

(i) $\sigma(r) < \sigma(a_i)$ holds for all i such that there is an east step in L from $(i-1, y)$ to (i, y) for some y;

(ii) $\sigma(r) < \sigma(b_j)$ holds for $j = 1, 2, \ldots, m$.

The weight of the pair (L, σ) is defined to be a function of the Delannoy path L: each northeast step contributes a factor of (-1), all other steps contribute a factor of 1.

In fact, a Delannoy path L from $(0,0)$ to $(n+\beta, n)$ with k north steps has $n-k$ northeast steps and $n+\beta-(n-k) = \beta+k$ east steps. After fixing L, condition (i) above requires $\sigma(r)$ to be less than $\beta+k$ elements in $\{\sigma(a_1), \ldots, \sigma(a_{n+\beta})\}$. Conditions (i) and (ii) together force exactly $\beta+k+m$ elements of $\{1, \ldots, m+n+\beta+1\}$ to be more than $\sigma(r)$. Thus the number of bijections σ forming a valid pair (L, σ) with this fixed L is $(n+m+\beta+1)!/(k+m+\beta+1)$ and the total contribution of all pairs (L, σ) for this fixed L is $(-1)^{n-k}(n+m+\beta+1)!/(k+m+\beta+1)$. Let us call the pairs (L, σ) satisfying (i) and (ii) *valid pairs*.

Let us now eliminate the contribution of some valid pairs (L, σ) by introducing the following involutions τ_i for $i = 1, 2, \ldots, n+\beta$. If the Delannoy path L contains a northeast step from some $(i-1, y)$ to $(i, y+1)$ and $\sigma(r) < \sigma(a_i)$ then we define $\tau_i(L, \sigma)$ as (L', σ) where L' is obtained from L by replacing the northeast step from $(i-1, y)$ to $(i, y+1)$ with an east step from $(i-1, y)$ to (i, y) followed by a north step from (i, y) to $(i, y+1)$. (Note that the condition $\sigma(r) < \sigma(a_i)$ guarantees that (L', σ) is a valid pair.) If L contains an east step from $(i-1, y)$ to (i, y) followed by a north step from (i, y) to $(i, y+1)$, we define $\tau_i(L, \sigma)$ as (L', σ) where L' is obtained from L by replacing the sequence of one east step and one north step from (i, y) to $(i, y+1)$ by a diagonal step from (i, y) to $(i, y+1)$. (Note that condition (i) for (L, σ) requires $\sigma(r) < \sigma(a_i)$ in this case.) In all other situations we define $\tau_i(L, \sigma) := (L, \sigma)$. Obviously, the τ_i's are involutions on the set of valid pairs, and they commute pairwise, since they change disjoint parts of the underlying Delannoy paths. Thus they induce a $\mathbb{Z}_2^{n+\beta}$-action on the set of valid pairs. If $\tau_i(L, \sigma) \neq (L, \sigma)$ then the weight of $\tau_i(L, \sigma)$ is the negative of the weight of (L, σ). Thus the weight of all (L, σ) pairs that belong to the same $\mathbb{Z}_2^{n+\beta}$-orbit cancels, unless the orbit in question contains a single fixed point.

We have shown that

$$(n+m+\beta+1)! \cdot \int_0^1 x^{m+\beta} \cdot \widetilde{P}_n^{(0,\beta)}(x)\, dx$$

equals the total weight of all valid pairs (L, σ) satisfying $\tau_i(L, \sigma) = (L, \sigma)$ for $i = 1, 2, \ldots, n+\beta$. These pairs may be characterized by the following two conditions:

(iii) $\sigma(r) > \sigma(a_i)$ holds for all i such that there is a northeast east step in L from $(i-1, y)$ to $(i, y+1)$ for some y;

(iv) there is no east step immediately followed by a north step.

Condition (iv) is a statement about the Delannoy path L only. It may be stated equivalently by requiring that the only way to go before the first, after the last, or between two consecutive northeast steps is to use all the north steps before all the east steps. Such a Delannoy path is uniquely determined by the set of its diagonal steps. A sequence of diagonal steps can be completed to a Delannoy path satisfying (iv) if and only if the first coordinates and the second coordinates of the starting points both form a strictly increasing sequence. Thus there are $\binom{n+\beta}{k}\binom{n}{k}$ Delannoy paths satisfying (iv). Given a Delannoy path L satisfying (iv) having k diagonal steps, conditions (i), (ii), and (iii) are equivalent to stating that $\sigma(r) = k+1$, $\sigma(a_i) \le k$ for all i such that there is a northeast east step in L from $(i-1, y)$ to $(i, y+1)$ for some $y+1$, and the image of all remaining elements under σ belongs to $\{k+1, k+2, \ldots, n+m+\beta+1\}$. There are $k!(n+m+\beta-k)!$ ways to find such a σ.

Therefore we obtain the following equality:

$$(n+m+\beta+1)! \cdot \int_0^1 x^{m+\beta} \cdot \widetilde{P}_n^{(0,\beta)}(x)\, dx = \sum_{k=0}^{n} (-1)^k \binom{n+\beta}{k}\binom{n}{k} \cdot k!(n+m+\beta-k)! \tag{17}$$

We are left to show that the right hand side is zero for $m < n$. This is known in the theory of rook polynomials (see Section 4), so here we indicate a short proof for completeness sake only. The right hand side is $(m+\beta)!$ times

$$p(m) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n+\beta)_k (n+m+\beta-k)_{n-k} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (n+\beta)_k (-m-\beta-1)_{n-k}.$$

Consider $p(m)$ as a polynomial function of m. The number $(-1)^n p(-m)$ is then the number of ways to select a k-element subset of an n-element set and injectively color its elements using $n+\beta$ colors, then color the remaining $n-k$ elements injectively, using a disjoint set of $m-\beta-1$ colors. Thus we have

$$(-1)^n p(-m) = \binom{n+m-1}{n} \quad \text{implying} \quad p(m) = (-1)^n \binom{n-m-1}{n}.$$

This is zero for $m < n$. □

The rest of the proof of Theorem 3.2 is a consequence of the following Proposition.

Proposition 3.4. *For all $\alpha, \beta \in \mathbb{N}$, the shifted Jacobi polynomials satisfy*

$$(x-1)^{\alpha} \widetilde{P}_n^{(\alpha,\beta)}(x) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i x^{\alpha-i} \widetilde{P}_n^{(0,\alpha+\beta-i)}(x).$$

Proof. First we show using Proposition 2.4 that $(x-1)^{\alpha}\widetilde{P}_n^{(\alpha,\beta)}(x)$ is the total weight of all Delannoy paths from $(0,0)$ to $(n+\alpha+\beta, n+\alpha)$ subject to the restriction that none of the first α steps is an east step. Here each east step contributes a factor of 1, each north step contributes a factor of x and each northeast step contributes a factor of -1. In fact, if such a Delannoy path has k north steps then it has $n+\alpha-k$ northeast steps and $\beta+k$ east steps. There are $\binom{n+\alpha}{k}$ ways to determine the order of the north and the northeast steps among themselves and then there are $\binom{n+\beta}{n}$ ways to determine the order of the $\beta+k$ east steps with respect to the n other steps (we subtracted α from $n+\alpha$ for the first α steps which can not be east steps).

Every Delannoy path subject to the above restriction may be uniquely decomposed as follows. The first α steps form a Delannoy path with no east step from $(0,0)$ to some (i,α) where $0 \leq i \leq \alpha$ is the number of northeast steps among the first α steps. These contribute a factor of $(-1)^i x^{\alpha-i}$. Given i, there are $\binom{\alpha}{i}$ ways to determine of the order of the first α steps. The rest of the path is an unrestricted Delannoy path from (i,α) to $(n+\alpha+\beta, n+\alpha)$, shifting its start to the origin yields a Delannoy path from $(0,0)$ to $(n+\alpha+\beta-i, n)$. This part of the path may be chosen independently among all Delannoy paths from $(0,0)$ to $(n+\alpha+\beta-i,n)$ and, by Corollary 3.1, their total contribution is $\widetilde{P}_n^{(0,\alpha+\beta-i)}(x)$. □

Using Proposition 3.4 we may conclude the proof of Theorem 3.2 as follows. Assume $\alpha, \beta \in \mathbb{N}$ and $m < n$. Then

$$\begin{aligned}\int_0^1 x^m \cdot \widetilde{P}_n^{(\alpha,\beta)}(x) \cdot (1-x)^\alpha x^\beta \, dx &= (-1)^\alpha \int_0^1 x^{m+\beta} \cdot \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i x^{\alpha-i} \widetilde{P}_n^{(0,\alpha+\beta-i)}(x) \, dx \\ &= \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^{\alpha+i} \int_0^1 x^{m+(\alpha+\beta-i)} \cdot \widetilde{P}_n^{(0,\alpha+\beta-i)}(x) \, dx.\end{aligned}$$

All integrals in the last sum are zero by Proposition 3.3.

4. Connections to the classical theory of rook polynomials

For $\beta = 0$ equation (17) takes the form

$$(n+m+1)! \cdot \int_0^1 x^m \cdot \widetilde{P}_n(x) \, dx = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \cdot k!(n+m-k)! \tag{18}$$

The right hand side here is $\mathcal{L}(x^m l_n(x))$ where $l_n(x)$ is the Laguerre polynomial defined in (11). The orthogonality of these Laguerre polynomials with respect to the inner product induced by $\mathcal{L}$ is equivalent to stating

$$\mathcal{L}(x^m l_n(x)) = \int_0^\infty x^m l_n(x) e^{-x} \, dx = 0 \quad \text{for } m < n.$$

Thus the combinatorial orthogonality of the shifted Legendre polynomials reduces to the combinatorial orthogonality of the Laguerre polynomials used in the theory of rook polynomials. At a hasty look, the connection is "almost obvious" visually, since in both combinatorial situations the term $\binom{n}{k}^2$ expresses the number of ways to select k rows and k columns on some $n \times n$ "chess-board", as the rows and columns where diagonal steps, respectively, rooks are located. The main difference is in the next step of the selection process: in the case of the valid pairs (L,σ) that are fixed points of the involutions τ_i the diagonal steps must follow a strictly increasing order in the picture, and a factor of $k!$ arises by selecting the restriction of an injective map σ to the columns associated to the diagonal steps. In rook polynomial theory, however, the selected rooks may have any permutation pattern, thus a factor of $k!$ arises in a completely different way. It should be also noted that the condition $m < n$ was not used in the proof of (17), thus we may state in general

$$(n+m+1)! \cdot \int_0^1 x^m \cdot \widetilde{P}_n(x)\,dx = \int_0^\infty x^m l_n(x) e^{-x}\,dx \quad \text{for all } m,n \in \mathbb{N}. \tag{19}$$

For general $\beta \in \mathbb{N}$, equation (17) may be rewritten as

$$(n+m+\beta+1)! \cdot \int_0^1 x^{m+\beta} \cdot \widetilde{P}_n^{(0,\beta)}(x)\,dx = \int_0^\infty x^m l_n^{(\beta)}(x) x^\beta e^{-x}\,dx \quad \text{for all } m,n \in \mathbb{N}. \tag{20}$$

Here

$$l_n^{(\beta)}(x) := \sum_{k=0}^{n} (-1)^k \binom{n+\beta}{k}\binom{n}{k} k! x^{n-k}$$

is the n-th *generalized Laguerre polynomial* associated to the rectangular board $[n+\beta]\times[n]$ (as a subset of itself). These generalized Laguerre polynomials are related to the usually normalized generalized Laguerre polynomials $L_n^{(\beta)}(x)$ by the formula

$$l_n^{(\beta)}(x) = (-1)^n n! L_n^{(\beta)}(x),$$

and form an orthogonal basis with respect to the weight function $x^\beta e^{-x}$.

5. Shifted Jacobi polynomials $\widetilde{P}_n^{(0,\beta)}(x)$ with negative integer β

Using weighted Delannoy numbers it is easy to connect most shifted Jacobi polynomials $\widetilde{P}_n^{(0,\beta)}(x)$ with negative integer β to the Jacobi polynomials with positive integer β. In fact, we have the following formula.

Proposition 5.1. *For $\beta \in \mathbb{N}$ and $n \geq \beta$ we have*

$$\widetilde{P}_n^{(0,-\beta)}(x) = x^\beta \widetilde{P}_{n-\beta}(x).$$

Proof. By Corollary 3.1 we have

$$\widetilde{P}_n^{(0,\beta)}(x) = d_{n+\beta,n}^{1,x,-1}.$$

Dividing the weight of all north and northeast steps by x we may take out a factor of x^n and get

$$\widetilde{P}_n^{(0,\beta)}(x) = x^n d_{n+\beta,n}^{1,1,-1/x}.$$

Since each east and each north step has the same weight, we may swap the horizontal and vertical axis and get

$$\widetilde{P}_n^{(0,\beta)}(x) = x^n d_{n,n+\beta}^{1,1,-1/x}.$$

Finally, multiplying both sides by x^β and using the arising factor of $x^{n+\beta}$ on the right hand side to multiply the weight of each north and each northeast step by x yields

$$x^\beta \widetilde{P}_n^{(0,\beta)}(x) = d_{n,n+\beta}^{1,x,-1}.$$

Here the right hand side is equal to $\widetilde{P}_{n+\beta}^{(0,\beta)}(x)$ by Corollary 3.1. The statement follows by replacing n with $n-\beta$. □

Proposition 5.1 does not apply to $\widetilde{P}_n^{(0,-\beta)}(x)$ when $n < \beta$. Table 1 contains these polynomials and $\widetilde{P}_6^{(0,-6)}(x)$ for $\beta = 6$.

n	0	1	2	3	4	5	6
$\widetilde{P}_n^{(0,-6)}(x)$	1	$5-4x$	$3x^2-12x+10$	$3x^2-12x+10$	$5-4x$	1	x^6

TABLE 1. $\widetilde{P}_n^{(0,-6)}(x)$ for $n \leq 6$

The list of polynomials exhibits a certain symmetry which is most easily shown for another shifted version of the Jacobi polynomials.

Definition 5.2. *We define the* transformed Jacobi polynomials $\widehat{P}_n^{(\alpha,\beta)}(x)$ *by the formula*

$$\widehat{P}_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(2x+1).$$

As an immediate consequence of (4) we obtain the following.

$$\widehat{P}_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} \binom{n+\alpha+\beta+j}{j} \binom{n+\alpha}{n-j} x^j. \tag{21}$$

In particular, substituting $\alpha = 0$ and $-\beta$ for β yields

$$\widehat{P}_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} \binom{n-\beta+j}{j} \binom{n}{j} x^j,$$

which may be rewritten as

$$\widehat{P}_n^{(0,-\beta)}(x) = \sum_{j=0}^{n} \binom{\beta-1-n}{j} \binom{n}{j} (-x)^j. \tag{22}$$

As an immediate consequence of this last equation we obtain

Corollary 5.3. *For $\beta \in \mathbb{N}$ and $0 \leq n \leq \beta - 1$ we have*

$$\widehat{P}_n^{(0,-\beta)}(x) = \widehat{P}_{\beta-1-n}^{(0,-\beta)}(x),$$

implying also

$$P_n^{(0,-\beta)}(x) = P_{\beta-1-n}^{(0,-\beta)}(x) \quad \textit{and} \quad \widetilde{P}_n^{(0,-\beta)}(x) = \widetilde{P}_{\beta-1-n}^{(0,-\beta)}(x).$$

Because of Corollary 5.3 the Jacobi polynomials $\{P_n^{(0,-\beta)}(x)\}_{n\geq 0}$ can not form an orthogonal polynomial sequence. There is a chance to having such a sequence only up to $n = \lfloor (\beta-1)/2 \rfloor$. Surprisingly, up to that value of n we do have a finite sequence of orthogonal polynomials! This is perhaps most easily shown for the transformed Jacobi polynomials.

Theorem 5.4. *Let $\beta \geq 2$ be any positive integer and let $\mathcal{L}$ be the linear functional defined defined on the vector space $\{p(x) \in \mathbb{C}[x] \,:\, \deg(p) \leq (\beta-2)/2\}$ by*

$$\mathcal{L}(x^k) = k! \cdot (\beta-2-k)! \quad \textit{for } 0 \leq k \leq \beta - 2.$$

Then the transformed Jacobi polynomials $\{\widehat{P}_n^{(0,-\beta)}(x) : 0 \le n \le (\beta-2)/2\}$ *form an orthogonal basis in the with respect to inner product* $\langle f, g\rangle := \mathcal{L}(f\cdot g)$*. For odd* β *we may extend* $\mathcal{L}$ *and the induced inner product to polynomials of degree at most* $(\beta-1)/2$ *by making* $\mathcal{L}(x^{\beta-1})$ *large enough to make the determinant of the* $(\beta+1)/2\times(\beta+1)/2$ *matrix*

$$M_\beta := \begin{pmatrix} \mathcal{L}(x^0) & \mathcal{L}(x^1) & \cdots & \mathcal{L}(x^{(\beta-1)/2}) \\ \mathcal{L}(x^1) & \mathcal{L}(x^2) & \cdots & \mathcal{L}(x^{(\beta-1)/2+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}(x^{(\beta-1)/2}) & \mathcal{L}(x^{(\beta-1)/2+1}) & \cdots & \mathcal{L}(x^{\beta-1}) \end{pmatrix}$$

positive. The polynomial $\widehat{P}_{(\beta-1)/2}^{(0,-\beta)}(x)$ *may then be added to the orthogonal basis.*

Proof. First we show that the bilinear operation $\langle f, g\rangle := \mathcal{L}(f\cdot g)$ in an inner product on the vector space of polynomials of degree at most $(\beta-2)/2$. In fact, for $0\le k\le \beta-2$ we have

$$\mathcal{L}(x^k) = (\beta-1)!B(k+1, \beta-1-k).$$

Here $B(z,w)$ is the beta function [1, 6.2.1] satisfying [1, 6.2.2]

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

for the gamma function [1, 6.1.1] $\Gamma(z)$ which satisfies [1, 6.1.15] $\Gamma(z+1) = z!$ for all $z\in\mathbb{N}$. By the definition of the beta function [1, 6.2.1] we have

$$\mathcal{L}(x^k) = (\beta-1)!\int_0^1 \left(\frac{t}{1-t}\right)^k (1-t)^{\beta-2}\, dt \quad \text{for } 0\le k\le\beta-2. \tag{23}$$

Using this equation it is easy deduce

$$\langle f, g\rangle = (\beta-1)!\int_0^1 f\left(\frac{t}{1-t}\right)\cdot g\left(\frac{t}{1-t}\right)\cdot(1-t)^{\beta-2}\, dt \quad \text{if } \deg(f)+\deg(g)\le\beta-2. \tag{24}$$

Note that the condition $\deg(f)+\deg(g)\le\beta-2$ implies that the degree of $(1-t)$ in the denominator of $f(t/(1-t))\cdot g(t/(1-t))$ is at most $\beta-2$, and so the integrand on the left hand side is a polynomial function. Thus the bilinear function is in deed an inner product. Thus, to prove the first half of the theorem, it is sufficient to show

$$\langle x^m, \widehat{P}_n^{(0,-\beta)}(x)\rangle = 0 \quad \text{for } m<n\le \tfrac{\beta-2}{2}.$$

By the definition of $\mathcal{L}$ this is equivalent to

$$\sum_{j=0}^n \binom{\beta-1-n}{j}\binom{n}{j}(-1)^j(m+j)!(\beta-2-m-j)! = 0 \quad \text{for } m<n\le\tfrac{\beta-2}{2}.$$

Dividing both sides by $(\beta-1-n)!m!(n-m-1)!$ yields the equivalent form

$$\sum_{j=0}^n (-1)^j\binom{n}{j}\binom{m+j}{m}\binom{\beta-2-m-j}{n-m-1} = 0 \quad \text{for } m<n\le\tfrac{\beta-2}{2}. \tag{25}$$

The left hand side of (25) is the total weight of all triplets (X, A, B) where

(i) X is a subset of $\{1, 2, \ldots, n\}$;
(ii) $A = \{a_1, \ldots, a_m\}$ is an m-element multiset such that each a_i belongs to $X \cup \{0\}$;
(iii) $B = \{b_1, \ldots, b_{n-m-1}\}$ is an $(n-m-1)$-element multiset such that each b_j belongs to $\{1, \ldots, \beta - n\} \setminus X$.

The weight of the triplet (X, A, B) is $(-1)^{|X|}$. In fact, for a fixed j there are $\binom{n}{j}$ ways to select X. Once X is fixed, there are

$$\binom{(j+1)+m-1}{m} = \binom{m+j}{m}$$

ways to select the multiset A on $X \cup \{0\}$ and

$$\binom{(\beta-n-j)+(n-m-1)-1}{n-m-1} = \binom{\beta-m-2-j}{n-m-1}$$

ways to select the multiset B on $\{1, \ldots, \beta - n - 1\} \setminus X$. Let us now fix the multisets A and B first. Since $|A| + |B| = n - 1$, there is at least one $c \in \{1, \ldots, n\}$ that does not appear in A, nor in B. Then, for a subset $X \subset \{1, \ldots, n\} \setminus \{c\}$, the triplet (X, A, B) satisfies the above conditions if and only if the triplet $(X \cup \{c\}, A, B)$ does. The weight of these triplets cancel each other.

For the case when β is odd, observe first that the above proof of orthogonality can be adapted to $m < n \leq (\beta-1)/2$ without any changes since we only need to use the value of $\mathcal{L}(x^k)$ for $k \leq \beta - 2$. The only thing left to show is that we can set the value of $\mathcal{L}(x^{\beta-1})$ in such a way that we get a positive definite inner product. It is well known that a bilinear form defined by a symmetric real matrix is positive definite if the principal minors of the matrix are positive. Thus we only need to make sure that the principal minors of M_β are positive. The fact that the restriction of $\mathcal{L}$ to the vector space of polynomials of degree at most $(\beta-3)/2$ is positive definite implies the positivity of all principal minors of M_β except for $\det(M_\beta)$. Finally, consider $\mathcal{L}(x^\beta)$ as a variable, and expand $\det(M_\beta)$ by its last column. The determinant becomes a linear function of $\mathcal{L}(x^\beta)$ whose coefficient is the $(\beta-3)/2 \times (\beta-3)/2$ principal minor of M_β, a positive number. Therefore $\det(M_\beta) > 0$ holds for any sufficiently large $\mathcal{L}(x^\beta)$. □

6. Weighted Schröder numbers and iterated antiderivatives of Legendre polynomials

We define a *Schröder path* from $(0,0)$ to (n,n) as a Delannoy path not going above the line $y = x$. In analogy to Definition 2.1 we may introduce weighted Schröder numbers as follows.

Definition 6.1. *Let u, v, w be commuting variables. We define the* weighted Schröder numbers $s_n^{u,v,w}$ *as the total weight of all Schröder paths from $(0,0)$ to (n,n), where each east step $(0,1)$ has weight u, each north step has weight v, and each northeast step has weight w. The weight of a lattice path is the product of the weights of its steps.*

Introducing the *Schröder polynomials* by

$$S_n(x) := s_n^{1,x,-1} \quad \text{for } n \geq 0, \tag{26}$$

we have the following formula.

Proposition 6.2. *For $n \geq 1$ the Schröder polynomials are given by*

$$S_n(x) = \frac{x-1}{(n+1)x} \widetilde{P}_n^{(1,-1)}(x).$$

Proof. First we show that the Schröder polynomials satisfy

$$S_n(x) = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{j+1} \binom{2j}{j} \binom{n+j}{n-j} x^j \quad \text{for } n \geq 1. \tag{27}$$

Let $j \in \{0, \ldots, n\}$ be the number of east steps in a Schröder path, the number of norths steps must be the same, the number of northeast steps is $(n-j)$. The path is a Schröder path if and only if at any given step the number of north steps thus far does not exceed the number of east steps. The number of ways to arrange j east steps and j north steps satisfying this criterion is the Catalan number $\binom{2j}{j}/(j+1)$, and there are $\binom{n+j}{n-j}$ ways to determine the order of the northeast steps with respect to the east and north steps. Since

$$\binom{2j}{j}\binom{n+j}{n-j} = \binom{n+j}{j, j, n-j} = \binom{n+j}{n}\binom{n}{j} \quad \text{and}$$

$$\frac{1}{j+1}\binom{n}{j} = \frac{1}{n+1}\binom{n+1}{j+1},$$

we may rewrite (27) as

$$S_n(x) = \frac{1}{n+1} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j+1} \binom{n+j}{n} x^j \quad \text{for } n \geq 1. \tag{28}$$

On the other hand, Proposition 2.4 gives

$$(x-1)\widetilde{P}_n^{(1,-1)}(x) = \sum_{k=0}^{n+1} (-1)^{n+1-k} x^k \binom{n+1}{k} \binom{n-1+k}{n} \quad \text{for } n \geq 1.$$

Observe that for $n \geq 1$ the constant term in the last sum is zero, thus we may write

$$(x-1)\widetilde{P}_n^{(1,-1)}(x) = \sum_{k=1}^{n+1} (-1)^{n+1-k} x^k \binom{n}{k} \binom{n-1+k}{n}.$$

The statement now follows from substituting $j+1$ into k in the last equation and comparing the result with (28). □

In analogy to Lemma 2.2 we have the following

Lemma 6.3. *The weighted Schröder numbers are linked to the Schröder polynomials by the formula*

$$s_n^{u,v,w} = (-w)^n S_n\left(-\frac{uv}{w}\right)$$

As a consequence of Proposition 6.2 and Lemma 6.3 we obtain

Corollary 6.4. *For $n \geq 2$, the weighted Schröder numbers are linked to the shifted Jacobi polynomials by the formula*

$$s_n^{u,v,w} = \frac{(-w)^n}{n+1}\left(1+\frac{w}{uv}\right)\widetilde{P}_n^{(1,-1)}\left(-\frac{uv}{w}\right).$$

In particular, substituting $u = v = w = 1$ gives that the ordinary Schröder numbers are given by

$$s_n := s_n^{1,1,1} = \frac{(-1)^n 2}{n+1}\widetilde{P}_n^{(1,-1)}(-1) = \frac{(-1)^n 2}{n+1}P_n^{(1,-1)}(-3) \quad \text{for } n \geq 2.$$

Using the swapping rule (5) we obtain the following remarkable variant of (2).

$$s_n = \frac{2}{n+1}\cdot P_n^{(-1,1)}(3) \quad \text{for } n \geq 1. \tag{29}$$

The same swapping rule for shifted Jacobi polynomials (16) allows to rewrite Corollary 6.4 as

$$s_n^{u,v,w} = \frac{w^n}{n+1}\left(1+\frac{w}{uv}\right)\widetilde{P}_n^{(-1,1)}\left(\frac{uv}{w}+1\right) \quad \text{for } n \geq 1. \tag{30}$$

Central Delannoy numbers and Schröder numbers satisfy an obvious recursion formula, which may be generalized to their weighted variants as follows.

Proposition 6.5. *The weighted central Delannoy numbers and the weighted Schröder numbers satisfy the following formula.*

$$d_{n,n}^{u,v,w} = 2uv\sum_{k=0}^{n-1} d_{k,k}^{u,v,w}\, s_{n-k-1}^{u,v,w} + w d_{n-1,n-1}^{u,v,w}.$$

Proof. For any Delannoy path from $(0,0)$ to (n,n) there is a unique (k,k) on the path where the path contains a lattice point of the form (i,i) for the last time before (n,n). Consider the contribution of all Delannoy paths associated to the same fixed (k,k), where $k \in \{0,1,\ldots,n-1\}$. The part of all such paths up to (k,k) contributes a factor of $d_{k,k}^{u,v,w}$. The rest must begin with an east step and end with a north step, contributing a factor of uv. For $k < n-1$ the remaining steps contribute a factor of $2s_{n-k-1}^{u,v,w}$. The factor of 2 represents the choice of staying strictly above or below the line $y = x$ for the rest of the path. For $k = n-1$, besides the contribution of $2uvs_0^{u,v,w} = 2uv$ the possibility of adding a northeast step from $(n-1,n-1)$ to (n,n) arises: this possibility is accounted for by the term $wd_{n-1,n-1}^{u,v,w}$. □

Using Corollary 3.1 and (26) we obtain:

Corollary 6.6. *The shifted Legendre polynomials satisfy the recursion formula*

$$\widetilde{P}_n(x) = 2x\sum_{k=0}^{n-1}\widetilde{P}_k(x)S_{n-k-1}(x) - \widetilde{P}_{n-1}(x).$$

Using Proposition 6.2 the last corollary may be rewritten as follows.

Corollary 6.7. *We have*

$$\widetilde{P}_n(x) = 2\sum_{k=0}^{n-2} \widetilde{P}_k(x)\frac{x-1}{n-k}\widetilde{P}^{(1,-1)}_{n-k-1}(x) + (2x-1)\widetilde{P}_{n-1}(x) \quad \text{and}$$

$$P_n(x) = \sum_{k=0}^{n-2} P_k(x)\frac{x-1}{n-k}P^{(1,-1)}_{n-k-1}(x) + xP_{n-1}(x) \quad \text{for } n\geq 1.$$

Comparing (27) with (8) we find immediately that the Schröder polynomials are closely related to the the antiderivatives of the shifted Legendre polynomials:

$$S_n(x) = \frac{1}{x}\int_0^x \widetilde{P}_{n-1}(t)\ dt \quad \text{holds for } n\geq 1. \tag{31}$$

By Proposition 6.2 this implies

$$\frac{1}{n+1}(x-1)\widetilde{P}^{(1,-1)}_n(x) = \int_0^x \widetilde{P}_n(t)\ dt \quad \text{for } n\geq 1. \tag{32}$$

This equation may be generalized to the following statement.

Proposition 6.8. *Let n and α be positive integers. Applying the antiderivative operator*

$$f(x)\mapsto \int_0^x f(t)\ dt$$

to $\widetilde{P}_n(x)$ exactly α times yields the polynomial $\frac{1}{(n+\alpha)_\alpha}(x-1)^\alpha\widetilde{P}^{(\alpha,-\alpha)}_n(x)$.

Proof. By Proposition 2.4 we have

$$(x-1)^\alpha\widetilde{P}^{(\alpha,-\alpha)}_n(x) = \sum_{k=0}^{n+\alpha}(-1)^{n+\alpha-k}x^k\binom{n+\alpha}{k}\binom{n-\alpha+k}{n}.$$

Here the binomial coefficient $\binom{n-\alpha+k}{n}$ is zero for $k<\alpha$. Thus, after introducing $j:=k-\alpha$, we may write

$$(x-1)^\alpha\widetilde{P}^{(\alpha,-\alpha)}_n(x) = \sum_{j=0}^{n}(-1)^{n-j}x^{j+\alpha}\binom{n+\alpha}{j+\alpha}\binom{n-j}{n}.$$

Thus

$$\frac{(x-1)^\alpha\widetilde{P}^{(\alpha,-\alpha)}_n(x)}{(n+\alpha)_\alpha} = \sum_{j=0}^{n}(-1)^{n-j}\frac{x^{j+\alpha}}{(j+\alpha)_\alpha}\binom{n}{j}\binom{n-j}{n}. \tag{33}$$

Taking the derivative on both sides yields a right hand side of the same form, with the value of α decreased by one. Thus we obtain

$$\frac{d}{dx}\frac{(x-1)^\alpha\widetilde{P}^{(\alpha,-\alpha)}_n(x)}{(n+\alpha)_\alpha} = \frac{(x-1)^{\alpha-1}\widetilde{P}^{(\alpha-1,-(\alpha-1))}_n(x)}{(n+\alpha-1)_{\alpha-1}} \quad \text{for } \alpha\geq 1.$$

The statement follows from applying this observation α times and from the fact that $\widetilde{P}^{(\alpha,-\alpha)}_n(0)=0$ holds for $n,\alpha>0$. □

We conclude this section by observing that the polynomials $\{S_n(x)\}_{n\geq 0}$ *almost* form an orthogonal polynomial sequence. In fact it is possible to show that the monic polynomials

$$p_n(x) := \frac{1}{\binom{2n}{n}} \frac{x-1}{x} \widetilde{P}_n^{(1,-1)}(x)$$

satisfy Favard's recursion formula (10)

$$p_n(x) = \left(x - \frac{1}{2}\right) p_{n-1}(x) - \frac{n(n-2)}{4(2n-1)(2n-3)} p_{n-2}(x) \quad \text{for } n \geq 2.$$

Unfortunately, substituting $n = 2$ yields $\lambda_2 = 0$ thus the moment functional will not be quasi-definite.

7. Concluding remarks

One of the main observations in this paper is that the connection (1) between the central Delannoy numbers and Legendre polynomials may be extended to all Delannoy numbers and the Jacobi polynomials $\{P_n^{(\alpha,0)}(x)\}_{n\geq 0}$ via (2). An eerily similar relation was found in [9, (3.2)] between the *modified Delannoy numbers* $\widetilde{d}_{m,n}$ and the Jacobi polynomials $\{P_n^{(0,\beta)}(x)\}_{n\geq 0}$. Formula (3.2) in [9] may be restated as

$$\widetilde{d}_{n+\beta,n} = P_n^{(0,\beta)}(3) \quad \text{for } \beta \in \mathbb{N}. \tag{34}$$

The modified Delannoy number $\widetilde{d}_{m,n}$ is the number of lattice paths from $(0,0)$ to $(m, n+1)$ whose steps belong to $\mathbb{N} \times \mathbb{P}$. (Here $\mathbb{P}$ denotes the set of positive integers.) It would be worth exploring whether the weight enumeration of modified Delannoy paths (defined as having steps from $\mathbb{N}\times\mathbb{P}$) could also be done using shifted Jacobi polynomials, and whether there is a *duality* between the two theories that extends the ("signless") swapping of the parameters observed above in (2) and (34).

The fact that the (shifted) Legendre and Jacobi polynomials are orthogonal with respect to *some* weight function is also a consequence of Favard's theorem. The question naturally arises whether Viennot's combinatorial theory [21] could be applied to find a moment functional with respect to which they are orthogonal. Laguerre polynomials (their usual monic variant) have a very nice combinatorial interpretation in Viennot's work. For (the monic version of the) Legendre and Jacobi polynomials, however, the coefficients $\{c_n\}_{n\geq 1}$ and $\{\lambda_n\}_{n\geq 1}$ are not all integers. For example, the monic variant of the Legendre polynomials is given by

$$p_n(x) := \frac{2^n P_n(x)}{\binom{2n}{n}}.$$

Favard's recursion formula (10) takes the form

$$p_n(x) = x p_{n-1} x - \frac{(n-1)^2}{(2n-1)(2n-3)} p_{n-2}(x).$$

Thus some weights $-c_n$ and $-\lambda_n$ used on Viennot's *Favard paths* have to be fractions, and the formula expressing the moments as total weights of Motzkin paths seems very hard to evaluate. For Legendre polynomials, the horizontal steps will have zero weight, the northeast steps $(1,1)$ will have weight 1, the southeast steps $(1,-1)$ will have weight $k^2/(4k^2-1)$ if they start at a point whose second coordinate k. Using the fact that Legendre polynomials are orthogonal with respect to the inner product

$$\langle f,g\rangle := \int_{-1}^{1} f(x)\cdot g(x)\ dx,$$

we know that the moments must be a constant multiple of

$$\int_{-1}^{1} \frac{1}{x^n}\ dx = \begin{cases} 0 & \text{if } x \text{ is odd,} \\ \frac{2}{n+1} & \text{if } x \text{ is even.} \end{cases}$$

Experimental evaluation of Viennot's Motzkin paths for the first few even values of n shows that the total weight of these paths is $1/(n+1)$. A direct combinatorial proof of this fact would be desirable. It should also be noted that the non-monic variant

$$q_n(x) := \frac{2^n(2n-1)!!P_n(x)}{\binom{2n}{n}}$$

of the Legendre polynomials satisfies the recursion formula

$$q_n(x) = (2n-1)xq_{n-1}(x) - (n-1)^2 q_{n-2}(x) \tag{35}$$

with integer connecting coefficients. Perhaps a non-monic version of Favard's theorem is worth considering, to which Viennot's combinatorial proof could be extended by replacing the weight x on the short vertical steps in his Favard paths by a weight of $(2n-1)x$ where n is determined by the height of the step.

As seen in Section 4, our current work relates the Legendre and Jacobi polynomials to the classical theory of rook polynomials. The level of this connection is not satisfying, since we only found a binomial identity to which the orthogonality of the Laguerre polynomials (rook theory variant) and the orthogonality of certain Jacobi polynomials may both be reduced. It would be desirable to find results directly relating colored Delannoy paths and rook placements as combinatorial structures.

Our proof of the orthogonality of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for nonzero α depends on the algebraic formula given in Proposition 3.4. A more direct combinatorial proof would be desirable, which can perhaps be found once the "duality" between $P_n^{(\alpha,0)}(x)$ and $P_n^{(0,\beta)}(x)$ is better understood. Although Proposition 6.8 seems to give a "philosophical reason" to focus on the polynomials $(x-1)^\alpha \widetilde{P}_n^{(\alpha,-\alpha)}(x)/(n+\alpha)_\alpha$, these have a combinatorial interpretation for $\alpha = 1$ only, for higher values we start having non-integer coefficients. It is remarkable though that the same polynomials $(x-1)^\alpha \widetilde{P}_n^{(\alpha,-\alpha)}(x)$, without being divided by $(n+\alpha)_\alpha$ play a key role in our work and have a combinatorial interpretation for all $\alpha \in \mathbb{N}$.

Finally we wish to note that there is a yet to be explored connection with the theory of *Riordan arrays* (see Sprugnoli [17] for definition and uses). The weighted Delannoy number $d_{m,n}^{u,v,w}$ is the coefficient of t^n in $(u+wt)^m/(1-vt)^{m+1}$. An immediate consequence of this observation is that the n-th row k-th column entry in the Riordan array $(1/(1-vt), t(u+wt)/(1-vt))$ is $d_{k,n-k}^{u,v,w}$. The numbers $d_{m,n}^{1,2,-1}$ appear as entry A1016195 in Sloane [16], listing the entries of the Riordan array $(1/(1-2t), t(1-t)/(1-2t))$. Our results should allow to write summation formulas for Jacobi polynomials using the theory of Riordan arrays.

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