The Linearity of the Ropelengths of Conway Algebraic Knots in Terms of Their Crossing Numbers

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ABSTRACT. For a knot or link $\mathcal{K}$, let $L(\mathcal{K})$ denote the ropelength of $\mathcal{K}$ and let $Cr(\mathcal{K})$ denote the crossing number of $\mathcal{K}$. An important problem in geometric knot theory concerns the relationship between $L(\mathcal{K})$ and $Cr(\mathcal{K})$ (or intuitively, the relationship between the length of a rope needed to tie a particular knot and the complexity of the knot). We show that there exists a constant $a > 0$ such that if a knot $\mathcal{K}$ allows a special knot diagram D (called Conway algebraic knot diagram) with n crossings, then $L(\mathcal{K}) \leq a \cdot n$. Furthermore, if D is alternating (but not necessarily reduced and in fact $\mathcal{K}$ may not have a minimal alternating diagram that is algebraic), then $L(\mathcal{K}) \leq a \cdot Cr(\mathcal{K})$. The approach used here can be applied to a larger class of knots, namely those formed by replacing single crossings in a Conway algebraic knot diagram by tangles whose crossing number is bounded by a constant. Interestingly, it has been shown by the same authors that the Jones polynomials of these knots can be computed in polynomial time.

1. Introduction

The focus of this paper is the relation between the crossing number of a thick knot and its arc length. Intuitively speaking, we are interested in the relationship between the minimum length of a rope (with uniform unit thickness) required to tie a particular knot and the topological complexity of that knot (which is very often measured by its minimum crossing number). Such a length is called the *ropelength* of the knot. More precisely, to define a "uniform rope" one would need to define the thickness of a smooth simple closed curve in $\mathbb{R}^3$ first. Let K be a smooth and simple closed curve in $\mathbb{R}^3$, its thickness, denoted by $\tau(K)$, is defined to be the supremal radius of the embedded normal tubes around K (self-intersections are not allowed in the tubes). See [**6, 10, 16**] (among others) for the various definitions of thickness. We use $\mathcal{K}$ for the set of all smooth simple closed curves that are topologically equivalent and call it a *knot type*. A member of $\mathcal{K}$ is often denoted by K. For any given knot type $\mathcal{K}$, a thick realization K of $\mathcal{K}$ is a member of $\mathcal{K}$ which has unit thickness (i.e. $\tau(K) = 1$). Let $L(K)$ be the length of K. The *ropelength*

$L(\mathcal{K})$ of $\mathcal{K}$ is then defined as the infimum of $L(K)$ taken over all thick realizations of $\mathcal{K}$. The existence of $L(\mathcal{K})$ is shown in [6]. Of course, if K is a thick realization of $\mathcal{K}$, then we have $L(K) \geq L(\mathcal{K})$. The above definition of ropelength can be made using any of the different thickness definitions, and the results obtained in this paper using one definition also hold for the other thickness definitions modulo a suitable change in the coefficients of the results.

For the lower bound of ropelength, it is shown in [1, 3] that the minimum crossing number of a knot of unit thickness is bounded by a constant times its ropelength to the four-thirds power, i.e., $L(\mathcal{K}) \geq c \cdot (Cr(\mathcal{K}))^{3/4}$ for some constant c. The constant c is estimated to be at least 2.135 [18]. This four-thirds power is also known to be achievable for torus knot families [5, 7]. That is, there exists a family of knot types $\{\mathcal{K}_n\}$, such that $L(\mathcal{K}_n) \leq c_0 \cdot (Cr(K))^{3/4}$ for some constant $c_0 > 0$. For the upper bound of ropelength, it has been shown in [12] recently that there exists a constant $d > 0$ such that $L(\mathcal{K}) \leq d \cdot (Cr(\mathcal{K}))^{3/2}$ for any knot type $\mathcal{K}$. A power p is called *realizable* by a knot family $\mathcal{K}_n$ (with $\lim_{n \to \infty} Cr(\mathcal{K}_n) = \infty$) if $L(\mathcal{K}_n) = O((Cr(\mathcal{K}_n))^p)$. It has been shown in [8, 11] that there exists a family of knots $\{\mathcal{K}_n\}$ (depending on p) with the property that $Cr(\mathcal{K}_n) \to \infty$ (as $n \to \infty$) such that $L(\mathcal{K}_n)$ grows as $Cr(\mathcal{K}_n)^p$ for any p with $3/4 \leq p \leq 1$. It is still an open question whether any power p such that $1 < p \leq 3/2$ is realizable by any knot family.

In [9] the ropelength problem was studied for a large class of knots called Conway algebraic knots. It was shown there that many knots or links with alternating and algebraic diagrams have ropelength upper bounds proportional to their crossing numbers. The approach used there relied on the algorithm given in [12] based on a Hamiltonian cycle in a the knot diagram. In this article we will extend the above results to cover all Conway algebraic knots. We show that if a knot diagram D admits a Conway algebraic structure and has n crossings, then $L(\mathcal{K})$ is bounded above by a constant times n, where $\mathcal{K}$ is the knot type that the diagram D represents. In addition, if a knot (or link) $\mathcal{K}$ admits an alternating diagram and is a Conway algebraic knot, then its ropelength is bounded above by $O(Cr(\mathcal{K}))$. The approach used in this paper is essentially different from that used in [9]. It does not depend on a Hamilton cycle in the knot diagram, instead it makes use of the fact that a degree-bounded tree structure can be associated to the Conway algebraic structure in the knot diagram.

This paper is organized as follows: Section 2 contains a brief introduction to Conway algebraic knots based on the structures of knot diagrams. Section 3 shows the existence of a binary tree within Conway algebraic knot diagrams, Section 4 explains how to embed a tree into the planar cubic lattice with edge crossings based on the work in [20]. Section 5 explains how the ropelength of a Conway algebraic knot is related to the embedding length of the binary tree obtained from a diagram of the knot that shows it Conway algebraic structure. Finally, Section 6 concludes the paper with some remarks for generalizations and open questions. Throughout this paper, we will use the word knots meaning knots or links.

2. Conway algebraic knots

In [2], Bonahon and Siebenmann established a method to classify some families of knots by decomposing a knot into simpler pieces in a canonical way. In this

work they extended the existing classifications of two bridge knots and Montesinos knots to a much larger class of knots called *Conway algebraic knots*. These knots are often also called *aborescent knots* and should not be confused with complex algebraic knots [**14, 21**]. Algebraic knots can be defined in several ways, for example by decompositions of the knot exterior or by decompositions of some particular diagrams of the knots. For our purpose in this article it is enough to concentrate on a particular diagram decomposition for such knots. For more details, see [**2**].

DEFINITION 2.1. Given a knot $\mathcal{K}$ with a regular diagram D in S^2, then D is *reduced* if there is no reduction in the number of crossing in D possible by flipping part of the diagram, see Figure 1.
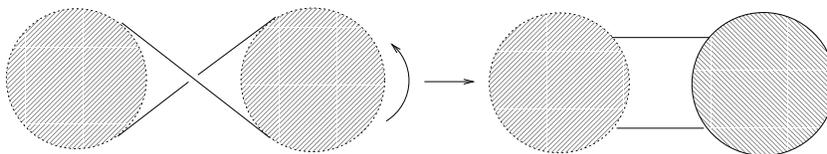

FIGURE 1. A crossing that can be reduced by a simple flipping.

We will assume that all knot diagrams in this paper are reduced and that a diagram contains at least 3 crossings.

DEFINITION 2.2. Given a knot $\mathcal{K}$ with a diagram D in S^2, then a *Conway circle* is a simple closed curve C in S^2 that intersects D transversely in exactly four non-crossing points of D.

DEFINITION 2.3. Given a knot diagram D and a finite collection of disjoint Conway circles C_1, C_2, ..., C_n in D, then we will call the components of D in $S^2 - \cup_i C_i$ *Conway regions*.

Notice that a Conway region is actually a connected region in S^2. A knot diagram D is called an *algebraic knot diagram* if it can be decomposed by a finite number of Conway circles C_1, C_2, ..., C_n into Conway regions of the following types (such a decomposition will be called an *algebraic decomposition*):

a. A Conway region that contains a single crossing, see Figure 2A.

b. A Conway region that contains exactly three Conway circles together with the arcs connecting them as shown in Figure 2B. There are no crossings in the Conway region.

DEFINITION 2.4. A knot $\mathcal{K}$ is called *Conway algebraic knot* or an *algebraic knot* if it admits an algebraic knot diagram, that is, there exists $K \in \mathcal{K}$ such that K has a regular projection diagram that is algebraic.

REMARK 2.5. The class of algebraic knots is very large. For example, all two-bridge knots (rational knots), Montesinos knots (for a classification of Montesinos knots, see for example [**4**]) and many knots in the existing knot tables are algebraic knots. Among the 249 prime knots with up to 10 crossings there are 207 which have a diagram with minimal crossing number that is algebraic. Of these 207 prime knots, 95 are rational (two bridge) knots and 112 are non-rational algebraic knots.
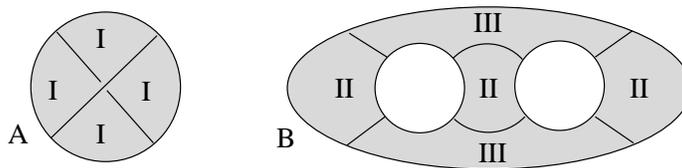
FIGURE 2. A. A Conway region containing a single Conway circle with a single crossing from the knot diagram in it. B. A Conway region containing three Conway circles without any crossings from the knot diagram.

REMARK 2.6. In general, it is not known which algebraic knots admit a minimal crossing diagrams that can be decomposed by Conway circles as described above. There exist algebraic knots whose minimal diagrams are not algebraic diagrams. For example, the minimum projection of the Borromean rings is not algebraic but it has a non-minimum projection which is algebraic. See Figure 3 and [19]. In the special case that a given knot or link $\mathcal{K}$ has a minimal alternating diagram D, then whether $\mathcal{K}$ is algebraic can be determined by an algorithm described in [19]. In the case that $\mathcal{K}$ is indeed algebraic, the algorithm would then produce an algebraic (possibly non minimal) diagram D' for $\mathcal{K}$. Moreover the number of crossings in D' is less or equal to $(4/3)Cr(\mathcal{K})$ (note that $Cr(\mathcal{K})$ is the number of crossings in D). However, if D is non-alternating then there is no known practical method to determine whether K is algebraic and in the case that it is, how many crossings an algebraic diagram of K would have. Hence knowing an algebraic knot diagram of a knot does not easily lead to the minimum crossing number of that knot. Of course, there are exceptions. For example, an algebraic knot diagram of a Montesinos knot (with some suitable conditions) [17] is a minimum crossing diagram of that knot.
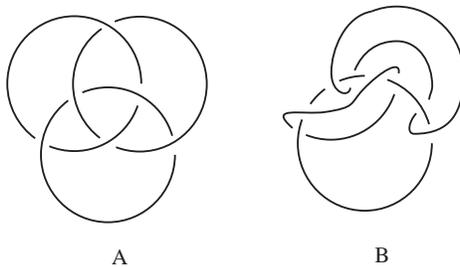

FIGURE 3. A. A minimum projection of the Borromean rings that does not admit an algebraic decomposition. B. A projection of the Borromean rings that is an algebraic diagram.

3. Algebraic knot diagrams and their preferred trees

Let D be an algebraic knot diagram with n crossings. The union of the algebraic knot diagram D and the Conway circles $C = \cup C_i$ which form its decomposition

define a 4-regular graph on S^2. Each Conway region can be further divided into one of the following three Conway sub-regions.

Type I: These are regions which arise from Conway regions containing one single crossing as shown in Figure 2 A. Each type I region has a single arc belonging to a Conway circle in its boundary and a type A Conway region contains four type I Conway sub-regions.

Type II: These are regions which arise from Conway regions containing three Conway circles. Each type II Conway sub-region contains two arcs belonging to two different Conway circles in its boundary. There are three type II Conway sub-regions in each Conway region of type B. See Figure 2 B.

Type III: These are regions which arise from Conway regions containing three Conway circles. Each type III Conway sub-region contains three arcs belonging to three different Conway circles in its boundary (namely the three Conway circles in the corresponding Conway region). There are two such type III Conway sub-regions in each Conway region of type B. We say that the two type III Conway sub-regions in each Conway region of type B form a *pair*. See Figure 2 B.

Note that the Conway sub-regions can be thought of as regions in $S^2 - (D \cup C)$. For our purpose some Conway sub-regions will be treated differently from the other ones and we will indicate those by shading them. A *shading* of the Conway sub-regions is a coloring of the Conway sub-regions of type III using only white and black colors. The ones colored black are said to be shaded. Note that only one Conway sub-region of each pair of type III regions will be shaded. A shading is constructed using the following rules: Start with a single type III Conway sub-region R_1 and shade it. If there is only one type B Conway region in the entire diagram D then we are done. Otherwise R_1 shares boundary (a point or an arc) with another type III Conway sub-region R_2 that is not paired with R_1 and we shade R_2. Note R_1 and R_2 either share a common boundary arc that is part of a Conway circle or share a single common boundary point that is an intersection point of the knot diagram D with a Conway circle. Continue the process by shading another type III Conway sub-region R_3 that is not paired with R_1 or R_2 and shares boundary (a point or an arc) with R_1 or R_2. This process can be continued until one type III Conway sub-region of each pair is shaded since every pair of type III Conway sub-regions is adjacent to at least one other pair of type III Conway sub-regions. A shading following the above rules is called a *preferred shading* and the above discussion assures that a preferred shading of the type III Conway sub-regions exists for any Conway algebraic knot diagram D. Note that usually a Conway algebraic knot diagram D has several different preferred shadings. Figure 4 shows a preferred shading of an algebraic diagram.

From a preferred shading we can now construct a plane graph as follows. First, place a vertex in each shaded type III Conway sub-region. Two such vertices are connected by an edge if their corresponding type III Conway sub-regions R_1 and R_2 intersect on their boundaries (a single point or an arc) and the edge passes from R_1 to R_2 through such a common boundary point (or close to a boundary point as drawn in Figure 4). For each crossing of D that is contained in a type A Conway region (the boundary of this region is a Conway circle C_i that is part of a type B Conway region), attach a leaf (a vertex and an edge) by placing the leaf-vertex

next to the crossing (within C_i) and by drawing an edge from the leaf to the vertex representing the shaded type III Conway sub-region adjacent to C_i, see Figure 4.

By the definition of a preferred shading, it is easy to see that the graph so constructed is a *binary tree* (a tree in which every vertex has degree three or less) whose number of leaves equals the crossing number n in the diagram D and the number of edges of the tree is $2n - 3$. We call this tree a *preferred tree* (associated with the preferred shading used for its construction) and will denote it by T. Figure 4 shows an example of a preferred tree T.
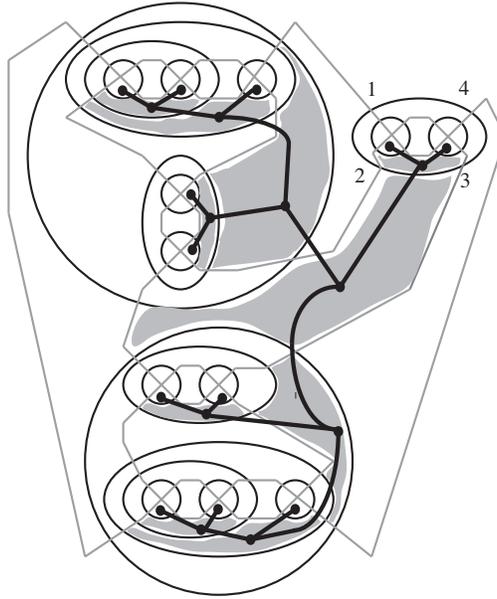

FIGURE 4.   An algebraic knot diagram D together with a set of Conway circles. Also shown is a preferred shading together with the corresponding preferred tree T. One of the Conway circles has a counterclockwise labeling of its intersection points with D.

The importance of T lies in the fact that D can be isotoped to a new diagram D' with at most $4n$ crossings that has exactly four strands "parallel" to each edge in the tree as shown in Figure 6. From this fact it will follow that the ropelength of a thick embedding of D is bounded above by the order of the lattice embedding of T. Let us describe the new diagram D' locally at the type I and type III Conway sub-regions first. Label the four intersection points of the Conway circles with the diagram D counterclockwise with labels 1, 2, 3, and 4 such that labels 2 and 3 appear right next to the edges of T, see Figure 4 for an example of such a labeling along a single Conway circle. The diagram D' is obtained in two steps: First one can draw the tree T in a nicer way by deforming T to a new tree T', see Figure 6 and drag the diagram D along by isotopy. In the second step one makes the diagram D "parallel" to the tree T' such that for each intersection of a Conway circle with an edge in T' the labels 1 through 4 of the Conway circle are in that order close to T' and in the order 1 to 4. This can be seen locally by looking at the different

Conway sub-regions. For a type I Conway sub-region the local picture is shown in Figure 5A and for a type III Conway sub-region the local picture is shown in Figure 5B. The figures show one vertex of T together with one or three "half-edges" of T. These local structures then need to be glued together to obtain a diagram D' that is ambient isotopic to D. This "glueing operation" is straight forward when a type I local structure is glued to a type III local structure or two type III local structures are glued together when the corresponding type III Conway sub-regions share an arc of a Conway circle on their boundary. However, a complication arises for shaded type III Conway sub-regions that intersect each other at a single point and extra care needs to be taken in such cases. Let C_i be such a Conway circle where two shaded type III Conway sub-regions intersect each other at a single point. Observe that an arc of the diagram D crosses T' (or T) at the Conway circle C_i in these instances and thus the arcs of the diagram cannot be as nicely parallel to T as before. Consider an annulus with C_i as its center line and with two boundary circles C_{i_1} and C_{i_2} parallel to C_i. As before we label the intersection points of D with the circles C_{i_1} and C_{i_2} counterclockwise with labels 1, 2, 3, and 4 such that labels 2 and 3 appear right next to the edges of T, see Figure 5C on the very left. We can now see that the 4 arcs of D that intersect the annulus are labeled from the inside out as 21, 32, 43, and 14. There is no problem to make the arcs 21 and 43 parallel to T, while the arc 32 can be made "almost parallel" to T since it has an intersection point with T. The arc 14 needs to be isotoped as shown in Figure 5C in the middle. More specifically, we lift the arc 14 up above D and drag it over the part of the diagram that is inside the Conway circle C_i and put it back down as shown in Figure 5C in the middle. Note that the arc 14 now crosses over the other three arcs 21, 32, and 43. Now we can deform 14 to be "parallel" to the tree T' as well and we obtain the configuration as shown in Figure 5C on the right. Figure 5D shows a similar situation where the arc 14 has to cross the other three arcs from right to left.
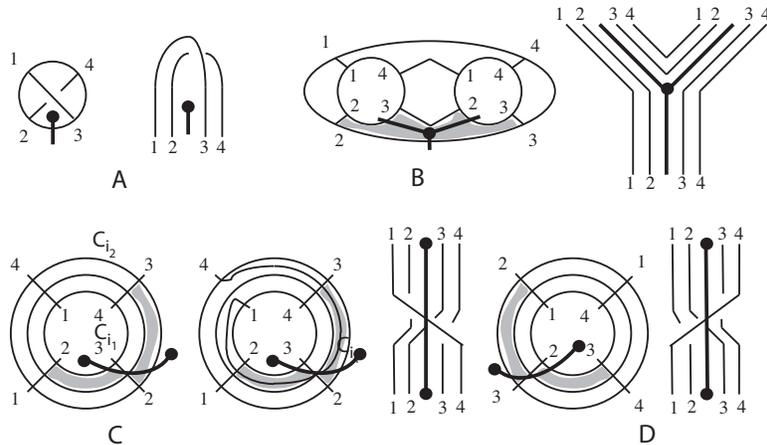

FIGURE 5. The local changes one need to make to an algebraic diagram to make it "parallel" to its tree.

The diagram D' can now be assembled locally with the condition that at each point where two shaded type III Conway sub-regions meet in a single point an

overpass occurs that crosses the other three strings. This introduces at most 3 crossing for each interior edge of T and thus D' has at most $4n$ crossings. Notice that each overpass created at two shaded type III Conway sub-regions that meet in a single point can be removed since it cuts the tree T into two pieces and we can slide it across one of these pieces to get rid of it. If we do this for all overpasses then we obtain a new diagram D'' with n crossings that is isotopic to D using only deformations. (No Reidemeister moves will be required in this isotopy.) Figure 6 shows a preferred tree of the knot diagram in Figure 4 and how the diagram can be deformed to a structure which is parallel to the tree. We summarize this in the following lemma.

LEMMA 3.1. *Let D be an algebraic knot diagram of $\mathcal{K}$ with n crossings, then $\mathcal{K}$ has a diagram D' with at most $4n$ crossings that is "parallel" to a preferred tree T with less than $2n$ edges defined by the algebraic knot diagram D.*
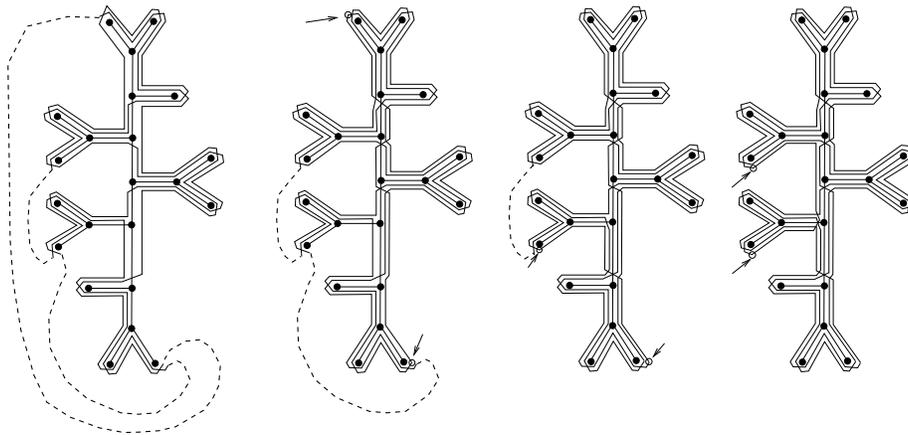

FIGURE 6. Deforming the knot diagram in Figure 4 so that it is "parallel" to a preferred tree: the over/under information at crossings are omitted. Initially there are three edges that cannot be made parallel to the tree (under the requirement that two strands are on each side of the tree) as shown in the left most figure. The other three figure show how these three edges are deformed: the starting and ending points are marked with an open circle and an arrow. The edge currently deformed crosses over all the edges that have already been deformed (so that the last deformed edge is over every other edge).

4. Embedding a binary tree in the plane.

The arguments given here follow the development given in [**15, 20**]. Before we describe our algorithm of how to embed a tree into the lattice let us provide some preliminary results first. For a tree T let us denote its number of edges by $|T|$.

LEMMA 4.1. *Let T be a tree with $n \geq 3$ edges whose maximal degree is less or equal to three, then there exists an edge $e \in T$ such that $T - e$ consists of two trees T_1 and T_2 with $|T_1|/|T| = c$ where $c \in [1/2 - 1/(2n), 2/3] \subseteq [1/3, 2/3]$.*

PROOF. Assign each edge $e = vu$ two integers at the vertices. The edge endpoint v gets a label z_{vu} if the component of $T - e$ that contains v has z_{vu} edges. Let $n = |T|$. Then $z_{vu} + z_{uv} = n - 1$ for each pair of integer labels at an edge and at a vertex of degree k the labels of the edge endpoints add to $(k-1)n$. If there is an edge $e = uv$ with equal labels $z_{uv} = z_{vu}$ then n is odd and deleting e splits the tree into two pieces with $|T_1|/|T| = |T_2|/|T| = 1/2 - 1/(2n) = c \geq 1/3$. If no edge with equal labels exists orient all edges from the smaller label to the larger label. Note the edges at the leaves are oriented away from the leaf. Thus there must be a vertex v that has only edges directed towards it (since otherwise there would be a path from a leaf to another leaf where each edge on the path has the same orientation, which would be a contradiction).

If $deg(v) = 2$, then we can delete either edge that has vertex v as an endpoint. Assume that there exist edges uv and wv that are oriented towards v. If $z_{vu} = z_{vw}$ then n is even and deleting the edge uv results into two pieces with $|T_1|/|T| = 1/2$ and $|T_2|/|T| = 1/2 - 1/n$. If the labels are not equal assume $z_{vu} < z_{vw}$. In this case $z_{vw} - 1 = z_{uv} \geq z_{vu}$ and the edge wv cannot be oriented towards v and therefore this is not possible.

If $deg(v) = 3$, let the labels of the three edges incident to v be $e_1 = vu$, $e_2 = vw$ and $e_3 = vy$. We have $z_{vu} + z_{vw} + z_{vy} = 2n$. Without loss of generality we assume that $z_{vw} + z_{vy} = \max\{z_{vu} + z_{vw}, z_{vw} + z_{vy}, z_{vu} + z_{vy}\} \leq 2n/3$. Thus $2n = z_{vu} + z_{vw} + z_{vy} = (z_{vu} + z_{vw})/2 + (z_{vu} + z_{vy})/2 + (z_{vw} + z_{vy})/2 \leq 3(z_{vw} + z_{vy})/2$. So $3(z_{vw} + z_{vy}) \geq 4n$ and $3z_{vu} = 6n - 3(z_{vw} + z_{vy}) \leq 2n$. So $z_{vu}/n \leq 2/3$. On the other hand, by the property of v that the three edges are oriented toward it, the label z_{uv} at u satisfies the condition $z_{uv} = n - 1 - z_{vu} < z_{vu}$. So $z_{vu} > (n-1)/2$. It follows that $z_{vu}/n \in (1/2 - 1/(2n), 2/3] \subseteq [1/3, 2/3]$. That is, the edge vu has the desired property.

Summarizing we have that T_1 can be chosen such that $|T_1|/|T| = c \in [1/2 - 1/(2n), 2/3] \subseteq [1/3, 2/3]$. $\square$

A plane graph G with maximum degree ≤ 4 can be drawn on the plane lattice so that its vertices are lattice points (points with integer coordinates) and its edges are lattice paths. In our case, we would like to draw the entire tree within a rectangle of the form $[0, m-1] \times [0, n-1]$ and we will allow the edges to intersect each other at lattice points. We call a graph realized on the lattice in this way an (m, n)-*embedding*.

Consider two binary trees T and T' realized on the lattice as a (m, n)-embedding and a (m', n)-embedding respectively. We can combine the two trees into a new binary tree T'' as an $(m + m' + 1, n + 1)$-embedding as follows: First shift $[0, m' - 1] \times [0, n-1]$ (together with T' in it) to the right by m units and then glue it with the right edge to the rectangle $[0, m-1] \times [0, n-1]$. The result is the rectangle $[0, m + m' - 1] \times [0, n-1]$ which contains an embedding of T and T'. Next pick any two vertices $v \in T$ and $w \in T'$ with degree(v), degree(w) ≤ 2. Vertically transform v and w by $1/2$ unit to the points v' and w'. If v' is in T, then v' will become a vertex replacing vertex v. If v' is not in T, connect v and v' with a vertical line segment of length $1/2$ and keep the vertex v as a vertex of T. Similarly, connect w

and w' with a vertical segment of length $1/2$ if w' is not in T'. Now connect v' to w' with a horizontal straight line segment joining v' and the point v'' on the vertical line $x = m - 1/2$, a horizontal straight line segment joining w' and the point w'' on the vertical line $x = m - 1/2$, and a vertical line segment joining v'' and w'' if needed. Now we expand $[0, m + m' - 1] \times [0, n - 1]$ to $[0, m + m'] \times [0, n]$ by adding a horizontal row each in $[0, m - 1] \times [0, n - 1]$ and $[m, m + m' - 1] \times [0, n - 1]$, and one vertical row such that v' and w' will become lattice vertices and the path connecting v' and w' is on the lattice in the resulting rectangle. This generates a new tree T'' embedded in $[0, m + m'] \times [0, n]$. See Figure 7 for an illustration of this. We summarize this in the following lemma.

LEMMA 4.2. *If two binary trees T and T' have (m, n)- and (m', n)-embeddings respectively (so the total length of the two embedding rectangles in the x-direction is $m + m'$), then a tree T'' obtained by connecting any two degree one or two vertices $v \in T$ and $v' \in T'$ has an $(m + m' + 1, n + 1)$-embedding.*
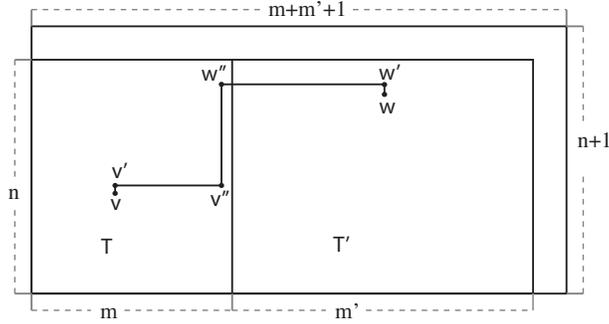

FIGURE 7.   The tree T'' generated by placing rectangular embeddings of the trees T and T' next to each other.

The *aspect ratio* of the rectangle $[0, m - 1] \times [0, n - 1]$ is defined as $r = m/n$. Let $A(k)$ be the minimum over all positive integers p with the property that if the area of a rectangle $[0, m - 1] \times [0, n - 1]$ with aspect ratio between 1 and 3 is equal to or greater than p, then any binary tree T with k edges has an (m, n)-embedding. $A(k)$ exists since the set of such integers p is non-empty. By experimenting one can see that $A(3) = A(4) = 6$, $A(5) = 8$, $A(6) = 9$, $A(7) = 10$, $A(8) = 12$, $A(9) = 15$, and so on. However, it is hard to get a general and explicit formula for $A(k)$. In the following, we will provide an upper bound for $A(k)$ instead. The idea is to apply Lemma 4.1 repeatedly to divide the tree T into smaller and smaller pieces until each small piece has at most 9 edges. By Lemma 4.1, at most $\lfloor \log_{3/2}(k) \rfloor$ such splits are needed. The pieces will then be re-assembled in a manner as outlined above (and shown in Figure 7) while keeping the area of the rectangle and its aspect ratio within the allowed range at each assembling step. Notice that in this reassembly we do not care if a vertex is the right or left child of a parent (i.e. the cyclic order of the edges around a vertex may change). In addition after repeated subdivisions

some of the rectangle splits may have to be rotated by 90 degrees when compared to Figure 7.

LEMMA 4.3. *If a function* $A_0(k) : \mathbb{Z}^+ \to \mathbb{R}^+$ *satisfies the following conditions, then* $A(k) \leq A_0(k)$:
(i) $A_0(3), A_0(4) \geq 6$, $A_0(5) \geq 8$, $A_0(6) \geq 9$, $A_0(7) \geq 10$, $A_0(8) \geq 12$, $A_0(9) \geq 15$;
(ii) $\forall k > 9$ *and* $\forall c$, *where* $1/3 \leq c \leq 2/3$ *and* ck *has an integer value*, $A_0(ck) \leq c(A_0(k) - 4\sqrt{A_0(k)})$.

In other words, if a rectangle $[0, m-1] \times [0, n-1]$ *has aspect ratio between 1 and 3 and* $mn \geq A_0(k)$, *then any binary tree* T *with* k *edges has an* (m, n)-*embedding.*

PROOF. We will use induction to prove this. Assume that we have shown $A(t) \leq A_0(t)$ for $t \leq k-1$ (where $k-1 \geq 9$), we need to show that $A(k) \leq A_0(k)$. In other words, we need to show that the rectangle $[0, m-1] \times [0, n-1]$ with $1 \leq r = m/n \leq 3$ and $mn \geq A_0(k)$ is large enough so that any binary tree T with $k = |T|$ edges can be embedded in it. Let T be such a tree. By Lemma 4.1, there is an edge e such that $T - e$ consists of two trees T_1 and T_2 with $c = |T_1|/k \in [1/2 - 1/(2k), 2/3]$ and thus $|T_2|/k = 1 - c - 1/k \in [1/3 - 1/k, 1/2 - 1/(2k)]$. We will distinguish two cases depending on the aspect ratio $r = m/n$.

Case 1: $r \in (1, 3]$. Consider the rectangle $[0, \lceil c(m-2) - 1 \rceil] \times [0, n-2]$. Since $3 \geq r = m/n > 1$, $m \geq n+1$ and $m \leq 3n$, so $\lceil c(m-2) \rceil/(n-1) \geq c(n-1)/(n-1) = c \geq 1/3$ and $\lceil c(m-2) \rceil/(n-1) \leq (c(3n-2)+1)/(n-1) < 3c+1 \leq 3$. Thus it is a rectangle with an aspect ratio between 1 and 3 (it needs to be viewed as being rotated 90 degrees in case that $\lceil c(m-2) \rceil/(n-1) < 1$). Furthermore, it can be shown that its area $\lceil c(m-2) \rceil (n-1)$ is bounded below by $c(m-2)(n-1) \geq c(mn - 4\sqrt{mn}) \geq c(A_0(k) - 4\sqrt{A_0(k)}) \geq A_0(ck) = A_0(|T_1|)$ since $1 < m/n \leq 3$. By our induction assumption, T_1 then has a $(\lceil c(m-2) \rceil, n-1)$-embedding. Similarly, the rectangle $[0, \lceil (1-c)(m-2) - 1 \rceil] \times [0, n-2]$ also has the right aspect ratio and its area is large enough so that T_2 can be embedded in it. That is, T_2 has a $(\lceil (1-c)(m-2) \rceil, n-1)$-embedding. Since the total length of the two rectangles in the x-direction is $\lceil c(m-2) \rceil + \lceil (1-c)(m-2) \rceil \leq m-1$, it follows that T has an (m, n)-embedding by Lemma 4.2.

Case 2: $r = 1$. Consider the rectangle $[0, \lceil c(m-2) - 1 \rceil] \times [0, n-3]$ in this case. It again has the right aspect ratio since $\lceil c(m-2) \rceil/(n-2) = \lceil c(m-2) \rceil/(m-2)$ is between $c \geq 1/3$ and $1 + c < 2$. Furthermore, its area $\lceil c(m-2) \rceil (n-2)$ is bounded below by $c(m-2)^2 \geq c(A_0(k) - 4\sqrt{A_0(k)}) \geq A_0(ck) = A_0(|T_1|)$. So again by the induction hypothesis, T_1 has a $(\lceil c(m-2) \rceil, n-2)$-embedding. Similarly, T_2 has a $(\lceil (1-c)(m-2) \rceil, n-2)$-embedding and T has an $(m, n-1)$-embedding (hence an (m, n)-embedding) by Lemma 4.2. $\square$

LEMMA 4.4. *The function* $F(n)$ *defined by* $F(n) = 40n - 113\sqrt{n}$ *for* $n \geq 10$ *and* $F(n) = A(n)$ *for* $3 \leq n \leq 9$ *satisfies the recursive relationship of Lemma 4.3.*

PROOF. Given a function $F(n) = an - 4b\sqrt{n}$ where a and b are constants. We choose a and b to satisfy the equation $\sqrt{a} \leq b(\sqrt{3/2} - 1)$. This inequality implies:

$$\sqrt{an - 4b\sqrt{n}} < \sqrt{an} \leq b\sqrt{n}(\sqrt{3/2} - 1) \leq b\sqrt{n}(\sqrt{1/c} - 1).$$

The inequality $\sqrt{an - 4b\sqrt{n}} \leq b\sqrt{n}(\sqrt{1/c} - 1)$ is equivalent to the recursive relation $F(cn) \leq c(F(n) - 4\sqrt{F(n)})$ for $F(n) = an - 4b\sqrt{n}$ and all $1/3 \leq c \leq 2/3$ and

$n \geq 10$. In addition one needs to choose a and b large enough so that $F(n)$ works for the small values $3 \leq n \leq 9$. For $a = 40$ and $b = 113/4$ we must for example satisfy: $15 \leq F(9) = 21$, for $c = 2/5$ and $n = 10$, $6 = A(4) = F(c10) \leq c(F(10) - 4\sqrt{F(10)}) \approx 6.614$, for $c = 1/2$, $8 = A(5) = F(c10) \leq c(F(10) - 4\sqrt{F(10)}) \approx 8.268$, and many similar relationships up to $c = 1/3$ and $n = 27$. The critical relationships are those where n is close to 10 and the details are left to the reader.   □

5. Estimating the embedding length of an Conway algebraic knot diagram

We are now in the position to combine the results of the last two sections:

THEOREM 5.1. *Let D be an algebraic knot diagram of $\mathcal{K}$ with n crossings and T be a preferred binary tree of D. Then there is a realization of $\mathcal{K}$ on the cubic lattice $\mathbb{Z}^3$ which is contained in a rectangular box whose volume is bounded above by $11520n$.*

PROOF. By Lemma 3.1 there is a diagram D' of $\mathcal{K}$ that is "parallel" to the tree T. Note that T has less than $2n$ edges. Assume that T has been embedded on the square lattice as described in the previous section and let us call this embedding T'. We will modify T' in stages to a lattice embedding of the diagram D'. First we subdivide the unit lattice plane $\mathbb{Z}^2$ to a lattice plane $(Z/12)^2$. This has the effect of producing a new embedding of T' with 144 times its area (let us still use T' for this embedding since only the scale has changed, not the structure). Next we "quadruple" T' as shown in Figure 8A for a straight edge, in Figure 8B for a 90 degree turn, and in Figure 8C a degree 3 vertex of T'.
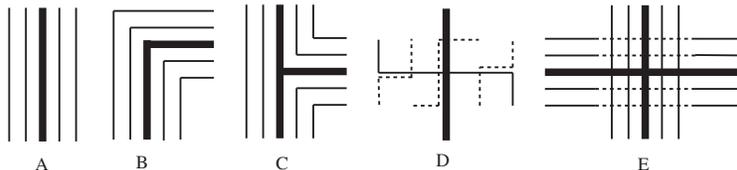

FIGURE 8. T' is quadrupled to make room for embedding D'.

For the vertices of degree 3 in T' there may be one more complication: the embedding algorithm in the last section did not care about the cyclic order of the edges around the vertex of degree 3, yet it is important that this is preserved in order to keep the topological structure of D'. If needed, this can be corrected by introducing one full twist in the braid of 4 strings that is parallel to T' at the said vertex, see Figure 9 for an illustration. Such a full twist will not change the knot topology since we can untwist these by untwisting the whole subtree connected at the vertex. Such a full twist can be realized in a lattice box of dimension $4 \times 4 \times 1$ as shown in Figure 10.

Denote the new lattice embedding of the quadrupled T' (now with the full four string twists where needed) by T''. Now we need to insert the additional crossings. Figure 8D shows how an overpass corresponding to the overpasses shown in Figures
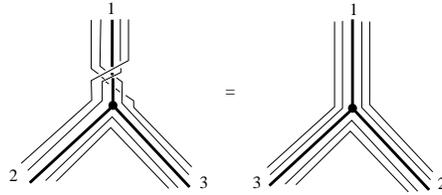
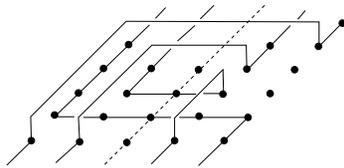FIGURE 9.   A full twist used to chance the cyclic order of edges connecting a vertex of degree three in T'.

FIGURE 10.   The realization of a full twist on the cubic lattice. The centerline in the figure is part of an edge incident to the vertex of T' where the correction is needed.

5C and D is realized on the lattice. The lattice segments shown in Figure 8D fit into a $4 \times 2 \times 1$ rectangular box and one can think of the dashed segments as having z-coordinate one while the others are in the plane $z = 0$. Of course, for the embedding to go between the planes $z = 0$ and $z = 1$, unit vertical segments will be needed wherever appropriate. Finally we need to take care of the edge crossing of the embedding of the tree (recall that we have allowed edges to intersect each other in the tree in the previous section). First we note that if there is such a crossings in T we now have 4 strings crossing 4 strings which will look like the configuration shown in Figure 8E. Again we can think of the dashed and the non-dashed strings to be in the planes $z = 1$ and $z = 0$ respectively. Since it does not matter for a single string of T to go over or under the other strings in its embedding (because this will not change its tree structure), we can always choose the vertical strings to go over the horizontal strings as shown in Figure 8E. Note that by subdividing the original lattice to $(Z/12)^2$, even if we have two consecutive crossings of the edges in T' there is enough room to insert a configuration that requires one string to cross the other three and to insert a full twist of the four strings. This results in a lattice embedding of D' into a rectangular lattice box with a height of one. Its volume is bounded above as follows: T' has at most $2n$ edges and its embedding uses an area bounded above by 40 times its edge number. Moving to the $(Z/12)^2$ lattice increased this area be a factor of 12^2 and so the volume of the box is bounded above by $11520n$.   □

COROLLARY 5.2. Let $\mathcal{K}$ be a knot with an algebraic knot diagram D having n crossings. Then there is a lattice embedding of $\mathcal{K}$ into the cubic lattice $\mathbb{Z}^3$ with a length at most $8376n$. This gives a ropelength $L(\mathcal{K}) \leq 16752n$.

PROOF. To estimate the length of the embedding we can use the volume of a box as follows. The lattice embedding of Theorem 5.1 is entirely contained between

the two lattice planes $z = 0$ and $z = 1$. We can estimate the length of the embedding contained in the planes $z = 0$ and $z = 1$ as follows. Before moving to the $(Z/12)^2$ lattice the length of the tree T' is at most $2 \times 2 \times 40n = 160n$. After moving to the $(Z/12)^2$ lattice this becomes $1920n$. Making four "parallel" copies of the tree increases this to $7680n$. Now we have to estimate the expense of moving some segments into the $z = 1$ plane. There are at most $80n$ crossings of edges in the original tree embedding due to the area constraint. Moving to $(Z/12)^2$ lattice does not increase this number and each such crossing of edges introduces at most 8 vertical steps (a step is a unit line segment length). Thus we can account for this with at most $640n$. Each instance of one edge crossing the other three edges adds 10 steps and there are at most $2n$ such passes needed. This adds another $20n$ steps. Each full twists adds 18 steps and there are at most $2n$ of them. This adds another $36n$ steps and we arrive at a length upper bound of $8376n$.   □

It is known that if $\mathcal{K}$ is an alternating knot and it has an algebraic diagram, then $\mathcal{K}$ has an algebraic diagram with at most $4/3Cr(\mathcal{K})$ crossings [**19**]. Thus we have the following corollary.

COROLLARY 5.3. Let $\mathcal{K}$ be a Conway algebraic knot that has a minimal alternating diagram. Then the ropelength of $\mathcal{K}$ is bounded above by $22336Cr(\mathcal{K})$.

6. Ending remarks

The arguments given here can be generalized by expanding the class of Conway algebraic knots as follows:

DEFINITION 6.1. A knot diagram D is called an *k-generalized algebraic knot diagram* if it can be decomposed by a finite number of Conway circles C_1, C_2, ..., C_n into Conway regions of the following types (such a decomposition will be called a *k-generalized algebraic decomposition*):

a. A Conway region that contains a tangle with at most k crossings.

b. A Conway region that contains exactly three Conway circles together with the arcs connecting them as shown in Figure 2B. There are no crossings in such a Conway region.

A knot $\mathcal{K}$ is called a *k-generalized Conway algebraic knot* or a *k-generalized algebraic knot* if it admits a *k*-generalized algebraic knot diagram, that is, there exists $K \in \mathcal{K}$ such that K has a regular projection diagram that is *k*-generalized algebraic.

For *k*-generalized algebraic knots we can essentially make the same arguments as for algebraic knot. The only difference will be that in the proof of Theorem 5.1 additional space must be provided to accommodate any tangle with up to k crossings. We state this as the following theorem without further proof.

THEOREM 6.2. *Let $\mathcal{K}$ be a knot with a k-generalized algebraic knot diagram D. Then the ropelength $L(\mathcal{K}) \leq aCr(D)$ where a is a constant that depends on k but is independent of $Cr(D)$.*

The investigations in this article were partly motivated by the following open questions:

A. Does there exist a family of knots $\mathcal{K}_n$ such that their ropelengths grow faster than $O(Cr(\mathcal{K}_n))$?

B. For any prime alternating knot $\mathcal{K}$, is it true that its ropelength is at least of the order $O(Cr(\mathcal{K}))$?

Here we have shown that we cannot find examples to questions A in the class of alternating Conway algebraic knots. In [**11**] examples of Conway algebraic knots were given for which the ropelength grows linearly with the crossing number. Thus it is possible that question B is true for Conway algebraic knots.

Numerical simulations by the authors in [**13**] have recently shown that the algorithm in [**12**] produces an average ropelength upper bound of $O(n \ln^2(n))$ for random knot diagrams with n crossings. Thus if an example to question A exists it would be likely that its ropelength does not grow much faster than linear growth with the crossing number.

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