MEIXNER POLYNOMIALS OF THE SECOND KIND AND QUANTUM ALGEBRAS REPRESENTING $SU(1,1)$

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Abstract. We show how Viennot's combinatorial theory of orthogonal polynomials may be used to generalize some recent results of Hodges and Sukumar on the matrix entries in powers of operators in a representation of $SU(1,1)$. Our results link these calculations to finding the moments and inverse polynomial coefficients of certain generalized Laguerre polynomials and Meixner polynomials of the second kind. For the related operators, substitutions into essentially the same Meixner polynomials of the second kind was used by Klimyk to express its eigenvectors. Our combinatorial approach explains and generalizes this "coincidence".

Introduction

In two recent papers [3], [6] Sukumar and Hodges introduced a one-parameter family of quantum algebras with generators R and L exhibiting a parity-dependent structure. These operators induce a representation of the algebra $SU(1,1)$. They observed that calculating the entries in the powers of $L+R$ lead to some interesting combinatorial questions. Almost at the same time, Klimyk [4] considered another representation of $SU(1,1)$ which turns out to be a homomorphic image of the quantum algebra constructed by Sukumar and Hodges, induced by restricting the actions to the even-indexed or odd-indexed basis vectors only. In Klimyk's setting, the operator $L+R$ corresponds to the momentum operator, and Klimyk showed that the eigenfunctions of this operator may be expressed via substitutions into *Meixner-Pollaczek polynomials* with the appropriate parameters.

The purpose of this paper is twofold. First we show that several combinatorial questions inspired by the work of Hodges and Sukumar [3], [6] may be systematically discussed using Viennot's combinatorial theory of orthogonal polynomials [7]. The application of this theory allows not only to generalize some of the results stated in [3] and [6] but seems to be applicable to a broader class of triplets of operators, which we call *Hodges-Sukumar triplets*. In particular, some matrix entries in the powers of $L+R$ in the Hodges-Sukumar model may be obtained by finding the moments and inverse polynomial coefficients of some *Meixner polynomials of the second kind*.

The Meixner-Pollaczek polynomials and the Meixner polynomials of the second kind are essentially the same class of polynomials. The second purpose of our paper is to show that the appearance of the same class of polynomials in calculating the matrix entries and the eigenvectors of $L+R$ is not a coincidence. By generalizing Klimyk's result to all Hodges-Sukumar triplets we show that the same "coincidence" appears in a much broader setting: for the same operator, calculating some

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matrix entries in its powers and finding its eigenvectors corresponds to finding the inverse polynomial coefficients and substituting into the same orthogonal polynomial system.

Our paper is structured as follows. In the Preliminaries we review some of Viennot's results [7] and the representations of $SU(1,1)$ constructed by Sukumar and Hodges [6] and Klimyk [4]. We also add a few observations that are necessary to their adaptation. The definition of the Hodges-Sukumar triplets and the generalization of some results from [3] and [6] using Viennot's theory [7] may be found in Section 2. The specialization of these results to the Hodges-Sukumar representation of $SU(1,1)$ may be found in Section 3. Section 4 contains the generalization of Klimyk's result to Hodges-Sukumar triplets and also the statement of an analogous result for the operator $L+S+R$. The concluding Section 5 contains remarks on the potential wider adaptability of the ideas presented in this paper.

1. Preliminaries

1.1. **Viennot's combinatorial proof of Favard's theorem.** Concerning orthogonal polynomials, in this paper we follow Viennot's [7] notation and terminology, by sometimes we complement the facts stated in [7] with results cited from Chihara's classical work [2]. The direct way to define an *orthogonal polynomial sequence (OPS)* $\{p_n(x)\}_{n\geq 0}$ is to provide a linear form $f:\mathbb{C}[x]\rightarrow\mathbb{C}$ and postulate the following three axioms:

(i) for all n, $p_n(x)$ is a polynomial of degree n,
(ii) $f(p_m(x)p_n(x))=0$ if $m\neq n$,
(iii) for all n, $f(p_n(x)^2)\neq 0$.

The map f is called a *moment functional*. Whenever an OPS exists, each of its elements is determined up to a non-zero constant factor [2, Ch. I, Corollary of Theorem 2.2].

An equivalent way define an OPS is by means of Favard's theorem [2, Ch. I, Theorems 4.1 and 4.4]. This states that a sequence of monic polynomials $\{p_n(x)\}_{n\geq 0}$ is an orthogonal polynomial sequence, if and only if it satisfies the initial conditions

$$p_0(x)=1, p_1(x)=x-b_0, \tag{1}$$

and a two-term recurrence formula

$$p_{n+1}(x)=(x-b_n)p_n(x)-\lambda_n p_{n-1}(x)\quad \text{for } n\geq 1, \tag{2}$$

where the numbers b_n and λ_n are constants and $\lambda_n\neq 0$ for $n\geq 1$. The above form appears in [7, Ch. I, Theorem 9], in [2] the indices are shifted. The original proof of Favard's theorem provides only a recursive description of the moment associated functional f. Viennot [7, Ch I, Proposition 17] gave an explicit combinatorial description by expressing the *moments* $\mu_n:=f(x^n)$ as sums of weighted *Motzkin paths*. A Motzkin path [7, Ch. I, Def. 15] of length n is a path $\omega=(s_0,\ldots,s_n)$ in $\mathbb{Z}\times\mathbb{Z}$ from s_0 to s_n such that the second coordinate of all s_i's is non-negative, and each step (s_i,s_{i+1}) is either a northeast step $(1,1)$ or an east step $(1,0)$ or a southeast step $(1,-1)$. Viennot introduces the following valuation: the weight $v(\omega)$ of ω is the product of the weights of its steps (s_{i-1},s_i), where each northeast step has weight 1, each east step at level k has weight b_k and each southeast step starting at level k has weight λ_k. He then defines $\mu_n:=\sum_\omega v(\omega)$ as the total weight of all Motzkin paths ω from $(0,0)$ to $(n,0)$.

Theorem 1.1 (Viennot). *Let $\{p_n(x)\}_{n\geq 0}$ be a sequence of monic polynomials, given by (1) and (2) and let $f:\mathbb{C}[x]\rightarrow\mathbb{C}$ be the linear map given by $f(x^n):=\mu_n$. Then for all $n,k,\ell\geq 0$ we have*

$$f(x^n p_k(x)p_\ell(x))=\lambda_1\cdots\lambda_\ell\sum_{\omega}v(\omega)$$

where the summation is over all Motzkin paths of length n from level k to level ℓ.

Substituting $n=0$ in Theorem 1.1 yields the more difficult implication of Favard's theorem. A generalization of Theorem 1.1 [7, Ch. III, Theorem 1] allows the computation of the *inverse polynomials* of a system of monic polynomials $\{p_n(x)\}_{n\geq 0}$ given by (1) and (2), even if some scalars λ_n are zero (and thus $\{p_n(x)\}_{n\geq 0}$ is not an OPS). The inverse polynomials $q_n(x):=\sum_{i=0}^n q_{n,i}x^i$ are defined by $x^n=\sum_{i=0}^n q_{n,i}p_i(x)$.

Theorem 1.2 (Viennot). *Let $\{p_n(x)\}_{n\geq 0}$ be a system of monic polynomials defined by (1) and (2) for some numbers $\{b_n\}_{n\geq 0}$ and $\{\lambda_n\}_{n\geq 1}$. The coefficient $\mu_{n,k}$ of x^k in the inverse polynomial $q_n(x)$ is then the total weight of all weighted Motzkin paths of length n starting at $(0,0)$ and ending at (n,k).*

1.2. **Viennot's "histoires", Laguerre and Meixner polynomials.** Sometimes even more combinatorial models may be built, using Viennot's second valuation of Motzkin paths, defined as follows. Given three sequences $\{a_k\}_{k\geq 0}$, $\{b_k\}_{k\geq 0}$ and $\{c_k\}_{k\geq 1}$ of numbers we define the weight $v_1(\omega)$ of a Motzkin path ω as the product of the weights $v_1(s_{i-1},s_i)$ of its steps, such that each northeast (resp. east, resp. southeast) step starting at level k has weight a_k (resp. b_k, resp. c_k). Setting

(3) $$\lambda_k=a_{k-1}c_k\quad\text{for } k\geq 1,$$

for a Motzkin path ω from $(0,0)$ to $(n,0)$, $v_1(\omega)$ is equal to $v_(\omega)$, as defined in Section 1.1, because each northeast step starting at some level $k-1$ may be matched to the first subsequent southeast step starting at level k. The combinatorial interest in v_1 arises when the sequences $\{a_k\}_{k\geq 0}$, $\{b_k\}_{k\geq 0}$ and $\{c_k\}_{k\geq 1}$ consist of non-negative integers, allowing us to think of these weights as making *choices* from a set of options. In such situations Viennot defines a *story ("histoire")* as a pair $(\omega;(p_1,\ldots,p_n))$ of a Motzkin path $\omega=(s_0,\ldots,s_n)$ and a sequence $(p_1,\ldots,p_n)$ of positive integers satisfying $1\leq p_i\leq v_1(s_{i-1},s_i)$ for $1\leq i\leq n$. Clearly the moment μ_n is the number of stories of length n.

Of particular interest to us is Viennot's bijection between his "histoires de Laguerre" of length n associated to the valuation

(4) $$a_k=k+1,\quad b_k=2k+2\quad\text{for } k\geq 0;\quad c_k=k+1\quad\text{for } k\geq 1,$$

and the permutations of the set $\{1,\ldots,n+1\}$. Here we only outline the construction, for details we refer to [7]. We represent each permutation by an increasing binary tree, recursively defined as follows: 1 becomes the root of the tree, the numbers preceding 1 become the nodes of the left descendants of 1, the remaining numbers become the right descendants of 1. We think of each labeled Motzkin path as the description of the process of "growing" such a tree from an unlabeled root: in step i the p_{i+1}th unlabeled node gets label i, and this node obtains two (resp. one resp. zero) unlabeled children if step i is a northeast (resp. east resp. southeast) step. For the east steps we write $b_k=b'_k+b''_k$ where $b'_k=k+1$ is the number of "red" east steps, $b''_k=k+1$ is the number of "blue" east steps, and the color red (blue) represents the options of adding a left (right) single child.

The *(monic) Laguerre polynomials* $L_n^{(\alpha)}(x)$ are the OPS defined by (1) and (2) where $b_k=2k+\alpha+1$ and $\lambda_k=k(k+\alpha)$, see [7, Ch. II, §5]. The associated moments are given by [7, Ch. II, (31')]

(5) $$\mu_n=(\alpha+1)_n=(\alpha+1)(\alpha+2)\cdots(\alpha+n).$$

The combinatorial model outlined above is associated to the case $\alpha = 1$ yielding *large Laguerre stories*. For general α we have *weighted Laguerre stories*, for $\alpha = 0$ we obtain *restricted Laguerre stories* by limiting b'_k to k above. The restricted Laguerre stories are thus a subset of the large Laguerre stories and correspond bijectively to the permutations $\sigma \in S_{n+1}$ with $\sigma(1) = n + 1$.

Another class of particular interest to us are the *(monic) Meixner polynomials of the second kind* $M_n(x; \delta, \eta)$, defined by (1) and (2) where $b_k = (2k + \eta)\delta$ and $\lambda_k = (\delta^2 + 1)k(k + \eta - 1)$. We will only be interested in the case when $\delta = 0$ thus $b_k = 0$ and $\lambda_k = k(k + \eta - 1)$. The moments associated to $\{M_n(x; \delta, \eta)\}_{n\geq 0}$ may be expressed using the Motzkin paths associated to the Laguerre polynomials $\{L_n^{(\eta-1)}(x)\}_{n\geq 0}$ subject to the restriction that *no east step occurs*. Motzkin paths with no east steps are also known as *Dyck paths*. For the nth moment $\mu_n(\delta, \eta)$ we have [7, Ch. II, Eq. (74)]

$$\mu_n(\delta, \eta) = \sum_{\sigma\in S_n} \eta^{\mathrm{s}(\sigma)} \delta^{\mathrm{dm}(\sigma)+\mathrm{dd}(\sigma)} (1 + \delta^2)^{\mathrm{c}(\sigma)}.$$

Here $\mathrm{dm}(\sigma)$ resp. $\mathrm{dd}(\sigma)$ denotes the number of double rises resp. double descents of σ with the convention that a descent $(\sigma(1), \sigma(2))$ or $(\sigma(n - 1), \sigma(n))$ counts as double. For $\delta = 0$, the only permutations that yield a nonzero contributions are *alternating or zig-zag* permutations starting and ending with a rise, whose set we denote by Z_n. Thus we obtain

$$\mu_n(0, \eta) = \sum_{\sigma\in Z_n} \eta^{\mathrm{s}(\sigma)}, \tag{6}$$

where $\mathrm{s}(\sigma)$ is the number of *prominent elements ("éléments saillants")* $\sigma(i)$, defined by the property $\sigma(i) = \min\{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$. In particular, $\mu_{2n+1}(\eta, 0) = 0$ for all $n \geq 0$. The case of a positive integer η was worked out by Carlitz [1]. He considered the polynomials $f_n^k(x) = (-i)^n M_n(ix; 0, k)$ but, using the substitution $x \mapsto x/i$ we may recover the moments $\mu_{2n}(0, \eta)$. In the next lemma we state Carlitz' formula modified for our purposes, together with the outline proof that extends its validity to all numbers η. It is very likely that by now this generalization is also known.

Lemma 1.3. *For all $\eta \neq 0$ the moments of $\{M_n(x; 0, \eta)\}_{n\geq 0}$ satisfy*

$$\mu_{2n}(0, \eta) = E_{2n}^{(\eta)} \quad \textit{where} \quad \sec^{\eta}(t) = \sum_{n\geq 0} E_{2n}^{(\eta)} \frac{t^{2n}}{(2n)!}.$$

Proof. Let $E_{2n,k}$ denote the number of those zig-zag permutations on $2n$ elements that start and end with a rise and have k prominent elements. Since 1 is the rightmost prominent element, using the decomposition $\sigma(1)\cdots\sigma(n) = \sigma(1)\cdots 1 \cdots\sigma(n)$ we have the recursion formula

$$E_{2n,k} = \sum_{m=0}^{n-1} \binom{2n-1}{2m} E_{2m,k} \cdot T_{2n-2m-1}. \tag{7}$$

Here $T_{2n-2m-1}$ counts the number of zig-zag permutations on $2n - 2m - 1$ elements, starting with a descent and ending with a rise. Introducing $\phi_k(t) := \sum_{n\geq 0} E_{2n,k}x^{2n}/(2n)!$ we obtain the recursion

$$\phi_k(t) = \int_0^t \phi_{k-1}(u) \tan(u)\, du.$$

Using this, we may show by induction that $\phi_k(t) = (-\ln\cos(t))^k$. Thus, by (6), $\mu_{2n}(0, \eta)$ is the coefficient of $t^{2n}/(2n)!$ in $\sum_{k\geq 0}(-1)^k \ln\cos(t)^k \cdot \eta^k/k! = \exp(-\eta \ln\cos(t)) = \sec(t)^{\eta}$. □

We conclude this section by reviewing the inverse polynomials of the polynomials $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ and $\{M_n(x;0,\eta)\}_{n\geq 0}$. Both OPS are examples of *Sheffer orthogonal polynomials* defined as polynomials $\{p_n(x)\}_{n\geq 0}$ given by a generating function

$$\sum_{n\geq 0} p_n(x)\frac{t^n}{n!} = f(t)\exp(xg(t)). \tag{8}$$

As noted by Viennot [7, Ch. III, (11) and (12)], the inverse polynomials $\{q_n(x)\}_{n\geq 0}$ of a Sheffer OPS $\{p_n(x)\}_{n\geq 0}$ form a Sheffer OPS, given by

$$\sum_{n\geq 0} q_n(x)\frac{t^n}{n!} = \frac{1}{f(g^{\langle -1\rangle}(t))}\exp(xg^{\langle -1\rangle}(t)). \tag{9}$$

Here $g^{\langle -1\rangle}(t)$ stands for the compositional inverse of $g(t)$.

1.3. **Representations of $SU(1,1)$ by Sukumar, Hodges, and Klimyk.** Following the notation used by Klimyk [4], the Lie algebra $SU(1,1)$ has the generators J_0, J_1, J_2 subject to the commutation relations $[J_0, J_1] = \mathbf{i}J_2$, $[J_1, J_2] = \mathbf{i}J_0$, $[J_2, J_0] = \mathbf{i}J_1$. Equivalently, we may use $J_+ := J_1 + \mathbf{i}J_2$ and $J_- := J_1 - \mathbf{i}J_2$ instead of J_1 and J_2. Then we have the commutation relations

$$[J_0, J_+] = J+, \quad [J_0, J_-] = -J_-, \quad [J_-, J_+] = 2J_0. \tag{10}$$

Both Sukumar and Hodges [6] and Klimyk [4] have studied representations of the above generators as operators acting on an infinite dimensional Hilbert space, thus obtaining models of a quantum oscillator.

In the work of Sukumar and Hodges [6] the Hilbert space has the orthonormal basis vectors $|0\rangle$, $|1\rangle$, $|2\rangle$, ... and α is a fixed real number. They define the operators L, R and S, by the following formulas.

$$\begin{array}{rcl} L|2n\rangle &=& \sqrt{(2n+1+\alpha)(2n+2))}/2|2n+2\rangle \\ L|2n-1\rangle &=& \sqrt{(2n+1+\alpha)(2n))}/2|2n+1\rangle \\ R|2n+2\rangle &=& \sqrt{(2n+1+\alpha)(2n+2))}/2|2n\rangle \\ R|2n+1\rangle &=& \sqrt{(2n+1+\alpha)(2n))}/2|2n-1\rangle \\ S|n\rangle &=& (2n+1+\alpha)/2|n\rangle \end{array} \tag{11}$$

Obviously, L represents J_+, R represents J_- and S represents $2J_0$. A deeper analysis of the special case $\alpha = 1$ appears in [3], by the same authors. In particular, they show that for $\alpha = 1$ we have

$$\langle 0|(L+R)^{2m}|0\rangle = E_{2m} \quad \text{and} \quad \langle 1|(L+R)^{2m}|1\rangle = T_{2m+1} \quad \text{for all } m \geq 0. \tag{12}$$

Here $\langle 0|$, $\langle 1|$, ... denote the appropriate dual basis vectors. The numbers $\{E_{2m}\}_{m\geq 0}$ resp. $\{T_{2m+1}\}_{m\geq 0}$ are the *secant* resp. *tangent* numbers, given by

$$\sec(x) = \sum_{m\geq 0}\frac{E_{2m}}{(2m)!}x^{2m} \quad \text{and} \quad \tan(x) = \sum_{m\geq 0}\frac{T_{2m+1}}{(2m+1)!}x^{2m+1}.$$

The representation considered by Klimyk [4] was introduced in [8, Ch. 6], and is over the Hilbert space obtained as the closure of the space of all polynomials in a single variable y. Fixing a parameter l, Klimyk introduces the basis

$$e_n^l(y) := \left(\frac{(2l+n-1)!}{n!}\right)^{1/2} y^n, \quad \text{where} \quad n = 0, 1, 2, \ldots$$

The operators J_0, J_1, and J_2 are represented by differential operators, such that the relations [4, (8), (9)]

$$J_0 e_n^l = (l+n)e_n^l, \quad J_+ e_n^l = \sqrt{(2l+n)(n+1)}e_{n+1}^l, \quad J_- e_n^l = \sqrt{(2l+n-1)n}e_{n-1}^l \tag{13}$$

are satisfied. Comparing (11) and (13), the following statement is immediate.

Lemma 1.4. *For each l, the representation of $SU(1,1)$ considered by Klimyk [4] is a homomorphic image of a quantum algebra constructed by Sukumar and Hodges [6], via either of the the following restrictions and identifications:*

- (i) *We set $\alpha := 4l-1$, $e_n^l(y)$ is identified with $|2n\rangle$ for $n = 0, 1, \ldots$ and we restrict the action of the operators to the closure of the subspace $\langle |2n\rangle : n \geq 0\rangle$.*
- (ii) *We set $\alpha := 4l-3$, $e_n^l(y)$ is identified with $|2n+1\rangle$ for $n = 0, 1, \ldots$ and we restrict the action of the operators to the closure of the subspace $\langle |2n+1\rangle : n \geq 0\rangle$.*

As noted above, L represents J_+, R represents J_-, and S represents $2J_0$.

One of Klimyk's [4, Eq. (17)] results is that the eigenfunctions of the momentum operator $P := J_1$ are of the form

$$\psi_p(y) = \sum_{n\geq 0} P_n^{(l)}(p; \pi/2)y^n. \tag{14}$$

Here the polynomials $P_n^{(l)}(p; \pi/2)$ are the *Meixner-Pollaczek polynomials* which are essentially Meixner polynomials of the second kind $M_n(x; \delta, \eta)$. The formula connecting them is [2, Ch. VI, (3.22)]

$$P_n^\lambda(x; \phi) = \frac{\sin^n \phi}{n!} M_n(2x; \delta, 2\lambda), \quad \delta = \cot \phi, \quad 0 < \phi < \pi. \tag{15}$$

Chihara [2] does not use the term "Meixner-Pollaczek polynomials", but mentions that they satisfy the recurrence [2, Ch. VI, Eq. (5.13)]

$$nP_n^\lambda(x) = 2[x \sin \phi + (n+\lambda-1)\cos \phi]P_{n-1}^\lambda(x) - (n+2\lambda-2)P_{n-2}^\lambda(x), \tag{16}$$

which is the same as (1.7.3) in the Askey-Wilson scheme published by Koekoek and Swarttouw [5] and cited by Klimyk [4]. Setting $\phi = \pi/2$ in (15) leads to $\delta = 0$.

2. Hodges-Sukumar triplets of operators

In this section we generalize Eq. (12) and the similar formulas in the work of Sukumar and Hodges [3], [6] to a reasonably large class of triplets of operators and to arbitrary dual basis indices.

Definition 2.1. *Consider an infinite dimensional Hilbert space with orthonormal basis vectors $|0\rangle$, $|1\rangle$, $|2\rangle$, ... We call a triplet (L, R, S) of operators a* Hodges-Sukumar triplet, *if $R|0\rangle = R|1\rangle = 0$, and for $n \geq 0$ we have*

$$\begin{aligned} L|n\rangle &= l_n|n+2\rangle \\ R|n+2\rangle &= r_n|n\rangle \\ S|n\rangle &= s_n|n\rangle \end{aligned} \tag{17}$$

for some real numbers $\{l_n\}_{n\geq 0}$, $\{r_n\}_{n\geq 0}$, $\{s_n\}_{n\geq 0}$.

We will show how Theorems 1.1 and 1.2 allow us to compute the values of $\langle j|(L+S+R)^n|0\rangle$, $\langle j|(L+S+R)^n|1\rangle$, $\langle j|(L+R)^{2n}|0\rangle$ and $\langle j|(L+R)^{2n}|1\rangle$ for all $n, j \geq 0$. We begin with the appropriate generalization of equations (4.2) and (4.3) in [3]. Observe that, for technical reasons we read the words X consisting of the letters L, R, S *right-to-left.*

Lemma 2.2. *Let X be a word of length n consisting of the letters L, R, S, where (L, R, S) is a Hodges-Sukumar triplet, and let $i \geq 0$ and $d \geq -i/2$ be integers. Associate to the pair (X, i) a lattice path starting at $(0, \lfloor i/2 \rfloor)$ such that each L is a northeast step, each R is a southeast step, each S is an east step, and we read the letters in X right-to-left. Then $\langle i+2d|X|i\rangle$ is not zero only if the lattice path associated to (X, i) is a Motzkin path of length n from $(0, \lfloor i/2 \rfloor)$ to $(n, \lfloor i/2 \rfloor + d)$. Here $\lfloor x \rfloor$ is the largest integer that is less than or equal to x.*

Proof. Applying X to the basis vector $|i\rangle$ involves applying the operators L, R, and S in it, reading X from right to left. While applying these operators, at every instance of the calculation our partial result is a scalar multiple of a single basis vector. Each L increases the index of this basis vector by 2, each L decreases it by 2 and each S leaves the index unchanged. Thus the end result is either zero or a scalar multiple of $|i+2d\rangle$ where d is the difference between the number of Ls and Rs in X (and may be negative). If at any instance the number of Rs read exceeds the number of Ls by more than $i/2$ then we get zero by $R|0\rangle = R|1\rangle = 0$. Therefore, if $\langle i+2d|X|i\rangle \neq 0$ then (X, i) must represent a Motzkin path starting at $(0, \lfloor i/2 \rfloor)$ and necessarily ending at $(n, \lfloor i/2 \rfloor + d)$. □

Next we derive the generalizations of the equations (4.6) and (4.7) in [3]. Our formulas look somewhat different due to our convention of reading X right-to-left.

Lemma 2.3. *For all $p, i \geq 0$ and $d \geq -i/2$ we have*

$$\langle i+2d|XSL^p|i\rangle = s_{2p+i} \cdot \langle i+2d|XL^p|i\rangle \quad \text{and} \tag{18}$$

$$\langle i+2d|XRL^{p+1}|i\rangle = l_{2p+i}r_{2p+i} \cdot \langle i+2d|XL^p|i\rangle. \tag{19}$$

The proof is immediate. Using Lemmas 2.2 and 2.3 we may express all $\langle i+2d|(L+S+R)^n|i\rangle$ and $\langle i+2d|(L+R)^n|i\rangle$ as total weights of weighted Motzkin paths.

Theorem 2.4. *Let $i \geq 0$ and $d \geq -i/2$ be integers. Then $\langle i+2d|(L+S+R)^n|i\rangle$ is the total weight of all weighted Motzkin paths of length n starting at $(0, \lfloor i/2 \rfloor)$, ending at $(n, \lfloor i/2 \rfloor + d)$ such that each northeast step has weight 1, each southeast step starting at level q has weight $\lambda_q = l_{2q-2+2\{i/2\}}r_{2q-2+2\{i/2\}}$, and each east step at level q has weight $s_{2q+2\{i/2\}}$. Similarly, $\langle i+2d|(L+R)^n|i\rangle$ is the total weight of all weighted Dyck paths of length n starting at $(0, \lfloor i/2 \rfloor)$, ending at $(n, \lfloor i/2 \rfloor + d)$ such that each northeast step has weight 1 and each southeast step starting at level q has weight $\lambda_q = l_{2q-2+2\{i/2\}}r_{2q-2+2\{i/2\}}$. Here $\{x\}$ is defined as $\{x\} := x - \lfloor x \rfloor$.*

As a consequence of Theorem 1.1, setting $d = 0$ and $i \in \{0, 1\}$ in Theorem 2.4 yields the following algebraic statements.

Corollary 2.5. *If $l_{2m}r_{2m} \neq 0$ for all $m \geq 0$ then for all n, $\langle 0|(L+S+R)^n|0\rangle$ is the nth moment of the functional associated to the OPS given by (1) and (2) where $b_p = s_{2p}$ and $\lambda_p = l_{2p-2}r_{2p-2}$, and $\langle 0|(L+R)^{2n}|0\rangle$ is the $2n$th moment of the functional associated to the OPS given by (1) and (2) where $b_p = 0$ and $\lambda_p = l_{2p-2}r_{2p-2}$. If $l_{2m+1}r_{2m+1} \neq 0$ for all $m \geq 0$ then for all n, $\langle 1|(L+S+R)^n|1\rangle$ is the nth moment of the functional associated to the OPS given by (1) and (2) where $b_p = s_{2p+1}$ and $\lambda_p = l_{2p-1}r_{2p-1}$, and $\langle 1|(L+R)^{2n}|1\rangle$ is the $2n$th moment of the functional associated to the OPS given by (1) and (2) where $b_p = 0$ and $\lambda_p = l_{2p-1}r_{2p-1}$.*

Remark 2.6. As noted in the Preliminaries, an OPS and the associated moments determine each other only up to a constant factor. In Corollary 2.5 we understand that the moments were defined as total weights of weighted Motzkin paths as in Theorem 1.1. If we obtain the moments by some other means, we always need to check whether an adjustment by a constant factor is necessary. It suffices to check whether $\mu_0 = 1$ is satisfied.

When we apply Theorem 2.4 to $d \neq 0$, Theorem 1.2 becomes useful, at least for the cases $i \in \{0, 1\}$.

Theorem 2.7. *For all $n, d \geq 0$, each of the expressions $\langle 2d|(L+S+R)^n|0\rangle$, $\langle 2d|(L+R)^n|0\rangle$, $\langle 2d+1|(L+S+R)^n|1\rangle$ and $\langle 2d+1|(L+R)^n|1\rangle$ is the coefficient $\mu_{n,d}$ of x^d in the degree n inverse polynomial for the appropriate system of monic polynomials given by (1) and (2). The results are summarized in the table below.*

Expression	b_p	λ_p
$\langle 2d|(L+S+R)^n|0\rangle$	s_{2p}	$l_{2p-2}r_{2p-2}$
$\langle 2d+1|(L+S+R)^n|1\rangle$	s_{2p+1}	$l_{2p-1}r_{2p-1}$
$\langle 2d|(L+R)^n|0\rangle$	0	$l_{2p-2}r_{2p-2}$
$\langle 2d+1|(L+R)^n|1\rangle$	0	$l_{2p-1}r_{2p-1}$

3. The representation of $SU(1,1)$ constructed by Sukumar and Hodges

The operators R, L, and S appearing in the work of Sukumar and Hodges [6] are the motivating example of a Hodges-Sukumar triplet, defined in Section 2. For the values of r_n, l_n and s_n indicated in (11), Corollary 2.5 may be used to obtain the following results.

Proposition 3.1. *For $\alpha \neq -1$ we have*

$$\langle 0|(L+S+R)^n|0\rangle = \left(\frac{\alpha+1}{2}\right)_n = \frac{\alpha+1}{2} \cdot \left(\frac{\alpha+1}{2}+1\right) \cdots \left(\frac{\alpha+1}{2}+n-1\right),$$

and for $\alpha \neq -3$ we have

$$\langle 1|(L+S+R)^n|1\rangle = \left(\frac{\alpha+3}{2}\right)_n = \frac{\alpha+3}{2} \cdot \left(\frac{\alpha+3}{2}+1\right) \cdots \left(\frac{\alpha+3}{2}+n-1\right).$$

Proof. By Corollary 2.5, the numbers $\langle 0|(L+S+R)^n|0\rangle$ are the moments of the functional associated to the OPS given by (1) and (2) where $b_p = s_{2p} = 2p + (\alpha+1)/2$ and $\lambda_p = l_{p-2}r_{p-2} = p(p+(\alpha-1)/2)$. This OPS is the set of Laguerre polynomials with parameter $(\alpha-1)/2$. Similarly, $\langle 1|(L+S+R)^n|1\rangle$ is the nth moment of the Laguerre polynomials with parameter $(\alpha+1)/2$. Both equations follow from (5). □

Remark 3.2. As a consequence of the proof of Proposition 3.1, a combinatorial interpretation of $\langle 0|(L+S+R)^n|0\rangle$ and $\langle 1|(L+S+R)^n|1\rangle$ may be found using Viennot's "Laguerre stories", outlined in the Preliminaries. The "best" combinatorial interpretations (linked to permutation enumeration with no special weights) is associated to the cases when $(\alpha-1)/2$ or $(\alpha+1)/2$ belongs to $\{0, 1\}$. More precisely, $\langle 0|(L+S+R)^n|0\rangle$ counts all permutations of $\{1, 2, \ldots, n\}$ when $\alpha = 1$ and it counts all permutations of $\{1, 2, \ldots, n+1\}$ when $\alpha = 3$. Similarly, $\langle 0|(L+S+R)^n|0\rangle$ counts all permutations of $\{1, 2, \ldots, n\}$ when $\alpha = -1$ and it counts all permutations of $\{1, 2, \ldots, n+1\}$ when $\alpha = 1$.

Similarly to the proof of Proposition 3.1, it is easy to show the following.

Proposition 3.3. *For $\alpha \neq -1$ we have*

$$\langle 0|(L+R)^{2n}|0\rangle = \mu_{2n}(0,(\alpha+1)/2)$$

and for $\alpha \neq -3$ we have

$$\langle 1|(L+R)^{2n}|1\rangle = \mu_{2n}(0,(\alpha+3)/2).$$

Here $\mu_{2n}(0,\eta)$ stands for the $(2n)$th moment associated to the Meixner polynomials of the second kind with parameters $(0,\eta)$.

For $\alpha = 1$, (6) takes a very simple form, as this is noted in [7, Ch. II, Example 21]. Thus we obtain that, in this case $\langle 0|(L+R)^{2n}|0\rangle$ is the secant number E_{2m}, as stated in [3, Eq. (5.2)]. We may also easily compute $\langle 1|(L+R)^{2n}|1\rangle$ using Lemma 1.3 which gives

$$\langle 1|(L+R)^{2n}|1\rangle = E^{(2)}_{2n} \quad \text{where} \quad \sec^2(t) = \sum_{n\geq 0} E^{(2)}_n \frac{t^n}{n!}.$$

As noted in [6, Eq. (3.2)], we have $E^{(2)}_n = T_{2n+1}$. Lemma 1.3 also allows to make the following generalizations.

Corollary 3.4. *For $\alpha \neq -1$ we have*

$$\langle 0|(L+R)^{2n}|0\rangle = E^{((\alpha+1)/2)}_{2n}$$

and for $\alpha \neq -3$ we have

$$\langle 1|(L+R)^{2n}|1\rangle = E^{((\alpha+3)/2)}_{2n}.$$

Remark 3.5. In analogy to Remark 3.2 we may observe that Viennot's theory offers the best combinatorial interpretations for the numbers $\langle 0|(L+R)^{2n}|0\rangle$ and $\langle 0|(L+R)^{2n}|0\rangle$ in the cases when $(\alpha+1)/2$ or $(\alpha+3)/2$ equals 1. In these cases we may find an appropriate model by further restricting the appropriate "Laguerre stories" which corresponds to enumerating certain zig-zag permutations. The case when $\alpha = -1$ or $\alpha = -3$ is special since setting $\delta = \eta = 0$ does not yield an OPS. (The requirement of all $\lambda_n \neq 0$ is violated.) For the handling of the case $\alpha = -1$ we refer to the work of Sukumar and Hodges [6]. We should stress however that, even beyond the issue of these "singularities", not all combinatorial questions raised by Sukumar and Hodges [6] may be handled using Viennot's theory in a straightforward manner. The most remarkable challenge is presented by the case $\alpha = 0$ where, by Viennot's theory, we could not avoid considering "weighted stories". Sukumar and Hodges handle this case by considering the enumeration of *transposition zig-zag classes*. It is an interesting question for future research whether this enumeration question may be related to Viennot's theory by constructing a highly nontrivial bijection, similarly to Viennot's bijection between all permutations and his "Laguerre stories".

As seen in the preceding proofs, the application of Theorem 2.4 is linked to the use of certain Laguerre polynomials and Meixner polynomials of the second kind. It is easy to show in a similar way that, to use Theorem 1.2 to calculate $\langle i+2d|(L+S+R)^n|i\rangle$ for $i \in \{0,1\}$ we need to find the inverse polynomials of the Laguerre polynomials $\{L^{((\alpha-1)/2)}_n(x)\}_{n\geq 0}$ and $\{L^{((\alpha+1)/2)}_n(x)\}_{n\geq 0}$, whereas to calculate $\langle i+2d|(L+R)^n|i\rangle$ for $i \in \{0,1\}$ we need to find the inverse polynomials of the Meixner polynomials of the second kind $\{M_n(x;,0,(\alpha+1)/2)\}_{n\geq 0}$ and $\{M_n(x;,0,(\alpha+3)/2)\}_{n\geq 0}$. We may find these inverse polynomials using (8) and (9).

The Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ satisfy [7, Ch. II, (29)]

$$\sum_{n\geq 0} L_n^{(\alpha)}(x)\frac{t^n}{n!} = (1+t)^{-\alpha-1}\exp\left(x\frac{t}{1+t}\right), \tag{20}$$

thus we must set $f(t)=(1+t)^{-\alpha-1}$ and $g(t)=t/(1+t)$ in (8). As indicated in [7, Ch. III, Table 4], this implies $g^{\langle -1\rangle}(t)=t/(1-t)$. Thus the inverse polynomials of $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ are given by

$$\sum_{n\geq 0} Q_n(x)\frac{t^n}{n!} = \left(\frac{1}{1-t}\right)^{\alpha+1}\exp\left(x\frac{t}{1-t}\right). \tag{21}$$

Hence we have

$$\mu_{n,d}=[x^d]Q_n(x)=\frac{n!}{d!}\sum_{j=0}^{n-d}(-1)^j\binom{-\alpha-1}{j}\binom{-d}{n-d-j}(-1)^{n-d-j}=\frac{(-1)^{n-d}n!}{d!}\binom{-d-\alpha-1}{n-d},$$

implying

$$\mu_{n,d}=\binom{n}{d}(\alpha+1+d)_{n-d}. \tag{22}$$

The Meixner polynomials of the second kind $\{M_n(x;,0,\eta\}_{n\geq 0}$ satisfy [7, Ch. II, (64)]

$$\sum_{n\geq 0} M_n(x;0,\eta)\frac{t^n}{n!} = (1+t^2)^{-\eta/2}\exp(x\arctan(t)),$$

thus we must set $f(t)=(1+t^2)^{-\eta/2}$ and $g(t)=\arctan(t)$ in (8). As indicated in [7, Ch. III, Table 4], this implies $g^{\langle -1\rangle}(t)=\tan(t)$, and the inverse polynomials of $\{M_n(x;,0,\eta\}_{n\geq 0}$ are given by

$$\sum_{n\geq 0} Q_n(x)\frac{t^n}{n!} = \sec(t)^{\eta}\exp\left(x\tan(t)\right). \tag{23}$$

Hence we have

$$\mu_{n,d}=\frac{n!}{d!}[t^n]\sec(t)^{\eta}\tan(t)^d. \tag{24}$$

Using (22) and (22) we have the following result.

Theorem 3.6. *For all $n,d\geq 0$ we have*

$$\langle 2d|(L+S+R)^n|0\rangle = (-1)^{n-d}\binom{n}{d}\left(\frac{\alpha+1}{2}+d\right)_{n-d}, \tag{25}$$

$$\langle 2d+1|(L+S+R)^n|1\rangle = (-1)^{n-d}\binom{n}{d}\left(\frac{\alpha+3}{2}+d\right)_{n-d}, \tag{26}$$

$$\langle 2d|(L+R)^n|0\rangle = \frac{n!}{d!}[t^n]\sec(t)^{(\alpha-1)/2}\tan(t)^d \quad \textit{and} \tag{27}$$

$$\langle 2d+1|(L+R)^n|1\rangle = \frac{n!}{d!}[t^n]\sec(t)^{(\alpha+1)/2}\tan(t)^d. \tag{28}$$

4. A generalization of Klimyk's result to Hodges-Sukumar triplets

In this section we return to general Hodges-Sukumar triplets, but we make the supplementary assumption that none of the coefficients r_n that appear in Definition 2.1 is zero. Since

$$J_1 = \frac{1}{2}(J_+ + J_-),$$

our generalization of Klimyk's result [4, Eq. (17)] is a statement about the eigenvectors of $L + R$.

Theorem 4.1. *Let (L, R, S) be a Hodges-Sukumar triplet satisfying $r_n \neq 0$ for all $n \geq 0$. Then all eigenvectors of $L + R$ may be written as the linear combination of a vector of the form*

$$\sum_{n\geq 0} \frac{p_n(r)}{\prod_{i=0}^{n-1} r_{2i}} |2n\rangle$$

and of a vector of the form

$$\sum_{n\geq 0} \frac{q_n(r)}{\prod_{i=0}^{n-1} r_{2i+1}} |2n+1\rangle.$$

Here r is any real number, $\{p_n(x)\}_{n\geq 0}$ and $\{q_n(x)\}_{n\geq 0}$ are sequences of monic polynomials defined by (1) and (2). For $\{p_n(x)\}_{n\geq 0}$ we have $b_n = 0$ and $\lambda_n = l_{2n-2}r_{2n-2}$, for $\{q_n(x)\}_{n\geq 0}$ we have $b_n = 0$ and $\lambda_n = l_{2n-1}r_{2n-1}$.

Proof. Let H_e resp. H_o be the closure of the subspace generated by the vectors $\{|2n\rangle : n \geq 0\}$ resp. $\{|2n+1\rangle : n \geq 0\}$. These are both invariant under L and R and the entire Hilbert space is the direct sum of H_e and H_o. Thus each eigenvector of $L + R$ my be written uniquely as the sum of an eigenvector belonging to H_e and an eigenvector belonging to H_o. Let us look for the eigenvector in H_e in the form $\sum_{n\geq 0} h_n \cdot |2n\rangle$. Assuming the eigenvalue is r, we have an eigenvector of $L + R$ if and only if

$$r_0 h_1 = r h_0 \quad \text{and} \tag{29}$$

$$r_{2n}h_{n+1} + l_{2n-2}h_{n-1} = rh_n \quad \text{holds for } n \geq 2. \tag{30}$$

Introducing $p_0 := h_0$ and

$$p_n := h_n \cdot r_0 r_2 \cdots r_{2n-2} \quad \text{for } n \geq 1$$

we may rewrite (29) as

$$p_1 = rp_0, \tag{31}$$

while multiplying both sides of (30) by $r_0 r_2 \cdots r_{2n-2}$ yields the equivalent equation

$$p_{n+1} = rp_n - l_{2n-2}r_{2n-2}p_{n-1} \quad \text{for } n \geq 1. \tag{32}$$

If $p_0 = 0$ then (31) and (32) implies $p_n = 0$ for all n. We obtain a nonzero eigenvector only if $p_0 \neq 0$ and then, without loss of generality we may assume $p_0 = 1$. Furthermore, (32) implies that p_n must be obtained by substituting r into a monic polynomial sequence $\{p_n(x)\}_{n\geq 0}$ satisfying Favard's recursion formula with $\lambda_n = l_{2n-2}r_{2n-2}$ for $n \geq 1$. Therefore any eigenvector belonging to H_e is of the form stated in the theorem.

Finding an eigenvector belonging to H_e may be performed similarly. Assuming it is of the form $\sum_{n\geq 0} h_n|2n+1\rangle$ we must have $r_1h_1 = rh_0$ and

$$r_{2n+1}h_{n+1} + l_{2n-1}h_{n-1} = rh_n \quad \text{for } n \geq 2.$$

Introducing $q_0 := h_0$ and

$$q_n := h_n \cdot r_1 r_3 \cdots r_{2n-1} \quad \text{for } n \geq 1$$

we may rewrite the conditions for h_n as $q_1 = rq_0$ and

$$q_{n+1} = rq_n - l_{2n-1} r_{2n-1} q_{n-1} \quad \text{for } n \geq 2.$$

□

Remark 4.2. Just like Klimyk's original result [4, Eq. (17)], Theorem 4.1 is only a necessary condition. Whether all vectors arising in the above form are eigenvectors (or even convergent sums) is subject to further investigation. For the model considered by Klimyk [4], he is able to show that the spectrum consists of all real numbers.

For the quantum algebras constructed by Sukumar and Hodges [6], Theorem 4.1 has the following consequence.

Corollary 4.3. *In the quantum algebras constructed by Sukumar and Hodges* [6]*, all eigenvectors of $L + R$ may be written as a linear combination of a vector of the form*

$$\sum_{n\geq 0} \frac{M_n(r;0,(\alpha+1)/2)}{\prod_{i=0}^{n-1}\sqrt{(i+(\alpha+1)/2)(i+1)}}|2n\rangle$$

and of a vector of the form

$$\sum_{n\geq 0} \frac{M_n(r;0,(\alpha+3)/2)}{\prod_{i=0}^{n-1}\sqrt{(i+(\alpha+3)/2)(i+1)}}|2n+1\rangle.$$

Here r is any real number and the $M_n(x;0,\eta)$ stand for the Meixner polynomials of the second kind.

Remark 4.4. For $\alpha = 1$ Corollary 4.3 takes the following form. Each eigenvector of $L + R$ may be written as a linear combination of a vector of the form

$$\sum_{n\geq 0} \frac{M_n(r;0,1)}{n!}|2n\rangle$$

and of a vector of the form

$$\sum_{n\geq 0} \frac{M_n(r;0,2)}{\sqrt{n!(n+1)!}}|2n+1\rangle.$$

A combinatorial model for the coefficients of the powers of x in $M_n(x;0,1)$ is mentioned in [7, Ch. II, Example 21]. Up to sign, the coefficient of x^k in $M_n(x;0,1)$ is the number of permutations of $\{1,2,\ldots,n\}$ having k odd cycles.

In analogy to Theorem 4.1 we may also express the eigenvectors of $L + S + R$ as follows.

Theorem 4.5. *Let (L, R, S) be a Hodges-Sukumar triplet satisfying $r_n \neq 0$ for all $n \geq 0$. Then all eigenvectors of $L + S + R$ may be written as the linear combination of a vector of the form*

$$\sum_{n\geq 0} \frac{p_n(r)}{\prod_{i=0}^{n-1} r_{2i}}|2n\rangle$$

and of a vector of the form

$$\sum_{n\geq 0} \frac{q_n(r)}{\prod_{i=0}^{n-1} r_{2i+1}}|2n+1\rangle.$$

Here r is any real number, $\{p_n(x)\}_{n\geq 0}$ and $\{q_n(x)\}_{n\geq 0}$ are sequences of monic polynomials defined by (1) and (2). For $\{p_n(x)\}_{n\geq 0}$ we have $b_n = s_{2n}$ and $\lambda_n = l_{2n-2}r_{2n-2}$, for $\{q_n(x)\}_{n\geq 0}$ we have $b_n = s_{2n+1}$ and $\lambda_n = l_{2n-1}r_{2n-1}$.

The proof is completely analogous to the proof of Theorem 4.1 and omitted. For the quantum algebras constructed by Sukumar and Hodges [6], Theorem 4.1 has the following consequence.

Corollary 4.6. *In the quantum algebras constructed by Sukumar and Hodges* [6]*, all eigenvectors of $L+S+R$ may be written as a linear combination of a vector of the form*

$$\sum_{n\geq 0} \frac{L_n^{((\alpha-1)/2)}(r)}{\prod_{i=0}^{n-1}\sqrt{(i+(\alpha+1)/2)(i+1)}}|2n\rangle$$

and of a vector of the form

$$\sum_{n\geq 0} \frac{L_n^{((\alpha+1)/2)}(r)}{\prod_{i=0}^{n-1}\sqrt{(i+(\alpha+3)/2)(i+1)}}|2n+1\rangle.$$

Here r is any real number and the $L_n^{(\alpha))}(x)$ stand for the generalized Laguerre polynomials.

Remark 4.7. The connecting coefficients $\{\lambda_n\}_{n\geq 1}$ resp. $\{b_n\}_{n\geq 0}$ for the polynomials $\{p_n(x)\}_{n\geq 0}$ and $\{q_n(x)\}_{n\geq 0}$ used in Theorems 4.1 resp. 4.5 are the same as the ones used in Corollary 2.5 and Theorem 2.7. Thus, the same systems of polynomials may be used to calculate some matrix entries of the powers of $L+R$ and $L+S+R$ as in the calculation of their eigenvectors.

5. Concluding remarks

Our first goal in this paper was to offer a systematic combinatorial framework for some questions arising in the work of Sukumar and Hodges [6] who studied linear operators exhibiting parity-violating properties. However, nothing prevents us from considering the restriction of Hodges-Sukumar triplets to the even-indexed basis vectors only, thus obtaining a parity-independent model. It seems also plausible that the definition could be further generalized and thus adaptable in an even wider range of situations. One obvious generalization is to replace the notion of parity by congruence modulo a given number. Viennot's [7] combinatorial results on calculating the weight of all weighted Motzkin paths between two given points remain applicable and the proofs of the generalizations of Klimyk's result [4, Eq. (17)] on the eigenvectors remain adaptable. What we obtain is the repetition of the same "coincidence" in a wide range of situations: the same sequence of orthogonal polynomials may be used to calculate some matrix entries in the powers of an operator and to calculate the eigenvectors of the same operator. We hope that this observation will be useful in the analysis of many future models.

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References

[1] L. Carlitz, Bernoulli and Euler numbers and orthogonal polynomials, *Duke Math. J.* **26** (1959), 1–16.

[2] T.S. Chihara, "An Introduction to Orthogonal Polynomials," Gordon and Breach Science Publishers, New York-London-Paris, 1978.

[3] A. Hodges and C. V. Sukumar, Bernoulli, Euler, permutations and quantum algebras, *Proc. R. Soc. A* **463** (2007) 2401–2414.

[4] A.U. Klimyk, The su(1,1)-models of quantum oscillator, *Ukr. J. Phys.* **51** (2006) 1019–1027.

[5] R. Koekoek and R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, `math.CA/9602214`

[6] C. V. Sukumar and A. Hodges, Quantum algebras and parity-dependent spectra, *Proc. R. Soc. A* **463** (2007) 2415–2427.

[7] X. Viennot, "Une théorie combinatoire des polynômes orthogonaux généraux," Lecture Notes LACIM, UQAM, 1983. Available online at the author's personal website `http://web.mac.com/xgviennot`

[8] N.Ja. Vilenkin, A.U. Klimyk, "Representation of Lie Groups and Special Functions," Vol 1., Kluwer, Dordrecht 1991.

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