On the Direct Path Problem of s-elementary Frame Wavelets

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Abstract

In this paper, we discuss the path-connectivity between two s-elementary normalized tight frame wavelets via the so-called direct paths. We show that the existence of such a direct path is equivalent to the non-existence of an atom of a σ-algebra defined over the defining sets of the corresponding frame wavelets, using a mapping defined by the natural translation and dilation operations between the sets. In particular, this gives an equivalent condition for the existence of a direct path between two s-elementary wavelets.

Key words: wavelets, frame wavelets, frame wavelet sets, path-connectivity of wavelets, direct path
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1. Introduction and Basic Terminology

The topological property of various families of wavelets is an interesting topic in the study of wavelet theory. The question concerning the path-connectedness of the set of all orthonormal wavelets was first raised in [8]. Similar questions were raised and studied in [5, 13, 15, 16, 17] about the set of all MRA-wavelets, tight frame wavelets, MRA tight frame wavelets and s-elementary frame wavelets. In [13, 17], it is shown that the set of MRA-wavelets is path-connected. In [16], it is shown that the set of s-elementary wavelets are path-connected. The proofs of these theorems were based on the complete characterizations of the MRA-wavelets and s-elementary wavelets. While the complete characterization of the s-elementary frame wavelets is still an open question, it has been shown that the set of s-elementary frame wavelets is path-connected as well [5]. In most works mentioned above, the connecting paths constructed to connect the two given s-elementary wavelets or s-elementary frame wavelets usually involve measurable sets other than the two sets defining the two s-elementary wavelets or s-elementary frame wavelets. The question whether such a path can always be constructed using only the two sets defining the two s-elementary wavelets or s-elementary frame wavelets is raised in [1]. A path with such a property is called a *direct path*. In this paper, we are mainly concerned with the direct path problem between two s-elementary normalized tight frame wavelets.

Let $\mathbb{H}$ be a Hilbert space. A set of elements $\{e_i\}$ is called a *frame* of $\mathbb{H}$ if there exist two positive constants $0 < A \leq B$ such that

$$A\|f\|^2 \leq \sum_i |\langle f, e_i \rangle|^2 \leq B\|f\|^2 \tag{1.1}$$

for all $f \in \mathbb{H}$. The supremum of all such numbers A and the infimum of all such numbers B are called the *frame bounds* of the frame and are denoted by A_0 and B_0 respectively. $\{e_i\}$ is called a *tight* frame when $A_0 = B_0$ and is called a *normalized tight* frame when $A_0 = B_0 = 1$. Frames can be regarded as the generalizations of orthonormal bases of Hilbert spaces.

Let T and D be the translation and dilation unitary operators acting on $L^2(\mathbb{R})$ defined by

$$(Tf)(t) = f(t-1) \text{ and } \quad (Df)(t) = \sqrt{2}f(2t), \forall f \in L^2(\mathbb{R})$$

and let $\mathcal{F}$ be the (unitary) Fourier-Plancherel transform defined by

$$(\mathcal{F}f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t)dt, \forall f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}).$$

$\mathcal{F}(f)$ is sometimes written as $\widehat{f}$ as well. An *orthonormal wavelet* of $L^2(\mathbb{R})$ is a function $\psi(t)$ in $L^2(\mathbb{R})$ with unit norm such that $\{2^{\frac{n}{2}}\psi(2^n t-\ell) : n,\ell\in\mathbb{Z}\} = \{D^nT^\ell\psi : n,\ell\in\mathbb{Z}\}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$. Similarly, $\psi(t)\in L^2(\mathbb{R})$ is called a normalized tight frame wavelet if $\{2^{\frac{n}{2}}\psi(2^n t-\ell) : n,\ell\in\mathbb{Z}\} = \{D^nT^\ell\psi : n,\ell\in\mathbb{Z}\}$ is a normalized tight frame of $L^2(\mathbb{R})$. Let us denote the set of all orthonormal wavelets of $L^2(\mathbb{R})$ by $\mathcal{W}$ and the set of all normalized tight frame wavelets by $\mathcal{N}$. Notice that the set of all MRA-wavelets is a subset of $\mathcal{W}$ and $\mathcal{W}$ is a subset of $\mathcal{N}$. Although it is shown that the set of MRA-wavelets is path-connected [13, 17], the path-connectedness of $\mathcal{W}$ and $\mathcal{N}$ remains an open question at this time.

Let E be a Lebesgue measurable set of finite measure and χ_E be the corresponding characteristic function. If the function $\psi_E\in L^2(\mathbb{R})$ defined by $\widehat{\psi_E} = \frac{1}{\sqrt{2\pi}}\chi_E$ is an orthonormal wavelet for $L^2(\mathbb{R})$, then the set E is called a *wavelet set* and the corresponding function ψ_E is called an *s-elementary* wavelet. Similarly, if the function $\psi_E\in L^2(\mathbb{R})$ is a normalized tight frame wavelet for $L^2(\mathbb{R})$, then the set E is called a *normalized tight frame wavelet set* (or NTFW set for short) and the corresponding function ψ_E is called an *s-elementary* normalized tight frame wavelet (or s-elementary NTF wavelet for short).

For the sake of convenience, let us introduce the following concepts and terms. More details can be found in [3].

Let E be a measurable set. $x,y\in E$ are $\stackrel{\delta}{\sim}$ equivalent if $x=2^n y$ for some integer n. The δ-index of a point x in E is the number of elements in its $\stackrel{\delta}{\sim}$ equivalent class and is denoted by $\delta_E(x)$. Let $E(\delta,k)=\{x\in E:\ \delta_E(x)=k\}$, then E is the disjoint union of the sets $E(\delta,k)$. Furthermore, each $E(\delta,k)$ $(k\geq 1)$ is Lebesgue measurable and is a disjoint union of k measurable sets $\{E^j(\delta,k)\}$, $1\leq j\leq k$, such that $E^j(\delta,k)\stackrel{\delta}{\sim}E^{j'}(\delta,k)$ for any $1\leq j,j'\leq k$. If we let $\Delta(E)=\cup_{k\in\mathbb{Z}}E^{(1)}(\delta,k)$, then every point in it has δ-index one (within $\Delta(E)$ itself). Furthermore, we have $\cup_{k\in\mathbb{Z}}2^kE=\cup_{k\in\mathbb{Z}}2^k\Delta(E)$. A set E is called a *2-dilation generator* of $\mathbb{R}$ if $E=E(\delta,1)$ and $\cup_{k\in\mathbb{Z}}2^kE=\mathbb{R}$. A 2-dilation generator for a subset of $\mathbb{R}$ that is invariant under 2-dilation can be similarly defined. Two sets E and F with $E=E(\delta,1)$ and $F=F(\delta,1)$ are said to be 2-dilation equivalent (and also denoted by $E\stackrel{\delta}{\sim}F$) if every point in E is $\stackrel{\delta}{\sim}$ to a point in F and vice versa.

In the case of translation, we say that $x,y\in E$ are 2π-translation equivalent, denoted by $x \overset{\tau}{\sim} y$, if $x = y + 2n\pi$ for some integer n. The τ-index of a point x in E is the number of elements in its $\overset{\tau}{\sim}$ equivalent class and is denoted by $\tau_E(x)$. Let $E(\tau, k) = \{x \in E : \tau_E(x) = k\}$. Then E is the disjoint union of the (measurable) sets $E(\tau, k)$. Two sets E and F with $E = E(\tau, 1)$ and $F = F(\tau, 1)$ are said to be 2π-translation equivalent (and denoted by $E \overset{\tau}{\sim} F$) if every point in E is $\overset{\tau}{\sim}$ to a point in F and vice versa. Finally, each $E(\tau, k)$ is a disjoint union of k measurable sets $\{E^{(j)}(\tau, k)\}$, $1 \le j \le k$, such that $E^{(j)}(\tau, k) \overset{\tau}{\sim} E^{j'}(\tau, k)$ for any $1 \le j, j' \le k$.

The following lemma (quoted from [3]) is needed for the proof of our main results. It characterizes the normalized tight frame wavelet sets (and the s-elementary wavelet sets). Part of the result can also be found in [12].

Lemma 1.1. *Let E be a Lebesgue measurable set with finite measure. Then E is a normalized tight frame wavelet set if and only if the following three conditions hold: (1) $E = E(\tau, 1)$; (2) $E = E(\delta, 1)$ and (3) $\cup_{k\in\mathbb{Z}} 2^k E = \mathbb{R}$. Furthermore, E is an s-elementary wavelet set if and only if (4) $\mu(E) = 2\pi$ in addition to the above three conditions (where μ is the Lebesgue measure).*

2. The Core of Two NTFW Sets

In this section we introduce a key concept, namely the *core* of a pair of NTFW sets.

Definition 2.1. Two measurable sets E, F are said to be *compatible* if they satisfy the following conditions:

(1) $E = E(\tau, 1) = E(\delta, 1)$, $F = F(\tau, 1) = F(\delta, 1)$;

(2) $E \overset{\tau}{\sim} F$, $E \overset{\delta}{\sim} F$;

(3) $E \cap F = \emptyset$.

Thus, if E and F are compatible, then both τ and δ are one-to-one and onto maps from E to F (modulo a null set). Let $f = \tau^{-1} \circ \delta$, then f is a one-to-one and onto map from E to itself. A measurable subset G of E is called invariant under the map f if we have $f(G) = G$. We are interested in such special subsets of E and will denote the collection of all subsets of E that are invariant under f by $\mathcal{A}$. We have the following lemmas.

Lemma 2.2. *If G is a measurable subset of E, then $f(G)$ is also measurable.*

Proof. Let G is a measurable subset of E, then

$$f(G) \quad = \quad \cup_{k\in\mathbb{Z}}((\delta(G)+2k\pi)\cap E)$$

where $\delta(G) = F \cap (\cup_{j\in\mathbb{Z}} 2^j E)$. Thus $f(G)$ is measurable since $\delta(G)$ (hence each $\delta(G)+2k\pi$) is measurable. □

Lemma 2.3. *$\mathcal{A}$ is a σ-algebra.*

Proof. Apparently, the empty set and the set E are in $\mathcal{A}$. If $G \in \mathcal{A}$, then $\overline{G} = E\setminus G \in \mathcal{A}$ since $\overline{G}$ is measurable and $f(\overline{G}) = f(E)\setminus f(G) = E\setminus G = \overline{G}$. If $\{G_i\}$ is a sequence of elements in $\mathcal{A}$, then $f(\cup G_i) = \cup f(G_i) = \cup G_i$. Since $\cup G_i$ is still measurable, it follows that $\cup G_i \in \mathcal{A}$. □

Let us now consider the case where E and F are two normalized tight frame wavelet sets. So we have $E = E(\tau,1) = E(\delta,1)$, $F = F(\tau,1) = F(\delta,1)$, $\delta(E) = F$ and $\cup_{j\in\mathbb{Z}} 2^j E = \cup_{j\in\mathbb{Z}} 2^j F = \mathbb{R}$. In case that $E \cap F \neq \emptyset$, we will remove $E\cap F$ from the discussion (since this part stays unchanged in any direct path connecting ψ_E to ψ_F). Keep in mind that even after we remove $E\cap F$, the remaining sets $E\setminus F$ and $F\setminus E$ may still not be compatible since we may not have $F = F\cap(\cup_{k\in\mathbb{Z}} E+2k\pi)$ or $E = E\cap(\cup_{k\in\mathbb{Z}} F+2k\pi)$ to begin with. Thus we will focus our discussions to disjoint sets E and F satisfying the conditions $E = E(\tau,1) = E(\delta,1)$, $F = F(\tau,1) = F(\delta,1)$, $\delta(E) = F$. Notice that as a function, $f = \tau^{-1}\circ\delta$ may no longer be defined on a subset of E (that is of positive measure) because $E\cap(\cup_{k\in\mathbb{Z}} F+2k\pi)$ may be a proper (or empty) subset of E. By a slight abuse of notation, let us define $\tau(G) = F\cap(\cup_{k\in\mathbb{Z}} G+2k\pi)$ for any subset $G \subseteq E$ and $\tau^{-1}(H) = E\cap(\cup_{k\in\mathbb{Z}} H+2k\pi)$ for any subset $H \subseteq F$.

Let us define the following two operations.

Operation 1. This operation only applies if $F_1 = F\setminus\tau(E)$ is of positive measure. Denote $\tau(E)$ by F_0. Let $E_0 = \tau^{-1}(F_0)$ and $\overline{E_0} = E\setminus E_0$, then $E = E_0 \cup \overline{E_0}$. $\mu(\delta^{-1}(F_1)) > 0$ since $\mu(F_1) > 0$. Let $\delta^{-1}(F_1)\cap E_0 = E_1$, $\delta^{-1}(F_1)\cap\overline{E_0} = E_1'$ and $F_1' = \delta(E_1)$, $F_1'' = \delta(E_1')$, then $F_1 = F_1' \cup F_1''$, $\delta^{-1}(F_1') = E_1$ and $\delta^{-1}(F_1'') = E_1'$. Similarly, let $F_2 = \tau(E_1)$ and split F_2 into $F_2 = F_2' \cup F_2''$ such that $\delta^{-1}(F_2') = \delta^{-1}(F_2)\cap E_0 = E_2$ and $\delta^{-1}(F_2'') =$

$\delta^{-1}(F_2) \cap \overline{E_0} = E_2'$. Continue in this fashion, we will then have defined E_j, E_j', F_j such that $E_j \cup E_j' = \delta^{-1}(F_j)$ and $F_j = \tau(E_{j-1})$. Finally, define $E' = \cup_{j\geq 1}(E_j \cup E_j')$ and $F' = \delta(E') = \cup_{j\geq 1}(F_j' \cup F_j'')$. See the top portion of Figure 1 for an illustration.

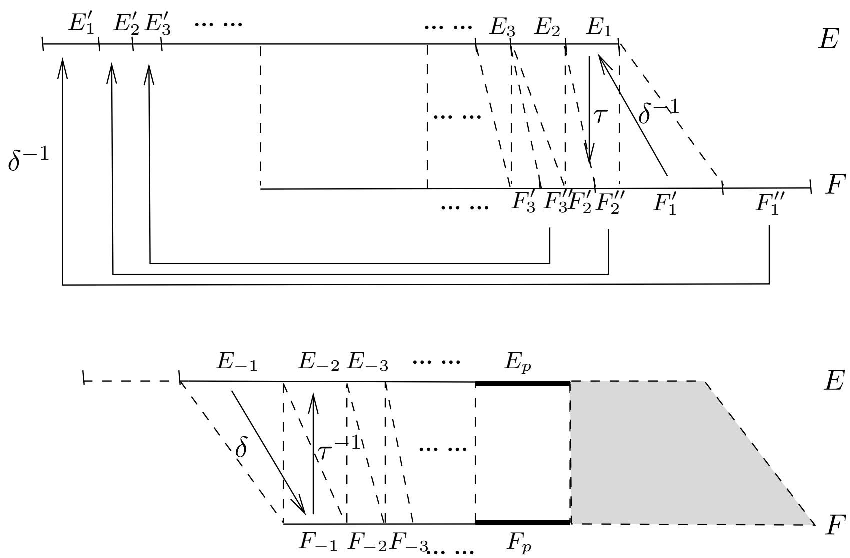

Figure 1: An illustration of the derivation of the sets E_p and F_p where the line segments stand for the corresponding subsets in E and F, not the actual structure of E and F in $\mathbb{R}$. However, in the figure, two subsets (in E and F respectively) represented by two line segments are translation equivalent if and only if their projections overlap.

Observations.

1. It is obvious from the construction of E' and F' that $F' = \delta(E')$ (so $\delta(E \setminus E') = F \setminus F'$ as well) and that $\tau^{-1}(\delta(E')) \subseteq E'$.

2. $F \setminus F' \subseteq \tau(E \setminus E')$. In fact, since $F_1 \subseteq F'$ and $F = \tau(E) \cup F_1$, we have $(F \setminus F') = (\tau(E) \setminus \cup_{j\geq 2}F_j)$, so $(F \setminus F') \setminus \tau(E \setminus E') = (\tau(E) \setminus \cup_{j\geq 2}F_j) \setminus (\tau(E) \setminus \cup_{j\geq 1}\tau(E_j)) = (\tau(E) \setminus \cup_{j\geq 2}F_j) \setminus (\tau(E) \setminus \cup_{j\geq 2}F_j) = \emptyset$. It follows that $\tau^{-1}(F \setminus F') \subseteq E \setminus E'$.

Operation 2. Operation 2 only applies if $\mu(\overline{E'} \setminus \tau^{-1}(F \setminus F')) > 0$. It is just like Operation 1 with $F \setminus F'$ taking the role of E and $E \setminus E'$ taking the role of F. As before, let $E_{-1} = (E \setminus E') \setminus \tau^{-1}(F \setminus F')$. If $\mu(E_{-1}) = 0$ then we stop here. Otherwise $\mu(E_{-1}) > 0$ and we let $F_{-1} = \delta(E_{-1})$, $E_{-2} = \tau^{-1}(E_1)$,

$F_{-2} = \delta(E_{-2})$, and so on. This defines two sequences of disjoint sets $\{E_{-j} : j \geq 1\}$ and $\{F_{-j} : j \geq 1\}$ in $E \setminus E'$ and $F \setminus F'$ respectively.

Now let $E'' = \cup_{j\geq 1} E_{-j}$, $F'' = \cup_{j\geq 1} F_{-j}$ and $E_p = E \setminus (E' \cup E'')$, $F_p = F \setminus (F' \cup F'')$. See Figure 1 for an illustration of the relationship among these sets. Similar to the claims we made earlier, we have (the details are left to the reader to verify):

3. $f(E'') \subseteq E''$.
4. $E_p \setminus \tau^{-1}(F_p) = \emptyset$.

Notice that $\delta(E'') = F''$ by its definition. On the other hand, $F_p \setminus \tau(E_p) = \emptyset$. To see this, recall that $\tau^{-1}(F \setminus F') \subseteq E \setminus E'$ so $F \setminus F' = F_p \cup F'' \subseteq \tau(E \setminus E') = \tau(E_p \cup E'') = \tau(E_p) \cup (\cup_{j\geq 2} \tau(E_{-j})) = \tau(E_p) \cup (\cup_{j\geq 1} F_{-j}) = \tau(E_p) \cup F''$ (since $\tau(E_{-1}) = \emptyset$). Since F'' and F_p are disjoint, it follows that $F_p \subseteq \tau(E_p)$. Thus we have $F_p = \tau(E_p)$ and $F_p = \delta(E_p)$. In other word, we have proved the following lemma.

Lemma 2.4. *Let E and F be two measurable sets satisfying the conditions $E = E(\tau, 1) = E(\delta, 1)$, $F = F(\tau, 1) = F(\delta, 1)$, $\delta(E) = F$ and $E \cap F = \emptyset$. Then the two sets E_p and F_p obtained from the above two operations are either of measure zero or they form a compatible pair.*

In the following two subsections, we will give two equivalent definitions of the core of a compatible pair (E, F).

2.1. The approach by an f-generator

Definition 2.5. Let (E, F) be a compatible pair. A measurable subset $G \subseteq E$ is said to be an *f-generator* if the union $\cup_{n\in Z} f^{(n)}(G)$ is a disjoint union (i.e., $\cup_{n\in Z} f^{(n)}(G) = \dot{\cup}_{n\in Z} f^{(n)}(G)$). Let X be the collection of all subsets of E which is generated by an f-generator.

It is possible that E is an element of X. If this is the case then it is easy to see that E is connected to F via a direct path, since we can just use the f-generator of E to construct the direct path. Let us assume that this is not the case for the time being. For any M_1, $M_2 \in X$, we say that $M_1 \leq M_2$ if $M_1 \subseteq M_2$.

Lemma 2.6. *In the case that X contains sets of positive measure, then the $\leq$ defined above is a partial order on X and every chain in X has an upper bound.*

Proof. It is obvious that the $\leq$ defined above via set inclusion is a partial order on X. Let $\{M_j\}_{j\geq 1}$ be an ascending sequence in X whose corresponding f-generators are G_j, then $M^\infty = \cup_{j\geq 1} M_j$ is also measurable and we claim that M^∞ can be generated by an f-generator as well. Let M_0 be the empty set so we can write $M^\infty = \cup_{j\geq 1} M_j = \cup_{j\geq 1}(M_j \setminus M_{j-1})$. For any $n \in \mathbb{Z}$, $f^{(n)}$ is a 1-1 mapping so we have $f^{(n)}(\cup_{j\geq 1}(G_j \setminus M_{j-1})) = \cup_{j\geq 1} f^{(n)}(G_j \setminus M_{j-1}) = \cup_{j\geq 1}(f^{(n)}(G_j) \setminus f^{(n)}(M_{j-1}))$. It follows that $\cup_{n\in Z} f^{(n)}(\cup_{j\geq 1}(G_j \setminus M_{j-1})) = \cup_{j\geq 1}(\cup_{n\in Z} f^{(n)}(G_j) \setminus \cup_{n\in Z} f^{(n)}(M_{j-1})) = \cup_{j\geq 1}(M_j \setminus M_{j-1}) = M^\infty$. For any $m \neq n$, $f^{(n)}(\cup_{j\geq 1}(G_j \setminus M_{j-1})) \cap f^{(m)}(\cup_{j\geq 1}(G_j \setminus M_{j-1})) = \cup_{j\geq 1, k\geq 1}\{(f^{(n)}(G_k \setminus M_{k-1})) \cap (f^{(m)}(G_j \setminus M_{j-1}))\}$. If $j = k$, then $f^{(m)}(G_j) \cap f^{(n)}(G_k) = f^{(m)}(G_j) \cap f^{(n)}(G_j) = \emptyset$ by the definition of an f-generator, which then implies that $f^{(m)}(G_j \setminus M_{j-1}) \cap f^{(n)}(G_j \setminus M_{j-1}) = \emptyset$. On the other hand, if $j > k$, then $M_k \subseteq M_{j-1}$ and it follows that $f^{(m)}(G_j \setminus M_{j-1}) \subseteq f^{(m)}(G_j \setminus M_k) \subseteq \overline{M_k}$. Since $f^{(n)}(G_k \setminus M_{k-1}) \subseteq f^{(n)}(G_k) \subseteq M_k$, we also have $f^{(m)}(G_j \setminus M_{j-1}) \cap f^{(n)}(G_k \setminus M_{k-1}) = \emptyset$. That is, $f^{(n)}(\cup_{j\geq 1}(G_j \setminus M_{j-1})) \cap f^{(m)}(\cup_{j\geq 1}(G_j \setminus M_{j-1})) = \emptyset$ if $m \neq n$, so $M^\infty = \dot{\cup}_{n\in Z} f^{(n)}(\cup_{j\geq 1}(G_j \setminus M_{j-1}))$. It is obvious that $\cup_{j\geq 1}(G_j \setminus M_{j-1})$ is a measurable subset of E. Therefore, $\cup_{j\geq 1}(G_j \setminus M_{j-1})$ is an f-generator of M^∞ hence M^∞ is in X. This proves that M^∞ is an upper bound for the chain. $\square$

By Zorn's lemma, X has a maximal element M. It is necessary that $0 < \mu(M) < \mu(E)$ since E is not an element of X. The set $E \setminus M$ is called the *core* of E and is denoted by E_c. Since $\tau(E_c) = \tau(E \setminus M) = \tau(E) \setminus \tau(M) = F \setminus \tau(M)$, $\delta(E_c) = \delta(E \setminus M) = \delta(E) \setminus \delta(M) = F \setminus \delta(M)$, but we know that $f(M) = M$, i.e., $\tau(M) = \delta(M)$, so $\tau(E_c) = \delta(E_c)$ and we call $F_c = \tau(E_c) = \delta(E_c)$ the core of F and the pair (E_c, F_c) is called the core of (E, F). In case that X contains only the empty set (or sets of measure zero), the core of (E, F) would be itself.

Finally, for any two measurable sets E and F satisfying the conditions of Lemma 2.4, we will define the core of (E, F) to be the core of (E_p, F_p) and still denote it by (E_c, F_c). Keep in mind that we have $F_c = \tau(E_c) = \delta(E_c)$ so $f(E_c) = E_c$.

Since in general the maximal element in a partial ordered set may not be unique, we may wonder if the core E_c of E rely on the maximal element M of the partial ordered set X. The following theorem answers this question.

Theorem 2.7. *The maximal element in X is unique and hence the core (E_c, F_c) is unique for the compatible pair (E, F).*

Proof. Suppose M_1 is a maximal element with f-generator G_1 and M_2 is another maximal element with f-generator G_2, i.e, $M_1 = \dot{\cup}_{n\in Z} f^{(n)}(G_1)$ and $M_2 = \dot{\cup}_{n\in Z} f^{(n)}(G_2)$. Then $M_1 \cup M_2 = \dot{\cup}_{n\in Z} f^{(n)}(G_1 \cup (G_2 \setminus M_1))$, so $M_1 \cup M_2$ is in X and $M_1 \subseteq M_1 \cup M_2$ and $M_1 \neq M_1 \cup M_2$, which contradicts the fact that M_1 is a maximal element in X. $\square$

Lemma 2.8. *E_c does not contain any element of X other than measure zero sets.*

Proof. Assume the contrary, that is, there exists $A \subseteq E_c$ such that $A \in X$ and $0 < \mu(A) < \mu(E_c)$. Then there exists an f-generator G' such that $A = \dot{\cup}_{n\in Z} f^{(n)}(G')$. Let G be an f-generator corresponding to the maximal element M in X, i.e, $M = \dot{\cup}_{n\in Z} f^{(n)}(G)$. Then $A \cup M = \cup_{n\in Z} f^{(n)}(G \cup G')$, since $f^{(n)}(G \cup G') = f^{(n)}(G) \cup f^{(n)}(G')$. Using the given conditions that $f^{(n)}(G) \subseteq M$, $f^{(n)}(G') \subseteq A \subseteq E_c$ and $E_c \cap M = \emptyset$, it is easy to see that for $n \neq m$, $f^{(n)}(G \cup G') \cap f^{(m)}(G \cup G') = \emptyset$. Thus $A \cup M = \dot{\cup}_{n\in Z} f^{(n)}(G \cup G')$. We then have $M \subseteq A \cup M$ and $\mu(M) < \mu(A) + \mu(M) = \mu(A \cup M)$. Thus $M \leq A \cup M$, which contradicts the fact that M is a maximal element in X. $\square$

2.2. The approach by passable pairs

Definition 2.9. Let (E, F) be a compatible pair. A measurable subset $A \subseteq E$ is said to be *passable* if $A \subseteq f(A)$ and the remaining set is of measure zero after Operation 1 (defined in the last section) is applied to A (with A playing the role of E, $\delta(A)$ playing the role of F in the definition of Operation 1). Let Y be the collection of all passable sets. For the sake of convenience, we will assume that the empty set is also an element in Y.

It is possible that Y contains only the empty set. By definition, E is not an element of Y since $F_1 = F \setminus \tau(E) = \emptyset$ so a non-trivial Operation 1 does not apply. For any A_1, $A_2 \in Y$, we say that $A_1 \leq A_2$ if $A_1 \subseteq A_2$ and

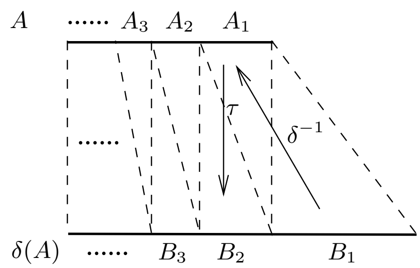

Figure 2: An illustration of a passable set.

$f(A_1) \setminus A_1 \subseteq f(A_2) \setminus A_2$ (modulo a measure zero set). See Figure 2 for an illustration of a set $A \in Y$.

Lemma 2.10. *In the case that Y contains sets of positive measure, then the $\leq$ defined above is a partial order on Y and every chain in Y has an upper bound.*

Proof. It is obvious that $A \leq A$ for any $A \in Y$ by the definition of Y. If $A_1 \leq A_2$ and $A_2 \leq A_1$, then $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$ so we must have $A_1 = A_2$. If $A_1 \leq A_2 \leq A_3$, then $A_1 \subseteq A_2 \subseteq A_3$ and $f(A_1) \setminus A_1 \subseteq f(A_2) \setminus A_2 \subseteq f(A_3) \setminus A_3$ so $A_1 \leq A_3$. Let $\{A_j\}_{j \geq 1}$ be an ascending sequence in Y, then $A^\infty = \cup_{j \geq 1} A_j$ is also measurable and we claim that $f(A_n) \setminus A_n \subseteq f(A^\infty) \setminus A^\infty$. Let $x \in f(A_n) \setminus A_n$, then $x = f(y)$ for some $y \in A_n$ but $x \notin A_n$ (so $x \notin A_k$ for any $k \leq n$ since such $A_k \subseteq A_n$). $x \in f(A^\infty)$ since $y \in A^\infty$. If $x \in A^\infty$, then $x \in A_k$ for some $k > n$. However, $x \in f(A_n) \setminus A_n \subseteq f(A_k) \setminus A_k$ (since $A_n \leq A_k$) implies that $x \notin A_k$, which is a contradiction. Thus $x \notin A^\infty$. This proves $f(A_n) \setminus A_n \subseteq f(A^\infty) \setminus A^\infty$ for any $n \geq 1$. Finally we need to prove that A^∞ is in X. First it is easy to see that $A^\infty \subsetneqq f(A^\infty)$. For any $k \geq 1$, let $\{A_n^k\}$ and $\{B_n^k\}$ be the sets defined in operation 1 (refer to Figure 2 with A^k playing the role of A in the figure). Similarly, let $\{A_n^\infty\}$ and $\{B_n^\infty\}$ be the sets defined in operation 1 for A^∞. We need to show that for any $x \in A^\infty$, $x \in A_n^\infty$ for some n. Since $x \in A^\infty = \cup_{j \geq 1} A_j$, $x \in A_k$ for some $k \geq 1$. Thus $x \in A_n^k$ for some $n \geq 1$. But one can show inductively that $B_j^k \subseteq B_j^\infty$ for any $j \geq 1$ and it follows that $A_n^k \subseteq A_n^\infty$. Hence A^∞ is an upper bound for the chain. □

By Zorn's lemma, Y has a maximal element M. It is necessary that $0 < \mu(M) < \mu(E)$ since E is not an element of Y. Notice that in this case, it is necessary that $\mu(f(M) \setminus M) > 0$ since otherwise M would not be in Y. We then perform Operation 2 (defined in Section 2) on $\overline{M} = E \setminus M$ as shown in Figure 3. The remaining set (denoted by E_c') after the operation is defined as the core of E and the corresponding $F_c' = \delta(E_c')$ is defined as the core of F.

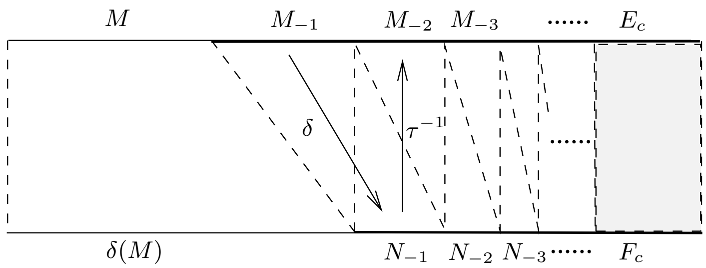

Figure 3: An illustration of the core of E.

Theorem 2.11. $E_c' = E_c$.

Proof. It is easy to show that M_{-1} (as shown in Figure 3) is an f-generator. That is, $L = \dot{\cup}_{n\in z} f^{(n)}(M_{-1})$ is an element in X and $E_c' = E \setminus L$. If L is not maximal in X, then there exists an element N in X such that $L \subseteq N$ with $\mu(L) < \mu(N)$. $N = \dot{\cup}_{n\in z} f^{(n)}(Q)$ for some f-generator Q. Thus $N \setminus L = \cup_{n\in z} f^{(n)}(Q) \setminus \cup_{n\in z} f^{(n)}(L) \subseteq \cup_{n\in z} f^{(n)}(Q \setminus L)$. It follows that $\cup_{n\in z} f^{(n)}(Q \setminus L)$ is of positive measure and so is $\dot{\cup}_{n<0} f^{(n)}(Q \setminus L)$. Notice that $\cup_{n\in z} f^{(n)}(M_{-1} \cup (Q \setminus L))$ is a disjoint union since $f^{(n)}(M_{-1}) \subseteq L$ for any n and $f^{(m)}(Q \setminus L) \cap L = \emptyset$ for any m (because $f(L) = L$ so $f^{(m)}(L) = L$ for any m). Now let $M' = \cup_{n<0} f^{(n)}(M_{-1} \cup (Q \setminus L)) = (\cup_{n<0} f^{(n)}(M_{-1})) \dot{\cup} (\cup_{n<0} f^{(n)}(Q \setminus L)) = M \dot{\cup} (\cup_{n<0} f^{(n)}(Q \setminus L))$, then $M' \subseteq f(M')$. The remaining set after Operation 1 is applied to M' is of measure zero. Thus $M' \in Y$, which contradicts the fact that M is a maximal element in Y since $M \subseteq M'$ and $\mu(M') = \mu(M \dot{\cup} (\cup_{n<0} f^{(n)}(Q \setminus L)) = \mu(M) + \mu(\cup_{n<0} f^{(n)}(Q \setminus L)) > \mu(M)$. $\square$

Lemma 2.8 can be stated now in a different way as the following lemma.

Lemma 2.12. *E_c does not contain any subset D with the property $D \subseteq f(D)$ and $\mu(f(D) \setminus D) > 0$.*

3. Examples

In this section we will address the problem regarding the existence of cores. We do so by giving a few examples

Example 3.1. Let $E = [-8\pi/3, -4\pi/3) \cup [2\pi/3, 4\pi/3)$ and $F = [-4\pi/3, -2\pi/3) \cup [4\pi/3, 8\pi/3)$. For any $t \in [0,1]$, define $E_t = [-4\pi(2-t)/3, -2\pi(2-t)/3) \cup [2\pi(1+t)/3, 4\pi(1+t)/3)$. In this example, we have $f^2(x) = x$ for any $x \in E$. Thus the set X contains only the empty set. Consequently, $E_c = E$. In fact, for any compatible pair (E, F), if there exists some $k \in \mathbb{Z}$ such that $f^k(x) = x$ for all $x \in E$, then $E_c = E$.

Example 3.2. For any pair of NTFW sets E, F such that $\tau(E) \cap F = \emptyset$, $E_c = \emptyset$ by definition (by applying Operation 1). For example $E = [-\frac{\pi}{2}, -\frac{\pi}{4}) \cup [\frac{\pi}{4}, \frac{\pi}{2})$ and $F = [-\frac{\pi}{4}, -\frac{\pi}{8}) \cup [\frac{\pi}{8}, \frac{\pi}{4})$.

Example 3.3. Let $E = [-\pi, -\frac{\pi}{2}) \cup [\frac{\pi}{2}, \pi)$ and $F = [-2\pi, -\pi) \cup [\pi, 2\pi)$. Then $F_1 = [-2\pi, -\frac{3\pi}{2}) \cup [\frac{3\pi}{2}, 2\pi)$ (see Figure 1). Using the fact that for any $x \in [\frac{3\pi}{2}, 2\pi)$, $f^{(-2)}(x) = x/4 + \pi/4 + \pi$, one can show that Operation 1 will remove $[-2\pi, -\frac{4\pi}{3}) \cup [\frac{4\pi}{3}, 2\pi)$ from F and $[-\pi, -\frac{2\pi}{3}) \cup [\frac{2\pi}{3}, \pi)$ from E. The remaining sets are $F_{-1} = [-\frac{4\pi}{3}, -\pi) \cup [\pi, \frac{4\pi}{3})$ (in F) and $E_{-1} = [-\frac{2\pi}{3}, -\frac{\pi}{2}) \cup [\frac{\pi}{2}, \frac{2\pi}{3})$ (in E) respectively. Since E_{-1} and F_{-1} are translation redundancy free (namely $\tau(E_{-1}) \cap F_{-1} = \emptyset$), applying Operation 2 will remove them completely. So in this case we all have $E_c = \emptyset$.

We should point out constructing an example with a nonempty core in a way different from Example 3.1 is a non-trivial problem. In fact, we cannot even prove at this point that such examples exist.

4. The Directed Path of Two NTFW Sets

Definition 4.1. A subset A of E is called an *atom* of the σ-algebra $\mathcal{A}$ if $A \in \mathcal{A}$, $\mu(A) > 0$ and $\mu(B) = 0$ for any $B \subset A$ in $\mathcal{A}$ such that $\mu(B) < \mu(A)$.

We have the following immediate lemma.

Lemma 4.2. *Let $\mathcal{A}$ be as defined in Lemma 2.3 and assume that $\mathcal{A}$ has no atoms. Then for any A_1, $A_2 \in \mathcal{A}$ such that $A_1 \subset A_2$ and $\mu(A_1) < \mu(A_2)$, there exists $A' \in \mathcal{A}$ such that $A_1 \subset A' \subset A_2$ and $\mu(A_1) < \mu(A') < \mu(A_2)$.*

Proof. First, we have $A_2 \setminus A_1 = A_2 \cap \overline{A_1} \in \mathcal{A}$, where $\overline{A_1}$ is the complement of A_1 in E. Furthermore, $\mu(A_2 \setminus A_1) = \mu(A_2) - \mu(A_1) > 0$ by the given condition. Since $\mathcal{A}$ does not have any atoms, $A_2 \setminus A_1$ is not an atom. It follows that there exists $B \in \mathcal{A}$ such that $B \subset A_2 \setminus A_1$ and $0 < \mu(B) < \mu(A_2 \setminus A_1)$. Thus $A' = A_1 \cup B$ satisfies the required conditions. □

Definition 4.3. Let E and F be two measurable sets such that $E = E(\tau, 1) = E(\delta, 1)$, $F = F(\tau, 1) = F(\delta, 1)$, $\delta(E) = F$ and $E \cap F = \emptyset$. Furthermore, let $(E_c, F_c, \mathcal{A})$ be the σ-algebra defined using the (compatible) core pair (E_c, F_c). A continuous set direct-path connecting E and F is defined as a parameterized set Ψ_t such that for each $t \in [0, 1]$, Ψ_t has the following properties: (i) $\Psi_t \subseteq (E \cup F)$; (ii) Ψ_t is measurable; (iii) $\Psi_t = \Psi_t(\tau, 1) = \Psi_t(\delta, 1)$; (iv) $\Psi_0 = E$ and $\Psi_1 = F$; (v) $E \subseteq \cup_{k \in \mathbb{Z}} 2^k \Psi_t$ and (vi) For any $\epsilon > 0$, there exists $\delta > 0$ such that $\mu((\Psi_{t_1} \setminus \Psi_{t_2}) \cup (\Psi_{t_2} \setminus \Psi_{t_1})) < \epsilon$ whenever $|t_1 - t_2| < \delta$.

Lemma 4.4. *Let E and F be two measurable sets such that $E = E(\tau, 1) = E(\delta, 1)$, $F = F(\tau, 1) = F(\delta, 1)$, $\delta(E) = F$ and $E \cap F = \emptyset$. Furthermore, let $(E_c, F_c, \mathcal{A})$ be the σ-algebra defined on the core pair (E_c, F_c) and let Ψ_t be a continuous set direct-path connecting E and F. Then for each $t \in [0, 1]$, $\Psi_t \cap E_c \in \mathcal{A}$.*

Proof. First let us keep in mind that we have $\delta(E_c) = F_c$, $\tau^{-1}(F_c) = E_c$ (so $f(E_c) = E_c$). Let $E_t = \Psi_t \cap E_c$. Then $E_c \setminus E_t$ is not a subset of Ψ_t hence we must have $\delta(E_c \setminus E_t) \subseteq \Psi_t \cap F_c$ by condition (v) of Definition 4.3. On the other hand, $\delta(E_c \setminus E_t) \subseteq F_c$ and $E_t \subseteq E_c$ are both subsets of Ψ_t so they must be translation redundancy free by condition (iii) of Definition 4.3. It follows that $\tau^{-1}(\delta(E_c \setminus E_t)) \cap E_t = \emptyset$, that is $f(E_c \setminus E_t) \cap E_t = \emptyset$. So $f(E_c \setminus E_t) \subseteq E_c \setminus E_t$ since $f(E_c \setminus E_t) \subseteq E_c$. Equivalently, $E_t \subseteq f(E_t)$. If $\mu(f(E_t) \setminus E_t) > 0$, it would contradict Lemma 2.12. Thus we must have $f(E_t) = E_t$ (modulo a measure zero set). □

Let E and F be two NTFW sets. Without loss of generality, we will assume that $E \cap F = \emptyset$, since otherwise we can simply apply the following arguments to the sets $E \setminus F$ and $F \setminus E$. By the definitions of the core of (E,F) and Operations 1 and 2, we have $E = E' \cup E'' \cup M \cup E_c$ and $F = F' \cup F'' \cup \delta(M) \cup F_c$ where M is a maximal element in X with an f-generator Q. So if we could construct set direct-paths Ψ_t^1 connecting E' to F', Ψ_t^2 connecting E'' to F'', Ψ_t^3 connecting M to $\delta(M)$ and Ψ_t^4 connecting E_c to F_c, then $\Psi_t^1 \cup \Psi_t^2 \cup \Psi_t^3 \cup \Psi_t^4$ is obviously a set direct-path connecting E to F. It turns out that Ψ_t^1, Ψ_t^2 and Ψ_t^3 can be similarly constructed. The following is a brief description (readers familiar with this research area know that this is a rather standard procedure).

Recall that $E' = \cup_{j\geq 1}(E_j \cup E'_j)$, $F' = \cup_{j\geq 1}F_j$, $E'' = \cup_{j\geq 1}E_{-j}$, $F'' = \cup_{j\geq 1}F_{-j}$ and there exists an f-generator G such that $M = \cup_{n\in z}f^{(n)}(G)$. Let $g = \tau \circ \delta^{-1}$, $h = \delta \circ \tau^{-1}$. Then $F_j = g^{(j-1)}(F_1)$ and $F' = \cup_{j\geq 1}g^{(j-1)}(F_1)$. Similarly, $E_{-j} = h^{(j-1)}(E_{-1})$ and $E'' = \cup_{j\geq 1}h^{(j-1)}(E_{-1})$. For any $t \in [0,1]$, let $F_t^1 = F_1 \cap (-\infty, \tan(t-\frac{1}{2})\pi)$, $E_t^{-1} = E_{-1} \cap (-\infty, \tan(t-\frac{1}{2})\pi)$, $G_t = G \cap (-\infty, \tan(t-\frac{1}{2})\pi)$, then F_t^1, E_t^{-1} and G_t are all continuous. Define $\Psi_t^1 = E' \backslash \delta^{-1}(\cup_{j\geq 1}g^{(j-1)}(F_t^1)) \cup (\cup_{j\geq 1}g^{(j-1)}(F_t^1))$, $\Psi_t^2 = E'' \backslash \cup_{j\geq 1}h^{(j-1)}(E_t^{-1}) \cup \delta(\cup_{j\geq 1}h^{(j-1)}(E_t^{-1}))$ and $\Psi_t^3 = M \setminus \cup_{n\in z}f^{(n)}(G_t) \cup \delta(\cup_{n\in z}f^{(n)}(G_t))$. We leave the details for our readers to verify that these are indeed continuous set direct-paths with the desired properties. Thus E is path connected to F via a direct path if and only if a set direct-path Ψ_t^4 connecting E_c to F_c exists.

Theorem 4.5. *Let E and F be two NTFW sets and let $(E_c, F_c, \mathcal{A})$ be the σ-algebra defined using the (compatible) core pair (E_c, F_c). Then E and F are path connected via a directed path if and only if $(E_c, F_c, \mathcal{A})$ contains no atoms.*

Proof. First, let us assume that there exists an atom $A \in \mathcal{A}$. We need to prove that in this case there exists no direct path from ψ_E to ψ_F. Assume the contrary, then for each $t \in [0,1]$, there exists a measurable set $E_t \subset E \cup F$ such that: (1) $E_t = E_t(\tau, 1) = E_t(\delta, 1)$ for each t; (2) $E_0 = E$ and $E_1 = F$; (3) E_t is continuous in t. Let $a = \mu(A) > 0$ and subdivide $[0,1]$ into small intervals $[t_j, t_{j+1}]$ $0 \leq j \leq n-1$ for some large integer n with $t_0 = 0$, $t_n = 1$ such that $|\mu(E_{t_{j+1}} \Delta E_{t_j})| < a$, this can be done by the definition of E_t. We claim that if $A \subset E_{t_j}$ (modulo a measure zero set), then we must also have $A \subset E_{t_{j+1}}$. If $A \subset E_{t_j}$, then A is contained in the set $E_{t_j} \cap E_c$, which is an element of $\mathcal{A}$ by Lemma 4.4. Since $E_{t_{j+1}} \cap E_c$ is also an element of $\mathcal{A}$, we have $E_{t_j} \cap E_{t_{j+1}} \cap E_c \in \mathcal{A}$ and $B = A \cap E_{t_j} \cap E_{t_{j+1}} \cap E_c \in \mathcal{A} = A \cap E_{t_{j+1}}$. Since $B \subseteq A$, we must have $\mu(B) = \mu(A)$ or $\mu(B) = 0$. If $\mu(B) = 0$, then A must be contained in $E_{t_j} \setminus E_{t_{j+1}}$ (modulo a measure zero set). But that is a contradiction since $\mu(E_{t_j} \setminus E_{t_{j+1}}) < \mu(A) = a$ by our choice of the t_j's. Thus we have $\mu(B) = \mu(A)$. Hence $A \subset E_{t_{j+1}}$ (modulo a measure zero set). Since $A \subset E_{t_0}$, it follows that $A \subset E_{t_j}$ for any j by induction. In particular, $A \subset E_{t_n} = E_1 = F$ (so $A \subseteq F_c$), contradicting the fact that $A \subset E_c$ (since $E_c \cap F_c = \emptyset$). This finishes this part of the proof.

We now assume that $\mathcal{A}$ contains no atoms. We need to show that there exists a continuous E_t which is a set direct-path connecting E_c to F_c. Assume that $\mu(E_c) = p > 0$ (otherwise there is nothing to prove). We will first show that there exists $A \in \mathcal{A}$ with the property that $\mu(A) = p/2$. If $\{A_j\}$ is an ascending sequence in $\mathcal{A}$ with the property that $\lim_{j\to\infty} \mu(A_j) = p/2$, then apparently $A = \cup_{j=1}^{\infty} A_j \in \mathcal{A}$ and $\mu(A) = p/2$. Similarly, if $\{B_j\}$ is a descending sequence in $\mathcal{A}$ with the property that $\lim_{j\to\infty} \mu(B_j) = p/2$, then $A = \cap_{j=1}^{\infty} B_j \in \mathcal{A}$ and $\mu(A) = p/2$.

Consider the collection $\mathcal{C}$ of all elements A of $\mathcal{A}$ with the property $\mu(A) \leq p/2$. $\mathcal{C}$ is partially ordered under the set inclusion operation. Furthermore, every totally ordered subset of $\mathcal{C}$ has an upper bound. By Zorn's Lemma, there exists a maximal element S of $\mathcal{C}$. We claim that $\mu(S) = p/2$. Assume that this is not the case, i.e., $0 < \mu(S) < p/2$. Let $b = p/2 - \mu(S)$ and consider $\overline{S} \in \mathcal{A}$. Since $\overline{S}$ is not an atom, there exists $S_1 \in \mathcal{A}$, $S_1 \subset \overline{S}$ such that $0 < \mu(S_1) < \mu(\overline{S})$. If $\mu(S_1) \leq b$ or $\mu(\overline{S} \setminus S_1) \leq b$, then we would have $S \cup S_1 \in \mathcal{C}$ or $S \cup (\overline{S} \setminus S_1) \in \mathcal{C}$ with a larger measure. This contradicts the fact S is a maximal element in $\mathcal{C}$. However, if this is not the case, then $\mu(S_1) < \mu(\overline{S}) - b$ and the same argument can be applied to S_1 (instead of $\overline{S}$) and we will reach a subset S_2 of S_1 that is also an element of $\mathcal{A}$ with the property that $0 < \mu(S_2) < \mu(S_1) - b < \mu(\overline{S}) - 2b$. This process can then be repeated. However, this process has to terminate at some point since $b > 0$ is a constant. In other word, at some point, we will get an element S_j of $\mathcal{A}$ that is contained in $\overline{S}$ with the property $0 < \mu(S_j) < b$. This will again lead us to a contradiction.

The above argument can be immediately extended so that for any number of the form $j/2^k$, where j, k are integers such that $0 < j < 2^k$, $k \geq 1$ and j is odd, there exists $A_{\frac{j}{2^k}} \in \mathcal{A}$ such that $\mu(A_{\frac{j}{2^k}}) = \frac{jp}{2^k}$. Furthermore, the sets $\{A_{\frac{j}{2^k}}\}$ are totally ordered in terms of set inclusion. That is, $A_{\frac{j}{2^k}} \subset A_{\frac{j'}{2^{k'}}}$ whenever $\frac{j}{2^k} < \frac{j'}{2^{k'}}$. Since the set $\{\frac{j}{2^k}\}$ is dense in $[0, 1]$, it follows we can extend this to a totally ordered set $\{A_t\}$ for each $t \in [0, 1]$ such that $A_t \in \mathcal{A}$ for each t, $\mu(A_t) = tp$ and $A_t \subset A_{t'}$ whenever $t < t'$. We leave the details of this part of the proof for our reader to verify.

We can now construct our direct path. For each $t \in [0,1]$, A_t is a subset of E_c with the property that $f(A_t) = A_t$. This means that A_t generates the same subset of $\mathbb{R}$ as that of $\tau(A_t)$ (under the 2-dilation), which is a subset of F_c. It follows that $E_t = (E \setminus A_t) \cup \tau(A_t)$ has the property $E_t = E_t(\tau, 1) = E_t(\delta, 1)$. Apparently, we have $E_0 = E_c$ and $E_1 = F_c$. Furthermore, for any $t_2 > t_1$, we have $E_{t_1} \Delta E_{t_2} = (A_{t_2} \setminus A_{t_1}) \cup \tau(A_{t_2} \setminus A_{t_1})$. It follows that $\mu(E_{t_1} \Delta E_{t_2}) = 2(t_2 - t_1)p$ and the continuity of E_t follows. □

5. An Ending Remark on the Case of Wavelet Sets

Wavelet sets are special NTFW sets with the extra condition that their Lebesgue measures have to be exactly 2π. Let E and F be two wavelet sets and let us consider the similar direct path problem regarding E and F. If Ψ_t is a set direct-path connecting E to F, then for each $t \in [0,1]$, it is necessary that $E \cap \Psi_t$ has the property $f(E \cap \Psi_t) = E \cap \Psi_t$. In other word, if we define a new σ-algebra $(E, F, \mathcal{A})$ such that $A \subseteq E$ is an element of $\mathcal{A}$ if and only if $f(A) = A$, then it is necessary that $E \cap \Psi_t \in \mathcal{A}$ for each t. The proof of Theorem 4.5 can be easily modified to prove the following corollary.

Corollary 5.1. *Two wavelet sets E and F are path connected via a directed path if and only if the σ-algebra $(E, F, \mathcal{A})$ contains no atoms.*

Notice that the σ-algebra $(E, F, \mathcal{A})$ in the case of wavelet sets is imposed by the need to preserve the 2π measure of the sets involved. The NTFW sets do not have such a measure preservation requirement. It is interesting to see that at the end, a similar σ-algebra structure has to be considered nonetheless.

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