

# Intrinsic $s$ -elementary Parseval Frame Multiwavelets in $L^2(\mathbb{R}^d)$

Zhongyan Li\*, Xingde Dai<sup>†</sup>, and Yuanan Diao<sup>†</sup>

ABSTRACT. A countable family  $\{x_j\}, j \in J$ , in a separable Hilbert space  $H$ , is a Parseval frame for  $H$  if  $\|f\|^2 = \sum_{j \in J} |\langle f, x_j \rangle|^2$  holds for all  $f \in H$ . In the case that  $H = L^2(\mathbb{R}^d)$  and the affine system  $\{D_A^j T_k \psi_i | j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq i \leq n\}$  obtained from a finite subset  $\Psi = \{\psi_1, \psi_2, \dots, \psi_n\}$  of  $L^2(\mathbb{R}^d)$  is a Parseval frame,  $\Psi$  is called an  $A$ -dilation Parseval frame multiwavelet (of length  $n$ ). Here  $A$  stands for a  $d \times d$  expansive matrix, and  $T, D_A$  are the translation and  $A$ -dilation unitary operators acting on  $L^2(\mathbb{R}^d)$  defined by  $(T^\ell f)(\mathbf{t}) = f(\mathbf{t} - \ell)$ ,  $(D_A f)(\mathbf{t}) = |\det A|^{\frac{1}{2}} f(A\mathbf{t})$ ,  $\forall f \in L^2(\mathbb{R}^d), \ell \in \mathbb{Z}^d, \mathbf{t} \in \mathbb{R}^d$ . In the special case that there exist disjoint measurable sets  $\{E_1, E_2, \dots, E_n\}$  such that  $\hat{\psi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{E_i}$  for each  $i$ ,  $\Psi$  is called an  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet. A measurable set  $E$  is called a frame multiwavelet set of multiplicity  $m$  (under  $A$ -dilation) if  $E$  can be written as a disjoint union of measurable sets  $\{E_1, E_2, \dots, E_m\}$  such that  $\hat{\psi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{E_i}$  defines a Parseval frame multiwavelet  $\Psi = \{\psi_1, \psi_2, \dots, \psi_n\}$ , and that this cannot be done for any integer less than  $m$ . An  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet with length  $m$  that is defined on a frame multiwavelet set of multiplicity  $m$  is said to be intrinsic. It is known that single  $A$ -dilation wavelets exist in  $L^2(\mathbb{R}^d)$  for any expansive matrix  $A$ . In this paper, we show that for any  $d \times d$  expansive matrix  $A$  and any given  $m \in \mathbb{N}$ , the family of intrinsic  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet with length  $m$  is not empty, and is path-connected under the norm topology of  $(L^2(\mathbb{R}^d))^m$ . The same result holds for the family of all intrinsic  $A$ -dilation  $s$ -elementary Parseval frame multiwavelets of length  $m$ .

## 1. Introduction

A sequence  $\{x_n\}$  in a Hilbert space  $H$  is called a *frame* for  $H$  if there exist constants  $C_1, C_2 > 0$  such that

$$C_1 \|x\|^2 \leq \sum_{n \in \mathbb{N}} |\langle x, x_n \rangle|^2 \leq C_2 \|x\|^2, \forall x \in H.$$

In the case that  $C_1 = C_2 = C$ ,  $\{x_n\}$  is called a *tight frame* and the constant  $C$  is called the frame bound for  $\{x_n\}$ . In particular, if  $C_1 = C_2 = 1$ , then  $\{x_n\}$  is called a *normalized tight frame*, or a *Parseval frame*.

---

1991 *Mathematics Subject Classification.* 42C15, 42C40.

*Key words and phrases.* multiwavelets, Parseval frames,  $s$ -elementary Parseval frame multiwavelets, path-connectivity.

In this paper, we consider a special kind of Parseval frames for the space  $L^2(\mathbb{R}^d)$ . A  $d \times d$  matrix  $A$  is called an *expansive matrix* if all eigenvalues of  $A$  have modulus greater than one. In this paper, we will restrict ourselves to such expansive matrices. Let  $T, D_A$  be the translation and dilation unitary operators acting on  $L^2(\mathbb{R}^d)$  defined by  $(T^\ell f)(\mathbf{t}) = f(\mathbf{t} - \ell)$ ,  $(D_A f)(\mathbf{t}) = |\det A|^{\frac{1}{2}} f(A\mathbf{t})$ ,  $\forall f \in L^2(\mathbb{R}^d)$ ,  $\ell \in \mathbb{Z}^d$ ,  $\mathbf{t} \in \mathbb{R}^d$ . Throughout this article, we will use  $\widehat{f}$  or  $\mathcal{F}f$  to denote the Fourier transform of a function  $f \in L^2(\mathbb{R}^d)$ . It is defined as

$$(\mathcal{F}f)(\mathbf{s}) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i(\mathbf{s} \circ \mathbf{t})} f(\mathbf{t}) d\mathbf{m},$$

for all  $f \in L^2(\mathbb{R}^d)$ , where  $\mathbf{s} \circ \mathbf{t}$  denotes the real inner product. For any bounded linear operator  $P$  on  $L^2(\mathbb{R}^d)$ , we define  $\widehat{S} := \mathcal{F}S\mathcal{F}^{-1}$ . Furthermore, for any subset  $X$  of  $L^2(\mathbb{R}^d)$ , we will use  $\widehat{X}$  to denote the collection of Fourier transforms of all elements in  $X$ .

Let  $\Psi = \{\psi_1, \psi_2, \dots, \psi_n\}$  be a finite subset of  $L^2(\mathbb{R}^d)$ .  $\Psi$  is called an *A-dilation Parseval frame multiwavelet (of length n)* if  $\{D_A^j T_k \psi_i | j \in \mathbb{Z}, k \in \mathbb{Z}^d, \psi_i \in \Psi\}$  is a Parseval frame for  $L^2(\mathbb{R}^d)$ . In the case that  $n = 1$ ,  $\Psi$  is called a *single Parseval frame* and the corresponding  $\psi_1$  is called a *Parseval frame wavelet*.

A Parseval frame wavelet and multiwavelet system is a redundant wavelet system and has many desirable properties that are of interest in applications [1, 10, 11, 12, 13]. Much effort has been devoted the study of various aspects of multiwavelet systems. For a few examples of recent studies related to multiwavelets, see [16, 18, 19].

In this paper, we are mainly concerned with another (rather theoretical) aspect of a multiwavelet system, namely the topological property of such a system. In particular, we are interested in the path-connectedness of the set of all (multi)wavelets in our study. The question concerning the path-connectedness of the set of all orthonormal wavelets was first raised in [7]. Similar questions were raised and studied in [2, 6, 22, 23, 24, 25] concerning the sets of all MRA-wavelets, tight frame wavelets, MRA tight frame wavelets and a special class of frame wavelets called *s*-elementary frame wavelets. In [22, 25], it is shown that the set of all MRA-wavelets is path-connected. In [24], it is shown that the set of all *s*-elementary orthonormal wavelets is path-connected. This result is extended to the set all *s*-elementary tight frame wavelets (with any given frame bound) in [2]. The proofs of these theorems were based on the complete characterizations of the corresponding wavelets. Interestingly, while the complete characterization of the *s*-elementary frame wavelets is still an open question, it has been shown that the set of *s*-elementary frame wavelets is path-connected as well [6]. These results all deal with the one dimensional case. In the higher dimensional case, the situation is complicated by the fact that various dilation matrices have to be considered. Nonetheless, the path-connectedness of the set of all MRA *A*-dilation wavelets has been established affirmatively for the special case that  $A$  has integer entries with  $|\det(A)| = 2$  [20, 21].

Let  $\Psi = \{\psi_1, \psi_2, \dots, \psi_m\}$  be an *A*-dilation Parseval frame multiwavelet. We say that  $\Psi$  is *s*-elementary if there exists disjoint measurable sets  $E_1, E_2, \dots, E_m$  such that  $\widehat{\psi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{E_i}$  for each  $i$ . The corresponding set  $E = \cup_{i=1}^m E_i$  is called an (*A*-dilation) *Parseval frame multiwavelet set*. Note that it is possible that  $E$  may

admit a different partition  $\{E'_1, E'_2, \dots, E'_n\}$  with  $n < m$  such that  $\{E'_1, E'_2, \dots, E'_n\}$  also defines an  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet. In particular, one can easily (and rather artificially) make an  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet of any length  $m$  from any wavelet set  $E$  by partitioning it into  $m$  disjoint subsets  $E_1, E_2, \dots, E_m$  of positive measures and define  $\Psi = \{\psi_1, \psi_2, \dots, \psi_m\}$  by  $\widehat{\psi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{E_i}$ . Thus the following definition makes sense.

**DEFINITION 1.1.** A measurable set  $E$  is called a *frame multiwavelet set of multiplicity  $m$*  (under  $A$ -dilation) if  $E$  can be written as a disjoint union of measurable sets  $\{E_1, E_2, \dots, E_m\}$  such that  $\widehat{\psi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{E_i}$  defines an  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet  $\Psi = \{\psi_1, \psi_2, \dots, \psi_m\}$  of length  $m$ , and that this cannot be done for any integer less than  $m$ . An  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet with length  $m$  that is defined on a frame multiwavelet set of multiplicity  $m$  is said to be *intrinsic*.

We aim to prove the following two main results in this paper. First, we show that for any expansive  $d \times d$  matrix  $A$  and any given integer  $m \geq 1$ , there exists a frame multiwavelet set of multiplicity  $m$ . In other words, there exists intrinsic  $s$ -elementary Parseval frame multiwavelets of any given length  $m$ . Second, we will prove that the set of all  $A$ -dilation  $s$ -elementary Parseval frame multiwavelets with the same length  $m$  is path-connected (the same is true for the set of all intrinsic  $A$ -dilation  $s$ -elementary Parseval frame multiwavelets of length  $m$  for any given  $m$ ).

We will introduce some basic definitions, terminologies in the next section, as well as some known and preliminary results needed later in the paper. In Section 3, we will prove the existence of intrinsic  $A$ -dilation  $s$ -elementary Parseval frame multiwavelets with any given length. In Section 4 we will prove that the set of all intrinsic  $A$ -dilation  $s$ -elementary Parseval frame multiwavelets with the same length is path-connected. In the last section, we generalize the concept of intrinsic  $A$ -dilation  $s$ -elementary Parseval frame multiwavelets to general  $A$ -dilation Parseval frame multiwavelets. However, the path-connectivity of the set of all intrinsic  $A$ -dilation Parseval frame multiwavelets with the same length remains open at this time.

## 2. Frame multiwavelet sets in $\mathbb{R}^d$

For any  $Y \subseteq \mathbb{R}^d$  and any  $\mathbf{t} \in \mathbb{R}^d$ , let  $Y + \mathbf{t}$  denote the set  $\{\mathbf{y} + \mathbf{t} : \mathbf{y} \in Y\}$ . In the case that  $\ell \in \mathbb{Z}^d$ , we will also let  $I_\ell$  denote the  $d$ -cube  $[-1/2, 1/2]^d + \ell$ . For a Lebesgue measurable set  $E$  in  $\mathbb{R}^d$  with finite measure, define

$$\tau(E) = \bigcup_{\ell \in \mathbb{Z}^d} (E \cap 2\pi I_\ell - 2\pi\ell).$$

Notice that  $\tau(E) \subseteq 2\pi I_0 = D_0$ , where  $I_0$  is the unit  $d$ -cube  $[-1/2, 1/2]^d$ . If the above union is a disjoint union, we say that  $E$  is *translation equivalent* to  $\tau(E)$ . If  $E$  and  $F$  are translation equivalent to the same subset in  $2\pi I_0$ , then we say that  $E$  and  $F$  are *translation equivalent*. This defines an equivalent relation among measurable sets of  $\mathbb{R}^d$  that are equivalent to some subsets in  $D_0$ . Let us denote this equivalent relation by  $\sim$ . Let  $\mu(\cdot)$  be the Lebesgue measure. It is clear that  $\mu(E) \geq \mu(\tau(E))$ . The equality holds iff  $E \sim \tau(E)$  modulo a measure zero set. If

$E \overset{\tau}{\sim} F$ , then  $\mu(E) = \mu(F)$ . Two points  $x, y \in E$  are said to be translation equivalent if  $x - y = 2\pi\ell$  for some  $\ell \in \mathbb{Z}^d$ . The translation redundancy index of a point  $x$  in  $E$  is the number of elements in its equivalent class. We write  $E(\tau, k)$  for the set of all points in  $E$  with translation redundancy index  $k$ . In general,  $E(\tau, k)$  could be an empty set, a proper subset of  $E$  or the set  $E$  itself. For  $k \neq m$ ,  $E(\tau, k) \cap E(\tau, m) = \emptyset$ , so  $E = E(\tau, \infty) \cup (\cup_{n \in \mathbb{N}} E(\tau, n))$ .

It can be shown that if  $E$  is a Lebesgue measurable set in  $\mathbb{R}^d$ , then  $E(\tau, k)$  is measurable for each  $k$ . It follows that  $E(\tau, \infty)$  is also measurable. (For a proof of this in the case of  $d = 1$ , see [3].) Furthermore,  $E(\tau, k)$  can be partitioned into a disjoint union of  $k$  measurable sets  $E^j(\tau, k)$ ,  $j = 1, 2, \dots, k$  such that  $E^{(j)}(\tau, k) \overset{\tau}{\sim} E(\tau, k)$ . This partition is not unique. If  $E$  is of finite measure, then it is rather obvious that  $E(\tau, \infty)$  must have measure zero.

Similarly, two non-zero points  $x, y \in E$  are said to be  $A$ -dilation equivalent if  $y = A^k x$  for some  $k \in \mathbb{Z}$ . This is also an equivalence relation on  $E$ . The  $A$ -dilation redundancy index of a point  $x$  in  $E$  is the cardinality in its equivalence class. The set of all points in  $E$  with  $A$ -dilation redundancy index  $k$  is denoted by  $E(\delta_A, k)$ . For  $k \neq m$ ,  $E(\delta_A, k) \cap E(\delta_A, m) = \emptyset$ . So  $E = E(\delta_A, \infty) \cup (\cup_{n \geq 1} E(\delta_A, n))$ . In the case that  $E = E(\delta_A, 1)$  and  $\cup_{k \in \mathbb{Z}} A^k E = \mathbb{R}^d$  (modulo a measure zero set),  $E$  is said to be an  $A$ -dilation generator of  $\mathbb{R}^d$ .

The following characterization of an  $s$ -elementary Parseval frame multiwavelet is given in [14].

LEMMA 2.1. [14] *Let  $E_1, E_2, \dots, E_p$  be measurable sets in  $\mathbb{R}^d$  and  $\Psi = \{\psi_1, \psi_2, \dots, \psi_p\}$  is defined by  $\widehat{\psi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{E_i}$  ( $i = 1, \dots, p$ ), then  $\Psi$  is an  $s$ -elementary Parseval frame multiwavelet iff the following conditions hold (modulo measure zero sets):*

- (1)  $E_1, E_2, \dots, E_p$  are mutually disjoint;
- (2)  $E_i = E_i(\tau, 1)$ ;
- (3)  $E = \bigcup_{i=1}^p E_i$  is an  $A^t$ -dilation generator of  $\mathbb{R}^d$  (where  $A^t$  is the transpose of  $A$ ).

PROPOSITION 2.1. *Let  $E$  be a measurable set of finite measure. Then  $E$  is a frame multiwavelet set of multiplicity  $m$  if and only if the following conditions hold:*

- (1)  $E$  is an  $A^t$ -dilation generator of  $\mathbb{R}^d$  (modulo a measure zero set);
- (2)  $\mu(E(\tau, m)) > 0$  and  $\mu(E(\tau, n)) = 0$  for any  $n > m$ .

PROOF. “ $\implies$ ” Assume that  $E$  is a frame multiwavelet set of multiplicity  $m$ , then (1) holds by definition. Let  $E_1, E_2, \dots, E_m$  be the sets that define the corresponding  $s$ -elementary Parseval frame multiwavelet  $\Psi = \{\psi_1, \psi_2, \dots, \psi_m\}$ . By Lemma 2.1,  $E_i = E_i(\tau, 1)$  for each  $i$ . Thus each point in a set  $E_i$  can only be translation equivalent to at most one point in each  $E_j$  where  $j \neq i$ . It follows that  $E(\tau, n)$  is of measure zero for any  $n > m$ . If  $\mu(E(\tau, m)) = 0$  then we must have  $\mu(E(\tau, k)) \neq 0$  for some  $k < m$ . In that case we can partition  $E$  into  $E_1, E_2, \dots, E_k$  such that  $E_i = E(\tau, 1)$ . By Lemma 2.1,  $\Psi = \{\psi_1, \psi_2, \dots, \psi_k\}$  defined by  $\widehat{\psi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{E_i}$  ( $1 \leq i \leq k$ ) is an  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet of length  $k$ , contradicting the definition of  $E$ .

“ $\Leftarrow$ ” We will first partition  $E$  into  $E_1, E_2, \dots, E_m$  such that  $E_i = E_i(\tau, 1)$ . This can be easily done by partition each  $E(\tau, j)$  into  $E^{(i)}(\tau, j)$  ( $1 \leq i \leq j$ ) and then let  $E_k = \cup E^{(k)}(\tau, j)$  (let  $E^{(k)}(\tau, j) = \emptyset$  if  $k > j$ ). It is apparent that  $\mu(E_k) > 0$  for each  $1 \leq k \leq m$ . By Lemma 2.1,  $\Psi = \{\psi_1, \psi_2, \dots, \psi_m\}$  defined by  $\widehat{\psi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{E_i}$  ( $1 \leq i \leq m$ ) is an  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet of length  $m$ . Now assume that this can be done for some integer  $1 \leq m_0 < m$ , that is,  $E$  can be partitioned into  $E'_1, E'_2, \dots, E'_{m_0}$  such that  $\Psi' = \{\psi'_1, \psi'_2, \dots, \psi'_{m_0}\}$  defined by  $\widehat{\psi}'_i = \frac{1}{(2\pi)^{d/2}} \chi_{E'_i}$  ( $1 \leq i \leq m_0$ ) is an  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet of length  $m_0$ . Again by Lemma 2.1, we must have  $E'_i = E'(\tau, 1)$  for  $1 \leq i \leq m_0$ . Which leads to  $\mu(E(\tau, m)) = 0$ , again a contradiction.  $\square$

EXAMPLE 2.2. Let  $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ ,  $A$  is expansive with  $|\det(A)| = 5$ . Let  $F = [-\pi, \pi]^2$  and  $E = A^t F \setminus F$ . We have  $E = E(\delta, 1)$ ,  $\bigcup_{n \in \mathbb{Z}} (A^\tau)^n E = \mathbb{R}^2$  and  $E = E(\tau, 4)$ . By Proposition 2.1,  $E$  is an intrinsic Parseval frame multiwavelet set of multiplicity 4. (In fact, it is an intrinsic orthonormal multiwavelet set of multiplicity 4.) One particular decomposition of  $E$  into  $E^{(i)}(\tau, 4)$  ( $1 \leq i \leq 4$ ) is shown in Figure 1.

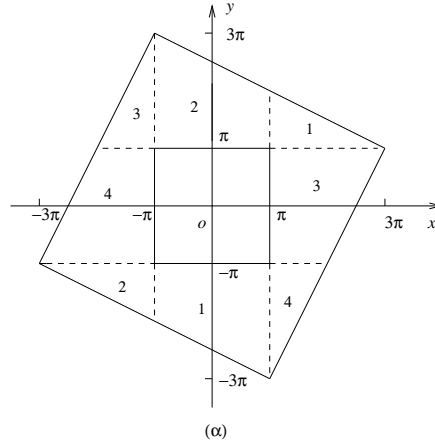


FIGURE 1. An intrinsic  $A$ -dilation frame multiwavelet set of multiplicity 4.

EXAMPLE 2.3. Let  $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ ,  $F = [-\frac{\pi}{2}, \frac{\pi}{2}]^2$  and  $E = A^t F \setminus F$ .  $E = E(\tau, 1)$  since  $E \subseteq [-\pi, \pi]^2$ , and  $E = E(\delta, 1)$ ,  $\bigcup_{n \in \mathbb{Z}} (A^\tau)^n E = \mathbb{R}^2$ . Partition  $E$  into 4 disjoint subsets  $E_1, E_2, E_3$  and  $E_4$  as shown in Figure 2. Then  $\Psi = \{\psi_1, \psi_2, \psi_3, \psi_4\}$  defined by  $\widehat{\psi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{E_i}$  is an  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet of length 4. However  $\Psi$  is not intrinsic since  $E$  is a Parseval frame wavelet set itself.

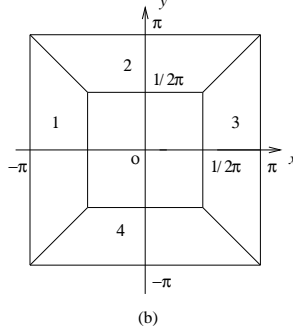


FIGURE 2. An  $A$ -dilation frame multiwavelet set that is not intrinsic.

### 3. The existence of intrinsic $A$ -dilation $s$ -elementary Parseval frame multiwavelets with any given length

We will need the following two lemmas in order to prove the existence of intrinsic  $A$ -dilation  $s$ -elementary Parseval frame multiwavelets with any given length.

LEMMA 3.1. [4] *Let  $A$  be a real expansive matrix. Then  $\lim_{k \rightarrow \infty} \|A^{-k}\| = 0$  and  $\lim_{k \rightarrow \infty} \|A^k\| = \infty$ .*

LEMMA 3.2. [15] *Let  $A$  be a real  $d \times d$  expansive matrix. Then there exists an open and bounded neighborhood  $F$  of the origin such that  $F \subseteq A^t F$ .*

REMARK 3.3. In fact  $F$  can be chosen to be  $(A^t)^{-1}B$  where  $B$  is the open unit ball of  $\mathbb{R}^d$  centered at the origin. The proof of  $F \subseteq A^t F$  in this case is just a simple exercise of linear algebra.

THEOREM 3.4. *For any real expansive matrix  $A$  and any  $m \in \mathbb{N}$ , there exists at least one intrinsic  $A$ -dilation Parseval frame multiwavelet set with length  $m$ .*

PROOF. By Remark 3.3, there exists an open neighborhood  $F$  of the origin such that  $F \subseteq A^t F$ , and that  $F$  and  $A^t F$  are both bounded in  $D_0$ . It follows that the set  $E = A^t F \setminus F$  is an  $A^t$ -dilation generator of  $\mathbb{R}^d$  (modulo a measure zero set). Furthermore, we have  $E = E(\tau, 1)$  since  $F$  and  $A^t F$  are both bounded in  $D_0$ . Let  $\ell_1 \in \mathbb{Z}^d$  be the vector whose first entry is 1 and all other entries are zero and let  $D$  be the union of the sets  $D_1 = D_0 + 2\pi\ell_1$ ,  $D_2 = D_0 + 4\pi\ell_1$ , ...,  $D_{m-1} = D_0 + 2(m-1)\pi\ell_1$ . Since  $F$  is an open set,  $E$  contains interior points. Let  $x_0$  be an interior point of  $E$  and consider a ball  $B_\epsilon$  of radius  $\epsilon$  in  $\mathbb{R}^d$  centered at  $x_0$ . Since  $x_0$  is an interior point,  $B_\epsilon \subseteq E$  when  $\epsilon$  is small enough. Let us fix such an  $\epsilon$ . By Lemma 3.1, there exists an  $n$  that is large enough such that  $(A^t)^n B_\epsilon$  contains a  $2\pi$ -translation copy of  $D$ , hence  $m-1$   $2\pi$ -translation copies of  $E$  (that are translation equivalent to each other). Denote these (equivalent)  $2\pi$ -translation copies of  $E$  by  $E_2, E_3, \dots, E_m$  and let  $E' = (A^t)^{-n}(\cup_{2 \leq i \leq m} E_i) \subseteq B_\epsilon \subseteq E$ . Apparently we have  $\mu(E^n) < \mu(E)$ . Now define  $E_1 = E \setminus E'$ . Then  $E_1, E_2, \dots, E_m$  are disjoint sets of positive measure.  $G = \cup_{1 \leq j \leq m} E_j$  is an  $A^t$ -dilation generator of  $\mathbb{R}^d$ ,  $G(\tau, m)$  is of positive measure and  $G(\tau, p) = \emptyset$  for any  $p > m$ . By Proposition 2.1,  $G$  is a frame multiwavelet set of multiplicity  $m$ . It follows that  $\Psi = \{\psi_1, \psi_2, \dots, \psi_m\}$  defined by  $\widehat{\psi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{E_i}$

$(1 \leq i \leq m)$  is an intrinsic  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet of length  $m$ .  $\square$

REMARK 3.5. In fact, we can construct intrinsic  $A$ -dilation Parseval frame multiwavelet sets  $E$  of any given length  $m$  with one more special condition:  $E$  is partitioned into  $E_1, E_2, \dots, E_m$  such that  $\mu(E_j) = (2\pi)^d$  for each  $j$ . The following is an outline of this construction. Start from any  $A$ -dilation wavelet set  $W$  (i.e.,  $\psi$  defined by  $\widehat{\psi} = \frac{1}{(2\pi)^{d/2}} \chi_W$  is an orthogonal  $A$ -dilation wavelet). For any given positive integer  $m$ , let  $\{W_1, W_2, \dots, W_m\}$  be a partition of  $W$  such that  $\mu(W_j) > 0$  for each  $j$ . For each  $j \leq m$  we denote  $\Omega_j = \cup_{n \in \mathbb{Z}} (A^t)^n W_j$ . Then  $L^2(\Omega_j)$ ,  $1 \leq j \leq m$  are the Fourier Transforms of  $m$  reducing subspaces under  $A$ -dilations and translations. The direct sum of the  $L^2(\Omega_j)$ 's is  $L^2(\mathbb{R}^d)$ . Apply the same techniques from [9], one can show that an  $A$ -dilation wavelet set  $E_j$  exists in each  $\Omega_j$  (it is necessary that  $\mu(E_j) = (2\pi)^d$  for a wavelet set). Apparently the sets  $E_j$ 's are disjoint so  $E = \cup_j E_j$  satisfies all the desired conditions.

#### 4. Path-connectivity of intrinsic $A$ -dilation $s$ -elementary Parseval frame multiwavelets in $L^2(\mathbb{R}^d)$

A set  $S \subseteq L^2(\mathbb{R}^d)$  is said to be path-connected under norm topology of  $L^2(\mathbb{R}^d)$  if for any two members  $f, g \in S$ , there exists a mapping  $\gamma : [0, 1] \rightarrow S$  such that  $\gamma(t)$  is continuous (with respect to the norm topology of  $L^2(\mathbb{R}^d)$ ) and  $\gamma(0) = f$ ,  $\gamma(1) = g$ . The same definition applies when  $S$  is a subset of  $L^2(\mathbb{R}^d)$ . That is, a subset  $S \subseteq (L^2(\mathbb{R}^d))^k$  is said to be path-connected under the norm topology of  $(L^2(\mathbb{R}^d))^k$  if for any two members  $(\varphi_1, \dots, \varphi_k)$  and  $(\psi_1, \dots, \psi_k)$  of  $S$ , there exists a mapping  $\gamma : [0, 1] \rightarrow S$  such that  $\gamma(t)$  is continuous (with respect to the norm topology of  $L^2(\mathbb{R}^d)$ ) and  $\gamma(0) = (\varphi_1, \varphi_2, \dots, \varphi_k)$ ,  $\gamma(1) = (\psi_1, \psi_2, \dots, \psi_k)$ .

Let us first consider the set of all  $A$ -dilation  $s$ -elementary Parseval frame multiwavelets of length  $m$ , not just the set of all intrinsic  $A$ -dilation  $s$ -elementary Parseval frame multiwavelets of length  $m$ . In this case it is a littler easier to show that this set is path-connected. We first state the result as the following theorem.

THEOREM 4.1. *Let  $A$  be an expansive  $d \times d$  matrix and  $m$  a fixed positive integer, then the set  $S_m$  of all  $A$ -dilation  $s$ -elementary Parseval frame multiwavelets of length  $m$  is path-connected.*

The proof of Theorem 4.1 relies on some results obtained in [4]. Let  $X$  be a closed subspace of  $L^2(\mathbb{R}^d)$ .  $X$  is called a *reducing subspace* for  $\{D_A, T_\ell : \ell \in \mathbb{Z}^d\}$  if  $D_A X = X$  and  $T_\ell X = X$  for each  $\ell \in \mathbb{Z}^d$ . We have the following two lemmas.

LEMMA 4.2. [4, Theorem 1] *Let  $A$  be a  $d \times d$  expansive matrix. A closed subspace  $X$  of  $L^2(\mathbb{R}^d)$  is a reducing subspace for  $\{D_A, T_\ell : \ell \in \mathbb{Z}^d\}$  if and only if  $\widehat{X} = L^2(\mathbb{R}^d) \cdot \chi_\Omega$  for some Lebesgue measurable subset  $\Omega$  of  $\mathbb{R}^d$  with the property that  $\Omega = A^t \Omega$ .*

LEMMA 4.3. [4, Theorem 2] *Let  $A$  be a  $d \times d$  expansive matrix and  $X$  be a reducing subspace. Then a measurable subset  $E \subset \mathbb{R}^d$  is an  $A$ -dilation frame wavelet set for  $X$  (meaning that  $\psi$  defined by  $\widehat{\psi} = \frac{1}{(2\pi)^{d/2}} \chi_E$  is a Parseval frame*

wavelet for  $X$ ) if and only if  $E = E(\delta_A, 1)$ ,  $E = E(\tau, 1)$  and  $\widehat{X} = L^2(\mathbb{R}^d) \cdot \chi_\Omega$  where  $\Omega = \cup_{k \in \mathbb{Z}} (A^t)^k E$  (modulo null sets).

LEMMA 4.4. [4, Theorem 4] *Let  $A$  be a  $d \times d$  expansive matrix and  $X$  be a reducing subspace. Then the set of all  $A$ -dilation  $s$ -elementary Parseval frame wavelets for  $X$  is path connected in the norm topology of  $X$ .*

We are now ready to give an outline of the proof of Theorem 4.1.

PROOF. Let  $A$  be the given expansive matrix and let  $m$  be a fixed positive integer. By Remark 3.3, there exists a bounded open neighborhood  $F$  of the origin such that  $F \subseteq A^t F \subseteq D_0 = [-\pi, \pi]^d$ . Let  $E = A^t F \setminus F$ . Then  $E = E(\tau, 1) = E(\delta_A, 1)$  and  $\cup_{k \in \mathbb{Z}} (A^t)^k E = \mathbb{R}^d$  (modulo a null set). Divide  $E$  into  $m$  disjoint measurable subsets of equal measure  $E_1, E_2, \dots, E_m$  and let  $\Omega_i = \cup_{k \in \mathbb{Z}} (A^t)^k E_i$ . Then  $\Omega_1, \Omega_2, \dots, \Omega_m$  are disjoint measurable subsets of  $\mathbb{R}^d$  whose union is  $\mathbb{R}^d$ . Furthermore,  $A^t \Omega_i = \Omega_i$  for each  $i$ . By Lemma 4.2, the closed subspace  $X_i$  of  $L^2(\mathbb{R}^d)$  defined by  $\widehat{X}_i = L^2(\mathbb{R}^d) \cdot \chi_{\Omega_i}$  is a reducing subspace. Furthermore,  $E_i$  is an  $A$ -dilation frame wavelet set for  $X_i$  by Lemma 4.3. Notice that by Lemma 2.1,  $\Psi = \{\psi_1, \psi_2, \dots, \psi_m\}$  defined by  $\widehat{\psi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{E_i}$  is an  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet of length  $m$  (however it is not intrinsic).

Now let  $\Psi' = \{\psi'_1, \psi'_2, \dots, \psi'_m\}$  be any  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet of length  $m$  such that each  $\psi'_i$  is defined by  $\widehat{\psi}'_i = \frac{1}{(2\pi)^{d/2}} \chi_{E'_i}$  for some measurable set  $E'_i$ . It suffices to show that  $\Psi'$  and  $\Psi$  can be connected by a path in  $S_m$ . This path can be constructed in two steps.

First let  $E' = \cup_{1 \leq j \leq m} E'_j$  (which is a disjoint union by Lemma 2.1). By Lemma 2.1, we have  $E'_j = E'_j(\tau, 1)$  for each  $j$  and  $E'$  is an  $A^t$ -dilation generator of  $\mathbb{R}^d$  (so we must have  $E'_j = E'_j(\delta_A, 1)$  as well). Let  $\Omega'_i = \cup_{k \in \mathbb{Z}} (A^t)^k E'_i$  and  $G_i = \Omega'_i \cap E$ . Notice that we must have  $E = \cup_{1 \leq i \leq m} G_i$ . Again by Lemmas 4.2 and 4.3, the closed subspace  $X'_i$  of  $L^2(\mathbb{R}^d)$  defined by  $\widehat{X}'_i = L^2(\mathbb{R}^d) \cdot \chi_{\Omega'_i}$  is a reducing subspace with  $E'_i$  and  $G_i$  both being  $A$ -dilation frame wavelet sets for  $X'_i$ . Let  $\varphi_i$  be the function defined by  $\widehat{\varphi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{G_i}$  and  $\varphi'_i$  be the function defined by  $\widehat{\varphi}'_i = \frac{1}{(2\pi)^{d/2}} \chi_{E'_i}$ . By Lemma 4.4, for each  $i$ , there exists a continuous path  $\gamma_i$  such that  $\gamma_i(0) = \varphi'_i$ ,  $\gamma_i(1) = \varphi_i$  and  $\gamma_i(t)$  is an  $A$ -dilation  $s$ -elementary Parseval frame wavelet for  $X'_i$  for each  $t \in (0, 1)$ . We will then define  $\gamma(t) = \{\gamma_1(t), \gamma_2(t), \dots, \gamma_m(t)\}$ . It is obvious that  $\gamma(t)$  is continuous in the norm topology of  $(L^2(\mathbb{R}^d))^m$  and it is easy to verify that  $\gamma(t)$  is an  $A$ -dilation Parseval frame multiwavelet of length  $m$  for each  $t$  using Lemma 2.1 and the definition of each  $\gamma_i$ . The details are left to the reader.

We have now shown that any  $A$ -dilation Parseval frame multiwavelet  $\Psi'$  of length  $m$  is path-connected to an  $A$ -dilation Parseval frame multiwavelet  $\Psi''$  of length  $m$  defined on  $m$  disjoint subsets  $G_1, G_2, \dots, G_m$  of  $E$  whose union is  $E$ . What remains to be shown is that  $\Psi''$  can be connected to  $\Psi$  by a path using only  $A$ -dilation Parseval frame multiwavelets of length  $m$  defined on subsets of  $E$  (it is necessary that the union of these sets is always  $E$ ).

For each  $t \in [0, 1]$ , let  $t_i = \sup\{t'_i : \mu(E_i \cap (t'_i D_0)) \leq t\mu(E_i)\}$ . Let  $E_i^t = E_i \cap (t_i D_0)$ . By this definition, we have  $E_i^{t_1} \subseteq E_i^{t_2}$  whenever  $t_1 < t_2$ . Furthermore, it is obvious that  $E_i^t$  is continuous in  $t$  in terms of its Lebesgue measure. Now let  $E^t = \cup_{1 \leq i \leq m} E_i^t$  and let  $G_i^t = E_i^t \cap G_i$ . Since the  $G_i$ 's are disjoint and



$\cup_{1 \leq i \leq m} G_i = E$  as well, the  $G_i^t$ 's are also disjoint and  $\cup_{1 \leq i \leq m} G_i^t = E^t$ . Now let  $H_i^t = (E_i \setminus E_i^t) \cup G_i^t$  for each  $t$ .  $H_i^t$  is continuous in  $t$  in terms of its Lebesgue measure hence  $\chi_{H_i^t}$  is a continuous function in  $t$ . We will leave it to our reader to verify that the  $H_i^t$ 's are disjoint,  $\cup_{1 \leq i \leq m} H_i^t = E$ , so  $\Psi_t = \{\psi_1^t, \psi_2^t, \dots, \psi_m^t\}$  defined by  $\widehat{\psi}_i^t = \frac{1}{(2\pi)^{d/2}} \chi_{H_i^t}$  is an  $A$ -dilation  $s$ -elementary Parseval multiwavelet of length  $m$ . By its definition,  $\Psi_0$  is the  $A$ -dilation  $s$ -elementary Parseval multiwavelet defined on  $\{E_1, E_2, \dots, E_m\}$  and  $\Psi_1$  is the  $A$ -dilation  $s$ -elementary Parseval multiwavelet defined on  $\{G_1, G_2, \dots, G_m\}$ . The continuity of  $\Psi_t$  follows from the continuity of each  $\widehat{\psi}_i^t = \frac{1}{(2\pi)^{d/2}} \chi_{H_i^t}$  in  $t$ .  $\square$

Finally, let us consider path-connectedness of the set of all intrinsic  $A$ -dilation Parseval frame multiwavelets.

**THEOREM 4.5.** *Let  $A$  be an expansive  $d \times d$  matrix and  $m$  a fixed positive integer, then the set  $\mathbf{S}_m$  of all intrinsic  $A$ -dilation  $s$ -elementary Parseval frame multiwavelets of length  $m$  is path-connected.*

**PROOF.** Recall from the proof of Theorem 3.4 that there exists an open neighborhood  $Q$  of the origin such that  $Q \subseteq A^t Q \subseteq D_0$ . Furthermore, there exist  $E_1, E_2, \dots, E_m$  such that  $E_j = E_2 + 2(j-2)\pi\ell_1$  ( $2 \leq j \leq m$ ),  $E_j \overset{\tau}{\sim} E = A^t Q \setminus Q$  for  $2 \leq j \leq m$  and  $E_1 = E \setminus (\cup_{k \in \mathbb{Z}} (A^t)^k (\cup_{2 \leq i \leq m} E_i))$ .  $\{E_1, E_2, \dots, E_m\}$  is an  $A$ -dilation frame multiwavelet of multiplicity  $m$  as shown in the proof of Theorem 3.4. Now let  $\Psi = \{\psi_1, \psi_2, \dots, \psi_m\}$  be an intrinsic  $A$ -dilation  $s$ -elementary Parseval frame multiwavelet of length  $m$  defined by  $\widehat{\psi}_i = \frac{1}{(2\pi)^{d/2}} \chi_{F_i}$  by a set of measurable sets  $F_i$  ( $1 \leq i \leq m$ ). To prove the theorem, it suffices to show that there exists  $\{E_1^t, E_2^t, \dots, E_m^t\}$  satisfying the following conditions:

- (1)  $E_i^0 = F_i$  and  $E_i^1 = E_i$  for each  $i$ ;
- (2)  $\{E_1^t, E_2^t, \dots, E_m^t\}$  satisfies the conditions of Proposition 2.1 for each  $t \in [0, 1]$ ;
- (3)  $\mu(E_i^t)$  is continuous in  $t$  for each  $i$ .

We will construct  $\{E_1^t, E_2^t, \dots, E_m^t\}$  in several steps.

**Step 1.** Let  $F = \cup_{1 \leq i \leq m} F_i$ . By Proposition 2.1,  $\mu(F(\tau, m)) > 0$  (and  $\mu(F(\tau, n)) = 0$  for any  $n > m$ ).  $\{F_1 \cap F(\tau, m), F_2 \cap F(\tau, m), \dots, F_m \cap F(\tau, m)\}$  gives a partition of  $F(\tau, m)$  into  $2\pi$ -translation equivalent subsets. In general, one of these subsets may have a part that is also  $2\pi$ -translation equivalent to a subset in  $E$ . Without loss of generality, let us assume that  $K_1 = \tau(F_1 \cap F(\tau, m)) \cap E$  has a positive measure and let  $H_1$  be the subset of  $F_1 \cap F(\tau, m)$  that is  $2\pi$ -translation equivalent to  $K_1$ . Also, let  $H_i$  be the corresponding subset in  $F_i \cap F(\tau, m)$  that is  $2\pi$ -translation equivalent to  $H_1$ . Let  $E' = (E \setminus K_1) \cup (A^t)^{-1} K_1$ . Since  $E'$  is still contained in  $D_0$ , we still have  $E' = E'(\tau, 1)$ . Of course,  $E'$  is still an  $A$ -dilation generator of  $\mathbb{R}^d$ . However,  $E'$  no longer shares  $2\pi$ -translation equivalent points with the  $H_i$ 's.

Choose a subset  $H'_1$  of  $H_1$  such that the set  $G = E \setminus (\tau(H'_1) \cup (\cup_{k \in \mathbb{Z}} (A^t)^k H'))$  has the property  $\mu(G(\tau, m)) > 0$ , where  $H' = \cup_{1 \leq i \leq m} H'_i$  and  $H'_i \subseteq H_i$  is  $2\pi$ -translation equivalent to  $H'_1$ .  $H'_1$  exists since as  $\mu(H'_1) \rightarrow 0$ ,  $\mu(\tau(H'_1)) \rightarrow 0$  and  $\mu(\cup_{k \in \mathbb{Z}} (A^t)^k H')$

Let  $E'_i = E' \cap (\cup_{k \in \mathbb{Z}} (A^t)^k (F_i \setminus H'_i))$ . As in the proof of the last theorem,  $\{F_1 \setminus H'_1, F_2 \setminus H'_2, \dots, F_m \setminus H'_m\}$  can now be continuously changed to  $\{E'_1, E'_2, \dots, E'_m\}$ , while keeping the  $H_i$ 's intact and the conditions of Proposition 2.1 satisfied. We have now arrived at  $\{E'_1 \cup H'_1, E'_2 \cup H'_2, \dots, E'_m \cup H'_m\}$ , which of course still defines an  $A$ -dilation frame multiwavelet set of multiplicity  $m$ .

Step 2. Choose  $H'_{it} \subseteq H'_i$  be such that  $\mu(H'_{it})$  is continuous in  $t$ ,  $\mu(H'_{it}) > 0$  for any  $t \in [0, 1)$ ,  $H'_{i0} = H'_i$  and  $H'_{i1} = \emptyset$ . (The proof of the existence of such sets is left to the reader as an exercise.) Let  $J'_i = (\cup_{k \in \mathbb{Z}} (A^t)^k (H'_i \setminus H'_{it})) \cap E'$ . Then  $\{E'_1 \cup J'_1 \cup H'_{1t}, E'_2 \cup J'_2 \cup H'_{2t}, \dots, E'_m \cup J'_m \cup H'_{mt}\}$  continuously change  $\{E'_1 \cup H'_1, E'_2 \cup H'_2, \dots, E'_m \cup H'_m\}$  into  $\{E'_1 \cup J'_1, E'_2 \cup J'_2, \dots, E'_m \cup J'_m\}$ . Notice that  $\{E'_1 \cup J'_1 \cup H'_{1t}, E'_2 \cup J'_2 \cup H'_{2t}, \dots, E'_m \cup J'_m \cup H'_{mt}\}$  satisfies the conditions of Proposition 2.1, except when  $t = 1$ , where the union of the sets  $E'_i \cup J'_i$  no longer has any translation redundant points hence  $\{E'_1 \cup J'_1, E'_2 \cup J'_2, \dots, E'_m \cup J'_m\}$  does not define an intrinsic  $A$ -dilation  $s$ -elementary frame multiwavelet.

Step 3. Choose  $G_i \subseteq G \cap E_i$  such that  $G_i \overset{\tau}{\sim} G_j$ ,  $\mu(G_i) > 0$ , and let  $G'_i = E' \cap (\cup_{k \in \mathbb{Z}} (A^t)^k G_i)$ . Notice that by the construction of the  $E_i$ 's (as shown in the proof of Theorem 3.4) and the choice of  $G$ , it is necessary that  $G'_1 = G_1$ . Let  $\{G'_{1t}, G'_{2t}, \dots, G'_{mt}\}$  be a continuous set path (meaning that  $\mu(G'_{it})$  is continuous in  $t$  for each  $i$ ) such that  $G'_{i0} = \emptyset$ ,  $G'_{i1} = G'_i$ ,  $G'_{it} \subseteq G'_i$  and  $\mu(G'_{it}) > 0$  for any  $t \in (0, 1]$ . Let  $\Delta_t(G') = E' \cap (\cup_{k \in \mathbb{Z}} (A^t)^k (\cup_{1 \leq i \leq m} G'_{it}))$ . Now modify the set path defined in the last step as

$$\{(E'_1 \setminus \Delta_t(G')) \cup J'_1 \cup H'_{1t} \cup G'_{1t}, \dots, (E'_m \setminus \Delta_t(G')) \cup J'_m \cup H'_{mt} \cup G'_{mt}\}.$$

We will leave the details for our reader to verify that the above set path is indeed continuous and satisfies all the conditions in Proposition 2.1 for each  $t$ .

Step 4. Let  $\{E''_1, E''_2, \dots, E''_m\}$  be the sets obtained at the end of the last step. Then  $\cup_{1 \leq i \leq m} E''_i \subseteq E \cup (A^t)^{-1} K_1$ ,  $G'_i \subseteq E''_i \cap E_i$  and  $E''_i \setminus G'_i \subseteq D_0$ . Thus we can continuously change  $\{E''_1, E''_2, \dots, E''_m\}$  to  $\{(A^t Q \setminus Q \setminus \Delta(G)) \cup G'_1, G'_2, \dots, G'_m\}$  (where  $\Delta(G) = \Delta_1(G')$ ), while keeping the sets  $G'_i$ 's fixed.

Step 5. Finally let  $I^t_2$  be a continuous set path such that  $I^t_2 \subseteq E_2 \setminus G'_2$  for  $t \in [0, 1]$ ,  $I^0_2 = \emptyset$ ,  $I^1_2 = E_2 \setminus G'_2$  and let  $I^t_i$  be the  $2\pi$ -translation equivalent part of  $I^t_2$  in  $E_i$  ( $2 \leq i \leq m$ ). Let  $\Delta(I^t) = ((A^t Q \setminus Q \setminus \Delta(G)) \cup G'_1) \cap (\cup_{k \in \mathbb{Z}} (A^t)^k (\cup_{2 \leq i \leq m} I^t_i))$ .

$$\{((A^t Q \setminus Q \setminus \Delta(G)) \cup G'_1) \setminus \Delta(I^t), G'_2 \cup I^t_2, \dots, G'_m \cup I^t_m\}$$

is continuous, satisfies all the conditions of Proposition 2.1 and connects  $\{(A^t Q \setminus Q \setminus \Delta(G)) \cup G'_1, G'_2, \dots, G'_m\}$  to  $\{E_1, E_2, \dots, E_m\}$ . Some details are again left to our reader to verify.  $\square$

## 5. Generalizations

The main point of this section is to discuss how to generalize the concept of intrinsic  $A$ -dilation  $s$ -elementary Parseval frame multiwavelets to all  $A$ -dilation Parseval frame multiwavelets. Notice that for any single  $A$ -dilation Parseval frame wavelet  $\psi(t)$  for  $L^2(\mathbb{R}^d)$ ,  $\Psi = \{\frac{1}{\sqrt{n}}\psi(t), \dots, \frac{1}{\sqrt{n}}\psi(t)\}$  (with  $n$  copies of  $\frac{1}{\sqrt{n}}\psi(t)$ ) defines a Parseval multiwavelet for  $L^2(\mathbb{R}^d)$  with length  $n$  (in a rather artificial way, of course). An ‘‘intrinsic’’ Parseval frame multiwavelet should not be Parseval

frame multiwavelets constructed in ways similar to this. Notice that the (non-intrinsic) Parseval frame multiwavelet given in Example 2.3 is not (and cannot be) constructed this way. These considerations lead us to the following definition.

**DEFINITION 5.1.** Let  $\Psi = \{\psi_1, \psi_2, \dots, \psi_p\}$  be an  $A$ -dilation Parseval frame multiwavelet of length  $p$  and let  $\{\widehat{\psi}_1, \widehat{\psi}_2, \dots, \widehat{\psi}_p\}$  be the corresponding Fourier transforms of  $\{\psi_1, \psi_2, \dots, \psi_p\}$ . Furthermore, let  $E = \cup_{1 \leq j \leq p} E_j$  where  $E_j = \text{supp}(\widehat{\psi}_j)$ . Then  $\Psi$  is called an intrinsic  $A$ -dilation Parseval frame multiwavelet (of length  $p$ ) if the following conditions hold.

- (i) For any  $1 \leq i < j \leq p$ , if  $\mu(E_i \cap E_j) > 0$ , then  $\widehat{\psi}_i/\widehat{\psi}_j$  cannot be a constant function (in the a.e. sense) on  $E_i \cap E_j$ ;
- (ii) Let  $q$  be any positive integer that is less than  $p$ . Let  $\{F_{1j}, F_{2j}, \dots, F_{qj}\}$  be any partition of  $E_j$  ( $F_{ij}$  may be an empty set and  $1 \leq j \leq p$ ) such that  $\mu(F_{ij} \cap F_{ij'}) = 0$  for any  $j \neq j'$ , then  $\Psi' = \{\psi'_1, \psi'_2, \dots, \psi'_q\}$  defined by  $\widehat{\psi}'_i = \sum_{1 \leq j \leq p} \chi_{F_{ij}} \widehat{\psi}_j$  is not an  $A$ -dilation Parseval frame multiwavelet.

We leave it to our reader to verify that an intrinsic  $s$ -elementary  $A$ -dilation Parseval frame multiwavelet as defined in Section 1 is also intrinsic under this more general definition (and vice versa). Thus, Theorem 3.4 also implies the following theorem.

**THEOREM 5.2.** *For any real expansive matrix  $A$  and any  $m \in \mathbb{N}$ , there exist intrinsic  $A$ -dilation Parseval frame multiwavelets of length  $m$ .*

We end this paper with the following example showing the existence of an intrinsic  $A$ -dilation Parseval frame multiwavelet that is not  $s$ -elementary for a special case of  $A$  in the case of  $d = 2$  and  $m = 3$ . It remains an open question at this point whether there exist intrinsic  $A$ -dilation Parseval frame multiwavelets of any given length that are not  $s$ -elementary.

**EXAMPLE 5.3.** Let  $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ . By [11, Chapter 10.1],  $\{\psi_1, \psi_2, \psi_3\}$  defined by

$$\begin{aligned} \widehat{\psi}_1(A^\tau \mathbf{s}) &= e^{-is_1} m_0(\mathbf{s} + (\pi, 0)) \widehat{\phi}(\mathbf{s}) \text{ a.e.}, \\ \widehat{\psi}_2(A^\tau \mathbf{s}) &= e^{-i(s_1+s_2)} m_0(\mathbf{s} + (0, \pi)) \widehat{\phi}(\mathbf{s}) \text{ a.e.}, \\ \widehat{\psi}_3(A^\tau \mathbf{s}) &= e^{-is_2} m_0(\mathbf{s} + (\pi, \pi)) \widehat{\phi}(\mathbf{s}) \text{ a.e.} \end{aligned}$$

(where  $\mathbf{s} = (s_1, s_2)$ ) is an  $A$ -dilation MRA multiwavelet with length 3 for  $L^2(\mathbb{R}^2)$ . Here  $m_0$  (the low pass filter) is a real trigonometric polynomial satisfying the condition

$$|m_0(\mathbf{s})|^2 + |m_0(\mathbf{s} + (\pi, 0))|^2 + |m_0(\mathbf{s} + (0, \pi))|^2 + |m_0(\mathbf{s} + (\pi, \pi))|^2 = 1,$$

and the scaling function  $\phi$  satisfies  $\widehat{\phi}(A^\tau \mathbf{s}) = m_0(\mathbf{s}) \widehat{\phi}(\mathbf{s})$ . In this case, the support of  $m_0$  (hence each  $\widehat{\psi}_j$  is  $\mathbb{R}^2$ ). Thus we have  $E_j = \text{supp}(\widehat{\psi}_j) = \mathbb{R}^2$  ( $1 \leq j \leq 3$ ). It is not possible to obtain partitions  $\{F_{1j}, F_{2j}\}$  of  $\mathbb{R}^2$  such that  $\{F_{11}, F_{12}, F_{13}\}$  and  $\{F_{21}, F_{22}, F_{23}\}$  both contain pair-wise disjoint sets. Therefore condition (ii) in Definition 5.1 holds. It is also relatively easy to verify that condition (i) in Definition 5.1 holds. For example, if  $\widehat{\psi}_1/\widehat{\psi}_2 = c$  a.e. then we would get  $m_0(\mathbf{s} +$

$(\pi, 0) = cm_0(\mathbf{s} + (0, \pi))e^{is_2}$  a.e., which is impossible since  $m_0$  is a real trigonometric polynomial function.

Acknowledgement. Y. Diao is partially supported by NSF grants DMS-0712958 and DMS-0920880.

## References

- [1] C. K. Chui, W. He and J. Stckler, *Compactly supported tight and sibling frames with maximum vanishing moments*, Appl. Comput. Harmon. Anal. **13**(2002), 224-262.
- [2] X. Dai and Y. Diao, *The path-connectivity of s-elementary tight frame wavelets*, Journal of Applied Functional Analysis **2**(4) (2007), 39-48.
- [3] X. Dai, Y. Diao and Q. Gu, *Frame wavelet sets in  $\mathbb{R}$* , Proceedings of AMS **129**(7) (2000), 2045-2055.
- [4] X. Dai, Y. Diao, Q. Gu and D. Han, *Frame wavelets in subspaces of  $L^2(\mathbb{R}^d)$* , Proc. Amer. Math. Soc. **11** (2002), 3259-3267.
- [5] X. Dai, Y. Diao, Q. Gu and D. Han, *Wavelets with frame multiresolution analysis*, J. Fourier Anal. Appl. **19**(1) (2003), 39-48.
- [6] X. Dai, Y. Diao, Q. Gu and D. Han, *The s-elementary frame wavelets are path-connected*, Proc. Amer. Math. Soc., **132**(9) (2004), 2567-2575.
- [7] X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Memoirs. Amer. Math. Soc., **134** (1998).
- [8] X. Dai, D. R. Larson and D. M. Speegle, *Wavelet sets in  $\mathbb{R}^n$* , J. Fourier Anal. Appl. **3**(2) (1997), 451-456.
- [9] X. Dai and S. Lu, *Wavelets in subspaces*, Mich. J. Math. **43** (1996), 81-98.
- [10] I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inform. Theory **36** (1990), 961-1005.
- [11] I. Daubechies, *Ten lecture on wavelets*, CBMS**61** SIAM, 1992.
- [12] I. Daubechies and B. Han, *Pairs of dual wavelet frames from any two refinable functions*, Constr. Approx. **20** (2004), 325-352.
- [13] I. Daubechies, B. Han, A. Ron and Z.W. Shen, *Framelets: MRA-based constructions of wavelet frames*, Appl. Comput. Harmon. Anal. **14** (2003) 1-46.
- [14] D. E. Dutkay, *Some equations relating multiwavelets and multiscaling functions*, Journal of Functional Analysis **226** (2005), 1-20.
- [15] Q. Gu and D. Han, *On multiresolution analysis(MRA) wavelets in  $\mathbb{R}^n$* , J. Fourier Anal. Appl. **6**(4) (2000), 437-447.
- [16] B. Han, *Dual multiwavelet frames with high balancing order and compact fast frame transform*, Appl. Comput. Harmon. Anal. **26**(1) (2009), 14-42.
- [17] D. Han and D. Larson, *Basis, frames, and group representations*, Memoirs. Amer. Math. Soc. **147**(2000).
- [18] S. Kwon, *Characterization of orthonormal high-order balanced multiwavelets in terms of moments*, Bull. Korean Math. Soc. **46**(1) (2009), 183-198.
- [19] S. Li and Z. Liu, *Riesz multiwavelet bases generated by vector refinement equation*, Sci. China Ser. A **52**(3) (2009), 468-480.
- [20] Z. Li, X. Dai, Y. Diao and W. Huang, *The path-connectivity of MRA wavelets in  $L^2(\mathbb{R}^d)$* , preprint.
- [21] Z. Li, X. Dai, Y. Diao and J. Xin, *Multipliers, phases and connectivity of wavelets in  $L^2(\mathbb{R}^2)$* , J. Fourier Anal. Appl., to appear.
- [22] R. Liang, *Wavelets, their phases, multipliers and connectivity*, Ph.D. Thesis, University of North Carolina at Charlotte, 1998.
- [23] M. Paluszynski, H. Sikic, G. Weiss and S. Xiao, *Tight frame wavelets, their dimension functions, MRA tight frame wavelets and connectivity properties*, Adv. in Comp. Math. **18** (2003), 297-327.
- [24] D. Speegle, *The s-elementary wavelets are path-connected*, Proc. Amer. Math. Soc. **127**(1) (1999), 223-233.
- [25] Wutam Consortium, *Basic properties of wavelets*, J. Fourier Anal. Appl. **4**(4) (1998), 575-594.

\* DEPARTMENT OF MATHEMATICS AND PHYSICS, NORTH CHINA ELECTRIC POWER UNIVERSITY,  
BEIJING, 102206, CHINA

*E-mail address:* lzhongy@ncepu.edu.cn

† DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NORTH CAROLINA AT  
CHARLOTTE, CHARLOTTE, NC 28223

*E-mail address:* xdai@uncc.edu; ydiao@uncc.edu