

POLYNOMIAL INTEGRABILITY OF THE HAMILTONIAN SYSTEMS WITH HOMOGENEOUS POTENTIAL OF DEGREE -3

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ABSTRACT. In this paper we study the polynomial integrability of natural Hamiltonian systems with two degrees of freedom having a homogeneous potential of degree k given either by a polynomial, or by an inverse of a polynomial. For $k = -2, -1, \dots, 3, 4$ their polynomial integrability has been characterized. Here we have two main results. First we characterize the polynomial integrability of those Hamiltonian systems with homogeneous potential of degree -3 . Second we extend a relation between the nontrivial eigenvalues of the Hessian of the potential calculated at a Darboux point to a family of Hamiltonian systems with potentials given by an inverse of a homogeneous polynomial. This relation was known for such Hamiltonian system with homogeneous polynomial potentials. Finally we present three open problems related with the polynomial integrability of Hamiltonian systems with a rational potential.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Ordinary differential equations in general and Hamiltonian systems in particular play a very important part in many branches of the applied sciences. The question whether a differential model admits a first integral is of fundamental importance as first integrals give conservation laws for the model and that enables us to lower the dimension of the system. Moreover, knowing a sufficient number of first integrals allows to solve the system explicitly. Until the end of the 19th century the majority of scientists thought that the equations of classical mechanics were integrable and finding the first integrals was mainly a problem of computation. In fact, integrability is a rare phenomenon and in general it is very hard to determine whether a given Hamiltonian system is integrable or not.

In this work we are concerned with the polynomial integrability of the natural Hamiltonian systems defined by the Hamiltonian function

$$(1) \quad H = \frac{1}{2} \sum_{i=1}^2 p_i^2 + V(q_1, q_2),$$

where $V(q_1, q_2) \in \mathbb{C}(q_1, q_2)$ is a rational homogeneous potential of degree k given by either a polynomial or an inverse of a polynomial. Here $\mathbb{C}(q_1, q_2)$ as usual is the field of rational functions over \mathbb{C} in the variables q_1, q_2 . To be more precise we consider the following system of four differential equations

$$(2) \quad \dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad i = 1, 2.$$

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Case	Potential
V_i	$\alpha(q_2 - iq_1)^i(q_2 + iq_1)^{3-i}$ for $i = 0, \dots, 3$.
V_4	q_1^3
V_5	$q_1^3/3 + cq_2^3/3$
V_6	$q_1^2q_2/2 + q_2^3$
V_7	$q_1^2q_2/2 + 8q_2^3/3$
V_8	$\pm i\sqrt{3}q_1^3/18 + q_1^2q_2/2 + q_2^3$

TABLE 1. All nonequivalent integrable homogeneous polynomial potentials of degree 3.

Let $A = A(\mathbf{q}, \mathbf{p})$ and $B = B(\mathbf{q}, \mathbf{p})$ be two functions, where $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$. We define the *Poisson bracket* of A and B as

$$\{A, B\} = \sum_{i=1}^2 \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right).$$

The functions A and B are *in involution* if $\{A, B\} = 0$. A non-constant function $F = F(\mathbf{q}, \mathbf{p})$ is a *first integral* for the Hamiltonian system (2) if it is in involution with the Hamiltonian function H , i.e. $\{H, F\} = 0$. Since the Poisson bracket is antisymmetric it follows that H itself is always a first integral. A 2-degree of freedom Hamiltonian system (2) is *completely* or *Liouville integrable* if it has 2 functionally independent first integrals H and F . As usual H and F are *functionally independent* if their gradients are linearly independent at all points of \mathbb{C}^4 except perhaps in a zero Lebesgue set.

Let $\text{PO}_2(\mathbb{C})$ denote the group of 2×2 complex matrices A such that $AA^T = \alpha \text{Id}$, where Id is the identity matrix and $\alpha \in \mathbb{C} \setminus \{0\}$. The potentials $V_1(\mathbf{q})$ and $V_2(\mathbf{q})$ are *equivalent* if there exists a matrix $A \in \text{PO}_2(\mathbb{C})$ such that $V_1(\mathbf{q}) = V_2(A\mathbf{q})$. Therefore we divide all potentials into equivalent classes. In what follows a potential means a class of equivalent potentials in the above sense. This definition of equivalent potentials is motivated by the following simple observation (for a proof see [10]). Let V_1 and V_2 be two equivalent potentials. If the Hamiltonian system (2) with the potential V_1 is integrable, then it is also integrable with the potential V_2 .

In the 80's all integrable Hamiltonian systems (1) with homogeneous polynomial potential of degree at most 5 and having a second polynomial first integral up to degree 4 in the variables p_1 and p_2 were computed, see [18, 7, 5, 3, 8, 9]. We note that all these first integrals are polynomials in the variables p_1, p_2, q_1 and q_2 . Recently two mathematically rigorous approaches to the integrability problem have been proposed by Ziglin [22, 23] and Morales-Ruiz and Ramis [17]. They explain relations between the existence of first integrals and branching of solutions as functions of the complex time and give necessary integrability conditions for Hamiltonian systems. It appears that the integrability is related to properties of the monodromy group or the differential Galois group of variational equations along a particular solution.

One of the strongest necessary conditions for the complete meromorphic integrability of the Hamiltonian systems with a homogeneous potential was given by Morales and Ramis (see [17, p. 100] and references therein). Using these results, Maciejewski and Przybylska [13] gave necessary and sufficient conditions for the

Case	Potential
V_i	$\alpha(q_2 - iq_1)^i(q_2 + iq_1)^{4-i}$ for $i = 0, 1, 2, 3, 4$.
V_5	αq_2^4
V_6	$\alpha q_1^4/4 + q_2^4$
V_7	$4q_1^4 + 3q_1^2 q_2^2 + q_2^4/4$
V_8	$2q_1^4 + 3q_1^2 q_2^2/2 + q_2^4/4$
V_9	$(q_1^2 + q_2^2)^2/4$
V_{10}	$-q_1^2(q_1 + iq_2)^2 + (q_1^2 + q_2^2)^2/4$

TABLE 2. Nonequivalent integrable homogeneous polynomial potentials of degree 4.

complete meromorphic integrability of Hamiltonian systems with the homogeneous polynomial potential of degree 3. See Table 1 for the list of nonequivalent integrable homogeneous potentials of degree 3. In [14] the same authors studied the meromorphic integrability of the class of Hamiltonian systems with a homogeneous polynomial potential of degree 4. They proved that except for the family of potentials

$$(3) \quad V = \frac{1}{2} a q_1^2 (q_1 + i q_2)^2 + \frac{1}{4} (q_1^2 + q_2^2)^2,$$

only these systems with potentials V_i for $i = 0, 1, \dots, 8$ given in Table 2 are the nonequivalent integrable homogeneous potentials of degree 4. We proved in [11] that for the family (3) only the potentials V_9 and V_{10} of Table 2 are integrable.

We note that from the papers [11, 13, 14] it follows that all the Hamiltonian systems (2) with a potential V given by a homogeneous polynomial of degree 3 or 4 which have an additional independent meromorphic first integral with respect to the Hamiltonian, have also a polynomial first integral independent with the Hamiltonian. It is well known, see for instance [12], that if the potential V is a homogeneous polynomial or an inverse of a homogeneous polynomial of degree 2, 1, 0 or -1 , then the Hamiltonian system (2) has always a polynomial first integral independent with the Hamiltonian. The existence of a polynomial first integral independent with the Hamiltonian for the Hamiltonian systems (2) with a potential given by the inverse of a homogeneous polynomial of degree -2 was characterized in [12]. More precisely, in [12] we classified the polynomial integrability of the Hamiltonian systems (2) with a homogeneous potential of degree -2 of the form $V = 1/(a q_1^2 + b q_1 q_2 + c q_2^2)$. We proved that the study of the integrability for this potential is equivalent to studying the integrability of (2) with homogeneous potential

$$(4) \quad V = 1/(a q_1^2 + c q_2^2).$$

Moreover, we showed that it is polynomially integrable if and only if either $c = 0$, or $c \neq 0$ and $a \in \{0, c\}$. Moreover it is well known that these Hamiltonian systems with potentials (4) have always the additional rational first integral

$$I = \frac{1}{2} (q_1 p_2 - q_2 p_1)^2 + (q_1^2 + q_2^2) V(q),$$

see [2, 15, 11].

One of the main results of this paper is the characterization of the polynomial integrability of the natural Hamiltonian systems (2) with a homogeneous potential

of degree -3 of the form

$$(5) \quad V = \frac{1}{aq_1^3 + bq_1^2q_2 + cq_1q_2^2 + dq_2^3},$$

with $aq_1^3 + bq_1^2q_2 + cq_1q_2^2 + dq_2^3 \neq 0$.

Our first intention was to characterize the existence of meromorphic first integral of the Hamiltonian systems (2) with potential (5) using the Morales-Ramis theory. Notice that such meromorphic first integrals includes the rational ones, probably the more natural class for looking the second additional first integral because the Hamiltonian of our systems is a rational function. But this characterization looks at this moment computationally very hard, and in a first approach to this characterization we have classified the existence of polynomial first integrals.

In the following proposition we reduce the study of the existence or non-existence of polynomial first integrals of the Hamiltonian systems (2) with potential (5) to the study of seven Hamiltonian systems (2) with the potentials described in the statement of Proposition 1.

Proposition 1. *After an appropriate rotation and a rescaling we can transform the potential (5) into one of the reduced forms*

$$\begin{aligned} V_0 &= \frac{1}{q_1^3}; & V_1 &= \frac{1}{aq_1^3 + q_2^3}; & V_{2,3} &= \frac{1}{(q_2^2 + q_1^2)(q_2 \pm iq_1)}; \\ V_{4,5} &= \frac{1}{(q_2 \pm iq_1)^3}; & V_{gen} &= \frac{1}{aq_1^3 + q_1^2q_2 + dq_2^3}. \end{aligned}$$

Proposition 1 is proved in section 4.

In what follows we say that a Hamiltonian system (2) is not polynomially integrable if it does not admit a polynomial first integral independent of the Hamiltonian. The existence or non-existence of polynomial first integrals for these seven classes of potentials is characterized in the following theorem.

Theorem 2. *The Hamiltonian system (2) with the potential:*

- (a) V_{gen} does not admit an additional polynomial first integral.
- (b) V_0 is completely integrable with the additional polynomial first integral p_2 .
- (c) V_1 admits a polynomial first integral if and only if $a = 0$, in which case the first integral is p_1 .
- (d) V_2 or V_3 does not admit a polynomial first integral.
- (e) V_4 is completely integrable with the additional polynomial first integral $p_1 - p_2i$.
- (f) V_5 is completely integrable with the additional polynomial first integral $p_1 + p_2i$.

Theorem 2 is proved in section 5.

We say that a nonzero $\mathbf{d} \in \mathbb{C}^2$ is a *Darboux point* of system (2) if it is the solution of the gradient equation

$$V'(\mathbf{d}) = \gamma \mathbf{d},$$

where $\gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. It is clear that if \mathbf{d} is a Darboux point, then $\tilde{\mathbf{d}} = \alpha \mathbf{d}$, where $\alpha \in \mathbb{C}^*$ satisfies

$$V'(\tilde{\mathbf{d}}) = \alpha^{k-2} \gamma \tilde{\mathbf{d}} = \tilde{\gamma} \tilde{\mathbf{d}}.$$

Of course this implies that \mathbf{d} and $\tilde{\mathbf{d}}$ represent the same Darboux point. Thus, as it was explained in [14] a Darboux point \mathbf{d} can be considered as a point in a projective space $\mathbf{d} = [d_1 : d_2] \in \mathbb{CP}^1$.

Consider the Hessian matrix $V''(\mathbf{q})$ evaluated at the Darboux point \mathbf{d} , that is

$$(6) \quad V''(\mathbf{d}) = \begin{pmatrix} \frac{\partial^2 V(\mathbf{d})}{\partial q_1^2} & \frac{\partial^2 V(\mathbf{d})}{\partial q_1 \partial q_2} \\ \frac{\partial^2 V(\mathbf{d})}{\partial q_2 \partial q_1} & \frac{\partial^2 V(\mathbf{d})}{\partial q_2^2} \end{pmatrix}.$$

Denote by $\{\mu, \lambda\}$ the two eigenvalues of $V''(\mathbf{d})$. Since V is a homogeneous function of degree k , it is well-known that one of the eigenvalues, say μ , equals $\tilde{\gamma}(k-1)$ and we shall call it the *trivial* one. We shall also say that λ is the *non-trivial eigenvalue*.

The second result of this paper is:

Theorem 3. *Assume that a homogeneous potential $V(\mathbf{q})$ of degree $m \in \mathbb{Z}$ being a polynomial or an inverse of a polynomial has $\mathbf{p}_1, \dots, \mathbf{p}_m$ different Darboux points. Then*

$$(7) \quad \sum_{i=1}^m \frac{1}{\Lambda_i} = -1,$$

where $\Lambda_i = \lambda_i - 1$ and λ_i is a non-trivial eigenvalue of the Hessian matrix V'' evaluated at the Darboux point \mathbf{p}_i .

Theorem 3 is an extension of Theorem 1.2 of Maciejewski and Przybylska [14], where the authors proved it for Hamiltonian systems with a homogeneous polynomial potential.

There are many papers on Hamiltonian systems (2) with polynomial potentials, and very few with rational. For one of these papers with rational potential see [1] and references therein. A natural extension of our main result would be considering the integrability of potentials being an inverse of a polynomial of arbitrary degree. Related to that we would like to state some open problems.

Open problem 1. *Consider a natural Hamiltonian system (2) with the potential*

$$(8) \quad V(q_1, q_2) = \frac{1}{a_0 q_1^m + a_1 q_1^{m-1} q_2 + \dots + a_{m-1} q_1 q_2^{m-1} + a_m q_2^m}, \quad a_j \in \mathbb{C},$$

where $a_{m-1} = 0$ and $m \geq 2$. Show that if $a_1 \dots a_{m-2} a_m \neq 0$, then system (2) does not admit any polynomial first integral.

The open problem 1 has a positive answer for $m = 2, 3$ (see [12] and Proposition 14 of this paper).

For certain types of potentials it will be shown in the next proposition that the Hamiltonian system (2) is completely integrable. The proof follows easily by direct computations.

Proposition 4. *Hamiltonian systems (2) having a potential V given below is completely integrable with the given corresponding additional first integral I :*

$$(9) \quad \begin{aligned} V &= f(q_1) + g(q_2), & I &= p_1^2 + 2f(q_1), \\ V &= f(aq_1 + bq_2), & I &= bp_1 - ap_2, \\ V &= f(q_1^2 + q_2^2), & I &= q_2 p_1 - q_1 p_2, \end{aligned}$$

where $f : \mathbb{C} \rightarrow \mathbb{C}$ is any differentiable function.

Open problem 2. *Consider a natural Hamiltonian system (2) with the potential given by (8) with $m \geq 1$. Prove or disprove that system (2) is completely integrable with an additional polynomial first integral if and only if the potential $V(\mathbf{q})$ is equivalent to one of the potentials given in (9).*

The open problem holds for $m = 1, 2, 3$ (see [12] and Proposition 14 of this paper).

We think that Theorem 2 holds if instead of polynomial first integrals we write global analytic first integrals. Thus a more general open question is the following one.

Open problem 3. *Prove or disprove that if a Hamiltonian system (2) with a potential given by the inverse of a homogeneous polynomial does not admit an additional polynomial first integral, then it does not admit an additional analytic first integral globally defined.*

We note that the open problem 3 has a positive answer for potentials given by homogeneous polynomials.

Somehow similar questions of whether the non-integrability within a smaller class of functions implies automatically the non-integrability in a bigger class were considered by Gorni and Zampieri [6]. To be more precise motivated by the works of Taïmanov [19] and Butler [4] the authors asked whether for a polynomial Hamiltonian system with two degrees of freedom an analytic integrability implies \mathbb{C}^∞ integrability. They answered this question negatively constructing a very simple explicit example, which does not have a full set of analytic first integrals but possesses a full set of \mathbb{C}^∞ first integrals. The example of Gorni and Zampieri was further generalized by Yoshino [21] (see also [20]).

2. MORALES–RAMIS THEORY

We would like to mention some known integrability obstructions for homogeneous polynomial potentials.

In the case that Hamiltonian system (2) is completely integrable the eigenvalues of (6) are not arbitrary. The following theorem due to Morales and Ramis determines these values. For proof see, e.g. [17].

Theorem 5 (Morales–Ramis). *Assume that the Hamiltonian system defined by the Hamiltonian (1) with a homogeneous potential V of degree $k \in \mathbb{Z} \setminus \{0\}$ is completely integrable with holomorphic or meromorphic first integrals, then λ satisfies the following conditions:*

Degree	Eigenvalue λ	Degree	Eigenvalue λ
k	$p + p(p-1)\frac{k}{2}$	-3	$\frac{25}{24} - \frac{1}{24}(\frac{12}{5} + 6p)^2$
2	arbitrary	3	$-\frac{1}{24} + \frac{1}{24}(2 + 6p)^2$
-2	arbitrary	3	$-\frac{1}{24} + \frac{1}{24}(\frac{3}{2} + 6p)^2$
-5	$\frac{49}{40} - \frac{1}{40}(\frac{10}{3} + 10p)^2$	3	$-\frac{1}{24} + \frac{1}{24}(\frac{6}{5} + 6p)^2$
-5	$\frac{49}{40} - \frac{1}{40}(4 + 10p)^2$	3	$-\frac{1}{24} + \frac{1}{24}(\frac{12}{5} + 6p)^2$
-4	$\frac{9}{8} - \frac{1}{8}(\frac{4}{3} + 4p)^2$	4	$-\frac{1}{8} + \frac{1}{8}(\frac{4}{3} + 4p)^2$
-3	$\frac{25}{24} - \frac{1}{24}(2 + 6p)^2$	5	$-\frac{9}{40} + \frac{1}{40}(\frac{10}{3} + 10p)^2$
-3	$\frac{25}{24} - \frac{1}{24}(\frac{3}{2} + 6p)^2$	5	$-\frac{9}{40} + \frac{1}{40}(4 + 10p)^2$
-3	$\frac{25}{24} - \frac{1}{24}(\frac{6}{5} + 6p)^2$	k	$\frac{1}{2}(\frac{k-1}{k} + p(p+1)k)$

where p is an integer.

3. A FAMILY OF NATURAL HAMILTONIAN SYSTEMS WITH A HOMOGENEOUS POTENTIAL OF DEGREE m .

We shall derive formulas for the Darboux points and the corresponding non-trivial eigenvalues of the Hessian matrix V'' of the potential V given by (8). We

also show that the relation (7) introduced first by Maciejewski and Przybylska in [14] for the generic Hamiltonian systems with the homogeneous polynomial potential also holds for the family of Hamiltonian systems with the rational potential being the inverse of a polynomial.

Setting $z = q_2/q_1$ we rewrite potential (8) as

$$V = \frac{1}{q_1^m v(z)} \quad \text{where} \quad v(z) = a_0 + a_1 z + \dots + a_m z^m.$$

We define the polynomials

$$(10) \quad h(z) = mv(z) - zv'(z), \quad g(z) = (1 + z^2)v'(z) - mzv(z),$$

where as usual $v'(z)$ denotes the derivative of $v(z)$ with respect to z .

The following proposition shows the way to calculate the Darboux points and the non-trivial eigenvalue λ associated to (8).

Proposition 6. *Assume that $g \neq 0$. The Darboux points $\mathbf{d} = [1 : z^*]$ associated with potential (8) are given by the zeroes of $g(z) = 0$ for which $h(z) \neq 0$. Moreover the non-trivial eigenvalue of $V''(\mathbf{d})$ is given by $\lambda(z^*) = g'(z^*)/h(z^*) + 1$.*

Proof. It is clear that the Darboux points can be viewed as equilibrium points of the differential system

$$(11) \quad \dot{q}_1 = -q_1 + \frac{\partial V(q_1, q_2)}{\partial q_1}, \quad \dot{q}_2 = -q_2 + \frac{\partial V(q_1, q_2)}{\partial q_2}.$$

We shall make a change of the variables $(q_1, q_2) \rightarrow (q_1, z)$, where $z = q_2/q_1$. Clearly the equilibrium points of the new differential system define also Darboux points. Expressing the derivatives of the potential in the variables q_1 and z we get

$$\frac{\partial V}{\partial q_1} = -\frac{1}{q_1^{m+1}v^2(z)} [mv(z) - zv'(z)], \quad \frac{\partial V}{\partial q_2} = -\frac{1}{q_1^{m+1}v^2(z)} v'(z).$$

Thus

$$\dot{z} = \frac{\dot{q}_2 q_1 - q_2 \dot{q}_1}{q_1^2} = -\frac{1}{q_1^{m+2}v^2(z)} [(1 + z^2)v'(z) - mzv(z)].$$

In the variables (q_1, z) system (11) writes

$$(12) \quad \dot{q}_1 = -q_1 - \frac{1}{q_1^{m+1}v^2(z)} h(z), \quad \dot{z} = -\frac{1}{q_1^{m+2}v^2(z)} g(z).$$

Since the equilibrium points of (12) are Darboux points of V , they must satisfy the algebraic system

$$(13) \quad g(z) = 0, \quad h(z) + q_1^{m+2}v^2(z) = 0.$$

One way to calculate the Hessian matrix V'' in the variables (q_1, z) is to calculate the Jacobian matrix J of system (12) and use the relationship obtained from system (11)

$$J = V'' - Id,$$

where Id stands for the identity matrix. We have

$$(14) \quad J = \begin{bmatrix} (1+m)q_1^{-2-m}h(z)v^{-2}(z) - 1 & q_1^{-1-m}v^{-3}(z)[-v(z)h'(z) + 2g(z)] \\ (2+m)q_1^{-3-m}g(z)v^{-2}(z) & q_1^{-2-m}v^{-3}(z)[-v(z)g'(z) + 2g(z)] \end{bmatrix}.$$

Taking into account relations (13) we get V'' calculated at the Darboux point is

$$(15) \quad V'' = J + Id = \begin{bmatrix} -m-1 & * \\ 0 & \frac{g'(z^*)}{h(z^*)} + 1 \end{bmatrix}.$$

It is clear that one of the eigenvalues of V'' is always $-m-1$ and the non-trivial eigenvalue calculated at the Darboux point is

$$\lambda(z^*) = \frac{g'(z^*)}{h(z^*)} + 1.$$

□

Proposition 7. *The number of Darboux points of the potential (8) is at most m .*

Proof. We use the freedom of the choice of the potential from the equivalence class and take such one that has all Darboux points localized in the affine part of \mathbb{CP}^1 where $q_1 \neq 0$. Thus potential (8) cannot possess Darboux points in the infinity except if polynomial in denominator is factorizable by q_1 .

From Proposition 6 the Darboux points $\mathbf{d} = [1 : z^*]$ are given by the zeroes of $g(z) = 0$ for which $h(z) \neq 0$. We have

$$g(z) = (1 + z^2)v'(z) - mzv(z) = -a_{m-1}z^m + (ma_m - 2a_{m-2})z^{m-1} + l.o.t.$$

where *l.o.t.* stands for the lower order terms in z . Thus there are at most m Darboux points of the form $[1 : z]$, and since there cannot be additional Darboux points of the form $[0 : z]$, the proposition follows. □

Proposition 8. *The polynomials $g(z)$ and $h(z)$ have a common root z^* if and only if z^* is a multiple root of $v(z)$.*

Proof. If z^* is a multiple root of $v(z)$ if and only if $v(z^*) = v'(z^*) = 0$. Therefore $g(z^*) = h(z^*) = 0$. On the other hand if $g(z^*) = h(z^*) = 0$, then from (10) we obtain that $v'(z) = zh(z) + g(z)$, so we get $v'(z^*) = 0$. Finally again from (10) we get $v(z^*) = 0$. □

Proof of Theorem 3. We introduce the following 1-form

$$\omega(z) = \frac{h(z)}{g(z)} dz.$$

It is clear that $\mathbb{CP}^1 = U_1 \cup U_2$, where

$$U_1 = \{[q_1 : q_2] \in \mathbb{CP}^1 : q_1 \neq 0\}, \quad U_2 = \{[q_1 : q_2] \in \mathbb{CP}^1 : q_2 \neq 0\}.$$

The 1-form $\omega(z)$ is defined on U_1 and we shall extend it to \mathbb{CP}^1 . In order to do that we introduce the map $\xi : U_1 \rightarrow U_2$ defined as $\xi = 1/z$. Thus we have $dz = -d\xi/\xi^2$ and

$$\omega_\infty = \omega(1/\xi) = -\frac{h(1/\xi)}{g(1/\xi)} \frac{1}{\xi^2} d\xi.$$

Under our assumptions the Darboux points are simple roots of $g(z) = 0$, then we get

$$\text{res}_{z_i} \omega = \frac{h(z_i)}{g'(z_i)} = \frac{1}{\lambda_i - 1} = \frac{1}{\Lambda_i}.$$

Now the residue of $\omega(z)$ at infinity is $\text{res}_\infty \omega$. We have

$$\text{res}_\infty \omega = \text{res}_0 \omega_\infty = 1.$$

Then the theorem follows by the Residue Theorem, see the appendix. □

Proposition 9. *Consider a Hamiltonian system (2) with the following potential*

$$(16) \quad V = \frac{1}{aq_1^m + q_2^m},$$

where $m \in \mathbb{Z}$, $m > 2$. Then system (2) is completely integrable with a meromorphic first integral if and only if $a = 0$.

Proof. First we note that if $a = 0$, then potential (16) is trivially separable and therefore by Proposition 4 is integrable. To prove the non-integrability for $a \neq 0$, we shall use Theorem 5 of Morales and Ramis. Thus we shall calculate the non-trivial eigenvalue λ associated with potential (16). Since V is of the form (8), we can calculate λ using Proposition 6. The Darboux points z^* are given by the zeros of $g(z^*) = 0$ for which $h(z^*) \neq 0$. Since $g(z) = -amz + mz^{m-1}$ and $h(z) = am$ we obtain the Darboux point

$$z^* = a^{1/m-2},$$

and the corresponding non-trivial eigenvalue

$$(17) \quad \lambda = \frac{g'(z^*)}{h(z^*)} + 1 = m - 1.$$

Now we define $\mathcal{Z}_{-m}^a = \{p - p(p-1)\frac{m}{2} : p \in \mathbb{Z}\}$ and $\mathcal{Z}_{-m}^b = \{\frac{m+1}{2m} - p(p+1)\frac{m}{2} : p \in \mathbb{Z}\}$. We shall consider different cases depending on the degree of the homogeneity of V .

Case: $m = 3$. In order to prove that potential (16) is non-integrable when $a \neq 0$ consider the following sets

$$\begin{aligned} \mathcal{Z}_{-3}^1 &= \left\{ \frac{25}{24} - \frac{1}{24}(2+6p)^2 : p \in \mathbb{Z} \right\}, & \mathcal{Z}_{-3}^3 &= \left\{ \frac{25}{24} - \frac{1}{24}\left(\frac{6}{5} + 6p\right)^2 : p \in \mathbb{Z} \right\}, \\ \mathcal{Z}_{-3}^2 &= \left\{ \frac{25}{24} - \frac{1}{24}\left(\frac{3}{2} + 6p\right)^2 : p \in \mathbb{Z} \right\}, & \mathcal{Z}_{-3}^4 &= \left\{ \frac{25}{24} - \frac{1}{24}\left(\frac{12}{5} + 6p\right)^2 : p \in \mathbb{Z} \right\}. \end{aligned}$$

From (17) the nontrivial eigenvalue is $\lambda = 2$. Thus by Theorem 5 if potential (16) is completely integrable then

$$\lambda \in \{\mathcal{Z}_{-3}^1 \cup \mathcal{Z}_{-3}^2 \cup \mathcal{Z}_{-3}^3 \cup \mathcal{Z}_{-3}^4 \cup \mathcal{Z}_{-3}^a \cup \mathcal{Z}_{-3}^b\}.$$

A simple computation shows that the last condition is never fulfilled for $\lambda = 2$ and therefore the potential V given by (16) for $m = 3$ is not integrable.

Cases: $m = 4$ and $m = 5$. We prove the non-integrability in a similar manner.

Case: $m > 5$. By (17) the nontrivial eigenvalue is $\lambda = m - 1$. Theorem 5 for $m > 5$ implies that if potential (16) is completely integrable, then

$$\lambda = m - 1 \in \{\mathcal{Z}_{-m}^a \cup \mathcal{Z}_{-m}^b\}.$$

The last condition is fulfilled if and only if $m \in \{1, 2\}$, which is not compatible with our assumption that $m > 5$. \square

4. HOMOGENEOUS POTENTIALS OF DEGREE -3

In this section we prove all the results stated in the introduction for the homogeneous potentials of degree -3 given in (5).

Proof of Proposition 1. Consider the potential V given in (5) and let $P(q_1, q_2) = aq_1^3 + bq_1^2q_2 + cq_1q_2^2 + dq_2^3$ be its denominator. The integrability of the Hamiltonian system (2) is preserved under the rotation

$$(18) \quad q_1 = Q_1 \cos \alpha + Q_2 \sin \alpha, \quad q_2 = -Q_1 \sin \alpha + Q_2 \cos \alpha, \quad \alpha \in [0, 2\pi).$$

We shall use this fact in order to simplify the potential V .

We substitute (18) into the polynomial $P(q_1, q_2)$ and we get that the coefficient of $Q_1Q_2^2$ in the expression of $P(Q_1, Q_2)$ is

$$c \cos^3 \alpha + (2b - 3d) \cos^2 \alpha \sin \alpha + (3a - 2c) \cos \alpha \sin^2 \alpha - b \sin^3 \alpha.$$

Assume $c \neq 0$. Then dividing the previous expression by $\sin^3 \alpha$ we can vanish the coefficient of $Q_1Q_2^2$ if and only if there is α such that

$$(19) \quad c \cot^3 \alpha + (2b - 3d) \cot^2 \alpha + (3a - 2c) \cot \alpha - b = 0.$$

This is a cubic equation in $\cot \alpha$. Then it is easy to check that always there exists $\alpha^* \in (0, \pi)$ such that $\cot \alpha^*$ is a root of (19), except if all the roots of (19) are $\pm i$. So if we are not in this last case we can assume that $c = 0$.

We note that if all roots of (19) are $\pm i$, then there is no rotation to eliminate $Q_1 Q_2^2$. This is the case when the polynomial $P(q_1, q_2)$ is of the form $(q_1 \pm iq_2)^3$ or $(q_1^2 + q_2^2)(q_1 \pm iq_2)$, obtaining the potentials V_2, \dots, V_5 .

Now we recall that a $2n \times 2n$ matrix S is called *symplectic with multiplier μ* if

$$S^T J S = \mu S, \quad \text{where} \quad J = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}.$$

A symplectic change of variables takes a Hamiltonian system of equations into another Hamiltonian system (see [16] for details). When $c = 0$ and $b \neq 0$ the change

$$Q_1 = q_1, \quad Q_2 = q_2, \quad P_1 = \sqrt{b} p_1, \quad P_2 = \sqrt{b} p_2$$

is symplectic with multiplier \sqrt{b} and we get V_{gen} . When $b = c = 0$ and $d \neq 0$ with the symplectic change $Q_1 = q_1, Q_2 = q_2, P_1 = \sqrt{d} p_1, P_2 = \sqrt{d} p_2$ we get V_1 , and when $b = c = d = 0$ with the symplectic change $Q_1 = q_1, Q_2 = q_2, P_1 = \sqrt{a} p_1, P_2 = \sqrt{a} p_2$ we get V_0 . So the proposition is proved. \square

5. PROOF OF THEOREM 2

Statement (b) of Theorem 2 is trivial.

Statement (c) follows easily from Proposition 9 and it is immediate to check that its first integral is p_1 .

Now we prove statement (d). The nontrivial eigenvalue calculated at the simple Darboux point associated with V_2 and V_3 is $\lambda = 2$. Using the notation introduced in Proposition 9, it is easy to check that

$$2 \notin \{\mathcal{Z}_{-3}^1 \cup \mathcal{Z}_{-3}^2 \cup \mathcal{Z}_{-3}^3 \cup \mathcal{Z}_{-3}^4 \cup \mathcal{Z}_{-3}^a \cup \mathcal{Z}_{-3}^b\}.$$

Thus by Theorem 5 potentials V_2 and V_3 are not integrable.

Statements (e) and (f) of Theorem 2 follow easily from Proposition 4.

In the rest of this section we shall prove statement (a) of Theorem 2.

We consider a Hamiltonian system associated with the potential V_{gen} , that is

$$(20) \quad \dot{q}_1 = p_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_1 = \frac{3aq_1^2 + 2q_1q_2}{(aq_1^3 + q_1^2q_2 + dq_2^3)^2}, \quad \dot{p}_2 = \frac{q_1^2 + 3dq_2^2}{(aq_1^3 + q_1^2q_2 + dq_2^3)^2}.$$

Now we take the new independent variable τ defined by $dt = (aq_1^3 + q_1^2q_2 + dq_2^3)^2 d\tau$ and we write system (20) as

$$(21) \quad \begin{aligned} \dot{q}_1 &= p_1(aq_1^3 + q_1^2q_2 + dq_2^3)^2, & \dot{q}_2 &= p_2(aq_1^3 + q_1^2q_2 + dq_2^3)^2, \\ \dot{p}_1 &= 3aq_1^2 + 2q_1q_2, & \dot{p}_2 &= q_1^2 + 3dq_2^2. \end{aligned}$$

We separate the proof of the non-existence of polynomial first integrals of system (21) into three cases:

- $a = 0$,
- $a \neq 0$ and $d = 0$,
- $ad \neq 0$.

Proposition 10. *The Hamiltonian system (21) with $a = 0$ has no polynomial first integrals.*

Proof. Setting $a = 0$ equation (21) becomes

$$(22) \quad \begin{aligned} \dot{q}_1 &= p_1 q_2^2 (q_1^2 + dq_2^2)^2, & \dot{q}_2 &= p_2 q_2^2 (q_1^2 + dq_2^2)^2, \\ \dot{p}_1 &= 2q_1 q_2, & \dot{p}_2 &= q_1^2 + 3dq_2^2. \end{aligned}$$

We proceed by contradiction by assuming that there exists f a polynomial first integral of system (22). Then it satisfies

$$(23) \quad p_1 q_2^2 (q_1^2 + dq_2^2)^2 \frac{\partial f}{\partial q_1} + p_2 q_2^2 (q_1^2 + dq_2^2)^2 \frac{\partial f}{\partial q_2} + 2q_1 q_2 \frac{\partial f}{\partial p_1} + (q_1^2 + 3dq_2^2) \frac{\partial f}{\partial p_2} = 0.$$

We write f in sum of its homogeneous parts as $f = \sum_{j=0}^n f_j$ where each f_j is a homogeneous polynomial of degree j and we can assume that $f_n \neq 0$ and $n \geq 1$. Then computing the terms of degree $n + 6$ in (23) we have that f_n satisfies

$$q_2^2 (q_1^2 + dq_2^2)^2 \left(p_1 \frac{\partial f_n}{\partial q_1} + p_2 \frac{\partial f_n}{\partial q_2} \right) = 0.$$

The general solution of the linear partial differential equation is an arbitrary function in the variable $(q_1 p_2 - q_2 p_1)/p_1$ with coefficients arbitrary functions in the variables p_1 and p_2 . Since f_n must be a polynomial of degree n in the variables p_1, p_2, q_1 and q_2 , we get that f_n must be of the form as

$$(24) \quad f_n = \sum_{i+j+2l=0}^n f_{ijl} p_1^i p_2^j (q_1 p_2 - q_2 p_1)^l.$$

Now computing the terms of degree $n + 1$ we obtain

$$(25) \quad q_2^2 (q_1^2 + dq_2^2)^2 \left(p_1 \frac{\partial f_{n-6}}{\partial q_1} + p_2 \frac{\partial f_{n-6}}{\partial q_2} \right) + 2q_1 q_2 \frac{\partial f_n}{\partial p_1} + (q_1^2 + 3dq_2^2) \frac{\partial f_n}{\partial p_2} = 0.$$

Evaluating (25) on $q_2 = 0$ we get that

$$f_n = \sum_{i+j+2l=0}^n f_{ijl} (j+l) p_1^i p_2^{j+l-1} q_1^l = 0.$$

This implies that $j = l = 0$, and consequently $f_n = f_n(p_1)$. Now after simplifying equation (25) by q_2 and evaluating it on $q_2 = 0$, we get that $df_n/dp_1 = 0$. Hence f_n must be a constant. So $n = 0$ and we have a contradiction. \square

Proposition 11. *The Hamiltonian system (21) with $a \neq 0$ and $d = 0$ has no polynomial first integrals.*

Proof. Now equation (21) becomes

$$(26) \quad \begin{aligned} \dot{q}_1 &= p_1 q_1^4 (aq_1 + q_2)^2, & \dot{q}_2 &= p_2 q_1^4 (aq_1 + q_2)^2, \\ \dot{p}_1 &= 3aq_1^2 + 2q_1 q_2, & \dot{p}_2 &= q_1^2. \end{aligned}$$

We proceed by contradiction by assuming that there exists a polynomial first integral f of system (26). Then after removing a common q_1 , f satisfies

$$(27) \quad p_1 q_1^3 (aq_1 + q_2)^2 \frac{\partial f}{\partial q_1} + p_2 q_1^3 (aq_1 + q_2)^2 \frac{\partial f}{\partial q_2} + (3aq_1 + 2q_2) \frac{\partial f}{\partial p_1} + q_1 \frac{\partial f}{\partial p_2} = 0.$$

We write f in sum of its homogeneous parts as $f = \sum_{j=0}^n f_j$ where each f_j is a homogeneous polynomial of degree j and we have that $f_n \neq 0$ and $n \geq 1$. Then computing the terms of degree $n + 5$ in (27) we have that f_n satisfies

$$q_1^3 (aq_1 + q_2)^2 \left(p_1 \frac{\partial f_n}{\partial q_1} + p_2 \frac{\partial f_n}{\partial q_2} \right) = 0.$$

Solving it we obtain that f_n is of the form (24). Now computing the terms of degree n in (27) we obtain

$$(28) \quad q_1^3(aq_1 + q_2)^2 \left(p_1 \frac{\partial f_{n-5}}{\partial q_1} + p_2 \frac{\partial f_{n-5}}{\partial q_2} \right) + (3aq_1 + 2q_2) \frac{\partial f_n}{\partial p_1} + q_1 \frac{\partial f_n}{\partial p_2} = 0.$$

Evaluating (28) on $q_1 = 0$ and working as in the proof of Proposition 10 we get that $f_n = f_n(p_2)$. Now after simplifying equation (28) by q_1 and evaluating it on $q_1 = 0$ we get that $df_n/dp_2 = 0$. Hence f_n must be a constant, and thus we obtain a contradiction. \square

In the remainder of the paper we prove that system (21) with $ad \neq 0$ does not admit any polynomial first integrals.

Changing the variables $(q_1, q_2, p_1, p_2) \rightarrow (q_1, q_2, p_1, T)$, where $T = q_2p_1 - q_1p_2$, system (21) writes

$$(29) \quad \begin{aligned} \dot{q}_1 &= p_1(aq_1^3 + q_1^2q_2 + dq_2^3)^2, & \dot{q}_2 &= \frac{q_2p_1 - T}{q_1}(aq_1^3 + q_1^2q_2 + dq_2^3)^2, \\ \dot{p}_1 &= 3aq_1^2 + 2q_1q_2, & \dot{T} &= -q_1[q_1^2 - 3aq_1q_2 + (3d - 2)q_2^2]. \end{aligned}$$

If we denote by $F(q_1, q_2, p_1, p_2) \in \mathbb{C}[q_1, q_2, p_1, p_2]$ a polynomial first integral of (21), then in the variables (q_1, q_2, p_1, T) it writes

$$(30) \quad F(q_1, q_2, p_1, T) = \sum_{j=-n}^n f_j(q_2, p_1, T)q_1^j,$$

where $f_j(q_2, p_1, T) \in \mathbb{C}[q_2, p_1, T]$. We define the following differential operators that act on $f_j = f_j(q_2, p_1, T) \in \mathbb{C}[q_2, p_1, T]$:

$$\begin{aligned} \mathcal{A}[f_j] &:= jp_1f_j + (q_2p_1 - T)\frac{\partial f_j}{\partial q_2}, \\ \mathcal{B}[f_j] &:= q_2^2\mathcal{A}[f_j] - \frac{\partial f_j}{\partial T}, \\ \mathcal{C}[f_j] &:= 2dq_2^3\mathcal{A}[f_j] + 3\left(\frac{\partial f_j}{\partial p_1} + q_2\frac{\partial f_j}{\partial T}\right), \\ \mathcal{D}[f_j] &:= 2dq_2^3\mathcal{A}[f_j] + (2 - 3d)q_2\frac{\partial f_j}{\partial T} + 2\frac{\partial f_j}{\partial p_1}. \end{aligned}$$

Then by definition F is a first integral of (29) if and only if

$$(31) \quad a^2\mathcal{A}[f_{j-6}] + 2aq_2\mathcal{A}[f_{j-5}] + \mathcal{B}[f_{j-4}] + a\mathcal{C}[f_{j-3}] + q_2\mathcal{D}[f_{j-2}] + d^2q_2^6\mathcal{A}[f_j] = 0,$$

where $j = -n, \dots, n + 6$ and $f_j = 0$ if $j > n$ or $j < -n$. Thus system (21) admits a polynomial first integral if and only if there exist $2n + 1$ polynomial functions f_j satisfying system (31) of $2n + 7$ partial differential equations. The following two lemmas will address the problem of existence of such polynomial functions.

Lemma 12. *Let F be as in (30) and $ad \neq 0$. If F is first integral of (29), then $f_j(q_2, p_1, T) = 0$ for $j = 1, \dots, n$.*

Proof. Consider (31) for $j = n + 6$, that is, $a^2\mathcal{A}[f_n] = 0$. Using that $a \neq 0$, the solution is $f_n = \alpha/(T - q_2p_1)^n$, where $\alpha = \alpha(p_1, T)$. Since $f_n \in \mathbb{C}[q_2, p_1, T]$ we conclude that $f_n = 0$. If $n = 1$ we are done. If $n \geq 2$, using that $\mathcal{A}[f_n] = 0$ we get from (31) that $a^2\mathcal{A}[f_{n-1}] = 0$ and hence $f_{n-1} = 0$. If $n = 2$ we are done. If $n \geq 3$ then using that $\mathcal{B}[0] = 0$, and $f_n = f_{n-1} = 0$ the conditions in (31) imply that $a^2\mathcal{A}[f_{n-2}] = 0$. The arguments for solving $a^2\mathcal{A}[f_n] = 0$ imply that $f_{n-2} = 0$. Furthermore using that $\mathcal{C}[0] = 0$ together with (31) we get that as long as $n - 3 \geq 1$, then $f_{n-3} = 0$. Again, if $n = 4$ we are done. If $n \geq 5$, then we proceed by induction.

Assume that $f_n = f_{n-1} = \dots = f_{i+1} = 0$, where $i \geq 1$. We shall show that $f_i = 0$. Now we consider condition (31) and we get

$$(32) \quad a^2 \mathcal{A}[f_i] + 2aq_2^2 \mathcal{A}[f_{i+1}] + \mathcal{B}[f_{i+2}] + a\mathcal{C}[f_{i+3}] + q_2 \mathcal{D}[f_{i+4}] + d^2 q_2^6 \mathcal{A}[f_{i+6}] = 0.$$

Since $f_{i+6} = f_{i+4} = f_{i+3} = f_{i+2} = f_{i+1} = 0$, and using that $\mathcal{A}[0] = \mathcal{B}[0] = \mathcal{C}[0] = \mathcal{D}[0] = 0$, equation (32) reduces to $a^2 \mathcal{A}[f_i] = 0$. Since $a \neq 0$ and $i \geq 1$, the only polynomial solution of this differential equation is $f_i = 0$. This concludes the proof of the lemma. \square

Lemma 13. *Let F be as in (30) and $ad \neq 0$. If F is a first integral of (29), then $f_j(q_2, p_1, T) = 0$ for $j = -n, -n+1, \dots, -1$ and $f_0(q_2, p_1, T) = \text{constant}$.*

Proof. Consider (31) for $j = -n, -n+1$, i.e. $d^2 q_2^6 \mathcal{A}[f_{-n}] = 0$ and $d^2 q_2^6 \mathcal{A}[f_{-n+1}] = 0$. Since $d \neq 0$ this implies that $\mathcal{A}[f_{-n}] = \mathcal{A}[f_{-n+1}] = 0$ and solving it we get

$$(33) \quad f_{-n} = \alpha_{-n}(T - q_2 p_1)^n, \quad \text{and} \quad f_{-n+1} = \alpha_{-n+1}(T - q_2 p_1)^{n-1},$$

where $\alpha_{-n} = \alpha_{-n}(p_1, T)$ and $\alpha_{-n+1} = \alpha_{-n+1}(p_1, T)$ are polynomials. Now we consider (31) for $j = -n+2$, that is, $q_2 \mathcal{D}[f_{-n}] + d^2 q_2^6 \mathcal{A}[f_{-n+2}] = 0$ and using (33) we get

$$f_{-n+2} = (\alpha_{-n+2} + \beta_{-n+2})(T - q_2 p_1)^{n-2},$$

where $\alpha_{-n+2} = \alpha_{-n+2}(p_1, T)$ is an integration constant and

$$\beta_{-n+2} = \frac{1}{6d^2 q_2^4} \tilde{\beta}_{-n+2},$$

where

$$\tilde{\beta}_{-n+2} = 6ndq_2 \alpha_{-n} + (3d-2)q_2(2T-3q_2 p_1) \frac{\partial \alpha_{-n}}{\partial T} + (4q_2 p_1 - 3T) \frac{\partial \alpha_{-n}}{\partial p_1}.$$

Since α_{-n} is a polynomial, $\tilde{\beta}_{-n+2}$ is also a polynomial. We note that $\tilde{\beta}_{-n+2}$ is a polynomial if and only β_{-n+2} is divisible by q_2^4 . This is possible if and only if $\tilde{\beta}_{-n+2} = 0$ or equivalently $\alpha_{-n} = 0$, and from (33) $f_{-n} = 0$. Working with f_{-n+1} similarly as with f_{-n} we obtain

$$f_{-n+3} = (\alpha_{-n+3} + \beta_{-n+3})(q_2 p_1 - T)^{n-3},$$

where $\alpha_{-n+3} = \alpha_{-n+3}(p_1, T)$ and

$$\beta_{-n+3} = \frac{1}{6d^2 q_2^4} \tilde{\beta}_{-n+3},$$

where $\tilde{\beta}_{-n+3}$ is

$$6(n-1)dq_2 \alpha_{-n+1} + (3d-2)q_2(2T-3q_2 p_1) \frac{\partial \alpha_{-n+1}}{\partial T} + (4q_2 p_1 - 3T) \frac{\partial \alpha_{-n+1}}{\partial p_1}.$$

As in the previous case we conclude that $\alpha_{-n+1} = 0$. Following exactly the same steps we can prove that $f_{-n+3} = \dots = f_{-3} = 0$ and $f_{-2} = (T - q_2 p_1)^2 \alpha_{-2}$, $f_{-1} = (T - q_2 p_1) \alpha_{-1}$, where again $\alpha_{-2} = \alpha_{-2}(p_1, T)$ and $\alpha_{-1} = \alpha_{-1}(p_1, T)$ are polynomials.

We claim that $f_0(q_2, p_1, T) = f_0(p_1, T)$. To prove our claim recall that by Lemma 12 we have $f_j = 0$ for $j \geq 1$. Now the condition (31) for $j = 6$ becomes $\mathcal{A}[f_0] = 0$, so $\partial f(q_2, p_1, T)/\partial q_2 = 0$, which proves our claim.

In the rest of the proof we show that $f_{-1} = f_{-2} = 0$ and $f_0 = \text{constant}$. Consider (31) for $j = 0$, that is $q_2 \mathcal{D}[f_{-2}] = 0$. Since $f_{-2} = (T - q_2 p_1)^2 \alpha_{-2}$ the above condition takes the simplified form

$$\mathcal{D}[(T - q_2 p_1)^2 \alpha_{-2}(p_1, T)] = 0.$$

The general solution of this partial differential equation is

$$\alpha_{-2}(p_1, T) = \frac{g(T - q_2 p_1 + 3dq_2 p_1/2)}{4(T - q_2 p_1)^2},$$

where g is a function of the variable $T - q_2 p_1 + 3dq_2 p_1/2$. Since α_{-2} does not depend on q_2 and $d \neq 0$, we obtain that $\alpha_{-2} = 0$. So, $f_{-2} = 0$.

Considering (31) for $j = 1$ we get $\mathcal{D}[(T - q_2 p_1)\alpha_{-1}(p_1, T)] = 0$. Its general solution is

$$\alpha_{-1}(p_1, T) = \frac{g(T - q_2 p_1 + 3dq_2 p_1/2)}{2(T - q_2 p_1)}.$$

As before we get $\alpha_{-1} = 0$, and consequently, $f_{-1} = 0$.

Taking into account Lemma 12 and that $f_{-4} = \dots = f_{-1} = 0$, the condition (31) for $j = 2$ writes, after simplifying by q_2 ,

$$(2 - 3d)q_2 \frac{\partial f_0}{\partial T} + 2 \frac{\partial f_0}{\partial p_1} = 0.$$

Solving this equation and using that f_0 does not depend on q_2 , we obtain first that $\partial f_0 / \partial T = 0$, and then $\partial f_0 / \partial p_1 = 0$. So $f_0 = \text{constant}$. This concludes the proof of the lemma. \square

Proposition 14. *The Hamiltonian system (21) with $ad \neq 0$ has no polynomial first integrals.*

Proof. It follows immediately from Lemmas 12 and 13. \square

This completes the proof of statement (a) of Theorem 2.

APPENDIX A. RESIDUE THEOREM

Theorem 15 (Residue theorem). *Let $f(z)$ be a holomorphic function everywhere on the plane $\mathbb{C} \setminus \{z_1, \dots, z_n\}$, where $z_j \in \mathbb{C}$, $j = 1, \dots, n$. Then*

$$\sum_{j=1}^n \text{res}_{z_j} f + \text{res}_{\infty} f = 0,$$

where res_{z_j} is the residue $f(z)$ at z_j and res_{∞} is the residue of $f(z)$ at infinity.

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