On s-elementary Super Frame Wavelets and their Path-connectedness

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Abstract

A super wavelet of length n is an n-tuple $(\psi_1, \psi_2, ..., \psi_n)$ in the product space $\prod_{j=1}^n L^2(\mathbb{R})$, such that the *coordinated dilates* of all its *coordinated translates* form an orthonormal basis for $\prod_{j=1}^n L^2(\mathbb{R})$. This concept is generalized to the so-called super frame wavelets, super tight frame wavelets and super normalized tight frame wavelets (or super Parseval frame wavelets), namely an n-tuple $(\eta_1, \eta_2, ..., \eta_n)$ in $\prod_{j=1}^n L^2(\mathbb{R})$ such that the coordinated dilates of all its coordinated translates form a frame, a tight frame, or a normalized tight frame for $\prod_{j=1}^n L^2(\mathbb{R})$. In this paper, we study the super frame wavelets and the super tight frame wavelets whose Fourier transforms are defined by set theoretical functions (called s-elementary frame wavelets). An n-tuple of sets $(E_1, E_2, ..., E_m)$ is said to be τ-disjoint if the E_j's are pair-wise disjoint under the 2π-translations. We prove that a τ-disjoint n-tuple $(E_1, E_2, ..., E_m)$ of frame sets (i.e., η_j defined by $\widehat{\eta_j} = \frac{1}{\sqrt{2\pi}}\chi_{E_j}$ is a frame wavelet for $L^2(\mathbb{R})$ for each j) lead to a super frame wavelet $(\eta_1, \eta_2, ..., \eta_m)$ for $\prod_{j=1}^n L^2(\mathbb{R})$ where $\widehat{\eta_j} = \frac{1}{\sqrt{2\pi}}\chi_{E_j}$. In the case of super tight frame wavelets, we prove that $(\eta_1, \eta_2, ..., \eta_m)$, defined by $\widehat{\eta_j} = \frac{1}{\sqrt{2\pi}}\chi_{E_j}$, is a super tight frame wavelet for $\prod_{1\le j\le m} L^2(\mathbb{R})$ with frame bound k_0 if and only if each η_j is a tight frame wavelet for $L^2(\mathbb{R})$ with frame bound k_0 and that $(E_1, E_2, ..., E_m)$ is τ-disjoint. Denote the set of all τ-disjoint s-elementary super frame wavelets for $\prod_{1\le j\le m} L^2(\mathbb{R})$ by $\mathfrak{S}(m)$ and the set of all s-elementary super tight frame wavelets (with the same frame bound k_0) for $\prod_{1\le j\le m} L^2(\mathbb{R})$ by $\mathfrak{S}^{k_0}(m)$. We further prove that $\mathfrak{S}(m)$ and $\mathfrak{S}^{k_0}(m)$ are both path-connected under the $\prod_{1\le j\le m} L^2(\mathbb{R})$ norm, for any given positive integers m and k_0.

1 Introduction

Frame wavelets in the space $L^2(\mathbb{R})$ or $L^2(\mathbb{R}^d)$ have received much attention from mathematicians and scientists in recent years, motivated in part by their potentials in applications. In this paper, we are interested in frame wavelets in the product space $\prod_{1\le j\le m} L^2(\mathbb{R})$ instead. Let us begin by introducing the concept of a frame under the most general setting, i.e., a frame for a Hilbert space. Let H be a Hilbert space. A sequence $\{x_n\}$ in H is called a *frame* for H if there exist constants $C_1, C_2 > 0$ such that

$$C_1\|x\|^2 \le \sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \le C_2\|x\|^2, \forall x \in H.$$

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If $C_1 = C_2 = C$, $\{x_n\}$ is called a *tight frame* and the constant C is called the frame bound for $\{x_n\}$. In particular, if $C_1 = C_2 = 1$, then $\{x_n\}$ is called a *normalized tight frame*, or a *Parseval frame*. In the case that $H = L^2(\mathbb{R})$ and the sequence considered is of the form

$$\{D^k T^\ell \psi : k, \ell \in \mathbb{Z}\} = \{2^{\frac{k}{2}} \psi(2^k t - \ell : k, \ell \in \mathbb{Z}\}, \tag{1.1}$$

where $\psi(t) \in L^2(\mathbb{R})$, T and D are the translation and dilation unitary operators acting on $L^2(\mathbb{R})$ defined by

$$(Tf)(t) = f(t-1) \text{ and } \mathrm{(Df)(t) = \sqrt{2}f(2t),\ \forall f \in L^2(\mathbb{R})}, \tag{1.2}$$

then ψ is called a *frame wavelet* of $L^2(\mathbb{R})$ if sequence (1.1) is a frame of $L^2(\mathbb{R})$, i.e.,

$$C_1 \|f\|^2 \leq \sum_{n,\ell \in \mathbb{Z}} |\langle \{D^n T^\ell \psi, f\rangle|^2 \leq C_2 \|f\|^2, \ \forall f \in L^(\mathbb{R}^2), \tag{1.3}$$

Similarly, ψ is called a *tight frame wavelet* of $L^2(\mathbb{R})$ if (1.1) is a tight frame, and a *normalized tight frame wavelet* (or Parseval frame wavelet) for $L^2(\mathbb{R})$ if (1.1) is a normalized tight frame. In the special case that (1.1) forms an orthonormal basis for $L^2(\mathbb{R})$, ψ is simply called a wavelet for $L^2(\mathbb{R})$. If a frame, a tight frame, or a normalized tight frame wavelet ψ has the property $\widehat{\psi} = \frac{1}{\sqrt{2\pi}} \chi_E$ for some measurable set $E \subset \mathbb{R}$, then ψ is called an s-elementary frame, tight frame, or normalized tight frame wavelet respectively.

In this paper, we are interested in the case where the Hilbert space is the product (or direct sum) space $\prod_{1 \leq j \leq m} L^2(\mathbb{R})$ (or $\bigoplus_{1 \leq j \leq m} L^2(\mathbb{R})$) and the frame sequence is generated by a single element in the space via coordinated dilates and translates. More precisely, we are interested in the case when the frame of $\prod_{1 \leq j \leq m} L^2(\mathbb{R})$ is given by

$$\{D^k T^\ell (\eta_1, \eta_2, ..., \eta_m) : k, \ell \in \mathbb{Z}\} = \{(D^k T^\ell \eta_1, ..., D^k T^\ell \eta_m) : k, \ell \in \mathbb{Z}\} \tag{1.4}$$

where $(\eta_1, \eta_2, ..., \eta_m) \in \prod_{1 \leq j \leq m} L^2(\mathbb{R})$ is a fixed element. Following the terminology used before (as in [12]) and with a stress on the difference of the underline space, we will then call $(\eta_1, \eta_2, ..., \eta_m)$ a *super frame wavelet* (of length m) for $\prod_{1 \leq j \leq m} L^2(\mathbb{R})$. Similarly, $(\eta_1, \eta_2, ..., \eta_m)$ is called a *super tight frame wavelet* (of length m) for $\prod_{1 \leq j \leq m} L^2(\mathbb{R})$ if $\{D^k T^\ell (\eta_1, \eta_2, ..., \eta_m) : k, \ell \in \mathbb{Z}\} = \{(D^k T^\ell \eta_1, ..., D^k T^\ell \eta_m) : k, \ell \in \mathbb{Z}\}$ is a tight frame for $\prod_{j=1}^n L^2(\mathbb{R})$. In the special case that the tight frame bound is one, $(\eta_1, \eta_2, ..., \eta_m)$ is called a *normalized super tight frame wavelet*, or a*super Parseval frame wavelet*. The concept of the super Parseval frame wavelets was first introduced and studied in [12]. Notice that the notation $\bigoplus_{1 \leq j \leq m} L^2(\mathbb{R})$ is used in [12] instead of $\prod_{1 \leq j \leq m} L^2(\mathbb{R})$ (but they are equivalent since m is finite). See the remark following [12, Definition 5.5]. Notice that this is very different from the usual Parseval frame for $\prod_{1 \leq j \leq m} L^2(\mathbb{R})$ induced by the individual Parseval frame wavelets η_j, which is of the form $(0, ..., 0, D^k T^\ell \eta_j, 0, ..., 0)$ $(1 \leq j \leq n,\ k, \ell \in \mathbb{Z})$. Interested reader may refer to [1, 9, 10, 11] for more detailed discussions on this topic.

Our main goal in this paper is to study the special class of super (tight) frame wavelets whose Fourier transforms are defined by set theoretical functions. Following the terminologies used in prior publications concerning these subjects, we will call them s-elementary super (tight) frame wavelets. In the next section, we will lay out some basic definitions and state some known results needed for later sections. In Section 3, we will first introduce the concept of τ-disjoint super wavelet sets, and then prove that any m-tuple of such sets define an s-elementary super frame wavelet. This result then helps us to give a characterization of the super tight frame wavelets. In Section 4, we will prove the path-connectedness of the set of all τ-disjoint s-elementary super frame wavelets and the set of all s-elementary super tight frame wavelets, using the results obtained from Section 3.

2 Terminologies and prior results

Let E be a measurable set. $x, y \in E$ are δ-equivalent if $x = 2^n y$ for some integer n. The δ-index of a point x in E is the number of elements in its δ-equivalent class and denoted by $\delta_E(x)$. Let $E(\delta, k) = \{x \in E : \delta_E(x) = k\}$, then E is the disjoint union of the sets $E(\delta, k)$. Furthermore, each $E(\delta, k)\,(k \geq 1)$ is Lebesgue measurable and is a disjoint union of k measurable sets $\{E^j(\delta, k)\}$, $1 \leq j \leq k$, such that $E^j(\delta, k) \sim E^{j'}(\delta, k)$ for any $1 \leq j, j' \leq k$. If we let $\Delta(E) = \cup_{n\in\mathbb{Z}} 2^n E$, then we have $\Delta(E) = \Delta(\cup_{k\geq 1} E^1(\delta, k))$. A set E is called a 2-*dilation generator* of $\mathbb{R}$ if $E = E(\delta, 1)$ and $\Delta(E) = \mathbb{R}$.

In the case of translation, we say that $x, y \in E$ are 2π-*translation equivalent* if $x = y + 2n\pi$ for some integer n. The $\tau-$index of a point x in E is the number of elements in its 2π-translation equivalent class and denoted by $\tau_E(x)$. Let $E(\tau, k) = \{x \in E : \tau_E(x) = k\}$. Then E is the disjoint union of the (measurable) sets $E(\tau, k)$. Define $\tau(E) = \cup_{n\in\mathbb{Z}}(E \cap ([2n\pi, 2(n+1)\pi) - 2n\pi))$ and $\mathcal{T}(E) = \cup_{n\in\mathbb{Z}}(E - 2n\pi)$. $\tau(E)$ is a measurable subset of $[0, 2\pi)$ while $\mathcal{T}(E)$ is measurable with infinite measure whenever $\mu(E) > 0$. $\tau(E)$ is a disjoint union if and only if $E = E(\tau, 1)$. When $\tau(E)$ is a disjoint union, we say that E is translation equivalent to $\tau(E)$, which is a subset of $[0, 2\pi)$. If E and F are $2\pi-$translation equivalent to the same subset in $[0, 2\pi)$, then we say that E and F are *translation equivalent*. Finally, each $E(\tau, k)$ is a disjoint union of k measurable sets $\{E^{(j)}(\tau, k)\}$, $1 \leq j \leq k$, such that $E^{(j)}(\tau, k) \sim E^{(j')}(\tau, k)$ for any $1 \leq j, j' \leq k$.

The decompositions of $E(\delta, k)$ (resp. $E(\tau, k)$) into $E^j(\delta, k)$ (resp. $E^{(j)}(\tau, k)$) are not unique in general except the case of $k = 1$, see [5].

Let $\mathcal{F}$ be the Fourier-Plancherel transform defined by

$$\widehat{f}(s) = (\mathcal{F}f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t)dt, \ \forall f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}). \tag{2.1}$$

The Fourier transformation defined here is normalized so that it is a unitary operator. Throughout this paper, we use $\widehat{\psi}$ to denote the Fourier transformation of ψ. We will also use $\widehat{D}$, $\widehat{T}$ for the product $\mathcal{F}D\mathcal{F}^{-1}$ and $\mathcal{F}T\mathcal{F}^{-1}$. It is known that $\widehat{D} = D^{-1}$, $\widehat{T}f(t) = e^{-ist}f(t)$ and that (1.3) is equivalent to

$$C_1\|f\|^2 \leq \sum_{n,\ell\in\mathbb{Z}} |\langle f, \widehat{D}^n \widehat{T}^\ell \widehat{\psi}\rangle|^2 \leq C_2\|f\|^2, \ \forall f \in L^2(\mathbb{R}) \tag{2.2}$$

[7]. Throughout this paper, we will be working with (2.2) instead of (1.3).

A function ψ is called an *s-elementary frame wavelet* (for $L^2(\mathbb{R})$) if it is a frame wavelet and is defined by $\hat{\psi} = \frac{1}{\sqrt{2\pi}}\chi_E$ for some measurable set E. In this case E is also called a *frame wavelet set*. Similarly, ψ is called an *s-elementary tight frame wavelet* (for $L^2(\mathbb{R})$) if it is a tight frame wavelet and is defined by $\hat{\psi} = \frac{1}{\sqrt{2\pi}}\chi_E$ for some measurable set E. In this case E is called a *tight frame wavelet set*. In the special case that the frame bounds equal to one, ψ is called an *s-elementary normalized tight frame wavelet*, or an *s-elementary Parseval frame wavelet*.

Definition 2.1 *A measurable set E is called a* basic *set if there exists $M > 0$ such that $E(\delta, k) = E(\tau, k) = \emptyset$ for all $k \geq M$.*

For a measurable set E and any $f \in L^2(\mathbb{R})$, define

$$H_E^k(f) = \sum_{\ell\in\mathbb{Z}} \langle f, \widehat{D}^k \widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_E\rangle \widehat{D}^k \widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_E$$

and

$$H_E(f) = \sum_{k\in\mathbb{Z}} H_E^k(f).$$

Lemma 2.2 [6] *Let E be a basic set. For any $f \in L^2(\mathbb{R})$, define f^k_{mj} to be the $2^{k+1}\pi$ periodical extension of $f \cdot \chi_{2^k E^{(j)}(\tau,m)}$ over $\mathbb{R}$. Then $H^k_{E(\tau,m)} f$ converges to $\sum_{j=1}^m f^k_{mj} \cdot \chi_{E(\tau,m)}$ and $H_E f$ converges to $\sum_{k\in\mathbb{Z}} \sum_{j=1}^m f^k_{mj} \cdot \chi_{E(\tau,m)}$ under the $L^2(\mathbb{R})$ norm. Furthermore,*

$$\sum_{\ell\in\mathbb{Z}} |\langle f, \widehat{D}^k \widehat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_E \rangle|^2 = \langle f, H^k_E f\rangle \geq 0$$

for any $k \in \mathbb{Z}$ and

$$\sum_{k,\ell\in\mathbb{Z}} |\langle f, \widehat{D}^k \widehat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_E \rangle|^2 = \langle f, H_E f\rangle.$$

Lemma 2.3 [6] *If E and F are two basic sets such that $\tau(E) \cap \tau(F) = \emptyset$, then for any $f \in L^2(\mathbb{R})$ and any $k \in \mathbb{Z}$,*

$$\sum_{\ell\in\mathbb{Z}} \langle f, \widehat{D}^k \widehat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_E \rangle \widehat{D}^k \widehat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_F = 0 \ a.e.$$

It follows that

$$\sum_{k,\ell\in\mathbb{Z}} \langle f, \widehat{D}^k \widehat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_E \rangle \widehat{D}^k \widehat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_F = 0 \ a.e.$$

as well.

Theorem 2.4 [6] *Let E be a Lebesgue measurable set with finite measure. Then E is a frame wavelet set if (i) $\cup_{n\in\mathbb{Z}} 2^n E(\tau, 1) = \mathbb{R}$ and (ii) There exists $M > 0$ such that $\mu(E(\delta, m)) = 0$ and $\mu(E(\tau, m)) = 0$ for any $m > M$.*

Furthermore, in this case, the lower frame bound is at least 1, and the upper frame bound is at most $M^{\frac{5}{2}}$.

Theorem 2.5 [6] *Let E be a Lebesgue measurable set with finite measure. Then E is a tight frame set if and only if $E = E(\tau, 1) = E(\delta, k)$ for some $k \geq 1$ and $\cup_{n\in\mathbb{Z}} 2^n E = \mathbb{R}$. Furthermore, the frame bound of the corresponding tight frame wavelet η defined by $\hat{\eta} = \frac{1}{\sqrt{2\pi}} \chi_E$ equals k.*

Non-tight frame wavelets can be easily constructed using Theorem 2.4. However the lower frame bound of a frame wavelet so constructed is always 1. Examples of s-elementary frame wavelets with non-integer lower frame bounds can be found in [6].

3 τ-disjoint s-elementary super frame wavelets

In this section, we introduce the concept of *τ-disjoint s-elementary super frame wavelets.* We then show that s-elementary super tight frame wavelets are necessarily τ-disjoint. Thus we are able to give a characterization of the s-elementary super tight frame wavelets.

Theorem 3.1 *Let E_1, E_2, ..., E_m be measurable sets in $\mathbb{R}$ with finite measures and let η_1, η_2, ..., η_m be the functions defined by $\hat{\eta}_j = \frac{1}{\sqrt{2\pi}} \chi_{E_j}$, then $\{\eta_1, \eta_2, ..., \eta_m\}$ is a super frame wavelet (of length m) for $\prod_{1\leq j\leq m} L^2(\mathbb{R})$ if the following conditions hold:*
(1) Each E_j is a frame wavelet set in $\mathbb{R}$ $(1 \leq j \leq m)$;
(2) $\tau(E_i) \cap \tau(E_j) = \emptyset$ if $i \neq j$.

Furthermore, the lower frame bound of $\{\eta_1, \eta_2, ..., \eta_m\}$ is $a = \min\{a_1, a_2, ..., a_m\}$ and the upper frame bound of $\{\eta_1, \eta_2, ..., \eta_m\}$ is $b = \max\{b_1, b_2, ..., b_m\}$ where a_j, b_j are the lower and upper frame bounds of η_j respectively.

An s-elementary super frame wavelet defined by the sets $\{E_1, E_2, ..., E_m\}$ satisfying the condition of Theorem 3.1 is called a *τ-disjoint s-elementary super frame wavelet.*

Proof. For any $(f_1, f_2, ..., f_m) \in \prod_{1\leq j\leq m} L^2(\mathbb{R})$, define

$$H_E(f_1, f_2, ..., f_m) = \sum_{k,\ell\in\mathbb{Z}} \langle (f_1, f_2, ..., f_m), \widehat{D}^k\widehat{T}^\ell(\hat{\eta_1}, \hat{\eta_2}, ..., \hat{\eta_m})\rangle \widehat{D}^k\widehat{T}^\ell(\hat{\eta_1}, \hat{\eta_2}, ..., \hat{\eta_m}).$$

The i-th entry of $H_E(f_1, f_2, ..., f_m)$ is

$$\begin{aligned} & \sum_{k,\ell\in\mathbb{Z}} \left(\sum_{1\leq j\leq m} \langle f_j, \widehat{D}^k\widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_{E_j}\rangle \right) \widehat{D}^k\widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_{E_i} \\ = & \sum_{k,\ell\in\mathbb{Z}} \langle f_1, \widehat{D}^k\widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_{E_1}\rangle \widehat{D}^k\widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_{E_i} \\ + & \sum_{k,\ell\in\mathbb{Z}} \langle f_2, \widehat{D}^k\widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_{E_2}\rangle \widehat{D}^k\widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_{E_i} + \cdots \\ + & \sum_{k,\ell\in\mathbb{Z}} \langle \widehat{D}^k\widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_{E_m}, f_m\rangle \widehat{D}^k\widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_{E_i}. \end{aligned}$$

Since $\tau(E_j) \cap \tau(E_i) = \emptyset$ when $i \neq j$, by Lemma 2.3 we have

$$\sum_{k,\ell\in\mathbb{Z}} \langle f_j, \widehat{D}^k\widehat{T}^\ell\hat{\eta_j}\rangle \widehat{D}^k\widehat{T}^\ell\eta_i = 0$$

when $j \neq i$. Thus

$$H_E(f_1, f_2, ..., f_m) = (H_{E_1}f_1, H_{E_2}f_2, ..., H_{E_m}f_m).$$

and

$$\sum_{1\leq j\leq m} a_j\|f_j\|^2 \leq \langle (f_1, f_2, ..., f_m), H_E(f_1, f_2, ..., f_m)\rangle \leq \sum_{1\leq j\leq m} b_j\|f_j\|^2$$

by Lemma 2.2. Since

$$\sum_{1\leq j\leq m} a_j\|f_j\|^2 \geq a \sum_{1\leq j\leq m} \|f_j\|^2 = a\|(f_1, f_2, ..., f_m)\|^2,$$

$$\sum_{1\leq j\leq m} a_j\|f_j\|^2 \leq b \sum_{1\leq j\leq m} \|f_j\|^2 = b\|(f_1, f_2, ..., f_m)\|^2,$$

and

$$\sum_{k,\ell\in\mathbb{Z}} \langle (f_1, f_2, ..., f_m), \widehat{D}^k\widehat{T}^\ell(\hat{\eta_1}, \hat{\eta_2}, ..., \hat{\eta_m})\rangle = |\langle (f_1, f_2, ..., f_m), H_E(f_1, f_2, ..., f_m)\rangle\|^2,$$

the conclusion of the theorem follows. ■

Unfortunately the following theorem shows that the converse of Theorem 3.1 is not true in general because the second condition fails.

Theorem 3.2 *Let E_1, E_2, ..., E_m be measurable sets in $\mathbb{R}$ with finite measures and let η_1, η_2, ..., η_m be defined by $\hat{\eta_j} = \frac{1}{\sqrt{2\pi}}\chi_{E_j}$. If $\{\eta_1, \eta_2, ..., \eta_m\}$ is a super frame wavelet of length m for $\prod_{1\leq j\leq m} L^2(\mathbb{R})$, then each η_j is a frame wavelet for $L^2(\mathbb{R})$ $(1 \leq j \leq m)$. However, the condition that $\tau(E_i) \cap \tau(E_j) = \emptyset$ if $i \neq j$ does not necessarily hold.*

Proof. The first claim of the theorem is obvious by considering the elements of the form $(0,...,0,f_j,0,...,0)$. For the second part, it suffices to give a counter example for the case of $m=2$.

Let $E_1=[-\pi,-\pi/4)\cup[\pi/4,\pi)$, $F_1=[-\pi,-\pi/2)$, $E_2=[-\pi/4,-\pi/16)\cup[\pi/16,\pi/4)\cup[\pi,3\pi/2)$, $F_2=[\pi,3\pi/2)$. Then $E_1=E_1(\tau,1)=E_1(\delta,2)$, $E_2=E_2(\tau_1,1)=E_2(\delta,2)\cup E_2(\delta,3)$. It follows that η_1 defined by $\widehat{\eta_1}=\frac{1}{\sqrt{2\pi}}\chi_{E_1}$ is a tight frame wavelet with frame bound 2 and η_2 defined by $\widehat{\eta_2}=\frac{1}{\sqrt{2\pi}}\chi_{E_2}$ is a frame wavelet with frame bounds 2 and 3. Notice that $F_2=F_1+2\pi$ and $\tau(E_1\backslash F_1)\cap\tau(E_2\backslash F_2)=\emptyset$. We claim that (η_1,η_2) is a super frame wavelet for $L^2(\mathbb{R})\times L^2(\mathbb{R})$.

Let $(f_1,f_2)\in L^2(\mathbb{R})\times L^2(\mathbb{R})$. The first coordinate of $H_E(f_1,f_2)$ is

$$\begin{aligned}
&\sum_{k,\ell\in\mathbb{Z}}\left(\langle f_1,\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{E_1}\rangle+\langle f_2,\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{E_2}\rangle\right)\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{E_1}\\
=\;& H_{E_1}f_1+\sum_{k,\ell\in\mathbb{Z}}\langle f_2,\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{E_2}\rangle\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{E_1}\\
=\;& H_{E_1}f_1+\sum_{k,\ell\in\mathbb{Z}}\langle f_2,\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{F_2}\rangle\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{F_1}\\
=\;& H_{E_1}f_1+\sum_{k,\ell\in\mathbb{Z}}\left(\frac{1}{\sqrt{2\pi}}\int f_2(s)\chi_{F_2}(\frac{s}{2^k})e^{-\frac{i\ell s}{2^k}}ds\right)\frac{1}{\sqrt{2\pi}}\chi_{F_1}(\frac{s}{2^k})e^{-\frac{i\ell s}{2^k}}.
\end{aligned}$$

Let $\frac{s}{2^k}=\frac{s}{2^k}-2\pi$, we get

$$\begin{aligned}
&\int f_2(s)\chi_{F_2}(\frac{s}{2^k})e^{-\frac{i\ell s}{2^k}}ds\\
=\;&\int f_2(u+2^k2\pi)\chi_{F_1}(\frac{u}{2^k})e^{-\frac{i\ell u}{2^k}}du\\
=\;&\int f_2(s+2^k2\pi)\chi_{F_1}(\frac{s}{2^k})e^{-\frac{i\ell s}{2^k}}ds.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{\ell\in\mathbb{Z}}\left(\frac{1}{\sqrt{2\pi}}\int f_2(s)\chi_{F_2}(\frac{s}{2^k})e^{-\frac{i\ell s}{2^k}}ds\right)\frac{1}{\sqrt{2\pi}}\chi_{F_1}(\frac{s}{2^k})e^{-\frac{i\ell s}{2^k}}\\
=\;&\sum_{\ell\in\mathbb{Z}}\left(\frac{1}{\sqrt{2\pi}}\int f_2(s+2^k2\pi)\chi_{F_1}(\frac{s}{2^k})e^{-\frac{i\ell s}{2^k}}ds\right)\frac{1}{\sqrt{2\pi}}\chi_{F_1}(\frac{s}{2^k})e^{-\frac{i\ell s}{2^k}}\\
=\;&H^k_{F_1}(f_2(s+2^k2\pi))\\
=\;&f_2(s+2^k2\pi)\chi_{2^kF_1}(s),
\end{aligned}$$

by Lemmas 2.2 and 2.3. Since $f_2(s+2^k2\pi)\chi_{2^kF_1}(s)$ is really just the $2^k2\pi$ translation of f_2 from 2^kF_2 to 2^kF_1 and $F_1=F_1(\delta,1)$, $F_2=F_2(\delta,1)$, it follows that

$$\sum_{k,\ell\in\mathbb{Z}}\left(\frac{1}{\sqrt{2\pi}}\int f_2(s)\chi_{F_2}(\frac{s}{2^k})e^{-\frac{i\ell s}{2^k}}ds\right)\frac{1}{\sqrt{2\pi}}\chi_{F_1}(\frac{s}{2^k})e^{-\frac{i\ell s}{2^k}}$$

is a translation of $f_2\chi_{\Delta(F_2)}$ from $\chi_{\Delta(F_2)}$ to $\chi_{\Delta(F_1)}$. Denote this function by f_2'. Observe that $\Delta(F_2)\subseteq(0,\infty)$, then $\|f_2'\|^2\leq\int_0^\infty|f_2(s)|^2ds\leq\|f_2\|^2$. It follows that

$$\langle f_1,H_{E_1}f_1+f_2'\rangle\geq 2\|f_1\|^2-\frac{1}{2}(\|f_1\|^2+\|f_2\|^2).$$

Similarly, we can show that the inner product of the second coordinate of $H_E(f_1, f_2)$ with f_2 is bounded below by $2\|f_2\|^2 - \frac{1}{2}(\|f_1\|^2 + \|f_2\|^2)$. Combining the two inequalities we obtain

$$\langle (f_1, f_2), H_E(f_1, f_2)\rangle \geq \|f_1\|^2 + \|f_2\|^2 = \|(f_1, f_2)\|^2.$$

We leave it to our reader to prove the inequality $\langle (f_1, f_2), H_E(f_1, f_2)\rangle \leq 3\|(f_1, f_2)\|^2$. Thus we have shown that (η_1, η_2) so constructed is a super frame wavelet (of length 2) with frame lower and upper bounds 1 and 3 respectively. ■

With a slight modification to the proof of Theorem 3.1 and Theorem 2.5, it is now easy to see the following theorem. We leave the details to the reader.

Theorem 3.3 *Let E_1, E_2, ..., E_m be measurable sets in $\mathbb{R}$ with finite measures and let η_1, η_2, ..., η_m be defined by $\hat{\eta_j} = \frac{1}{\sqrt{2\pi}}\chi_{E_j}$, then $\{\eta_1, \eta_2, ..., \eta_m\}$ is a super tight frame wavelet of length m and frame bound k_0 for $\prod_{1\leq j\leq m} L^2(\mathbb{R})$ if the following conditions hold:*
(1) Each η_j is a tight frame wavelet for $L^2(\mathbb{R})$ $(1 \leq j \leq m)$ of frame bound k_0;
(2) $\tau(E_i) \cap \tau(E_j) = \emptyset$ if $i \neq j$.

Furthermore, in this case the frame bound of $\{\eta_1, \eta_2, ..., \eta_m\}$ is the same as the frame bound k_0 of each η_j (which is a positive integer by Theorem 2.5).

However, in the case of super tight frame wavelet, the converse of the above theorem is also true. We state it as the following theorem.

Theorem 3.4 *Let E_1, E_2, ..., E_m be measurable sets in $\mathbb{R}$ with finite measures and let η_1, η_2, ..., η_m be defined by $\hat{\eta_j} = \frac{1}{\sqrt{2\pi}}\chi_{E_j}$, then $\{\eta_1, \eta_2, ..., \eta_m\}$ is a super tight frame wavelet of length m and frame bound k_0 for $\prod_{1\leq j\leq m} L^2(\mathbb{R})$ only if the following conditions hold:*
(1) Each η_j is a tight frame wavelet for $L^2(\mathbb{R})$ $(1 \leq j \leq m)$ of frame bound k_0;
(2) $\tau(E_i) \cap \tau(E_j) = \emptyset$ if $i \neq j$.

Proof. Considering functions of the form $(0, ..., 0, f_j, 0, ..., 0)$. We have $\|(0, ..., 0, f_j, 0, ..., 0)\|^2 = \|f_j\|^2$ so

$$\begin{aligned} k_0\|f_j\|^2 &= k_0\|(0, ..., 0, f_j, 0, ..., 0)\|^2 \\ &= \sum_{k,\ell\in\mathbb{Z}} |\langle (0, ..., 0, f_j, 0, ..., 0), \widehat{D}^k\widehat{T}^\ell(\hat{\eta_1}, \hat{\eta_2}, ..., \hat{\eta_m})\rangle|^2 \\ &= \sum_{k,\ell\in\mathbb{Z}} |\langle f_j, \widehat{D}^k\widehat{T}^\ell\hat{\eta_j}\rangle|^2, \end{aligned}$$

for any $f_j \in L^2(\mathbb{R})$. Thus each E_j is a tight frame wavelet set with frame bound k_0 and condition (1) holds.

Assume that condition (2) does not hold. Without loss of generality, assume that $F = \tau(E_1) \cap \tau(E_2)$ is of positive measure. Notice that from (a) we know that $E_1 = E_1(\tau, 1)$, $E_1 = E_1(\delta, k_0)$ and $E_2 = E_2(\tau, 1)$, $E_2 = E_2(\delta, k_0)$. Let $F_1 = \mathcal{T}(F) \cap E_1$, $F_2 = \mathcal{T}(F) \cap E_2$ and $f_1 = \chi_{F_1}$, $f_2 = -\chi_{F_2}$. Notice that there exists a mapping ρ from F_2 to F_1 such that ρ is measurable and that for each $s \in F_1$, $\rho(s) = s + 2q\pi$ for some integer q. By [6, Lemma 6], we have $H^k_{E_1}(f_1) = f_1\chi_{2^kE_1}$ and $H^k_{E_2}(f_2) = f_2\chi_{2^kE_2}$. Now consider

$$H_E(f_1, f_2, 0, ..., 0) = \sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}}\langle (f_1, f_2, 0, ..., 0)\widehat{D}^k\widehat{T}^\ell(\hat{\eta_1}, \hat{\eta_2}, ..., \hat{\eta_m})\rangle\widehat{D}^k\widehat{T}^\ell(\hat{\eta_1}, \hat{\eta_2}, ..., \hat{\eta_m}).$$

The first coordinate in the summation

$$\sum_{\ell\in\mathbb{Z}}\langle(f_1,f_2,0,...,0),\widehat{D}^k\widehat{T}^\ell(\hat{\eta_1},\hat{\eta_2},...,\hat{\eta_m})\rangle\widehat{D}^k\widehat{T}^\ell(\hat{\eta_1},\hat{\eta_2},...,\hat{\eta_m})$$

is

$$\sum_{\ell\in\mathbb{Z}}\left(\langle f_1,\widehat{D}^k\widehat{T}^\ell(\frac{1}{\sqrt{2\pi}}\chi_{E_1}\rangle+\langle f_2,\widehat{D}^k\widehat{T}^\ell(\frac{1}{\sqrt{2\pi}}\chi_{E_2}\rangle\right)\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{E_1}.$$

By [6, Lemma 5], it can be simplified to

$$\begin{aligned}
&H^k_{E_1}(f_1)+\sum_{\ell\in\mathbb{Z}}\langle f_2,\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{E_2}\rangle\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{E_1}\\
=\;&f_1\chi_{2^kE_1}+\sum_{\ell\in\mathbb{Z}}\langle f_2,\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{F_2}\rangle\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{F_1}\\
=\;&f_1\chi_{2^kE_1}+\sum_{\ell\in\mathbb{Z}}\langle f_2^{(k)},\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{F_1}\rangle\widehat{D}^k\widehat{T}^\ell\frac{1}{\sqrt{2\pi}}\chi_{F_1}\\
=\;&f_1\chi_{2^kE_1}+f_2^{(k)}\chi_{2^kF_1}\\
=\;&\chi_{F_1}\chi_{2^kE_1}-\chi_{F_1'}\chi_{2^kF_1},
\end{aligned}$$

where $f_2^{(k)}(s)=\begin{cases} f_2\left(2^k\rho^{-1}(\frac{s}{2^k})\right) & s\in F_2\cap 2^kF_2\\ 0 & \text{otherwise}\end{cases}$ and $F_1'=2^k\rho^{-1}\left(2^{-k}(F_2\cap 2^kF_2)\right)$ is a subset of 2^kF_1. It follows that $0\le\chi_{F_1}\chi_{2^kE_1}-\chi_{F_1'}\chi_{2^kE_1}\le\chi_{F_1}\chi_{2^kE_1}=H^k_{E_1}f_1$ for any k. In the special case that $k=0$, we have $F_1'=F_1$ so $0=\chi_{F_1}\chi_{E_1}-\chi_{F_1'}\chi_{E_1}$ but $\|\chi_{F_1}\chi_{E_1}\|=\|f_1\|>0$. It follows that

$$\left\langle f_1,\sum_{k,\ell\in\mathbb{Z}}\langle(f_1,f_2,0,...,0),\widehat{D}^k\widehat{T}^\ell(\hat{\eta_1},\hat{\eta_2},...,\hat{\eta_m})\rangle\widehat{D}^k\widehat{T}^\ell\hat{\eta_1}\right\rangle<k\|f_1\|^2$$

since it is known that $\sum_{k\in\mathbb{Z}}\langle f_1,H^k_{E_1}\rangle=k\|f_1\|^2$ [6]. Similarly, we can shows that

$$\left\langle f_2,\sum_{k,\ell\in\mathbb{Z}}\langle(f_1,f_2,0,...,0),\widehat{D}^k\widehat{T}^\ell(\hat{\eta_1},\hat{\eta_2},...,\hat{\eta_m})\rangle\widehat{D}^k\widehat{T}^\ell\hat{\eta_2}\right\rangle<k\|f_2\|^2.$$

Thus

$$\begin{aligned}
&\left\langle(f_1,f_2,0,...,0),\sum_{k,\ell\in\mathbb{Z}}\langle(f_1,f_2,0,...,0),\widehat{D}^k\widehat{T}^\ell(\hat{\eta_1},\hat{\eta_2},...,\hat{\eta_m})\rangle\widehat{D}^k\widehat{T}^\ell(\hat{\eta_1},\hat{\eta_2},...,\hat{\eta_m})\right\rangle\\
<\;&k\left(\|f_1\|^2+\|f_2\|^2\right)=k\|(f_1,f_2,0,...,0)\|^2.
\end{aligned}$$

This contradicts that fact that $(\eta_1,\eta_2,...,\eta_m)$ is a super tight frame of frame bound k. ∎

Remark 3.5 The special case of $k=1$ in Theorems 3.3 and 3.4 was obtained in [12].

Example 3.6 By Theorem 3.1, it is easy to construct super frame wavelets of any given length whose lower and upper frame bounds are positive integers. For example, to construct a super frame wavelet of length 3 with lower frame bound 1 and upper frame bound 3, we can simply choose $E_1=[-\pi,-\pi/2)\cup[\pi/2,\pi)$, $E_2=[-\pi/2,-\pi/4)\cup[\pi/4,\pi/2)$ and $E_3=[-\pi/4,-\pi/32)\cup[\pi/32,\pi/4)$. It is slightly harder to construct super frame wavelets with non-integer frame bounds. Interested reader may refer to the examples given in [6].

Example 3.7 Let $I_j = [-\frac{\pi}{2^{(j-1)k_0}}, -\frac{\pi}{2^{jk_0}}) \cup [\frac{\pi}{2^{jk_0}}, \frac{\pi}{2^{(j-1)k_0}})$, $1 \le j \le m$ and define $(\phi_1, \phi_2, ..., \phi_m)$ by $\widehat{\phi_j} = \frac{1}{\sqrt{2\pi}}\chi_{I_j}$, then $(\phi_1, \phi_2, ..., \phi_m)$ is a super tight frame wavelet for $\prod_{1\le j\le m} L^2(\mathbb{R})$ with frame bound k_0. Notice that m and k_0 can be any positive integers in this example.

4 Path-connectivity of s-elementary super frame wavelets

The topological property of various families of wavelets is an interesting topic in the study of wavelet theory. The question concerning the path-connectedness of the set of all orthonormal wavelets was first raised in [7]. Similar questions were raised and studied in [2, 4, 8, 13, 14, 16, 17, 18, 19] concerning the sets of all MRA-wavelets, tight frame wavelets, MRA tight frame wavelets and, in particular, the special case of s-elementary frame wavelets. In [13, 14, 16, 19], it is shown that the set of all single MRA-wavelets is path-connected. In [18], it is shown that the set of all s-elementary orthonormal wavelets is path-connected. This result is extended to the set all s-elementary tight frame wavelets (with the same frame bound) in [2]. The proofs of these theorems were based on the complete characterizations of the corresponding wavelets. Also, while the complete characterization of the s-elementary frame wavelets is still an open question, it has been shown that the set of s-elementary frame wavelets is path-connected as well [4]. Recently, the same result is extended to the set of all multiwavelets of length n in $L^2(\mathbb{R}^d)$ ([15]).

Let $\mathfrak{S}(m)$ denote the set of all s-elementary super frame wavelets with length m ($m \ge 1$) that are τ-disjoint and let $\mathfrak{S}^k(m)$ denote the set of all tight s-elementary super frame wavelets with length m and frame bound k. By Theorem 3.4, we see that $\mathfrak{S}^k(m)$ is a proper subset of $\mathfrak{S}(m)$ (which justifies the use of the same notation $\mathfrak{S}$). The path-connectedness of $\mathfrak{S}(m)$ and $\mathfrak{S}^k(m)$ were established in [4] and [2] respectively for the case of $m = 1$. In this section, we will prove that these two sets are also path-connected for any $m > 1$. The path-connectedness of the set $\mathfrak{S}(m)$ means that for any two s-elementary super frame wavelets $\Psi_E = (\psi_{E_1}, \cdots, \psi_{E_m})$ and $\Psi_F = (\psi_{F_1}, \cdots, \psi_{F_m})$ with super frame wavelet sets $E = (E_1, \cdots, E_m)$ and $F = (F_1, \cdots, F_m)$, respectively, there exists a continuous (in the norm of $\prod_{1\le j\le m} L^2(\mathbb{R})$) mapping $\gamma : [0, 1] \to \mathfrak{S}(m)$ such that $\gamma(0) = \Psi_E$ and $\gamma(1) = \Psi_F$.

Theorem 4.1 *$\mathfrak{S}(m)$ is path-connected under the $\prod_{1\le j\le m} L^2(\mathbb{R})$ norm.*

Proof. Let $(E_1, E_2, ..., E_m)$ be a super frame wavelet set and let $E = \cup_{1\le j\le m} E_j$. By [6, Theorem 3], there exists a number $M > 0$ such that $\mu(E(\tau, n)) = \mu(E(\delta, n)) = 0$, for all $n > M$. Let a_j be the lower frame bound corresponding to E_j and let $a = \min\{a_1, ..., a_m\}$. We have $a\|f_j\|^2 \le \langle f_j, H_{E_j} f_j\rangle \le M^{5/2}\|f_j\|^2$ for all $f_j \in L^2(\mathbb{R})$ (by Theorem 2.4). Let p be a positive integer large enough so that $M/2^p < 1/4$. Let $F_j = [-\frac{2\pi}{2^{p+j}}, -\frac{\pi}{2^{p+j}}) \cup [\frac{\pi}{2^{p+j}}, \frac{2\pi}{2^{p+j}})$ and $F = \cup_{1\le j\le m} F_j$. For any $s \in E$, there is a unique integer $k(s)$ such that $s/2^{k(s)} \in F_1$. Thus $h_j(s) = s/2^{k(s)+j-1}$ defines a mapping from E to F_j. We leave it to our reader to prove that the image of each measurable subset in E under h_j is measurable. For any subset E' of E, define $h(E) = \cup_{1\le j\le m} h_j(E')$. Of course, $h(E')$ is a subset of F. Furthermore, if E' is a subset of $E \cap \mathbb{R}\backslash[-\pi, \pi]$, then $\mu(h_j(E')) < \frac{1}{2^{p+j}}\mu(E')$

and it follows that $\mu(h(E')) < \frac{1}{2^p}\mu(E')$. Define

$$\begin{aligned}
F_t^0 &= \cup_{1\le j\le m}\left([-\frac{2\pi}{2^{p+j}}, -\frac{(2-t)\pi}{2^{p+j}}] \cup [\frac{\pi}{2^{p+j}}, \frac{(1+t)\pi}{2^{p+j}}]\right) \\
F_t^1 &= h(\mathcal{T}(F_t^0) \cap (E\backslash F_t^0)), \\
F_t^2 &= h(\mathcal{T}(F_t^1) \cap (E\backslash F_t^1)), \\
\cdots & \\
F_t^n &= h(\mathcal{T}(F_t^{n-1}) \cap (E\backslash F_t^{n-1})), \\
\cdots & \\
F_t &= \bigcup_{k\ge 0} F_t^k, t\in[0,1].
\end{aligned}$$

Notice that the set F_t is a measurable subset of F. Let $E_t = \mathcal{T}(F_t)\cap E$. Then E_t contains all points of E that are 2π-translation equivalent to points in F_t hence $E\backslash E_t$ contains no points that are 2π-translation equivalent to points in F_t. That is, the sets E_t and $E\backslash E_t$ are 2π-translation disjoint.

Now define $W_t = F_t \cup (E\backslash E_t)$. We will first show that the mapping $t \longrightarrow \chi_{W_t}$ is continuous in the $L^2(\mathbb{R})$ norm.

Step 1: We first show that the mapping $t \to \chi_{F_t}$ is continuous in norm. For $0 \le t \le 1$, we have $\mu(F_t^0) \le 2\pi/2^p$. By the property of E, for a point $s \in F_t^0$, the set $\{s + 2k\pi : k \in \mathbb{Z}\} \cap E$ has at most M points. This implies that

$$\mu(\mathcal{T}(F_t^0) \cap (E\backslash F_t^0)) \quad \le \quad M\mu(F_t^0). \tag{4.1}$$

Since $\mathcal{T}(F_t^0) \cap (E\backslash F_t^0) \subset \mathbb{R}\backslash[-\pi,\pi]$, it follows from (4.1) that

$$\begin{aligned}
\mu(F_t^1) &\le \frac{1}{2^p}\mu(\mathcal{T}(F_t^0) \cap (E\backslash F_t^0)) \\
&\le \frac{M}{2^p}\mu(F_t^0) \le \frac{1}{4}\mu(F_t^0).
\end{aligned}$$

By induction, we have

$$\mu(F_t^n) \le \frac{M}{2^p}\mu(F_t^{n-1}) \le \frac{1}{4^n}\mu(F_t^0). \tag{4.2}$$

Therefore, the convergence of $\chi_{\cup_{0\le k\le n}F_t^k}$ to χ_{F_t} is uniform with respect to $t \in [0,1]$. $\forall\epsilon > 0$, choose $N > 0$ large enough such that $\pi/4^N < \epsilon/4$, then for any $t \in [0,1]$, we have

$$\begin{aligned}
&|\chi_{\cup_{0\le k\le N}F_t^k} - \chi_{F_t}| \le \sum_{k>N}\frac{1}{4^k}\mu(F_t^0) \\
\le\ & \frac{\mu(F_t^0)}{4^N} < \frac{\pi}{4^N} < \frac{\epsilon}{4},
\end{aligned}$$

since $\mu(F_t^0) \le \pi$ for any t. If the mapping $t \to \chi_{F_t^n}$ is continuous in norm for each n, then $\chi_{\cup_{0\le k\le N}F_t^k}$ is uniformly continuous on $[0,1]$. Thus, there exists $\delta(\epsilon) > 0$ such that $|\chi_{\cup_{0\le k\le N}F_{t_2}^k} - \chi_{\cup_{0\le k\le N}F_{t_1}^k}| < \epsilon/2$ whenever $|t_2 - t_1| < \delta(\epsilon)$. It follows that

$$\begin{aligned}
&|\chi_{F_{t_2}} - \chi_{F_{t_1}}| \le |\chi_{\cup_{0\le k\le N}F_{t_2}^k} - \chi_{\cup_{0\le k\le N}F_{t_1}^k}| \\
+\ & |\chi_{\cup_{0\le k\le N}F_{t_2}^k} - \chi_{F_{t_2}}| + |\chi_{\cup_{0\le k\le N}F_{t_1}^k} - \chi_{F_{t_1}}| \\
\le\ & \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

That is, χ_{F_t} is also uniformly continuous on $[0,1]$. Therefore, it suffices for us to prove that the mapping $t \to \chi_{F_t^n}$ is continuous in norm for each n. We will prove this by induction. Clearly, the mapping $t \to \chi_{F_t^0}$ is continuous. Assume that it is true for n. We will show that it is true for $n+1$. For this purpose, we write $K\Delta L = (K\backslash L)\cup(L\backslash K)$ for any sets K and L, and let $D_t^n = \mathcal{T}(F_t^n)\cap(E\backslash F_t^n)$. For any $t, t' \in [0,1]$, we claim that $D_t^n \Delta D_{t'}^n \subset \mathcal{T}(F_t^n \Delta F_{t'}^n) \cap E$. Let $s \in D_t^n \Delta D_{t'}^n$. We can assume that $s \in D_t^n \backslash D_{t'}^n$. Then there is an integer k such that $s + 2k\pi \in F_t^n$. However $s \notin F_t^n$. It follows that $k \neq 0$. Thus $s \notin F_{t'}^n$, for otherwise we would have both s and $s+2k\pi \in F_{t'}^n \cup F_t^n \subset F \subset [-\pi, \pi)$ which is impossible since $k \neq 0$. Therefore $s \in E\backslash F_{t'}^n$. Since $s \notin D_{t'}^n = \mathcal{T}(F_{t'}^n) \cap (E\backslash F_{t'}^n)$, it follows that $s \notin \mathcal{T}(F_{t'}^n)$. Hence $s + 2k\pi \in F_t^n \Delta F_{t'}^n$ and therefore $s \in \mathcal{T}(F_t^n \Delta F_{t'}^n) \cap E$, as expected.

We now have

$$F_t^{n+1} \Delta F_{t'}^{n+1} \subset h(D_t^n \Delta D_{t'}^n) \subset h(\mathcal{T}(F_t^n \Delta F_{t'}^n) \cap E). \tag{4.3}$$

Therefore,

$$\begin{aligned} \mu(F_t^{n+1} \Delta F_{t'}^{n+1}) &\leq \mu(h((F_t^n \Delta F_{t'}^n)^+ \cap E)) \\ &\leq M 2^p \mu(F_t^n \Delta F_{t'}^n). \end{aligned} \tag{4.4}$$

(4.4) implies that the mapping $t \to \chi_{F_t^{n+1}}$ is continuous since the mapping $t \to \chi_{F_t^n}$ is. This completes the proof that the mapping $t \to \chi_{F_t^n}$ is continuous in norm for all n. Hence the mapping $t \to \chi_{F_t}$ is continuous, as claimed.

Step 2: We now show that the mapping $t \to \chi_{E_t}$ is also continuous. In fact, this follows from the inclusion $E_t \Delta E_{t'} \subset \mathcal{T}(F_t \Delta F_{t'}) \cap E$, which implies that

$$\mu(E_t \Delta E_{t'}) \leq \mu(\mathcal{T}(F_t \Delta F_{t'}) \cap E) \leq M\mu(F_t \Delta F_{t'}).$$

Step 3: Finally, the continuity of $t \to \chi_{W_t}$ follows from the continuity of the mappings $t \to \chi_{F_t}$ and $t \to \chi_{E\backslash E_t}$ and the fact that $F_t \cap (E\backslash E_t) = \emptyset$.

Observe that F_t is the disjoint union of the sets $F_t \cap F_j$ $(1 \leq j \leq m)$. By our construction, $\frac{1}{2}(F_t \cap F_j) = F_t \cap F_{j+1}$ for $1 \leq j \leq m-1$ and every point in E_t is δ-equivalent to a point in $F_t \cap F_j$ for each j. Denote $(F_t \cap F_j) \cup (E_j \backslash E_t)$ by G_t^j, then W_t is the disjoint union of the sets G_t^j. It is easy to see that $\chi_{G_t^j}$ is continuous for each j hence $(\chi_{G_t^1}, \chi_{G_t^2}, ..., \chi_{G_t^m})$ is continuous in the $\prod_{1\leq j\leq m} L^2(\mathbb{R})$ norm. Since F_t is 2π-translation disjoint from $E\backslash E_t$, E_j's are 2π-translation disjoint from each other and F_j's are also 2π-translation disjoint from each other, it follows that the sets G_t^j's are 2π-translation disjoint from each other. From the definition we also have $G_0^j = E_j$ and $G_1^j = F_j$. Thus the only thing remained to be proved is that each G_t^j is a frame wavelet set.

By the property of the number M and [4, Lemma 1], we have $\langle f, H_{G_t^j} f\rangle \leq M^{5/2}\|f\|^2, \forall f \in L^2(\mathbb{R})$. So we are done once we can establish that $\langle H_{G_t^j} f, f\rangle \geq c\|f\|^2$, $\forall f \in L^2(\mathbb{R})$ for some constant $c > 0$. By the definitions of E_j and a_j, we have $\langle H_{E_j} f, f\rangle \geq a_j\|f\|^2$, $\forall f \in L^2(\mathbb{R})$. Thus

$$\langle f, H_{E_j} f\rangle = \langle f, H_{E_j \cap E_t} f\rangle + \langle f, H_{E_j\backslash E_t} f\rangle \geq a_j\|f\|^2, \forall f \in L^2(\mathbb{R}) \tag{4.5}$$

by [4, Lemma 4]. Similarly,

$$\langle f, H_{(F_t\cap F_j)\cup(E_j\backslash E_t)} f\rangle = \langle f, H_{F_t\cap F_j} f\rangle + \langle f, H_{E_j\backslash E_t} f\rangle, \ \forall f \in L^2(\mathbb{R}). \tag{4.6}$$

Also, since $E_t \subseteq \cup_{k\in\mathbb{Z}} 2^k(F_t \cap F_j)$, [4, Lemma 3] can be applied. Which leads to

$$\langle f, H_{F_t\cap F_j} f\rangle \geq M^{-5/2}\langle f, H_{E_j\backslash E_t} f\rangle, \ \forall f \in L^2(\mathbb{R}). \tag{4.7}$$

Combining the above equations and inequalities, we get

$$\begin{aligned}
& \langle f, H_{(F_t\cap F_j)\cup(E_j\setminus E_t)}f\rangle \\
= & \langle f, H_{F_t\cap F_j}f\rangle + \langle f, H_{E_j\setminus E_t}f\rangle \\
\geq & M^{-\frac{5}{2}}\langle f, H_{E_j\cap E_t}f\rangle + \langle f, H_{E_j\setminus E_t}f\rangle \\
\geq & M^{-\frac{5}{2}}\left(\langle f, H_{E_j\cap E_t}f\rangle + \langle f, H_{E_j\setminus E_t}f\rangle\right) \\
= & M^{-\frac{5}{2}}\langle f, H_{E_j}f\rangle \\
\geq & a_j M^{-\frac{5}{2}}\|f\|^2, \forall f \in L^2(\mathbb{R}).
\end{aligned}$$

Therefore, G_t^j is a frame wavelet set for each $t \in [0,1]$. ■

Theorem 4.2 *$\mathfrak{S}^k(m)$ is path-connected under the $\prod_{1\leq j\leq m} L^2(\mathbb{R})$ norm.*

Proof. The proof of Theorem 4.2 can be modified from the proof of [2, Theorem 3.4] without significant changes. We will give an outline here and interested reader please refer to [2].

Let E_1, E_2, ..., E_m be measurable sets in $\mathbb{R}$ with finite measures such that $(\eta_1, \eta_2, ..., \eta_m)$ defined by $\hat{\eta_j} = \frac{1}{\sqrt{2\pi}}\chi_{E_j}$ is a super tight frame wavelet of length m with frame bound k_0 for $\prod_{1\leq j\leq m} L^2(\mathbb{R})$. Then $\tau(E_i)\cap\tau(E_j) = \emptyset$ if $i \neq j$ and $E_j = E_j(\delta, k_0)$, $E_j = E_j(\tau, 1)$ for each j. It follows that $E = \cup_{1\leq j\leq m} E_j$ is a tight frame wavelet for $L^2(\mathbb{R})$ with frame bound mk_0. It shown in [2] that for each $t \in [0,1]$, there exist measurable sets J_t, F_t with the following properties:

(a) χ_{J_t} and χ_{F_t} are continuous in t in the $L^2(\mathbb{R})$ norm;

(b) $J_0 = \emptyset$, $J_1 = [-\pi, -\frac{\pi}{2^{mk_0}}] \cup [\frac{\pi}{2^{mk_0}}, \pi]$;

(c) $J_t = J_t(\tau, 1) = J_t(\delta, mk_0)$;

(d) $F_t = E \cap \Delta(J_t)$ and $\tau(F_t) \cap (E\backslash F_t) = \emptyset$.

Denote the set $[-\frac{\pi}{2^{(j-1)k_0}}, -\frac{\pi}{2^{jk_0}}) \cup [\frac{\pi}{2^{jk_0}}, \frac{\pi}{2^{(j-1)k_0}})$ by I_j (as we did in Example 3.7) and let $G_t^j = (J_t \cap I_j) \cup (E_j \Delta(F_t))$, then each G_t^j is a tight frame wavelet set (with frame bound k_0), $\chi_{G_t^j}$ is continuous in t and $\tau(G_t^i) \cap \tau(G_t^j) = \emptyset$ for $i \neq j$. Therefore, $(\eta_1(t), \eta_2(t), ..., \eta_m(t))$ defined by $\hat{\eta_j(t)} = \frac{1}{\sqrt{2\pi}}\chi_{G_t^j}$ is a super tight frame wavelet of length m with frame bound k_0 for $\prod_{1\leq j\leq m} L^2(\mathbb{R})$. It is a path in $\mathfrak{S}^k(m)$ connecting $(\eta_1, \eta_2, ..., \eta_m)$ to the fixed super tight frame wavelet $(\phi_1, \phi_2, ..., \phi_m)$ given in Example 3.7. The details are left to the reader. ■

We end our paper by two remarks. The first remark is that the difficulties that one faces in trying to characterize the s-elementary super frame wavelet are very similar to the ones one encounters when trying to characterize the s-elementary frame wavelet (for the $L^2(\mathbb{R})$ case), since the defining frame wavelet sets do not have to be τ-disjoint, as the example given in the proof of Theorem 3.2 shows. It is thus plausible that the solution to one of them may lead to the solution of the other. The second remark is that it is likely all the results here can be extended to the case where the space under consideration is $\prod_{1\leq j\leq m} L^2(\mathbb{R}^d)$ for some $d > 1$, though one may still have to overcome some technical difficulties.

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