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NULLIFICATION OF KNOTS AND LINKS

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ABSTRACT

It is known that a knot/link can be nullified, i.e., can be made into the trivial knot/link, by smoothing some crossings in a projection diagram of the knot/link. The nullification of knots/links is believed to be biologically relevant. For example, in DNA topology, the nullification process may be the pathway for a knotted circular DNA to unknot itself (through recombination of its DNA strands). The minimum number of such crossings to be smoothed in order to nullify the knot/link is called the nullification number. It turns out that there are several different ways to define such a number, since different conditions may be applied in the nullification process. We show that these definitions are not equivalent, thus they lead to different nullification numbers for a knot/link in general, not just one single nullification number. Our aim is to explore the mathematical properties of these nullification numbers. First, we give specific examples to show that the nullification numbers we defined are different. We provide detailed analysis of the nullification numbers for the well known 2-bridge knots and links. We also explore the relationships among the three nullification numbers, as well as their relationships with other knot invariants. Finally, we study a special class of links, namely those links whose general nullification number equals one. We show that such links exist in abundance. In fact, the number of such links with crossing number less than or equal to n grows exponentially with respect to n.

Keywords: knots, links, crossing number, unknotting number, nullification, nullification number.

Mathematics Subject Classification 2000: 57M25

1. Introduction and basic concepts

Although knot theory is a branch of pure mathematics, many questions and studies there are motivated by questions from sciences. In fact, the original study of knots by Lord Kelvin and Tait was motivated by problems from physics. The discovery of DNA knots in recent decades is a late example of a source of many interesting new questions and research topics in knot theory. It turned out that the topology of the circular DNA plays a very important role in the properties of the DNA. Various geometric and topological complexity measures of DNA knots that are believed to be biologically relevant, such as the knot types, the 3D writhe, the average crossing numbers, the average radius of gyration, have been studied. In this paper, we are interested in another geometric/topological measure of knots and links called the *nullification number*, which is also believed to be biologically relevant [2,11]. Intuitively, this number measures how easily a knotted circular DNA can unknot itself through recombination of its DNA strands. It turns out that there are several different ways to define such a number. These different definitions lead to different versions of a nullification number that are related. Our aim is to explore the mathematical properties of the different versions of the nullification number. In this section, we will outline a brief introduction to basic knot theory concepts. In Section 2, we will give precise definitions for three different versions of a nullification number. In Section 3, we will study the nullification number for a well known class of knots called the class of Montesinos knots and links. In Section 4, we explore the relationships among the three versions of a nullification number, as well as their relationships with other knot invariants. In particular, we give examples to show that the three versions of a nullification number defined here are indeed different. In Section 5, we study a special class of links, namely the links whose general nullification number equals one. There we show that such links exist in abundance. In fact, the number of such links with crossing number less than or equal to n grows exponentially with respect to n.

Let K be a tame link, that is, K is a collection of several piece-wise smooth simple closed curves in $\mathbb{R}^3$. In the particular case that K contains only one component, it is called a knot instead. However through out this paper a link always includes the special case that it may be a knot, unless otherwise stated. A link is oriented if each component of the link has an orientation. Intuitively, if one can continuously deform a tame link K_1 to another tame link K_2 (in $\mathbb{R}^3$), then K_1 and K_2 are considered equivalent links in the topological sense. The corresponding continuous deformation is called an *ambient isotopy*, and K_1, K_2 are said to be ambient isotopic to each other. The set of all (tame) links that are ambient isotopic to each other is called a *link type*. For a fixed link (type) $\mathcal{K}$, a link diagram of $\mathcal{K}$ is a projection of a member $K \in \mathcal{K}$ onto a plane. Such a projection $p: \ K \subset R^3 \to D \subset R^2$ is regular if the set of points $\{x \in D : |p^{-1}(x)| > 1\}$ is finite and there is no x in D for which $|p^{-1}(x)| > 2$. In other words, in the diagram no more than two arcs of D cross at any point in the projection and there are only finitely many points where the arcs cross each other. A point where two arcs of D cross each other is called a crossing point, or just a crossing of D. The number of crossings in D not only depends on the link type $\mathcal{K}$, it also depends on the geometrical shape of the member K representing $\mathcal{K}$ and the projection direction chosen.

The minimum number of crossings in all regular projections of all members of $\mathcal{K}$ is called the *crossing number* of the link type $\mathcal{K}$ and is denoted by $Cr(\mathcal{K})$. For any member K of $\mathcal{K}$, we also write $Cr(K) = Cr(\mathcal{K})$. Of course, by this definition, if K_1 and K_2 are of the same link type, then we have $Cr(K_1) = Cr(K_2)$. However, it may be the case that for a member K of $\mathcal{K}$, none of the regular projections of K has crossing number $Cr(\mathcal{K})$. A diagram D of a link $K \in \mathcal{K}$ is *minimum* if the number of crossings in the diagram equals $Cr(\mathcal{K})$. We will often call D a *minimum projection diagram.* A link diagram is *alternating* if one encounters over-passes and under-passes alternatingly when traveling along the link projection. A diagram D is said to be *reducible* if there exists a crossing point in D such that removing this crossing point makes the remaining diagram two disconnected parts. D is *reduced* if it is not reducible. A link is *alternating* if it has a reduced alternating diagram. A famous result derived from the Jones polynomial is that the crossing number of an alternating link $\mathcal{K}$ equals the number of crossings in any of its reduced alternating diagram since each diagram is minimum. For example the diagram of the knot $\mathcal{K}_1$ in Figure 1 is minimum.

A link $\mathcal{K}$ is called a *composite link* if a member of it can be obtained by cutting open two nontrivial links K_1 and K_2 and reconnecting the strings as shown in Figure 1. The resulting link is written as $K = K_1\#K_2$ and K_1, K_2 are called the *connected sum components* of K. Of course a link can have more than two connected sum components. A link $\mathcal{K}$ that is not a composite link is called a *prime link.*

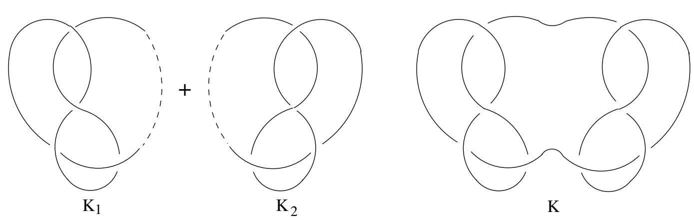

Fig. 1. A composite knot $K = K_1\#K_2$.

In the case of alternating links, any two minimum projection diagrams D and D' of the same alternating link $\mathcal{K}$ are flype equivalent, that is, D can be changed to D' through a finite sequence of flypes [19,20] (see Figure 2).

Let c be a crossing in an alternating diagram D. The *flyping circuit of* c is defined as the unique decomposition of D into crossings $c_1, c_2, \ldots, c_m$, $m \geq 1$ and tangle diagrams $T_1, T_2, \ldots, T_r$, $r \geq 0$ joined together as shown in Figure 3 such that (i) $c = c_i$ for some i and (ii) the T_i are minimum with respect to the pattern.

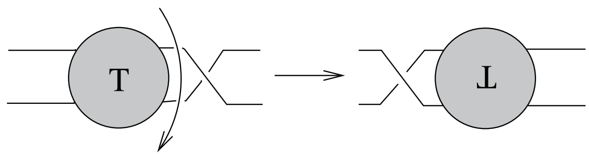

Fig. 2. A single flype. T denotes a part of the diagram that is rotated by 180 degrees by the flype.

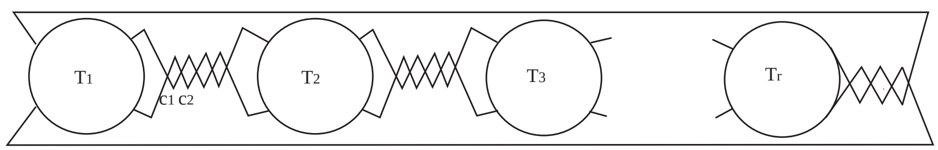

Fig. 3. A flyping circuit. Any crossing in the flyping circuit can be flyped to any position between tangles T_i and T_{i+1}.

2. Definitions of Nullification Numbers

Let D be a regular diagram of an oriented link $\mathcal{K}$. A crossing in D is said to be *smoothed* if the strands of D at the crossing are cut and re-connected as shown in Figure 4. If every crossing in D is smoothed, the result will be a collection of disjoint (topological) circles without self intersections. These are called the *Seifert circles* of D. Of course the set of Seifert circles of D represents a trivial link diagram. However, it is not necessary to smooth every crossing of D to make it a trivial link diagram. For example, if a diagram has only one crossing, or only two crossing with only one component, then the diagram is already a trivial link diagram. So the minimum number of crossings needed to be smoothed in order to turn D into a trivial link diagram is strictly less than the number of crossings in D. This minimum number is called the *nullification number* of the diagram D, which we will write as n_D. Notice that n_D is not a link invariant since different diagrams (of the same link) may have different nullification numbers. In order to define a number that is a link invariant, we would have to consider the set of all diagrams of a link. Depending on how we choose to smooth the crossings in the process, we may then get different versions of nullification numbers. This approach is in a way similar to the how different versions of unknotting numbers are defined in [9].

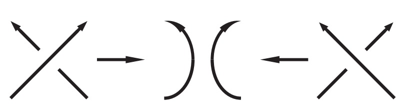

Fig. 4. The smoothing of a single crossing.

Let $\mathcal{K}$ be an oriented link and D be a (regular) diagram of $\mathcal{K}$. Choose some crossings in D and smooth them. This results in a new diagram D' which is likely of a different link type other than $\mathcal{K}$. Suppose we are allowed to deform D' (without changing its link type, of course) to a new diagram D_1. We can then again choose some crossings in D_1 to smooth and repeat this process. With proper choices of the new diagrams and the crossings to be smoothed, it is easy to see that this process can always terminate into a trivial link diagram. The minimum number of crossings required to be smoothed in order to make any diagram D of $\mathcal{K}$ into a trivial link diagram by the above procedure is then defined as the *general nullification number* of $\mathcal{K}$, or just the nullification number of $\mathcal{K}$. We will denote it by $n(\mathcal{K})$.

On the other hand, if in the above nullification procedure, we require that the diagrams used at each step be minimum diagrams (of their corresponding link types), then the minimum number of crossings required to be smoothed in order to make any minimum diagram D of $\mathcal{K}$ into a trivial link diagram is defined as the *restricted nullification number* of $\mathcal{K}$, which we will denote by $n_r(\mathcal{K})$.

In the case that D is a minimum diagram of $\mathcal{K}$, we have already defined the nullification number n_D for the diagram D, namely the minimum number of smoothing moves needed to change D into a trivial link diagram. If we take the minimum of n_D over all minimum diagrams of $\mathcal{K}$, then we obtain a third nullification number of $\mathcal{K}$, which we will call the *diagram nullification number* of $\mathcal{K}$ and will denoted it by $n_d(\mathcal{K})$.

By the above definitions, clearly we have

$$n(K) \leq n_r(\mathcal{K}) \leq n_d(\mathcal{K}).$$

We shall see later that these definitions of nullification numbers are indeed all different. Of the three nullification numbers, the diagram nullification number $n_d(\mathcal{K})$ has been studied in [6,24]. Specifically, in [24] it is shown that for an alternating link $\mathcal{K}$, the diagram nullification number $n_d(\mathcal{K})$ can be computed from any reduced alternating diagram D of $\mathcal{K}$ using the following formula:

$$n_d(\mathcal{K}) = Cr(D) - s(D) + 1, \tag{2.1}$$

where $s(D)$ is the number of Seifert circles in D. This allows us to express the genus $g(\mathcal{K})$ of an alternating link $\mathcal{K}$ in terms of the number of Seifert circles and the nullification number by

$$g(\mathcal{K}) = \frac{1}{2}(n_d(\mathcal{K}) - \mu + 1), \tag{2.2}$$

where μ is the number of components of $\mathcal{K}$. On the other hand, the diagram nullification number n_d for alternating links is closely related to the HOMFLY polynomial by the following lemma. This provides an expression of n_d without having to make reference to a particular diagram.

Lemma 2.1. *Let $\mathcal{K}$ be an alternating non-split link, then $n_D(\mathcal{K}) = \beta_z$, where β_z is the maximum degree of the variable z in the HOMFLY polynomial $P_{\mathcal{K}}(v, z)$ of $\mathcal{K}$.*

The result of Lemma 2.1 is given in [24]. In the following we give a short proof of the lemma, since a proof was not given in [24]. For more and detailed information regarding HOMFLY polynomial and other facts in knot theory, please refer to a standard text in knot theory such as [8].

Proof. For any link $\mathcal{K}$ with diagram D we have the inequality $\beta_z \leq Cr(D) - s(D)+1$. The Conway polynomial $C_{\mathcal{K}}(z)$ of $\mathcal{K}$ is related to $P_{\mathcal{K}}(v, z)$ by the equation $C_{\mathcal{K}}(z) = P_{\mathcal{K}}(1, z)$. It follows that the maximum degree α_z of z in $C_{\mathcal{K}}(z)$ is at most β_z. Since $\mathcal{K}$ is alternating, $\alpha_z = 2g(\mathcal{K}) + \mu - 1$, where $g(\mathcal{K})$ is the genus of $\mathcal{K}$ and μ is the number of components of $\mathcal{K}$. Using the Seifert algorithm on a reduced alternating diagram of $\mathcal{K}$ we get

$$g(\mathcal{K}) = \frac{1}{2}(Cr(D) - s(D) - \mu) + 1.$$

Now it follows that

$$\begin{aligned}\alpha_z &= 2g(\mathcal{K}) + \mu - 1 \\ &= (Cr(D) - s(D) - \mu) + \mu + 1 \\ &= Cr(D) - s(D) + 1 \\ &\geq \beta_z.\end{aligned}$$

Thus $\beta_z = \alpha_z = Cr(D) - s(D) + 1 = n_D(\mathcal{K})$. □

In general, if D is a diagram of some non-alternating link, then we have $n_D \leq Cr(D) - s(D) + 1$, but the precise determination of n_D (hence $n_d(\mathcal{K})$) is far more difficult. In the following we propose a different inequality concerning n_D using the concept of parallel and anti-parallel crossings. A flyping circuit is said to be *nontrivial* if it either contains more than one crossing or more than one tangle. Otherwise it is called a *trivial* flyping circuit. If a crossing is part of a nontrivial flyping circuit then the crossing belongs to a unique flyping circuit [4]. If there is more than one crossing in a flyping circuit of an oriented link diagram, then the crossings in the flyping circuit are called *parallel* or *anti-parallel* as shown in Figure 5.

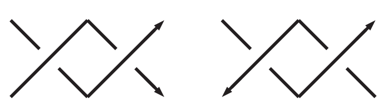

Fig. 5. Parallel (left) and anti-parallel (right) crossings in a nontrivial flyping circuit.

Note that we can assign the notion of parallel or anti-parallel even to a single crossing as long as the flyping circuit has at least two tangles. For trivial flyping circuits that consist of a single crossing and a single tangle there is no obvious way of assigning a notion of parallel or anti-parallel to it. A nontrivial flyping circuit has the special property that all the crossings in it can be eliminated by nullifying (i.e. smoothing) a single crossing if the crossings are anti-parallel, while all crossings in it have to be smoothed (in order to eliminate them with nullifying moves within the circuit) when the crossings are parallel. Let P_1, P_2, ..., P_m be the nontrivial flyping circuits with parallel crossings and let $|P_i|$ be the number of crossings in P_i. Let A be number of nontrivial flyping circuits with anti-parallel crossings and S be the total number of crossings in all trivial flyping circuits. We conjecture that for any link diagram D

$$n_D \leq \sum_{1\leq i\leq m} (|P_i| - 1) + A + S + c, \tag{2.3}$$

where $c \leq 1$ is an additional constant depending on the link type of D. For alternating diagrams, we expect an almost equality in (2.3), while for non-alternating diagrams (2.3) may still be a large overestimate. Figure 6 shows the case of a minimum diagram for the knot $11a_{263}$. There are four visible nontrivial flyping circuits (three of which have 3 crossings and one with two crossings) and all crossings in the circuits are parallel and each crossing belong to one such circuit. It follows that $A = S = 0$ and $\sum_{1\leq i\leq 4}(|P_i| - 1) + A + S = 7$. Thus (2.3) becomes an equality with the choice of $c = 1$ since $n_D = 8$. This example shows that it is necessary for us to have the constant c term in general.

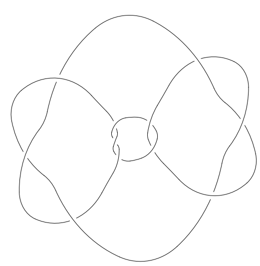

Fig. 6. A minimum diagram knot $11a_{263}$ with $n_D = 8$. It has four parallel nontrivial flyping circuits and each crossing belongs to one of these circuits.

3. Diagram Nullification Numbers of 4-plats and Montesinos Links

In this section we discuss the nullification number n_D of 4-plats and Montesinos links. The goal is to show that the inequality in (2.3) holds for these links.

A 4-plat is a link with up to two components that admits a minimum alternating diagram as shown in Figure 7 where a grey box marked by c_i indicates a row of c_i horizontal half-twists. Such a link is completely defined by such a vector $(c_1, c_2, \ldots, c_k)$ of positive integer entries. Obviously, two vectors of the form $(c_1, c_2, \ldots, c_k)$ and $(c_k, c_{k-1}, \ldots, c_1)$ define the same link. However, it is much less obvious that two such vectors define different 4-plats if they are not reversal of each other. For a detailed discussion on the classification of 4-plats see [3,8]. In a standard 4-plat diagram there is an obvious way to assign the notion of parallel or anti-parallel to a single crossing, based on if both strings move in the same right-left direction.

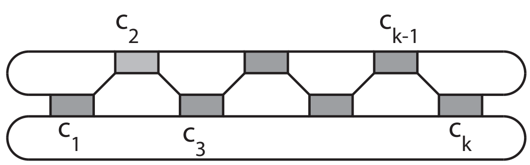

Fig. 7. A typical 4-plat template. A gray box with label c_i represents a horizontal sequence of c_i crossings.

A similar schema based on a vector $T = (a_1, a_2, \ldots, a_n)$ is used to classify rational tangles. A tangle T is part of a link diagram that consists of a disk that contains two properly embedded arcs. For a typical rational tangle diagram see Figure 8 where the rectangular box contains either horizontal or vertical half-twists. A horizontal (vertical) rectangle labeled a_i contains $|a_i|$ horizontal (vertical) half-twists and horizontal and vertical rectangles occur in an alternating fashion. All rational tangles end with a_n horizontal twists on the right. The a_i's are either all positive or all negative, with the only exception that a_n may equal to zero. For a classification and precise definition of such tangles see [3,8]. We assign the notion of parallel or anti-parallel to a single crossing, based on if both strings move in the same right-left direction for a horizontal crossing and based on if both strings move in the same up-down direction for a vertical crossing.

Each rational tangle $T = (a_1, a_2, \ldots, a_n)$ defines a rational number β/α using the continued fraction expansion:

$$\frac{\beta}{\alpha} = a_n + \frac{1}{a_{n-1} + \frac{1}{a_{n-1} + \cdots + \frac{1}{a_1}}}.$$

Rational tangles are the basic building blocks of a large family of links called Montesinos links. A Montesinos link admits a diagram that consists of rational tangles T_i strung together as shown in Figure 9 together with a horizontal number

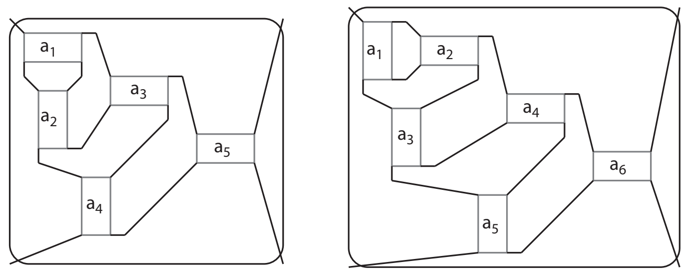

Fig. 8. A rational tangle diagram T given by the vector $T = (a_1, a_2, \ldots, a_n)$. On the left is a diagram with n odd ($n = 5$) and on the left is a diagram with n even ($n = 6$). A small rectangle with label a_i contains $|a_i|$ half-twists, where a_i's are either all positive or all negative, with the only exception that a_n may equal to zero.

of $|e|$ half-twists (as indicated by the rectangle in the figure). Such a diagram is called a *Montesinos diagram*. We say that a Montesinos diagram is of type I if the orientations between the two arcs connecting any two adjacent tangles in the diagram are parallel. In this case the orientations between the two arcs connecting any two other adjacent tangles in the diagram must be parallel as well. We say that diagram is of type II otherwise.

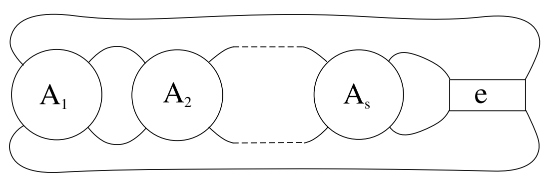

Fig. 9. An illustration of a Montesinos link diagram $D = K(T_1, T_2, \ldots, T_t, e)$ where each T_i is a rational tangle.

Let $\frac{\beta_i}{\alpha_i}$ be the rational number whose continued fraction expansion is the vector that defines the rational tangle T_i. We will sometimes write $K(T_1, T_2, \ldots, T_t, e)$ as $K(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_t}{\alpha_t}, e)$. It is known that a Montesinos link admits a Montesinos diagram satisfying the following additional condition: $|\frac{\beta_i}{\alpha_i}| < 1$ for each i (hence the continued fraction of $\frac{\beta_i}{\alpha_i}$ is of the form $(a_{i,1}, a_{i,2}, \cdots, a_{i,n_i}, 0)$, where $a_{i,j} > 0$). See [3] for an explanation of this and the classification of Montesinos links in general. For a Montesinos link $\mathcal{K}$, let $D_{\mathcal{K}}$ be a Montesinos diagram of $\mathcal{K}$ that satisfies this condition and let P_i be the set of indices i such that $a_{i,j}$ consists of parallel crossings and A_i be the set of indices i such that $a_{i,j}$ consists of anti-parallel crossings. We have the following theorem.

Theorem 3.1. *Let $\mathcal{K}$ be a Montesinos link with Montesinos diagram $D_{\mathcal{K}}$ =*

$K(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_t}{\alpha_t}, e)$, where $|\frac{\beta_i}{\alpha_i}| < 1$. Then the number of Seifert circles in $D_\mathcal{K}$ is given by the following formula

$$s(D_\mathcal{K}) = \begin{cases} \sum_{i=1}^{t}(\sum_{j\in A_i}(|a_{i,j}|-1)+|P_i|)+2 & \text{if } D_\mathcal{K} \text{ is of type I,} \\ \sum_{i=1}^{t}(\sum_{j\in A_i}(|a_{i,j}|-1)+|P_i|)+|e|+c & \text{if } D_\mathcal{K} \text{ is of type II,} \end{cases}$$

where $c = 2$ if $e = 0$ and all tangles end with anti-parallel vertical twists, and $c = 0$ otherwise.

Proof. Consider one of the tangle diagrams $\frac{\beta_i}{\alpha_i} = (a_{i,1}, a_{i,2}, \cdots, a_{i,n_i}, 0)$. If the crossings corresponding to $a_{i,j}$ have parallel orientation, then the crossings correspond to $a_{i,j-1}$ and $a_{i,j+1}$ must have anti-parallel orientation. This can be seen as follows: Assume that $a_{i,j}$ represents $|a_{i,j}|$ vertical twists with a parallel orientation, see Figure 10. Assume further that both strings are oriented upwards. Then we have two strands entering the tangle marked by the dashed oval in Figure 10 from below. Therefore the other two strands of that tangle must have an exiting orientation. This implies that the half-twists at $a_{i,j-1}$ and $a_{i,j+1}$ must be anti-parallel. A similar argument holds if $a_{i,j}$ represents horizontal twists with a parallel orientation.

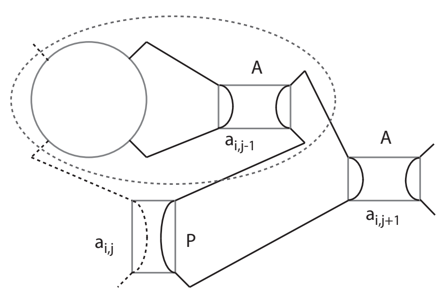

Fig. 10. The local structure of parallel and anti-parallel orientations and the Seifert circle structure resulting from nullification under the assumption that $a_{i,j}$ is vertical and has parallel orientation .

Moreover, there is a Seifert circle that uses the boxes of all three entries $a_{i,j-1}$, $a_{i,j}$ and $a_{i,j+1}$ as shown in the Figure 10. This implies that the dashed arcs at the top left and bottom left of the tangle marked by the dashed oval in Figure 10 must belong to the same Seifert circle. It is easy to see that these properties are the same if $a_{i,j}$ consists of horizontal twists.

Note that after nullification the two arcs in the tangle T_i are changed to a set of disjoint Seifert circles and two disjoint arcs connecting two of the four endpoints of the tangle. The Seifert circles generated be nullification can be grouped into three different categories. The first group consists of the *small Seifert circles*, namely those generated by two consecutive half-twists that are anti-parallel. Clearly, if

$|a_{i,j}| > 1$ and the corresponding crossings are anti-parallel, then there are $|a_{i,j}| - 1$ such small circles. The second group consists of the *medium sized Seifert circles*, namely the Seifert circles that are not small but are contained within one of tangles T_i. The two Seifert circles shown in Figure 10 are medium sized ones since they are contained in a tangle and involve parallel crossings. The third group consists of the rest of the Seifert circles. These are Seifert circles that involve more than one tangle and are called *large Seifert circles*. Figure 10 also shows that for each $a_{i,j}$ that is parallel, one of the two arcs after nullification belongs to a medium Seifert circle and the other belongs to a large Seifert circle. The only exception occurs when $a_{i,j}$ consists of parallel half twists and is the last nonzero entry of the tangle, that is $j = n_i$. In this case both arcs belong to large Seifert circles.

We have thus shown the following:

(i) There are $\sum_{i=1}^{t} \sum_{j\in A_j} (|a_{i,j}| - 1)$ small Seifert circles.

(ii) There are $-p + \sum_{i=1}^{t} |P_i|$ medium Seifert circles where p is the number of tangles T_i where a_{i,n_i} consists of parallel half twists.

If $D_{\mathcal{K}}$ is of type I, then each a_{i,n_i} consists of anti-parallel twists and $p = 0$. Furthermore, e consists of parallel twists as well and it is easy to see that there are exactly two large Seifert circles. (One passes through all the tangles at the bottom and the other weaves through all tangles in a more complex path). This proves the first case of the theorem.

Now assume that $D_{\mathcal{K}}$ is of type II. If a tangle T_i ends with parallel (anti-parallel) vertical twists then after nullification the arc with one end at the NW corner will connect to the SW corner (NE) corner. If we assume that $e = 0$ then we can see that after nullification there will be exactly p of the large Seifert circles when p is nonzero and exactly 2 if $p = 0$. If $e \neq 0$ then there will be an additional $|e| - 1$ small circles and the number of large Seifert circles changes by minus one if $p = 0$ and increases by plus one if $p \neq 0$. This proves the second case of the theorem. □

Corollary 3.2. *Let $\mathcal{K}$ be the Montesinos link represented by the diagram $D_{\mathcal{K}} = K(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_t}{\alpha_t}, e)$, where $|\frac{\beta_i}{\alpha_i}| < 1$ and e is an integer. Then we have*

$$n_d(\mathcal{K}) \leq \begin{cases} \sum_{i=1}^{t} (\sum_{j\in P_i} (|a_{i,j}| - 1) + |A_i|) + |e| - 1 & \text{if } D_{\mathcal{K}} \text{is of type I,} \\ \sum_{i=1}^{t} (\sum_{j\in P_i} (|a_{i,j}| - 1) + |A_i|) - c + 1 & \text{if } D_{\mathcal{K}} \text{is of type II,} \end{cases}$$

where $c = 2$ if $e = 0$ and all tangles end with anti-parallel vertical twists, and $c = 0$ otherwise. Moreover if $D_{\mathcal{K}}$ is alternating, then we have equality.

Proof. It suffices to prove the statement of equality for an alternating Montesinos link. In this case the Corollary follows from Theorem 3.1 and the relationship $n_D(\mathcal{K}) = Cr(D) - s(D) + 1$ for any alternating reduced diagram D of the alternating $\mathcal{K}$. □

Since a 4-plat is a Montesinos link that contains only one rational tangle, we have the following. (Note that a 4-plat is also obtained if there are two rational tangles in the Montesinos link. However this is not important in this context, see [3].)

Corollary 3.3. *Let $\mathcal{K}$ be the 4-plat defined by the vector $(a_1, a_2, \ldots, a_n)$, P be the set of indices i such that a_i consists of parallel crossings and A be the set of indices i such that a_i consists of anti-parallel crossings. Then*

$$n_d(\mathcal{K}) = \sum_{i \in P}(|a_i| - 1) + |A|.$$

Proof. Consider the 4-plat as the Montesinos link given by $D_{\mathcal{K}} = K(\frac{\beta_1}{\alpha_1}, e)$, where $\frac{\beta_1}{\alpha_1} = (a_1, a_2, \ldots, a_{n-1}, 0)$ and $e = a_n$. If a_{n-1} is zero then $\mathcal{K}$ is just an $(e, 2)$ torus link and the statement is true. If a_{n-1} is not zero then we apply Corollary 3.2. If $D_{\mathcal{K}}$ is of type I then e is parallel and in the formula of Corollary 3.2 we count e as parallel and $|e| - 1 = |a_n| - 1$ in the formula of the Corollary. If $D_{\mathcal{K}}$ is of type II then e is anti-parallel and will be counted as the +1 in Formula in Corollary 3.2.□

4. The Nullification Numbers and other Link Invariants

In this section we explore further the relationships among the three nullification numbers $n(\mathcal{K})$, $n_r(\mathcal{K})$ and $n_d(\mathcal{K})$, as well as their relationships with some other link invariants.

4.1. *The case of alternating links.*

First let us consider the alternating links. We have the following theorem.

Theorem 4.1. *If $\mathcal{K}$ is an alternating link then we have $n_d(\mathcal{K}) = n_r(\mathcal{K})$.*

Proof. Since we already have $n_r(\mathcal{K}) \le n_d(\mathcal{K})$, it suffices to show that $n_r(\mathcal{K}) \ge n_d(\mathcal{K})$. Let $cr(\mathcal{K}) = n$. Assume that $n_r(\mathcal{K})$ is obtained by smoothing crossings c_1, c_2,..., c_m first in a reduced alternating diagram D of $\mathcal{K}$. This results in a diagram D_1 with $n - m$ crossings that is still alternating. Assume further that $n_r(\mathcal{K})$ is obtained by deforming D_1 to a minimum diagram D_2 and some crossings in it are then smoothed. D_2 is necessarily alternating since D_1 is alternating and D_2 share the same knot type with D_1. On the other hand, D_1 can be changed to a reduced alternating diagram D_1' by performing all possible reduction moves as shown in Figure 11. D_1' is also minimum since it is reduced and is alternating. Thus it is flype equivalent to the diagram D_2. Note that for each crossing reduction move as shown in Figure 11, one crossing is removed and the number of Seifert circles is reduced by one at the same time. Thus $m+n_{D_2} = m+Cr(D_2)-s(D_2)+1 = m+Cr(D_1')-s(D_1')+1 = s+Cr(D_1)-s(D_1)+1 = Cr(D)-s(D)+1 = n_d(\mathcal{K})$. Therefore no reduction in the number of nullification steps can be gained by moving to the diagram D_2. Since D_2 is still alternating, moving to other minimum diagrams after smoothing some crossings in it will not result in a nullification number reduction either by the same argument. □

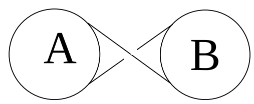

Fig. 11. A reducible alternating diagram contains a single crossing that splits the diagram into two parts. Rotation of one part (either A or B) in a proper direction by 180° will eliminate one crossing while preserving the alternating property of the diagram.

On the other hand, even for alternating links, the difference between $n(\mathcal{K})$ and $n_r(\mathcal{K})$ can be as large as one wants.

Theorem 4.2. *For any given positive integer m, there exists an alternating knot $\mathcal{K}$ such that $n_r(\mathcal{K}) - n(\mathcal{K}) = n_d(\mathcal{K}) - n(\mathcal{K}) > m$.*

Proof. As shown in Figure 12, the 4-plats of the vector form $(-k, -2, -1, 2, k)$ all have general nullification number one, where k is any positive integer.

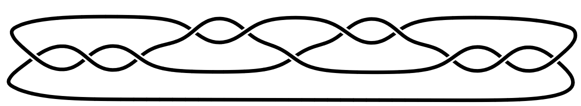

Fig. 12. The 4-plats $(-k, -2, -1, 2, k)$ have nullification number one. Here an example with $k = 3$ is shown. Smoothing the crossing in the center realizes $n(\mathcal{K}) = 1$.

Notice that $(-k, -2, -1, 2, k)$ can be isotoped to $(1, k-1, 3, 1, k)$ as shown in Figure 13, which is alternating. Thus we have an alternating knot $\mathcal{K}$ with general nullification number one. For the minimum diagram of $\mathcal{K}$ given in Figure 13 we have, by Corollary 3.3 or equation 2.1, $n_r(\mathcal{K}) = n_d(\mathcal{K}) = (2k+4) - 5 + 1 = 2k$. So the difference between the diagram nullification number and the general nullification number is $> m$ if $k > (m+1)/2$. □

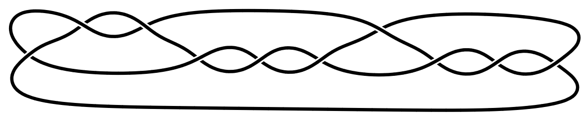

Fig. 13. The 4-plat of Figure 12 in a minimum diagram with vector form $(1, k-1, 3, 1, k)$ for $k = 3$.

4.2. *The case of non-alternating links.*

In order to show that $n_r(\mathcal{K})$ and $n_d(\mathcal{K})$ are indeed different in general, we need to demonstrate the existence of knots/links $\mathcal{K}$ such that $n_r(\mathcal{K}) < n_d(\mathcal{K})$. Because of Theorem 4.1, such examples can only be found in non-alternating knots and links. Worse, there are no known methods or easy approaches in finding such examples. This subsection is thus devoted to the construction of one single such example.

First, let us observe that the knot 8_{20} has the following special property. The left of Figure 14 shows that $n_d(8_{20}) = n_r(8_{20}) = n_d(8_{20}) = 1$. However, after applying a simply isotopy as shown on the right side of Figure 14, the new diagram can no longer be nullified by smoothing only one crossing. One has to smooth two crossings.

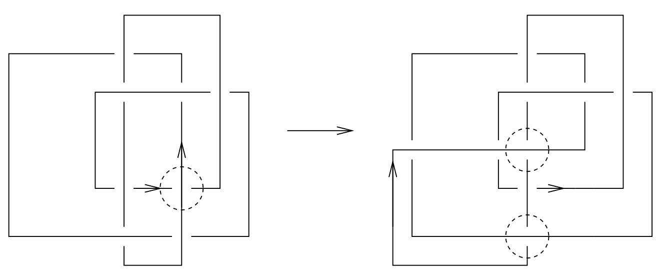

Fig. 14. Left: A minimum knot projection M of the knot 8_{20} that can be nullified by smoothing the crossing circled. Right: A non-minimum diagram N of 8_{20} that requires the smoothing of two crossings in order to be nullified (two such crossings are marked with circles).

Now we would like to construct an example using this observation. We construct a three component link L by adding two simple closed curves to N as shown in Figure 15 (drawn by thickened lines). The diagram D_L of Figure 15 can be shown to be adequate hence is minimum [8]. Assign the orientations to the components as shown in Figure 15.

It is easy to see that nullifying the two adjacent crossings marked in Figure 15 allows the new components to be removed by an ambient isotopy. The resulting diagram is N which can be further isotoped to M. Thus by the definition of n_r we

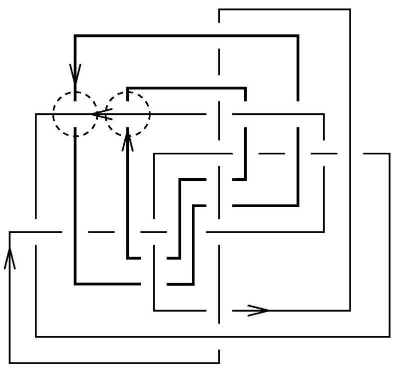

Fig. 15. A 3-component link L with $n_r(L) \leq 3$ and $n_d(L) = 4$.

have $n_r(L) \leq 3$.

Lacking a more elegant method, we took a programmatic approach for the confirmation that $n_D(L) > 3$. First, we observe that the diagram of Figure 15 is the only minimum diagram of L up to trivial isotopies. This follows from the fact that if we remove either one of the new components we obtain an alternating and hence minimum diagram that admits no flypes. Thus any diagram of such a two component link has to look like the one shown. The two additional components are "parallel" and therefore the only change we can make is to exchange them. This however does not change the diagram. We then implemented the nullification procedure which yielded all knot/link diagrams obtained by all possible combinations of 3 or less crossing smoothing steps on D_L. Using a Gauss code modification program and the Mathematica©KnotTheory package's Jones polynomial computation we were able to verify that none of these resulted in a knot/link with the polynomial of a trivial knot/link. Since $n_d(L) \leq 4$ by our construction of L, we have shown that $n_d(L) = 4$. Thus we have shown an example of a three component link L with $n_r(L) \leq 3$ and $n_d(L) = 4$.

4.3. *The genus, unknotting number and the signature vs* $n(\mathcal{K})$*.*

There is a simple inequality between the nullification number and the unknotting number:

Lemma 4.3. *Let* $\mathcal{K}$ *be any knot then* $n(\mathcal{K}) \leq 2u(\mathcal{K})$*, where* u *denotes the unknotting number of a knot.*

Proof: It suffices to show that a strand passage can be realized by two nullification moves. This is shown in Figure 16.

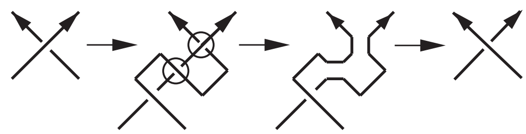

Fig. 16. Nullifying two crossings is equivalent to a strandpassage.

The next theorem shows that an inequality as given in Lemma 4.3 does not exist the other way around, that is the unknotting number can not be bound from above by a multiple of the general nullification number.

Theorem 4.4. *For any given positive integer m, (1) there exists an alternating knot $\mathcal{K}$ such that $g(\mathcal{K}) - n(\mathcal{K}) > m$ where $g(\mathcal{K})$ is the genus of $\mathcal{K}$ and (2) there exists a link $\mathcal{K}$ such that $u(\mathcal{K}) - n(\mathcal{K}) > m$. In other words, the general nullification number does not impose a general upper bound on the genus and the unknotting number of a link.*

Proof. Notice that the genus of the knot $\mathcal{K}$ in Figure 13 is k. Thus if $k > m + 1$, then $g(\mathcal{K}) - n(\mathcal{K}) = k - 1 > m$. For the second part of the theorem, consider the torus link $T(3n, 3)$ (n is an arbitrary positive integer) as shown in Figure 17, in which two of the three components are oriented in parallel (say clockwise) and the third component is oriented in the other direction (say counterclockwise). By nullifying any one of the crossings between two components with opposite orientations in Figure 17 we obtain the unlink. Thus $n(T(3n, 3)) = 1$. On the other hand, by the Bennequin Conjecture [1,27], the unknotting number of $T(3n, 3)$ is $3n$ and the result of the second part of the theorem follows. □

The signature $\sigma(\mathcal{K})$ of a link $\mathcal{K}$ is defined as the signature of the matrix $M + M^T$ where M is the Seifert matrix obtained from any regular diagram of $\mathcal{K}$. The definition and computation of a Seifert matrix is beyond the scope of this paper and we refer the reader to any standard text in knot theory such as [3,8].

Theorem 4.5. *For any oriented link $\mathcal{K}$ we have $n(\mathcal{K}) \geq |\sigma(\mathcal{K})|$.*

Proof. We use an approach similar to the one used in the proof of Theorem 6.8.2 in [8] (where the relationship between the signature and the unknotting number is investigated). We will need the following lemma 4.6 from [14]. A σ-series of an $n \times n$ matrix of rank r is a sequence of submatrices Δ_i such that (i) Δ_i is an $i \times i$ matrix; (ii) Δ_i is obtained from Δ_{i+1} by removing a single row and a single column; and (iii) no two consecutive matrices Δ_i and Δ_{i+1} are singular when $i < r$.

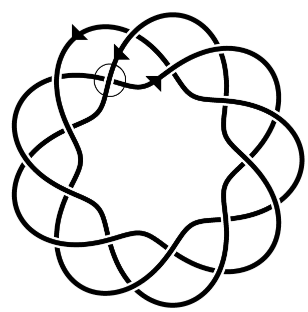

Fig. 17. The torus link $T(3n, 3)$ with one component of reverse orientation has nullification number one and unbounded unknotting number. The special case $n = 3$ is shown here.

Lemma 4.6. *[14] Let A be a symmetric $n \times n$ matrix with a σ-series as described above. Put $\Delta_i = 1$. Then the signature of A is given by*

$$\sigma(A) = \sum_{i=1}^{n} \operatorname{sgn}(\det(\delta_{i-1}) \det(\delta_i)),$$

where sgn() *is the sign function.*

Let D_+, D_- and D_0 be three link diagrams that are identical except at one crossing as shown in Figure 18 and let L_+, L_- and L_0 be their corresponding link types. We then have the following [22,18]:

$$|\sigma(L_\pm) - \sigma(L_0)| \leq 1.$$

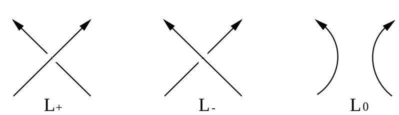

Fig. 18. Three diagrams that differ only at one crossing.

To prove this, let F_+, F_- and F_0 be the projection surfaces constructed from D_+, D_- and D_0, respectively. Let M_+, M_- and M_0 be the Seifert matrices constructed from these surfaces. If F_0 is disconnected then L_+ and L_- are isotopic links equivalent to a connected sum $L_1 \# L_2$ for some links L_1 and L_2.

Therefore in this case $\sigma(L_1 \# L_2) = \sigma(L_1) + \sigma(L_2) = \sigma(L_1 \sqcup L_2) = \sigma(L_0)$ and $\sigma(L_+) - \sigma(L_0) = \sigma(L_-) - \sigma(L_0) = 0$.

If F_0 is connected then F_+ and F_- are obtained from F_0 by adding a twisted rectangle at the crossing that is switched or eliminated. Therefore the Seifert matrices M_+ and M_- have one additional column and row added to the Seifert matrix M_0 as shown below for M_+ where b is an additional loop that passed through the added rectangle and back through the rest of the surface F_0 and $(a_1, \cdots, a_n)$ is a basis for $H_1(F_0)$.

$$\begin{array}{c|cccc} & a_1^+ & \cdots & a_n^+ & b^+ \\ \hline a_1 & & & & v_1 \\ \vdots & & M_0 & & \vdots \\ a_n & & & & v_1 \\ b & \lambda_1^+ & \cdots & \lambda_n^+ & \beta \end{array}$$

We write $A_* = M_* + M_*^T$ where $* = +, -$ or 0. The matrices A_* may not be singular, however they are non-singular if the corresponding L_* is a knot. In either case we can use Lemma 4.6 to obtain that:

$$\sigma(A_\pm) - \sigma(A_0) = sign(\det(A_\pm)\det(A_0)) = \delta,$$

where $* = +$ or $-$ and from this it follows that $\delta = \pm 1$ or 0.

This result implies that in any nullification sequence of $\mathcal{K}$, the smoothing of a crossing can only change the signature by at most one. Since the signature of the trivial link is zero and $n(\mathcal{K})$ is the minimum number of smoothing moves needed to change $\mathcal{K}$ to a trivial link, it follows that $|\sigma(\mathcal{K})| \le n(\mathcal{K})$. □

Corollary 4.7. *Let $\mathcal{K}$ be any knot then $|\sigma(\mathcal{K})| \le n(\mathcal{K}) \le 2u(\mathcal{K})$. In particular if $|\sigma(\mathcal{K})| = 2u(\mathcal{K})$, then $|\sigma(\mathcal{K})| = n(\mathcal{K}) = 2u(\mathcal{K})$.*

Note that the above Corollary is quite powerful if one wants to determine the actual general nullification number for small knots in the knot table. In [10] the general nullification number of all but two knots of up to 9 crossings is determined. The following corollary is an immediate consequence of Theorem 4.5 and the fact that all knots have even signatures [21].

Corollary 4.8. *If $\mathcal{K}$ is a knot and $n(\mathcal{K}) = 1$, then $\sigma(\mathcal{K}) = 0$.*

In [13] the signature of an oriented alternating link $\mathcal{K}$ is related to the writhe via the concept of nullification writhe. Let D be a reduced alternating diagram of $\mathcal{K}$ and in this diagram we have a nullification sequence of crossings c_1, c_2, ... , c_k, where $k = n_D = n_d(\mathcal{K})$, then the nullification writhe introduced in [6] is defined as the sum of the signs of the crossings c_1, c_2, ... , c_k and is denoted by $Wr(n_D)$. We have:

Theorem 4.9. *[13] For an alternating oriented link $\sigma(\mathcal{K}) + Wr(n_D) = 0$.*

The example of the 4-plat given in Figures 12 and 13 (with $k = 3$) represents the knot 10_{22}. This knot has $\sigma(10_{22}) = 0$ (In fact we can see that a nullification sequence for the diagram in Figure 13 has a nullification sequence of three positive and three negative crossings.) On the other hand we see that the inequality in Theorem 4.5 is strict since $n(10_{22}) = 1$. Note that the signature of a 4-plat can be computed by the following explicit formula.

Theorem 4.10. *[25] Let $\mathcal{K}$ be a 4-plat knot then*

$$\sigma(\mathcal{K}) = \sum_{i=1}^{2g} (-1)^{i-1} sign(a_i),$$

where the vector $(a_1, a_2, ..., a_{2g})$ is a continued fraction expansion of $\mathcal{K}$ using only even integers. (Such a continued fraction expansion is called an even continued fraction expansion*.)*

In fact, the length of the even continued fraction expansion is $2g$ where g is the genus of $\mathcal{K}$ [8]. We also have the following two corollaries:

Corollary 4.11. *If D is a reduced alternating diagram of an alternating oriented link $\mathcal{K}$ such that all crossings in a nullification sequence of D have the same sign, i.e., $|Wr(n_D)| = n_D$, then $n_d(\mathcal{K}) = n(\mathcal{K})$.*

Proof. We have $|\sigma(\mathcal{K})| = |Wr(n_D)| = n_D = n_d(\mathcal{K}) \geq n(\mathcal{K}) \geq |\sigma(\mathcal{K})|$. Thus the equality holds. □

Corollary 4.12. *For any even positive integer k the number of knots $\mathcal{K}$ with $Cr(\mathcal{K})$ crossings and $|n(\mathcal{K})| \geq k$ grows exponentially with $\sqrt{Cr(\mathcal{K})}$.*

Proof. Let us assume for simplicity that $m = Cr(\mathcal{K})$ is odd and consider the 4-plats $\mathcal{K}$ with a vector form $(a_1, a_2, ..., a_{2g})$ such that a_i is even and $a_i a_{i+1} < 0$ for all i. Then $\sigma(\mathcal{K}) = 2g$. The crossing number $Cr(\mathcal{K})$ is given by [8]

$$Cr(\mathcal{K}) = (\sum_{i=1}^{2g} |a_i|) - 2g + 1.$$

If $m = Cr(\mathcal{K})$ is sufficiently large then we can generate such a vector by partitioning the positive integer $m + 2g - 1$ into $2g$ even integers b_i where $b_i \geq 2$. Here we set

$a_i = (-1)^{i+1} \cdot b_i$. This is equivalent to partitioning the integer $m-2g-1$ into $2g$ even integers c_i where $c_i \geq 0$ which in turn is equivalent to the number of partitions of the integer $(m-1)/2-g$ into $2g$ integers d_i where $d_i \geq 0$. A standard result in number theory about the number of partitions then implies the result. □

5. Nullification Number One Links

In this section we explore the special family of links whose general nullification number is one. The first question is about the number of such links. In the following we will give a partial answer to this question. Let $(a_1, a_2, \ldots, a_k)$ be the standard vector form of a rational link $\mathcal{K}$ corresponding to the rational number $\frac{p}{q}$. Consider the rational link $\mathcal{K}'$ defined by the vector $(a_1, a_2, \ldots, a_k, \varepsilon, -a_k, \ldots, -a_1)$, where $\varepsilon = \pm 1$. Let $\frac{a}{b}$ be the fraction for $\mathcal{K}'$, then we can get $\frac{a}{b} = \frac{p^2}{(\varepsilon + pq)}$ by using a method from [23]. Furthermore, when $\mathcal{K}$ is a knot, $\mathcal{K}'$ is also a knot and has general nullification number 1.

Two fractions $\frac{p^2}{(\varepsilon_1+pq)}$ and $\frac{p^2}{(\varepsilon_2+pq')}$ represent the same knot iff $\varepsilon_1 + pq \equiv \varepsilon_2 + pq' \left(\operatorname{mod} p^2\right)$ or $(\varepsilon_1 + pq)(\varepsilon_2 + pq') \equiv 1 \left(\operatorname{mod} p^2\right)$. If $\varepsilon_1 = \varepsilon_2$ then this yields $pq \equiv pq' \left(\operatorname{mod} p^2\right)$ or $pq \equiv -pq' \left(\operatorname{mod} p^2\right)$. So $q \equiv q' \,(\operatorname{mod} p)$ or $q \equiv -q' \,(\operatorname{mod} p)$. This implies that different fractions $\frac{p}{q}$ result in different 4-plats $\frac{p^2}{(\varepsilon_1+pq)}$ (up to mirror images).

Thus we have shown that for each rational knot $\mathcal{K}$, there exists a rational knot $\mathcal{K}'$ (unique up to mirror image) with nullification number one. Furthermore, $Cr(\mathcal{K}') = 2\sum_{i=1}^{k} a_i = 2Cr(\mathcal{K})$. Since the number of rational knots grows exponentially, we have shown the following theorem.

Theorem 5.1. *The number of knots $\mathcal{K}$ with $n(\mathcal{K}) = 1$ and $Cr(\mathcal{K}) \leq n$ grows exponentially in terms of n.*

We would like to point out that the rational knots considered above do not contain all nullification number one rational knots. We give two additional example of 4-plat knot families with nullification number one. In the two examples we do not follow the usual vector notation for rational knots such as in [3]. Instead we adopt the sign assignment convention as shown in Figure 19 for the crossings that are in the boxes in Figures 20 and 21. Here the crossings have two ends marked as on the bottom and the other two ends as on the top and therefore the crossings in Figure 19 can not be rotated by 90 degrees.

Example 5.2. *Consider the rational knot defined by a vector of the form $(-2a, -2, -2b, 2, 2a, 2b)$, where the first $2a$ means a sequence of $2a$ positive crossings between the first and second strings using the sign convention given in Figure*

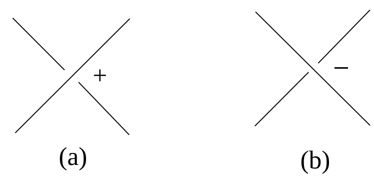

Fig. 19. Crossing (a) is positive and crossing (b) is negative. Notice that this is not how the sign of a crossing in a rational knot is assigned and is only used for the examples here.

19, and so on, as shown in the first diagram of Figure 20. Note that the actual sign convention of the crossings in the boxes does not matter, as long as boxes with opposite signs have twists that are mirror images of each other. By a rotation involving the two top boxes in the first diagram, followed by a proper flype involving the resulting two boxes and some other isotopes, one can see that the first diagram is equivalent to the second one. From there it is relatively easy to see the second diagram can be isotoped to the third diagram. If we smooth the crossing as marked in the third diagram, we end up with the fourth diagram. It is not too hard to see that the fourth diagram is the trivial knot. Thus this family of rational knots is also of general nullification number one. The details of the isotopies used are left to the reader as an exercise.

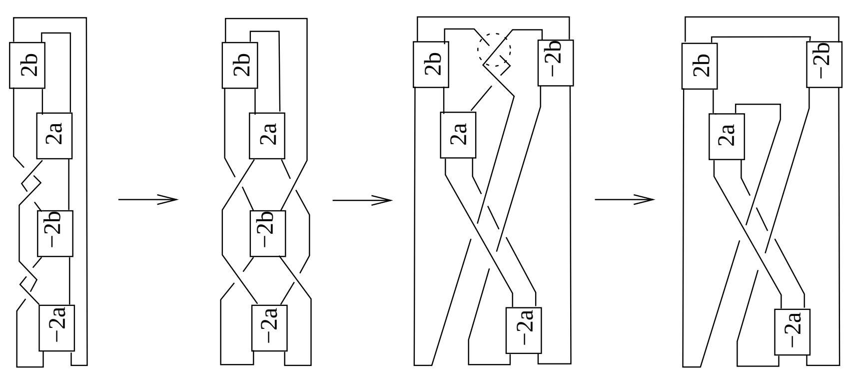

Fig. 20. The family of rational knots with the vector form $(-2a, -2, -2b, 2, 2a, 2b)$ using the sign convention as defined in Figure 19. These knots have general nullification number one, where the signed number on a box indicates the number of half twists with the shown sign.

Example 5.3. *The second example (of other nullification number one rational knots) is similarly constructed. Here the knot family consists of rational knots defined by vectors of the form* $(-2b, -2, -2a, 2b, 2, 2a)$ *as shown in the first diagram of Figure 21. The isotopes leading to the single nullification crossing of the knot are* *illustrated in the rest of the figure.*

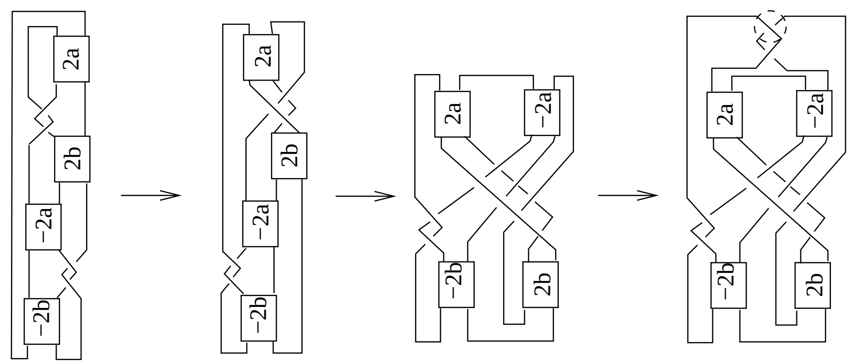

Fig. 21. Another family of rational knots (defined by vectors of form $(-2b,-2,-2a,2b,2,2a)$ using the sign convention as defined in Figure 19) with general nullification number one.

Note that the actual sign convention as shown in in Figure 19 does not matter as long as the the crossings in the two boxes with the same label have opposite handedness. These three families of rational knots have one interesting property in common, namely that they are all ribbon knots (to be defined next). In fact it was conjectured in [5] that these were the only rational ribbon knots. This conjecture was recently proven in [17].

Let K be a link in S^3, and $b : I \times I \to S^3$ be an imbedding such that $b\,(I \times I) \cap K = b\,(I \times \partial I)$. Let $K_b = \{K \setminus b(I \times \partial I)\} \cup b\,(\partial I \times I)$. If K_b is a link with an orientation compatible with K then K_b is called a *banding* of K, or is obtained from K by band surgery (along b). It turns out that nullification and banding are equivalent operations. See Figure 22.

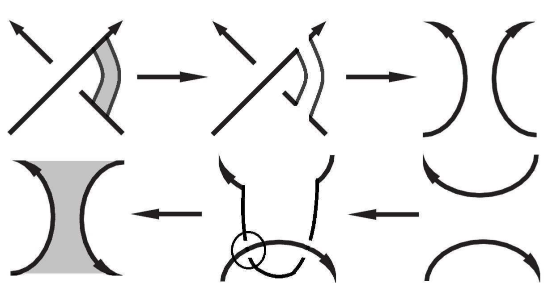

Fig. 22. Top: Nullification via banding. Bottom: Banding via nullification.

A knot $\mathcal{K}$ is a *ribbon knot* if it is a knot obtained from a trivial $(m+1)$-component link by band surgery along m bands for some m. The minimum of such number m is called the *ribbon-fusion number* of $\mathcal{K}$ and is denoted by $\mathrm{rf}(\mathcal{K})$. (This concept was introduced by Kanenobu [15].) If a knot can be nullified in one step then it yields the trivial 2-component link. So a knot has nullification number one if and only if it is a 1-fusion ribbon knot. Using this, one can easily find all nullification number one knots. This can be done by quickly referencing all ribbon presentations of ribbon knots with 10 or fewer crossings (such as in [16]). All such diagrams either explicitly have the necessary crossing, or will have one after applying a Reidemeister move of type II as shown in Figure 23. Furthermore, the fact that 1-fusion ribbon knots have nullification number one, together with the following theorem due to Tanaka [26], implies that the nullification number and the bridge number of a knot are unrelated in general.

Theorem 5.4. *[26] For any pair (p, q) of positive integers with $1 \leq p \leq q$, there exists a family of infinitely many composite ribbon knots K_i such that $2(p+q)+1 \leq b(K_i) \leq 2(p+q+1)$ and $rf(K_i) = p$, where $b(K_i)$ denotes the bridge number of the knot K_i.*

In particular if we let $(p, q) = (1, q)$ then we get an infinite family of 1-fusion ribbon knots (nullification number one knots) with arbitrarily large bridge number.

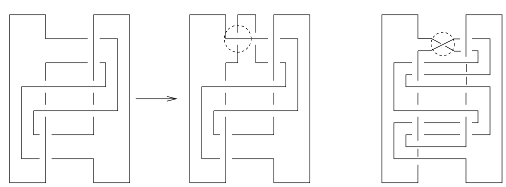

Fig. 23. Two examples of nullifying ribbon presentations: on the left is the knot 6_1 where the presentation requires a type II Reidemeister move first, on the right is the knot 8_9 where no Reidemeister moves are needed.

For higher fusion numbers it is not *a priori* clear whether nullification number is equal to the fusion number. Fusion number m relies on the separation of a ribbon knot into $(m + 1)$ trivial components, but nullification does not have such a restriction. So for a given ribbon knot $\mathcal{K}$, it is only obvious that $n(\mathcal{K}) \leq \mathrm{rf}(\mathcal{K})$.

Let us end our paper with a few open questions.

1. For a knot or link L, how big can the difference between $n_r(L)$ and $n_d(L)$ be? Can we find a class of knots/links such that $n_d(L) - n_r(L)$ is unbounded over all L from this family?

2. If D' is a diagram obtained from an alternating (reduced) diagram D by one crossing change (so D' is no longer alternating and it may even be the trivial knot/link), how much smaller is $n_{D'}$ compared to n_D? For alternating knots with unknotting number one, $n_{D'}$ is simply zero hence this difference can be as large as one wants. However, is there a way to relate this problem with the unknotting numbers in general?

3. By nullifying one crossing in a diagram D (not necessarily minimum) of a knot/link $\mathcal{K}$, we obtain a new knot/link. How many different knots/links can be obtained this way? In particular, is this number bounded above by $Cr(\mathcal{K})$?

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