BRANCHING PROCESSES IN RANDOM TREES

by

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ABSTRACT

YANJMAA JUTMAAN. Branching processes in random trees.
(Under the direction of DR. STANISLAV MOLCHANOV)

We study the behavior of the branching process in a random environment on trees in the critical, subcritical and supercritical case. We are interested in the case when both the branching and the step transition parameters are random quantities. We present quenched and annealed classifications for such processes and determine some limit theorems in the supercritical quenched case. Corollaries cover the percolation problem on the random trees. The main tools are the compositions of the random generating functions.

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TABLE OF CONTENTS

LIST OF FIGURES vi

CHAPTER 1: GALTON-WATSON PROCESS IN HOMOGENEOUS MEDIUM 1

- 1.1 Introduction 1
- 1.2 Simple Galton-Watson process 2
  - 1.2.1 Generating function of $n(t, \omega)$ 2
  - 1.2.2 Moments of the $n(t, \omega)$ 3

CHAPTER 2: CLASSICAL GALTON-WATSON BRANCHING PROCESSES. DIFFERENT APPROACHES TO THE LIMIT THEOREM IN THE SUPERCRITICAL CASE 6

- 2.1 Limit theorems 6
- 2.2 Moment of analysis approach to the limit theorem 9

CHAPTER 3: GALTON-WATSON IN RANDOM ENVIRONMENT 17

- 3.1 Construction of Random Environment 17
- 3.2 Annealed and quenched law 19
- 3.3 Moments and quenched classification of of Galton-Watson process in random environment 20

CHAPTER 4: PERCOLATION ON THE RANDOM TREE 30

- 4.1 Introduction and notation 30
- 4.2 Binary Random tree percolation 32

REFERENCES 38

LIST OF FIGURES

FIGURE 1.1 Moment generating function and extinction probability 5

FIGURE 4.1 The percolation probability $\theta(p)$ behaves roughly as indicated 31

FIGURE 4.2 Homogeneous binary tree 31

FIGURE 4.3 Percolation on the binary tree and the binary tree 32

FIGURE 4.4 Binary tree with random environment 33

CHAPTER 1: GALTON-WATSON PROCESS IN HOMOGENEOUS MEDIUM

1.1 Introduction

Branching Stochastic processes [23] have always been very intresting for mathematicians and physicists. They describe well a multitude of phenomena from chain reactions to population dynamics. [20]. Random walks in random environment on Z^d constitute one of the basic models of random motions in a random medium; see Bolthausen [2] Hughes [15], Molchanov [21] and Zeitouni [27], [28]. Several authors have considered branching random walk in random environment [18], [1], [3]-[6], [8], [15], [10] and [14]. On the other hand ordinary [9] and multiscale [19] percolaltion play a critical role in many applications [20].

There is significant literature on the random walk and the branching (reaction diffusion) process in the random environment on the lattice Z^d or the Euclidean space R^d, $d \geq 1$.

In most of these papers the classical Fourier analysis and the spectral theory of the Schröndinger type operators play the central role (explicitly and implicitly) corresponding to models and construction on the different graph, manifolds and groups. To the best of our knowledge branching random walks in random environment on trees have not been yet considered in the literature [18], [17]. The goal of this paper is the analysis of random processes in the random environments on the simplest graphs: the trees. We study branching processes and random walks in the situation when transition probabilities are random and homogeneous. The paper is organized as follows: in this chapter we begin by introducing some notations, terminologies and construction of random environment on the tree and some preliminary results as well as introducing basic examples and models. In Chapter 2 we will define the

Galton-Watson process in random environment, and its moment generating functions and moments. In Chapter 3 we prove several results on the annealed and quenched behavior of Galton-Watson random variable $n(t, \omega, \omega_m)$. We also discuss some limit theorems. In Chapter 4 we study Percolation problem on the random tree, critical probability of percolation and some properties.

1.2 Simple Galton-Watson process

Galton-Watson processes were introduced by Francis Galton in 1889 as a simple mathematical model for the propagation of family names. Galton-Watson processes continue to play a fundamental role in both the theory and applications of stochastic processes.

First, a description: A population of individuals (which may represent people, organisms, free neutrons, etc., depending on the context) evolves in discrete time $t = 0, 1, 2, \ldots$ according to the following rules. Each nth generation individual produces a random number (possibly 0) of individuals, called *offspring*, in the $(t+1)^{th}$ generation. The offspring counts n_t for distinct individual are mutually independent, and also independent of the offspring $X_{t,i}$ counts of individuals t, i from earlier generations. Furthermore they are identically distributed. The state $n(t)$ of the Galton-Watson process at time t is the number of individuals in the t^{th} generation.

$$n(t+1, \omega) = \sum_{i=1}^{n(t)} X_{t,i}(\omega) \tag{1.1}$$

where the law $X_{t,i}$ are independent identically distributed (i.i.d) random variables on some probability space (Ω, Γ, P).

1.2.1 Generating function of $n(t, \omega)$

$$\varphi(z) = Ez^{X_{n,i}} \tag{1.2}$$

and

$$\varphi'(z)|_{z=1} = a = EX.\ EX(X-1) = a_{[2]}$$

The iterated of the generating function φ_s will be defined by

$\varphi_0 = z$, $\varphi_{[1]} = \varphi(z)$,

$$\varphi_{[s+1]}(z) = \varphi(\varphi_{[s]}(z)) = \varphi_{[s]}(\varphi(z)), n = 1, 2, \dots \tag{1.3}$$

The following basic result was discovered by Watson (1874) and has been rediscovered a number of times since.

Theorem 1.1. *[13] [see Theorem 4.1] The generating function of $n(t)$ is the t-th iterate $\varphi_{[t]}(z)$.*

1.2.2 Moments of the $n(t, \omega)$

We can obtain the moments of $n(t, \omega)$ by differentiating (1.3) at $z = 1$. Thus differentiating (1.3) yields

$$\varphi'_{[t+1]}(1) = \varphi'_{[t]}\left(\varphi(1)\right)\varphi'(1) \tag{1.4}$$

whence by induction $\varphi'_{[t]}(1) = a^t$,$n = 0, 1, \dots$. If $\varphi''(1) < \infty$, we can differentiate (1.3) again, obtaining,

$$\varphi''_{[t+1]}(1) = \varphi'_{[t]}(1)\varphi''(1) + \varphi''_{[t]}(1)(\varphi')^2 \tag{1.5}$$

we obtain $\varphi''_{[t]}(1)$ by repeating application of (1.5) with $n = 0, 1, \dots$; thus

$$E(n_t(n_t-1)) = \begin{cases} \frac{a_{[2]}a^t(a^t-1)}{a(a-1)} & , a \neq 1 \\ t \cdot a_{[2]} & , a = 1 \end{cases} \tag{1.6}$$

Theorem 1.2. *Expected value of* $E\{n(t,\omega)\}$ *is* a^t, $t = 0, 1, \ldots$. *If second factorial moment of* X *is bounded, the the second factorial moment of* $n(t,\omega)$ *is given by* (1.6).

If higher moments of X exist, then higher moments of $n(t,\omega)$ can be found in a similar fashion.

We now consider the problem originally posed by Galton: find the probability of extinction of a family.

Definition 1.1. [13] By extinction we mean the event that the random sequence $n(t,\omega)$ consists of zeros for all but a finite number of values of n.

Since $n(t)$ is integer-valued, extinction is also the event that $n(t) \to 0$. Moreover, since $P(n(t+1,\omega) = 0|n(t,\omega) = 0) = 1$, we have the equalities

$$\begin{aligned} P(n(t) \to 0) &= P(n(t) = 0 \text{ for some } t) = P\left((n(1) = 0) \cup (n(2) = 0) \cup \ldots\right) \\ &= \lim_{t\to\infty} P\left((n(1) = 0) \cup (n(2) = 0) \cup \ldots (n(t) = 0)\right) \\ &= \lim_{t\to\infty} P(n(t) = 0) = \lim \varphi_{[t]}(0) \end{aligned} \tag{1.7}$$

Definition 1.2. Let $z*$ be the probability of extinction, i.e.,

$$z* = P(n(t) \to 0) = \lim \varphi_{[t]}(0) \tag{1.8}$$

Theorem 1.3. *If* $a = EX \leq 1$, *the extinction probability* $z*$ *is 1. If* $a > 1$, *the extinction probability is the unqique nonnegative solution less than 1 of the equation*

$$z = \varphi(z) \tag{1.9}$$

First proved in complete generality by Steffensen (1930,1932)

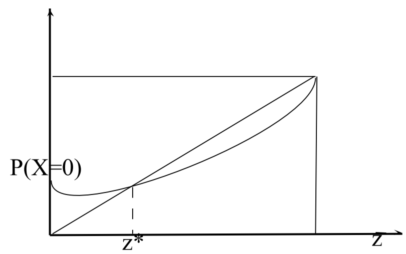


Figure 1.1: Moment generating function and extinction probability
Moment generating function $\varphi(z) = Ez^x$
where $z*$ is the solution of $z = \varphi(z)$ equation.

Solution and moment generating function are illustrated in Figure 1.1.

CHAPTER 2: CLASSICAL GALTON-WATSON BRANCHING PROCESSES. DIFFERENT APPROACHES TO THE LIMIT THEOREM IN THE SUPERCRITICAL CASE

2.1 Limit theorems

In this chapter we will prove several limit theorems for Branching process in random environment. To understand the idea of the method let us recall the classical result by N. A. Dmitriev and A.N. Kolmogorov [7]. Assume that $n(t)$, $t = 0, 1, \ldots$, $n(1) = 1$ is the classical Galton-Watson process in the fixed environment. If $\varphi(z) = Ez^X$ where X is the number of the offspring of single particle; then

$$\varphi_{[n]}(z) = \varphi(\varphi(\ldots(\varphi(z))\ldots)) \tag{2.1}$$

$$E\{n(t)\} = a^t,\ a = a_1 = \varphi(z = 1) = EX \tag{2.2}$$

Then, for $a > 1$ in the supercritical case we have

Theorem 2.1. *(N.Dmitriev, A.Kolmogorov, [7])*

$$P\left(\frac{n(t)}{a^t} < x\right) \to G(x) \tag{2.3}$$

where G is a limit distribution.

Remark. For continuous time branching Galton-Watson processes when the intensity of the duplication of a particle is λ, mortality rate is μ, $\lambda > \mu$ then the limit law for $\frac{n(t)}{a^t}$ also can be found explicitly. This limit law is the exponential distribution with additional atom at $x = 0$.

G is only known in the case of geometric distribution. Consider the generating function of the geometric distribution

$$P\{X=0\}=\alpha, P\{X=1)=\{1-\alpha\}q\ ,\ \dots\ P\{X_t=k\}=(1-\alpha)qp^{k-1},\dots \quad (2.4)$$

Then

$$\varphi(z)=Ez^{X_\tau}=\alpha+\frac{(1-\alpha)qz}{1-pz}=\frac{\alpha+z(q-\alpha)}{1-pz}$$
$$\varphi'(z)|_{z=1}=\frac{1-\alpha}{q}$$

The expansion of this generating function into Tailor series gives the geometric law for $k\geq 1$ and different probability for $\{X=0\}$, $P\{X=0\}=\alpha$. The composition $\varphi_{[n]}(z)=\varphi(\varphi(\dots(\varphi(z))\dots))$ has the same nature.
Note that

$$a=EX=\frac{1-\alpha}{q}$$

i.e process is supercritical for

$$\frac{1-\alpha}{q}>1$$
$$\Rightarrow 1>\alpha+q$$
$$\Rightarrow p>\alpha$$

The limiting distribution for $\frac{n(t)}{a^t}$, $t\to\infty$ is the combination of the exponential density and the atom at $x=0$, i.e it has density

$$\frac{n(t)}{a^t}\to g_\infty(x)=\beta\delta_0(s)+(1-\beta)\,\lambda e^{-\lambda x} \quad (2.5)$$

Let's find the $\beta,\ \lambda$. First

$$1 = E\frac{n(t)}{a^t} \longrightarrow \int_0^\infty x g_\infty(x)dx = \frac{1-\beta}{\lambda}$$

i.e $\lambda = 1-\beta$. Secondly

$$P\{n(t)=0\} \underset{t\to\infty}{\longrightarrow} s^* = \beta$$

, where $s^* = \varphi(s^*),\ s^* < 1$. i.e

$$\begin{aligned} s^* &= \frac{\alpha + s^*(q-\alpha)}{1-ps^*} \\ s^* - p(s^*)^2 &= \alpha + s^*(q-\alpha) \\ p(s^*)^2 - s^*(1-q+\alpha) + \alpha &= 0 \\ p(s^*)^2 - s^*(p+\alpha) + \alpha &= 0 \end{aligned}$$

One of the roots of this quadratic equation is $s^* = 1$, the second root which we need is

$$s^* = \frac{\alpha}{p} \tag{2.6}$$

Finally for β, λ we have the linear system

$$\begin{cases} \lambda &= 1-\beta \\ \alpha &= p\beta \end{cases} \Rightarrow \begin{cases} \lambda &= 1-\beta = \frac{p-\alpha}{p} \\ \beta &= \frac{\alpha}{p} < 1 \end{cases}$$

Let us return to main direction and give three proofs of this result (2.3) different from original work of Dimitriev and Kolmogorov.

Proof. 1 (Martingale approach).

Consider $\xi(t) = \frac{n(t)}{a^t}$ and $F_{\le t} = \sigma(n(1), n(2), \ldots, n(t))$ Then

$$n(t+1) = \sum_{i=1}^{n(t)} X_{t,i},$$

Here $X_{t,i}$ are independent identically distributed random variable with generating function $\varphi(z)$

As easy to see that $(n(t), F_{\le t})$ is a martingale. In fact

$$E\{n(t+1)|F_{\le t}\} = n(t) \cdot a$$

So

$$E\{\xi(t+1)|F_{\le t}\} = \frac{n(t)}{a^t} = \xi(t)$$

Due to Doob martingale convergence theorem P almost surely there exist

$$\lim_{t\to\infty} \xi(t) = \xi(\infty)$$

and the law of the $\xi(\infty)$is $G(x)$ *Q.E.D.*

This method can not give any information about $G(\cdot)$.

Remark. The same proof works, of course in the case $a \le 1$, but here the limit is equal to 0.

2.2 Moment of analysis approach to the limit theorem

The second method is based on the moments analysis.

Lemma 2.2. *Assume that*

$$P\{X = k\} \le cq^k,\ k \ge 0;\ 0 < q < 1 \tag{2.7}$$

then for any $1 > q_1 > q$, appropriate constant $M_1(q_1)$ and $n \geq 1$,

$$(a)_{(n)} = EX(X-1)\cdot\ldots\cdot(X-n+1) \leq M_1 \cdot n! \cdot q_1^n, \quad n \geq 1 \tag{2.8}$$

Proof. Lemma (2.2) Let's take arbitrary $\Lambda < \frac{1}{q}$, i.e $\Lambda q < 1$ and put $M = \max\limits_{|z|=\Lambda} \varphi(z)$ Due to Cauchy theorem for $|z| = \Lambda$

$$\varphi(z) = \frac{1}{2\pi i} \oint_{|z|=\Lambda} \frac{\varphi(z) d\lambda}{z-\lambda} \tag{2.9}$$

Then

$$\varphi^{(n)}(z) = \frac{1}{2\pi i} \oint_{|z|=\Lambda} \frac{\varphi(z) n!}{(z-\lambda)^{n+1}} d\lambda \tag{2.10}$$

i.e for $z = 0$

$$\varphi^{(n)}(0) = EX(X-1)\cdot\ldots\cdot(X-n+1) = (a)_{(n)} \leq \frac{M \cdot n!}{\Lambda^n}, \quad n \geq 1 \tag{2.11}$$

In other term for arbitrary $q_1 > q$ and appropriate $M_1 = M(q_1) < \infty$

$$(a)_{(n)} = EX(X-1)\cdot\ldots\cdot(X-n+1) \leq M_1 \cdot n! \cdot q_1^n, \quad n \geq 1 \tag{2.12}$$

Q.E.D.

We must finish Dmitriev-Kolmogorov's theorem 2.1 using moments.

Theorem 2.3. *In the assumption of lemma (2.2) For any $k \geq 1$ and $t \to \infty$*

$$\frac{En(t)(n(t)-1)\cdot\ldots\cdot(n(t)-k+1)}{a^{kt}} \to \nu_k \tag{2.13}$$

and $\nu_k \leq C_1 k! C^k$ for appropriate $C_1, C > 0$. Calculation of the limiting moments can be done recursively.

Corollary 2.4. *Due to well-known Carleman result the moment problem (reconstruct the distribution $G(\cdot)$ given moments $\nu_k, k \geq 1$) under the condition $\nu_k \leq C_1 k! C^k$ has unique solution and the Laplace transform*

$$\int_0^\infty e^{-\lambda x} dG(x) = \widehat{G}(\lambda)$$

has analytic continuation at domain $Re\lambda \geq -\delta$, $\delta > 0$ (for appropriate $\delta > 0$), i.e distribution function $G(x)$ has an exponential tail.

Proof. (the Theorem 2.3)

Let us use equation 1.2

$$\varphi_{[t]}(z) = \underbrace{\varphi(\varphi(\dots\varphi}_{t}(z))\dots) = \varphi_{[t-1]}(\varphi(z))$$

First,

$$\varphi_t'(1) = a_{[1]}(t) = a^t, \; a > 1, \text{where } \varphi'(1) = a \tag{2.14}$$

Then

$$\frac{E\{n(t)\}}{a^t} = 1 \xrightarrow[t\to\infty]{} \nu_1 = 1$$

Second,

$$\begin{aligned} \varphi_t''(1) = a_{[2]}(t) &= \varphi_{[t-1]}''(\varphi')^2 + \varphi_{[t-1]}'\varphi'' \\ &= a_{[2]}(t-1)a^2 + a^{t-1}a_{[2]}(1) \end{aligned} \tag{2.15}$$

Then

$$\begin{aligned}\frac{a_{[2]}(t)}{a^{2t}} &= \frac{a_{[2]}(t-1)}{a^{2(t-1)}} + \frac{a_{[2]}(1)}{a^{t+1}} \\ &= \frac{a_{[2]}(1)}{a^{t+1}} + \frac{a_{[2]}(1)}{a^{t}} + \frac{a_{[2]}(1)}{a^{t-1}} + \ldots + \frac{a_{[2]}(1)}{a^{2}} \\ &= a_{[2]}(1)\left(\frac{1}{a^2} + \frac{1}{a^3} + \ldots + \frac{1}{a^{t+1}}\right) \qquad (2.16)\end{aligned}$$

It gives

$$\lim_{t\to\infty} \frac{E\{n(t)(n(t)-1)\}}{a^{2t}} = \lim_{t\to\infty} \frac{E\{n(t)^2\}}{a^{2t}} = \frac{a_{[2]}(1)}{a^2(1-\frac{1}{a})} = \frac{a_{[2]}(1)}{a(a-1)} \qquad (2.17)$$

Assume that we already proved that for $l \leq k-1$

$$\frac{a_{[l]}(t)}{a^{lt}} = \frac{E\{n(t)(n(t)-1)\ldots(n(t)-l+1)\}}{a^{lt}} \leq C_0 l! C_1^l$$

and select such C_0, C_1 that recursively we can reconstruct the same estimate for $l = k$

Meanwhile, a higher order version of the chain rule (see e.g. [11] p. 33) states that

$$f^{(s)}(g) = \sum_{m_1,m_2,\ldots,m_s} \frac{s!}{m_1!m_2!\cdots m_s!}[f^{(m_1+m_2+\cdots+m_s)} \prod_{j=1}^{s}\left(\frac{g^{(j)}}{j!}\right)^{m_j} \qquad (2.18)$$

where $1 \cdot m_1 + \ldots + n \cdot m_n = n$

$$
\begin{aligned}
&\varphi_{[t]}^{(k)}(z)|_{z=1} = a_{[k]}(t) \\
&= \sum_{m_1,m_2,\ldots,m_k} \frac{k!}{m_1!m_2!\cdots m_k!}\left[\varphi_{t-1}^{(m_1+m_2+\cdots+m_k)}\prod_{j=1}^{k}\left(\frac{\varphi^{(j)}(1)}{j!}\right)^{m_j}\right] \\
&= \varphi_{t-1}^{(k)}(1)a_{[1]}^{k} + \\
&+ \sum_{\substack{m_1, m_2, \ldots, m_k \\ m_1 \neq k}} \frac{n!}{m_1!m_2!\cdots m_k!}\left[\varphi_{t-1}^{(m_1+m_2+\cdots+m_k)}\varphi'_{t-1}\prod_{j=1}^{k}\left(\frac{\varphi_t^{(j)}}{j!}\right)^{m_j}\right] \\
+ &\sum_{1\cdot m_1+\ldots+k\cdot m_{k-1}=k} \frac{k!}{1!^{m_1}m_1!2!^{m_1}m_2!\cdots(k)!^{m_1}m_{k-1}!}\times \\
&\times\left[\varphi_{t-1}^{(m_1+m_2+\cdots+m_{k-1})}\left(\sum_{j=1}^{n} m_j\frac{\varphi_t^{(j+1)}}{\varphi_t^{(j)}}\prod_{s=1}^{k-1}\left(\varphi_t^{(s)}\right)^{m_j}\right)\right]
\end{aligned}
$$

where $1 \cdot m_1 + \ldots + k \cdot m_k = k$ and highest order derivative in the last two sums is $k-1$ and $\varphi^{(j)}(1) \leq M_1 q_1^j j!$ and by the inductive assumption

$$\varphi_{[t-1]}^{(l)}(1) \leq C_0 \cdot l! \cdot C_1^l \cdot a^{l(t-1)}$$

Substitution of these inequalities in the long formula above gives, after some calculation $\varphi_{[t-1]}^{(l)}(1) \leq C_0 \cdot l! \cdot C_1^l \cdot a^{l(t-1)}$.

Q.E.D.

This is end of second proof. But let me give more calculation of higher order differential equation (2.18).

Let us use equation 1.2 again

$$\varphi_{[t]}(z) = \underbrace{\varphi(\varphi(\ldots\varphi}_{t}(z))\ldots) = \varphi_{[t-1]}(\varphi(z))$$

Let us introduce new notation

$$A^{(n)}(t) = \varphi_t^{(n)}(z)|_{z=1} = En(t)(n(t)-1)(n(t)-n+1) \tag{2.19}$$

$$A^{(n)}(1) = a^{[n]} = \varphi^{(n)}(1) = EX(X-1)(X-2)\dots(X-n+1) \tag{2.20}$$

$a = a^{[1]} = EX > 1$, X is the number of the offsprings of a single particle. The central recursive relation

$$\begin{aligned} A^{(n)}(t) &= \frac{\partial}{\partial z^n}\varphi_{[t-1]}(\varphi(z))|_{z=1} \\ &= \sum \underbrace{\frac{n!}{i!j!h!\cdots k!}}_{lfactor} \underbrace{\varphi_{[t-1]}^{(m)}}_{A^{(m)}(t-1)} \left(a^{[1]}\right)^i \left(\frac{a^{[2]}}{2!}\right)^j \left(\frac{a^{[3]}}{3!}\right)^h \dots \left(\frac{a^{[l]}}{l!}\right)^k \end{aligned} \tag{2.21}$$

Summation over $i, j, \dots, k$ such that

$$\begin{cases} i + 2j + 3h + \dots + lk = n \\ i + j + h + \dots + k = m \end{cases} \Rightarrow \begin{cases} j + 2h + \dots + (l-1)k = n - m \\ i + j + h + \dots + k = m \end{cases}$$

Several first terms

Case I. $n = 1$, $m = 1$, $i = 1$

$$A^{[1]}(t) = \frac{1!}{1!}A^{[1]}(t-1)\cdot a^{[1]} \Rightarrow A^{[1]}(t) = En(t) = a^t$$

Case II. $n = 2$

$$\begin{cases} i + 2j = 2 \\ i + j = m \end{cases} \Rightarrow \begin{cases} m = 2, i = 2, j = 0 \\ m = 1, i = 1, j = 1 \end{cases}$$

$$A^{[2]}(t) = \frac{2!}{2!0!}A^{[2]}(t-1)\cdot a^2 + \frac{2!}{1!0!}A^{[1]}(t-1)\cdot \frac{a^{[2]}}{2!}$$

$$A^{[2]}(t) = A^{[2]}(t-1) \cdot a^2 + a^{t-1} \cdot a^{[2]}$$

$$\begin{aligned}
\frac{A^{[1]}(t)}{a^{2t}} &= \frac{A^{[2]}(t-1)}{a^{2(t-1)}} + \frac{a^{[2]}}{a^{t+1}} \\
&= \frac{A^{[2]}(t-2)}{a^{2(t-2)}} + \frac{a^{[2]}}{a^{t+1}} + \frac{a^{[2]}}{a^{t}} \\
&= a^{[2]}\left(\frac{1}{a^2} + \frac{1}{a^3} + \cdots + \frac{1}{a^{t+1}}\right)
\end{aligned}$$

Case III. $n = 3$

$$\begin{cases} i + 2j + 3h = 3 \\ i + j + h = m \end{cases} \Rightarrow \begin{cases} m = 1, i = 0, j = 0, h = 1 \\ m = 2, i = 1, j = 1, h = 0 \\ m = 3, i = 3, j = 0, h = 0 \end{cases}$$

$$A^{[3]}(t) = \frac{3!}{3!0!0!}A^{[3]}(t-1) \cdot a^3 + \frac{3!}{1!1!0!}A^{[2]}(t-1) \cdot a \cdot a^{[2]} + \frac{3!}{1!0!0!}A^{[1]}(t-1) \cdot \frac{a^{[3]}}{3!}$$

$$\frac{A^{[3]}(t)}{a^{3t}} = \frac{A^{[3]}(t-1)}{a^{3(t-1)}} + 3\frac{A^{[2]}(t-1)}{a^{2(t-1)}} \cdot \frac{a^{[2]}}{a^{t+1}} + \frac{a^{[3]}}{a^{2t+1}}$$

Let us start third proof we may call it functional equational approach.

Proof.

$$\begin{aligned}
\varphi_t(\lambda) &= Ee^{-\frac{\lambda n(t)}{a^t}} = Ee^{-\frac{\lambda}{a}\sum_{0}^{n(1)} \frac{n(t-1)}{a^{t-1}}} = E\left[E\left(e^{-\frac{\lambda}{a}\sum_{0}^{n(1)} \frac{n(t-1)}{a^{t-1}}} |n(1)\right)\right] \\
&= E\varphi_{t-1}^{n(1)}\left(\frac{\lambda}{a}\right) = \varphi\left(\varphi_{t-1}\left(\frac{\lambda}{a}\right)\right)
\end{aligned} \tag{2.22}$$

$$\varphi_t(\lambda) = Ee^{\lambda\frac{n(t)}{a^t}} \xrightarrow[t\to\infty]{} \psi(\lambda) \tag{2.23}$$

Limiting Laplace transform satisfies the equation.

$$\psi(\lambda) = \varphi_X \left(\psi \left(\frac{\lambda}{a} \right) \right) \tag{2.24}$$

This is the functional equation for

$$\psi(\lambda) = \int_0^\infty e^{-\lambda x} dG(x)$$

where $G(x) = \lim_{t\to\infty} P\left(\frac{n(t)}{a^t} < x \right)$

Calculation of the moments

$$\psi'(\lambda)|_{\lambda=0} = En^* = 1 \ \text{ where } \ n^* = \frac{n(t)}{a^t} \tag{2.25}$$

$$\psi'(\lambda)|_{\lambda=0} = \varphi' \left(\psi \left(\frac{\lambda}{a} \right) \right) \frac{1}{a} \psi' \left(\frac{\lambda}{a} \right) |_{\lambda=0} = \varphi'(1) \frac{1}{a} = 1$$

$$\begin{aligned} \psi''(\lambda)|_{\lambda=0} &= \varphi'' \left(\psi \left(\tfrac{\lambda}{a} \right) \right) \tfrac{1}{a^2} \left(\psi' \left(\tfrac{\lambda}{a} \right) |_{\lambda=0} \right)^2 + \varphi' \left(\psi \left(\tfrac{\lambda}{a} \right) \right) \tfrac{1}{a^2} \psi'' \left(\tfrac{\lambda}{a} \right) |_{\lambda=0} \\ &= \varphi''(1) \tfrac{1}{a^2} \left(\psi'(0) \right)^2 + \varphi'(1) \tfrac{1}{a^2} \psi''(0) \\ &= \tfrac{a_{[2]}}{a^2} + \tfrac{1}{a} \psi''(0) \end{aligned}$$

$$\psi''(0) \left(1 - \frac{1}{a} \right) = \frac{a_{[2]}}{a^2}$$

$$\psi''(0) = E(n^*)^2 = \frac{a_{[2]}}{a(a-1)} \tag{2.26}$$

Q.E.D.

CHAPTER 3: GALTON-WATSON IN RANDOM ENVIRONMENT

3.1 Construction of Random Environment

Let us introduce now the Galton-Watson process in the random environment. For classical Galton-Watson branching processes it is assumed that individuals reproduce independently of each other according to some given offspring distribution. In the setting of this paper the offspring distribution varies in a random fashion, independently from one generation to the other. A mathematical formulation of the model is as follows. We replace the term $n(t,\omega)$ in the equation (1.1) by $n(t,\omega,\omega_m)$. Here the random variable $n(t,\omega,\omega_m)$ belongs to a new probability space ($\Omega_m \times \Omega$, $\Gamma_m \times \Gamma, P \times P_m$). More precisely we have new a model as defined:

Definition 3.1. Let $n(t,\omega,\omega_m)$ be the number of the particles at the moment $t = 0, 1, \ldots$. Then

$$n(t+1,\omega,\omega_m) = \sum_{i=1}^{n(t,\omega,\omega_m)} X_{t,i}(\omega_m) \tag{3.1}$$

where $X_{t,i}(\omega_m)$ are i.i.d random variable whose distribution depends on t and selected independently for different generations.

Let me give you several examples related to (3.1).

Example 3.1. Let

$$X_{t,\cdot} = \begin{cases} 2 & \text{with probability } \; p_t(\omega_m) \\ 0 & \text{with probability } \; 1 - p_t(\omega_m) \end{cases}$$

and p_t are i.i.d. random variable with values on $(0,1)$ and same density $f(p)$, $p \in (0,1)$

Moment generating function become

$$\varphi_t(z,\omega_m) = p_t(\omega_m) + (1 - p_t(\omega_m))z^2$$

Example 3.2. Let

$$p_0(\omega_m) + p_1(\omega_m) + \ldots + p_l(\omega_m) = 1 \;\; x_i \geq 0$$

be a symplex in R^{l+1} and $f(p)dp$ is density on this sympex. Let $\overrightarrow{P_t}$ be i.i.d vectors with the density f(p) and

$$X_t = \begin{cases} 0 & \text{with probability } p_{t,0}(\omega_m) \\ 1 & \text{with probability } p_{t,1}(\omega_m) \\ \vdots & \vdots \\ l & \text{with probability } p_{t,l}(\omega_m) \end{cases}$$

then moment generating function become

$$\varphi_t(z,\omega_m) = p_{t,0}(\omega_m) + p_{t,1}(\omega_m))z + \ldots + p_{t,l}(\omega_m)z^l$$

Example 3.3. Let us use the previous example in the chapter two that the generating function of the geometric distribution

$$P(X_t = 0) = \alpha, P(X_t = 1) = (1-\alpha)q \;, \;\ldots\; P(X_t = n) = (1-\alpha)qp^{n-1}, \ldots$$

Then

$$Ez^{X_\tau} = \alpha + \frac{(1-\alpha)qz}{1-pz} = \frac{\alpha + z(q-\alpha)}{1-pz}$$

We are going to change α into $\alpha_t, p_t \in (0,1)$. They are independent random vectors for different $t = 1, 2, \ldots$ with values on $(0,1)$ and for simplicity for fixed t, the parameters α_t, p_t these are also independent. Then it become

X_t is random variable get values $0, 1, 2, \ldots, n \ldots$ with corresponding probabilities $\alpha_t, (1-\alpha_t)q_t, (1-\alpha_t)q_t p_t, \ldots, (1-\alpha_t)q_t p_t^{n-1}, \ldots$ where $q_t = 1 - p_t$ Then

$$\alpha_t + (1-\alpha_t)(q_t z + q_t p_t z^2 + \ldots) = \alpha_t + (1-\alpha_t)\frac{q_t z}{1 - p_t z} = \frac{\alpha_t - \alpha_t z + q_t z}{1 - p_t z} = \frac{\alpha_t + (q_t - \alpha_t)z}{1 - (1 - q_t)z}$$

Assume also that $\alpha_t \leq 1 - \varepsilon$ and $p_t \leq 1 - \varepsilon$ for all t and some non-random $\varepsilon > 0$. This will give the existence of all moments of X_t with good estimations.

The generating functions $\varphi_t(z, \omega)$ in this case will be analytic in the circle $|z| < \frac{1}{1-\varepsilon}$ and uniformly bounded in each circle $|z| < \frac{1}{1-\varepsilon'}$, $\varepsilon' > \varepsilon$.

We will assume in general the same condition

$$|\varphi_t(z, \omega_m| \leq \varphi(z, \omega) \leq c_0$$

where $|z| \leq 1 + \varepsilon_0$, for some $\varepsilon_0 > 0$, i.e (due to Cauchy theorem).

$$EX_{t_j}^l \leq C_0 \cdot l! \cdot \frac{1}{\varepsilon_0^l}$$

Let $\varepsilon < \alpha_t < 1 - \varepsilon$ and $\varepsilon < p_t < 1 - \varepsilon$. Then the moment generating function becomes

$$\varphi_t(z, \omega_m) = \frac{\alpha_t + (q_t - \alpha_t)z}{1 - (1 - q_t)z}$$

where $(\alpha_t, q_t) \in (0, 1) \times (0, 1)$ and α_t, q_t are i.i.d random vectors.

3.2 Annealed and quenched law

The probability measure P_m that determines the distribution of the $n(t, \omega, \omega_m)$ in a given environment $\omega_m \in \Omega$ is referred to as the quenched law, while the full probability measure $P_m \times P$ is referred to as the annealed law.

Here, and in the future, the symbol $\langle\ \rangle$ or $\langle\ \rangle_{\omega_m}$ means the expectation with respect to the probability measure P_m of the random medium. The notation E or E_x will be used for the expectation over the quenched probability measure P for the random walk starting from x and the fixed environment ω.

3.3 Moments and quenched classification of of Galton-Watson process in random environment

Recall. Let $n(t, \omega, \omega_m)$ be the number of the particles at the moment $t = 0, 1, \ldots$. Then

$$n(t+1, \omega, \omega_m) = \sum_{i=1}^{n(t,\omega,\omega_m)} X_{t,i}(\omega_m) \quad (3.1)$$

where $X_{t,i}(\omega_m)$ is an i.i.d random variable whose distribution depends on t and selected independently for different generations.

Proposition 3.1.

$$Ez^{n(t,\omega,\omega_m)} = \sum_{\rho=0}^{\infty} z^{\rho} P\{n(t,\omega,\omega_m) = \rho | \omega_m\} = \varphi_1(\varphi_2(\ldots \varphi_t(z, \omega_m) \ldots)) =$$

$$= \varphi_1 \circ \cdots \circ \varphi_t(z, \omega_m) := \varphi_{[t]}(z, \omega_m) \quad (3.2)$$

where

$$\varphi_{t,\cdot}(z, \omega_m) = Ez^{X_t} = \sum_{\rho=0}^{\infty} P(X_t = \rho, \omega_m) z^l = \sum_{\rho=0}^{\infty} P_t(\rho, \omega_m) z^{\rho} \quad (3.3)$$

Moment generating function of $n(t, \omega, \omega_m)$ is

$$Ez^{n(t,\omega,\omega_m)} = \psi_t(z) = \psi_{t-1}(\varphi_t(z)) \quad (3.4)$$

Let us differentiate the above moment generating function (3.13) with respect to z, at $z = 1$. This will give

$$N_t(\omega_m) = E[n(t, \omega, \omega_m) | \omega_m] = a_1(\omega_m) \cdot \ldots \cdot a_t(\omega_m) \quad (3.5)$$

where

$$a_t(\omega_m) = \frac{d\ \varphi_t}{dz}(z,\omega_m)|_{z=1} = \sum_{\rho=0}^{\infty} \rho P_t(\rho,\omega_m) \tag{3.6}$$

Mean number, first moment of the offspring in the t^{th} generation:

$$N_t(\omega) = e^{\sum_{\rho=1}^{t} \ln a_\rho(\omega_m)} \tag{3.7}$$

where $a_t(\omega_m) = \sum_{\rho=0}^{\infty} \rho P_t(\rho,\omega_m)$

There are three cases of quenched classification:

1. Supercritical case

   $$\langle \ln\ a(\omega_m) \rangle > 0$$

   then by Strong Law of Large Numbers it satisfies:

   $$\frac{\ln N_t(\omega_m)}{t} \to \langle \ln\ a_. \rangle$$

2. Critical case

   $$\langle \ln\ a(\omega_m) \rangle = 0$$

3. Subcrictical case

   $$\langle \ln\ a(\omega_m) \rangle < 0$$

In the last case P_{w_m} is almost surely. $N_t(\omega) \to 0$ if $t \to \infty$ fast and P_{w_m} is a.s

$$P\{n_t \geq 1, \omega_m\} \leq \frac{1}{\langle n_t \rangle} \leq e^{-\delta t}$$

for some $\delta > 0$ and $t \geq t_0(\omega_m)$

The Borel-Cantelli lemma gives for fixed ω_m and P_{w_m} is almost surely.

$$n(t, \omega, \omega_m) \equiv 0 \ , \quad n \geq n_{.}(\omega_m)$$

In this case the branching process is degenerated.

Theorem 3.2. *Assume that* $\langle a_{.}(\omega_m)\rangle = \gamma > 0$, *then* P_{w_m} *almost surely for fixed* ω_m *the law of the random variable*

$$\frac{n_t(\omega, \omega_m)}{E[n(t, \omega, \omega_m)|\omega_m]} = \frac{n_t(\omega, \omega_m)}{\prod_{s=1}^{t} a_s(\omega_m)} \to n^*(\omega_m)$$

The random variable n^* *has non-trivial distribution and exponentially decreasing tails.*

This is main theorem of Chapter 3. We will prove it by two different way. First prove is using martingale approach but this method cannot give any information about n^*.

Proof. Consider

$$\zeta(t, \omega, \omega_m) = \frac{n_t(\omega, \omega_m)}{a_1(\omega_m) \cdot a_2(\omega_m) \cdot \ldots \cdot a_t(\omega_m)}$$

and $F_{\leq t} = \sigma(n_1(\omega_m), \ldots n_t(\omega_m)$

$E(n(t+1, \omega, \omega_m)|F_{\leq t}(\omega_m)) = n(t, \omega) \cdot a_{t+1}(\omega)$

$$E(\zeta(t+1, \omega, \omega_m)|F_{\leq t}(\omega_m)) = \frac{n(t, \omega) \cdot a_{t+1}(\omega)}{a_1(\omega_m) \cdot a_2(\omega_m) \cdot \ldots \cdot a_{t+1}(\omega_m)} = \zeta(t, \omega)$$

So $\left(\frac{n(t,\omega,\omega_m)}{\prod_{s=1}^{t} a_s(\omega_m)}, F_{\leq t}(\omega_m)\right)$ is martingale. Due to Doob martingale convergence theorem $p(\omega_m)$ almost surely there exist

$$\lim_{t\to\infty} \zeta(t, \omega, \omega_m) \to \zeta(\infty, \omega_m) \tag{3.8}$$

Q.E.D.

Let us find asymptotic formulas for the quenched moments

$$A^{(n)}(t,\omega_m) = E\{n(t,\omega_m)(n(t,\omega_m)-1)\dots(n(t,\omega_m)-n+1)\} \tag{3.9}$$

$$a^{[n]}(t,\omega_m) = EX_t(\omega_m)(X_t(\omega_m)-1)\dots(X_t(\omega_m)-n+1) \tag{3.10}$$

$$\nu_k(t,\omega_m) = \frac{E\{n(t,\omega_m)(n(t,\omega_m)-1)\dots(n(t,\omega_m)-k+1)}{\prod_{s=1}^{t} a_s^k(\omega_m)} \tag{3.11}$$

where

$$E\{X_t(\omega_m)(X_t(\omega_m)-1)\dots(X_t(\omega_m)-n+1)\}$$

and

$$E\{n(t,\omega_m)(n(t,\omega_m)-1)\dots(n(t,\omega_m)-n+1)\}$$

are the factorial moment of the random variable $X_t(\omega_m)$ and $n(t,\omega_m)$. For simplicity of notation, let us denote the first moment as $a^{[1]}(t,\omega_m) = a_t(\omega_m) = EX_t(\omega_m) > 1$.

Theorem 3.3.

$$\nu_k(t,\omega_m) = \frac{E\{n(t,\omega_m)(n(t,\omega_m)-1)\dots(n(t,\omega_m)-k+1)}{\prod_{s=1}^{t} a_s^k(\omega_m)} \to \nu_k(\omega_m) \tag{3.12}$$

and $\nu_k(\omega_m) \le C_1(\omega_m)k!C^k(\omega_m)$ for appropriate $C_1, C > 0$. Calculation of the limiting moments can be done recursively.

Proof. Let us proof Theorem (3.3).

$$Ez^{n(t,\omega,\omega_m)} = \psi_t(z) = \psi_{t-1}(\varphi_t(z)) = \varphi_1 \circ \dots \circ \varphi_t(z,\omega_m) \tag{3.13}$$

Let us find the second moment of $n(t,\omega,\omega_m)$ random variable.

Differentiating two times of (3.13) gives us

$$E\{n(t,\omega_m)[n(t,\omega_m)-1]z^{n(t,\omega_m)-2}\} = \psi''_{t-1}(\varphi(z,\omega_m))\cdot(\varphi'_t(z,\omega_m))^2 + \psi'_{t-1}(\varphi(z,\omega_m))\varphi''_t(z,\omega_m) \qquad (3.14)$$

When $z=1$ in (3.14) it gives us second order factorial moment

$$E\{n(t,\omega_m)[n(t,\omega_m)-1]\} = \psi''_{t-1}\cdot(\varphi'_t)^2 + \psi'_{t-1}\varphi''_t$$
$$= E\{n(t-1,\omega_m)[n(t-1,\omega_m)-1]\}\cdot a_t^2(\omega_m)$$
$$+E[n(t-1,\omega_m)]E[X_t(\omega_m)(X_t(\omega_m)-1)]$$

Let's divide second factorial moment to $\prod_{s=1}^{t-1} a_s^2(\omega_m)$ then we will get

$$\frac{E\{n(t,(\omega_m)[n(t,(\omega_m)-1]\}}{\prod_{s=1}^{t} a_s^2(\omega_m)} = \frac{E\{n(t-1,\omega_m)[n(t-1,\omega_m)-1]\}}{\prod_{s=1}^{t-1} a_s^2(\omega_m)} + \frac{a^{[2]}(t,\omega_m)}{a_1(\omega_m)\cdot\ldots\cdot a_{t-1}(\omega_m)a_t^2(\omega_m)}$$

By induction we will get

$$\frac{E\{n(t,\omega_m)[n(t,\omega_m)-1]\}}{\prod_{s=1}^{t} a_s^2(\omega_m)} = \frac{a^{[2]}(t,\omega_m)}{a_1(\omega_m)\cdot\ldots\cdot a_{t-1}(\omega_m)a_t^2(\omega_m)} + \frac{a^{[2]}(t-1,\omega_m)}{a_1(\omega_m)\cdot\ldots\cdot a_{t-2}(\omega_m)a_{t-1}^2(\omega_m)} + \ldots + \frac{a^{[2]}(1,\omega_m)}{a_1^2(\omega_m)}$$

Let us write it different way

$$\frac{E\{n(t,\omega_m)[n(t,\omega_m)-1]\}}{\prod_{s=1}^{t} a_s^2(\omega_m)} = \sum_{k=1}^{t} \frac{a^{[2]}(k,\omega_m)}{\left(\prod_{l=1}^{k} a_l(\omega_m)\right) a_k(\omega_m)} \tag{3.15}$$

Lemma 3.4. *The*(3.16) *last series converges and has non-random estimation from above.*

And second factorial moment becomes

$$E\{n(t,\omega_m)[n(t,\omega_m)-1]\} = \sum_{k=1}^{t} \frac{a^{[2]}(k,\omega_m)}{\left(\prod_{l=1}^{k} a_l(\omega_m)\right) a_k(\omega_m)} \cdot \prod_{s=1}^{t} a_s^2(\omega_m) \tag{3.16}$$

Let us now find the equation of third moment. In order to do that we need to differentiate three times and we will get

$$E\{n(t,\omega_m)[n(t,\omega_m)-1][n(t,\omega_m)-2]z^{n(t,\omega_m)-2}\} = \psi_{t-1}'''(\varphi(z)) \cdot (\varphi_t'(z))^3$$
$$+3\psi_{t-1}''(\varphi(z))\varphi_t'(z)\varphi_t''(z) + \psi_{t-1}'(\varphi(z))\varphi_t'''(z) \tag{3.17}$$

When $z = 1$ in (3.17) becomes

$$E\{n(t,\omega_m)[n(t,\omega_m)-1][n(t,\omega_m)-2]\} = \psi_{t-1}''' \cdot (\varphi_t')^3 + 3\psi_{t-1}''\varphi_t'\varphi_t'' + \psi_{t-1}'\varphi_t'''$$
$$= E\{n(t-1,\omega_m)[n(t-1,\omega_m)-1][n(t-1,\omega_m)-2]\} \cdot a_t^3(\omega_m)$$
$$+3E\{n(t-1,\omega_m)[n(t-1,\omega_m)-1]\}a_t(\omega_m)E[X_t(\omega_m)(X_t(\omega_m)-1)]$$
$$+E[n(t-1,\omega_m)]E[X_t(\omega_m)(X_t(\omega_m)-1)(X_t(\omega_m)-2)]$$

$$= E\{n(t-1,\omega_m)[n(t-1,\omega_m)-1][n(t-1,\omega_m)-2]\} \cdot a_t^3(\omega_m)$$
$$+3E\{n(t-1,\omega_m)[n(t-1,\omega_m)-1]\}a_t(\omega_m)a^{[2]}(t,\omega_m)$$
$$+E[n(t-1,\omega_m)]a^{[3]}(t,\omega_m)$$

i.e

$$\frac{E\{n(t,\omega_m)[n(t,\omega_m)-1][n(t,\omega_m)-2]\}}{\prod_{s=1}^{t} a_s^3(\omega_m)} = \nu_3(t-1,\omega_m) + 3\frac{\nu_2(t-1,\omega_m)a^{[2]}(t,\omega_m)}{(\prod_{s=1}^{t} a_s(\omega_m))a_t(\omega_m)} + \frac{a^{[3]}(t,\omega_m)}{(\prod_{s=1}^{t} a_s^2(\omega_m))a_t(\omega_m)}$$

$$= \nu_3(t-1,\omega_m) + 3\frac{a^{[2]}(t,\omega_m)\sum_{k=1}^{t-1}\frac{a^{[2]}(k,\omega_m)}{(\prod_{l=1}^{k} a_l(\omega_m))a_k(\omega_m)}}{(\prod_{s=1}^{t} a_s(\omega_m))a_t(\omega_m)} + \frac{a^{[3]}(t,\omega_m)}{(\prod_{s=1}^{t} a_s^2(\omega_m))a_t(\omega_m)}$$

$$= \nu_3(t-1,\omega_m) + 3\frac{a^{[2]}(t,\omega_m)\sum_{k=1}^{t-1}\frac{a^{[2]}(k,\omega_m)}{a_{[k,1]}(\omega_m)a_k(\omega_m)}}{a_{[t,1]}a_t(\omega_m)} + \frac{a^{[3]}(t,\omega_m)}{a_{[t,2]}(\omega_m)a_t(\omega_m)}$$

where

$$a_{[t,k]}(\omega_m) = \prod_{s=1}^{t} a_s^k(\omega_m)$$

By induction

$$\nu_3(t,\omega_m) = 3\frac{a^{[2]}(t,\omega_m)\sum_{k=1}^{t-1}\frac{a^{[2]}(k,\omega_m)}{(\prod_{l=1}^{k} a_l(\omega_m))a_k(\omega_m)}}{(\prod_{s=1}^{t} a_s(\omega_m))a_t(\omega_m)}$$

$$+3\frac{a^{[2]}(t-1,\omega_m)\sum_{k=1}^{t-2}\frac{a^{[2]}(k,\omega_m)}{(\prod_{l=1}^{k} a_l(\omega_m))a_k(\omega_m)}}{(\prod_{s=1}^{t-1} a_s(\omega_m))a_{t-1}(\omega_m)} + \ldots + 3\frac{\frac{a^{[2]}(2,\omega_m)a^{[2]}(1,\omega_m)}{a_1(\omega_m)a_1(\omega_m)}}{a_1(\omega_m)a_2(\omega_m)a_2(\omega_m)}$$

$$+\frac{a^{[3]}(t,\omega_m)}{(\prod_{s=1}^{t} a_s^2(\omega_m))a_t(\omega_m)} + \frac{a^{[3]}(t-1,\omega_m)}{(\prod_{s=1}^{t-1} a_s^2(\omega_m))a_{t-1}(\omega_m)} + \ldots + \frac{a^{[3]}(1,\omega_m)}{a_1^3(\omega_m)}$$

$$= \quad 3\frac{a^{[2]}(t,\omega_m)\sum_{k=1}^{t-1}\frac{a^{[2]}(k,\omega_m)}{a_{[k,1]}(\omega_m)a_k(\omega_m)}}{a_{[t,1]}(\omega_m)a_t(\omega_m)} + 3\frac{a^{[2]}(t-1,\omega_m)\sum_{k=1}^{t-2}\frac{a^{[2]}(k,\omega_m)}{a_{[k,1]}(\omega_m)a_k(\omega_m)}}{a_{[t-1,1]}(\omega_m)a_{t-1}(\omega_m)} +$$

$$\ldots + 3\frac{\frac{a^{[2]}(2,\omega_m)a^{[2]}(1,\omega_m)}{a_1(\omega_m)a_1(\omega_m)}}{a_1(\omega_m)a_2(\omega_m)a_2(\omega_m)} + \frac{a^{[3]}(t,\omega_m)}{a_{[t,2]}(\omega_m)a_t(\omega_m)}$$

$$+\frac{a^{[3]}(t-1,\omega_m)}{a_{[t-1,2]}(\omega_m)a_{t-1}(\omega_m)} + \ldots + \frac{a^{[3]}(1,\omega_m)}{a_1^3(\omega_m)}$$

$$\nu_3(t,\omega_m) = 3\sum_{j=1}^{t}\left(\frac{a^{[2]}(j,\omega_m)\sum_{k=1}^{j-1}\frac{a^{[2]}(k,\omega_m)}{a_{[k,1](\omega_m)}a_k(\omega_m)}}{a_{[j,1]}(\omega_m)a_j(\omega_m)}\right) + \sum_{j=1}^{t}\frac{a^{[3]}(j,\omega_m)}{a_{[j,2]}(\omega_m)a_j(\omega_m)} \tag{3.18}$$

Lemma 3.5. (3.18) *is converges and bounded above.*

The third factorial moment of $n(t, \omega, \omega_m)$ becomes

$$A^{(3)}(t, \omega_m) = E(n(t, \omega_m)n(t-1, \omega_m)n(t-2, \omega_m)) = \nu_3(t, \omega_m) \cdot a_{[t,3]}(\omega_m) \quad (3.19)$$

Assume that we already proved that for $l \leq k-1$

$$\nu_l(t, \omega_m) = \frac{A^{(l)}}{a_{[t,l]}} = \frac{E\{n(t, \omega_m)(n(t, \omega_m)-1)\dots(n(t, \omega_m)-l+1)}{\prod_{s=1}^{t} a_s^l(\omega_m)} \leq C_1(\omega_m) l! C^l(\omega_m) \quad (3.20)$$

and select such $C_1, C > 0$ that recursively we can reconstruct the same estimate for $l = k$

Meanwhile, a higher order version of the chain rule (see e.g. [11] p. 33) states that

$$f^{(s)}(g) = \sum_{m_1, m_2, \dots, m_s} \frac{s!}{m_1! m_2! \cdots m_s!} [f^{(m_1+m_2+\dots+m_s)} \prod_{j=1}^{s} \left(\frac{g^{(j)}}{j!}\right)^{m_j}]$$

where $1 \cdot m_1 + \dots + s \cdot m_s = s$ Let us find the k^{th} derivative. In order to do this, we need to differentiate k times and we will get

$$\begin{aligned} \psi_t^{(k)} &= \sum_{m_1, m_2, \dots, m_k} \frac{k!}{m_1! m_2! \cdots m_k!} [\psi_{t-1}^{(m_1+m_2+\dots+m_k)} \prod_{j=1}^{k} \left(\frac{\varphi_t^{(j)}}{j!}\right)^{m_j}] \\ &= \psi_{t-1}^{(k)} (\varphi_t')^k + \\ &+ \sum_{\substack{m_1, m_2, \dots, m_k \\ m_1 \neq k}} \frac{k!}{m_1! m_2! \cdots m_k!} [\psi_{t-1}^{(m_1+m_2+\dots+m_k)} \prod_{j=1}^{k} \left(\frac{\varphi_t^{(j)}}{j!}\right)^{m_j}] \end{aligned}$$

where $1 \cdot m_1 + \ldots + k \cdot m_k = k$

$$A^{(k)}(t, \omega_m) = A^{(k)}(t-1, \omega_m) a_t^k + \\ + \sum_{\substack{m_1, m_2, \ldots, m_k \\ m_1 \neq k}} \frac{k!}{m_1! m_2! \cdots m_k!} \left[(A^{(m_1+m_2+\cdots+m_k)}(t-1, \omega_m) \prod_{j=1}^{k} \left(\frac{a^{(j)}(t, \omega_m)}{j!} \right)^{m_j} \right]$$

where $1 \cdot m_1 + \ldots + n \cdot m_k = k$ and highest order derivative in the last sum is $k - 1$ and $\varphi^{(j)}(1) \leq M_1(\omega_m) j! q_1^j(\omega_m)$ and by inductive assumption

$$\psi_t^{(l)}(1) \leq C_1(\omega_m) l! C^l(\omega_m) \prod_{s=1}^{t} a_s^l(\omega_m) \tag{3.21}$$

Q.E.D.

CHAPTER 4: PERCOLATION ON THE RANDOM TREE

4.1 Introduction and notation

Suppose we immerse a large porous stone in a bucket of water. What is the probability that the centre of the stone is wetted? In formulating a simple stochastic model for such a situation Broadbent and Hammerslye (1957) gave birth to the "percolation model" [12] In this section we shall establish the basic definitions and notation of bond percolation on tree graphs. A graph is a pair G = (V, E), where V is a set of vertices and B is called the edge or the bond set. If this is finite for each vertex, we call the graph locally finite.

A path in a graph is a sequence of vertices where each successive pair of vertices is an edge in the graph; it is said to join its first and last vertices. A finite path with at least one edge and whose first and last vertices are the same is called a cycle. A graph is connected for each pair of different vertices there exist path. A graph with no cycles is called a forest; a connected forest is a tree. Our trees will usually be rooted, meaning that some vertex is designated as the root, denoted o. We imagine the tree as growing (upwards) away from its root. Each vertex then has branches leading to its children, which are its neighbors that are further from the root [16].

Suppose that we close or remove edges at random from a tree with probability p be a number satisfying $0 \leq p \leq 1$. We declare this edge to be open with probability p and closed otherwise, independent of all other edges. A principal quantity of interest is the percolation probability $\theta(p)$ being the probability that a given vertex belongs to an infinite open cluster. By the translation invariance of tree lattice and probability

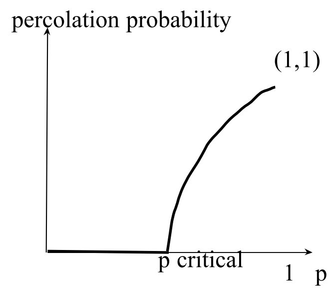


Figure 4.1: The percolation probability $\theta(p)$ behaves roughly as indicated

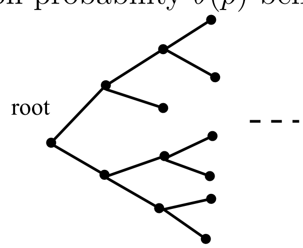


Figure 4.2: Homogeneous binary tree

measure, we lose no generality by taking this vertex to be the origin, thus we define

$$\theta(p) = P_o(|C| = \infty) = 1 - \sum_{n=1}^{\infty} P_o(|C| = n). \tag{4.1}$$

Clearly θ is a non-decreasing function of p with $\theta(0) = 0$ and $\theta(1) = 1$. It is fundamental to percolation theory that there exists a critical value p_c of p such that [12]

$$\theta(p) = \begin{cases} = 0 & \text{if } p \leq p_c; \\ > 0 & \text{if } p > p_c \ . \end{cases}$$

See figure 4.1 for a sketch of function θ.

For example if a tree binary and and every edge is either open or closed with probability p; then probability of percolation will be $\frac{1}{2}$.

$$\theta(p) = \begin{cases} = 0 & \text{if } p < \frac{1}{2}; \\ 1 - \left(\frac{q}{p}\right)^2 & \text{if } p \geq \frac{1}{2} \ . \end{cases}$$

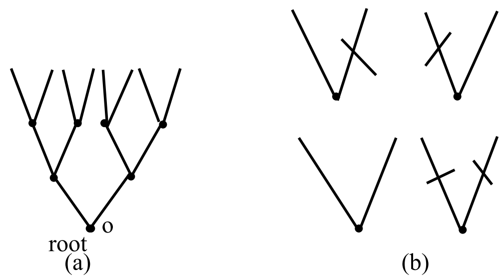


Figure 4.3: Percolation on the binary tree and the binary tree
a) Binary tree with random environment
b) Bond percolation on the binary tree

As we see, by Kolmogorovs 0-1 law, the probability that an infinite connected component remains in the tree is either 0 or 1. On the other hand, well see that this probability is monotonic in p, whence there is a critical value $p_c(T)$ where it changes from 0 to 1. It is also intuitively clear that the bigger the tree, the more likely it is that there will be an infinite component for a given p:

Theorem 4.1. *[16]*
For any tree, $p_c(T) = \frac{1}{brT}$. *Where* brT *is index of branching of* T.

4.2 Binary Random tree percolation

Let us consider instead of fixed p we change into p_{ω,ω_m}. What will happen to percolation? We consider an infinite directed connected binary tree and it is growing away from the root, see Figure 4.2.

We may consider a binary tree and every edge is either open or closed with probability $p_{\omega_m}(n,i,\omega) = p_{n,i}^{(\omega_m)}(\omega)$ and $q_{\omega_m}(n,i,\omega) = q_{n,i}^{\omega_m}(\omega) = 1 - p_{\omega_m}(n,i,\omega)$ and this happens independently of the other edges. We may say that our random environment depends on time, space and medium ω_m. At the nth level of the tree we have 2^n vertexes and edges and so $p_{n,i}^{\omega_m}(\omega)$ where $i \in [1, 2..., 2^n]$,$n = 1, 2....$

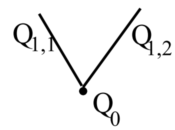


Figure 4.4: Binary tree with random environment
Where $Q^{\omega_m}_{i,j}(\omega) = P(C(i,j,\omega) = \infty)$
is probability that infinite cluster contains root (i, j)

Theorem 4.2. *Let $\langle\ \rangle$ or $\langle\ \rangle_{\omega_m}$ be the total expectation on ω_n and $C_0(\omega, \omega_m)$ is cluster contains root o.*

$$P_{\omega_m}\{|C_0(\omega,\omega_m)| < \infty\} = \begin{cases} 1 & \text{if } \langle p_{ij}\rangle_{\omega_m} \le \frac{1}{2}; \\ \rho(\omega,\omega_m) & \text{if } \langle p_{ij}\rangle_{\omega_m} > \frac{1}{2}. \end{cases}$$

and $\rho(\omega_m)$ is random variable and it is less than 1, $\rho(w_m) < 1$.

Proof. Let's fix ω_m then you can define new random variable

$$Q^{\omega_m}_{i,j}(\omega) = P(\text{There exist infinite cluster containing root } i, j$$
$$\text{where } i = 1, 2...n, j = 1, ..., 2^n)$$

$Q^{\omega_m}_{i,j}(\omega),\ \theta_{\omega_m}(\omega)$ is random variable and it is equal to probability that is infinite cluster contains the root 0. This can be written

$$\theta_{\omega_m}(\omega) = P_{\omega_m}(|C_0(\omega,\omega_m)| = \infty) = Q^{\omega_m}_0(\omega)$$

$$Q^{\omega_m}_0(\omega) = q_{1,1}(\omega)q_{1,2}(\omega)0 + p_{1,1}(\omega)q_{1,2}(\omega)Q_{11}(\omega) + p_{1,2}(\omega)q_{1,2}(\omega)Q_{1,2}(\omega)$$
$$+p_{1,1}(\omega)p_{1,2}(\omega)[1 - (1 - Q^{\omega_m}_{1,1}(\omega))(1 - Q^{\omega_m}_{1,2}(\omega)])$$

where

$$Q_0^{\omega_m} \overset{law}{=} Q_{1,1}^{\omega_m} \overset{law}{=} Q_{1,2}^{\omega_m}$$

and $\langle Q_0 \rangle^{\omega_m} = m_1$

Also $\langle p_0 \rangle_{\omega_m} = \langle p_{1,1} \rangle_{\omega_m} = \langle p_{1,2} \rangle_{\omega_m} = 1 - \langle q_{1,1} \rangle_{\omega_m} = 1 - \langle q_{1,2} \rangle_{\omega_m} = 1 - \langle q_0 \rangle_{\omega_m}$

Let us take total expectation

$$m_1 = 2\langle p_0 \rangle_{\omega_m} \langle q_0 \rangle_{\omega_m} m_1 + \langle p_0^2 \rangle_{\omega_m} (2m_1 - m_1^2) \tag{4.2}$$

In general

$$\begin{aligned} Q_{i,j}^{\omega_m}(\omega) = q_{i+1,2^j-1}(\omega) q_{i+1,2^j-1}(\omega) 0 + p_{i+1,2^j-1}(\omega) q_{i+1,2^j}(\omega) Q_{i+1,2^j-1}(\omega) \\ + p_{i+1,2^j}(\omega) q_{i+1,2^j}(\omega) Q_{i+1,2^j}(\omega) \\ + p_{i+1,2^j-1}(\omega) p_{i+1,2^j}(\omega) [1 - (1 - Q_{i+1,2^j-1}(\omega))(1 - Q_{i+1,2^j}(\omega)]) \end{aligned} \tag{4.3}$$

where

$$Q_{i,j}^{\omega_m} \overset{law}{=} Q_{i+1,2^j-1}^{\omega_m} \overset{law}{=} Q_{i+1,2^j}^{\omega_m}$$

and $\langle Q_{i,j}^{\omega_m} \rangle = m_1$

Also

$$\begin{aligned} \langle p_{i,j} \rangle_{\omega_m} &= \langle p_{i+1,2^j-1} \rangle_{\omega_m} = \langle p_{i+1,2^j} \rangle_{\omega_m} = 1 - \langle q_{i+1,2^j-1} \rangle_{\omega_m} \\ &= 1 - \langle q_{i+1,2^j} \rangle_{\omega_m} = 1 - \langle q_{i,j} \rangle_{\omega_m} \end{aligned}$$

Let us now take total expectation

$$m_1 = 2\langle p_{i,j} \rangle_{\omega_m} \langle q_{ij} \rangle_{\omega_m} m_1 + \langle p_{i,j}^2 \rangle_{\omega_m} (2m_1 - m_1^2)$$

$$m_1 = 2\langle p_{i,j}\rangle_{\omega_m} m_1 - \langle p_{i,j}\rangle^2_{\omega_m} m_1^2$$

Hence

$$m_1 = \frac{2\langle p_{i,j}\rangle_{\omega_m} - 1}{\langle p_{i,j}\rangle^2_{\omega_m}}$$

Q.E.D.

Theorem 4.3. *$|C_0(\omega, \omega_m)|$ is a volume of cluster containing a root.*

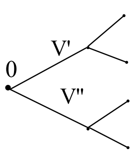
$$\langle E_{\omega_m}|C(0, w_m)|\rangle = \begin{cases} < \infty & \text{if } \langle p_{ij}\rangle < \frac{1}{2}; \\ \infty & \text{if } \langle p_{ij}\rangle \geq \frac{1}{2}. \end{cases}$$

V is event root 0 belongs to infinite cluster. V' event that vertex $(1, 1)$ belongs to infinite cluster V'' event that vertex (1,2) belongs to infinite cluster

Let's define function $\phi^{\omega_m}(z, \omega) = E_{\omega_m} Z^V$

It is started from zero. Let Z^V is generating function

$$Z^V = \begin{cases} (1 - p_{1,1})(1 - p_{1,2})Z^1 & \text{if both edges are closed} \\ p_{1,1}(1 - p_{1,2})Z^{1+V'} + (1 - p_{1,1})p_{1,2}Z^{1+V''} & \text{if one of the edges is closed} \\ p_{1,1}p_{1,2}Z^{1+V'+V''} & \text{if both edges are open} \end{cases}$$



Then,

$$\begin{aligned} E(Z^V) = E[(1 - p_{1,1})(1 - p_{1,2})Z] + E[p_{1,1}(1 - p_{1,2})]zE(Z^{V'}) \\ + E[(1 - p_{1,1})p_{1,2}]zEZ^{V''} + E[p_{1,1}p_{1,2}]z^1 EZ^{V'+V''} \end{aligned} \tag{4.4}$$

Let $\phi_{\omega_m}(z, w) = EZ^V = EZ^{V'} = EZ^{V''}$ because V,V' and V" have same law.

So we have the quadratic equation

$$\phi_{\omega_m}(z,\omega) = (1-2\langle p_{i,j}\rangle + \langle p_{i,j}\rangle^2)z + 2z\langle p_{i,j}\rangle\phi_{\omega_m}(z,w) - 2z\langle p_{i,j}^2\rangle\phi_{\omega_m}(z,w) + z\langle p_{i,j}^2\rangle\phi_{\omega_m}(z,w)^2$$

Let us take the derivative with respect to z,

$$\begin{aligned}\phi'_{\omega_m}(z,\omega) = 1 &- 2\langle p_{i,j}\rangle + \langle p_{i,j}\rangle^2 + 2\langle p_{i,j}\rangle\phi_{\omega_m}(z,\omega) - 2\langle p_{i,j}^2\rangle\phi_{\omega_m}(z,\omega)\\ &+2z\langle p_{i,j}\rangle\phi'_{\omega_m}(z,\omega) - 2z\langle p_{i,j}^2\rangle\phi'_{\omega_m}(z,\omega) + \langle p_{i,j}^2\rangle\phi_{\omega_m}(z,\omega)^2\\ &+2z\langle p_{i,j}^2\rangle\phi_{\omega_m}(z,\omega)\phi'_{\omega_m}(z,\omega)\end{aligned}$$

When $z = 1$ then

$$\phi_{\omega_m}(1,\omega) = 1$$

$$\begin{aligned} m_1 =& 1 + 2\langle p\rangle m_1\\ m_1 =& \frac{1}{1-2\langle p\rangle}\end{aligned}$$

Let us take the second derivative

$$\begin{aligned}\phi''_{\omega_m}(z,\omega) =& 4\langle p_{i,j}\rangle\phi'_{\omega_m}(z,\omega) - 4\langle p_{i,j}^2\rangle\phi'_{\omega_m}(z,\omega) + 2z\langle p_{i,j}\rangle\phi''_{\omega_m}(z,\omega)\\ &- 2z\langle p_{i,j}^2\rangle\phi''_{\omega_m}(z,\omega) + 4\langle p_{i,j}^2\rangle\phi_{\omega_m}(z,\omega)\phi'_{\omega_m}(z\omega)\\ &+ 2z\langle p_{i,j}^2\rangle\phi'_{\omega_m}(z,\omega)\phi'_{\omega_m}(z,\omega) + 2z\langle p_{i,j}^2\rangle\phi_{\omega_m}(z,\omega)\phi''_{\omega_m}(z,\omega)\end{aligned}$$

Let us take z equal to 1. We denote $m_2 = E(V(V-1))$. Then

$$m_2 = 4\langle p\rangle m_1 + 2\langle p^2\rangle m_1^2 + 2\langle p\rangle m_2$$

$$m_2 = \frac{4\langle p\rangle m_1 + 2\langle p^2\rangle m_1^2}{1 - 2\langle p\rangle}$$

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