A Unified Treatment of Derivative Pricing and Forward Decision Problems within HJM Framework

by

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Abstract

We study the HJM approach which was originally introduced in the fixed income market by David Heath, Robert Jarrow and Andrew Morton and later was implemented in the case of European option market by Martin Schweizer, Johannes Wissel, Rene Carmona and Sergey Nadtochiy. My main contribution is to apply HJM philosophy into the American option market. We derive the absense of arbitrage by a drift condition and a spot consistency condition. In addition, we introduce a forward stopping rule which is totally different from the classic stopping rule. When Ito stochastic differential equation are used to model the dynamics of underlying asset, we discover that the drift part instead of the volatility part will determine the value of option. As counterpart to the forward rate for the fixed income market and implied forward volatility and Local volatility for the European option market, we introduce the forward drift for the American option market.

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1 Chapter 1: HJM Philosophy

Modelling is a very important issue for the derivative market. We can do the pricing and hedging with a model. Because the initial value of the bond and option price for different maturies are obersvable from the market, the first requirment for a model is to be consistent with the initial obseravations. Since many spot rate models have some constant assumptions for their coefficients for example Vasicek Model for interest rate market and Black Schole Model for option market, they can't match the initial observation. Even some models letting coefficient depend on time, these models require to be recalibrated frequently. However, there is no theoretic solution to when to do the recalibration. Heath, Jarrow and Morton proposed to solve the problem by modelling directly the dynamics of the entire structure of the interest rate curve. Because initial price of European option price for different maturies are also obersvable from the market, HJM philosophy was used to model the dynamics of forward implied volatility by M.Schwerizer and J.Wissel(2008), R. Carmona, S.Nadtochiy (2009,2011). For the rest of this chapter, will talk HJM philosophy in detail for fixed income market and European option market.

1.1 Fixed Income Market

Given Probabiltiy space $(\Omega, (\mathcal{F})_{t\geq 0}, \mathcal{F}, P)$ where $(\mathcal{F}_{t\geq 0})$ satisfies the usual condition and P is the risk-netrual measure.

Definition 1.1 *Given adapted process (short rate) $\{r_t\}_{t\geq 0}$, Define:*

1. *$B(t,T) = \mathbb{E}_t e^{-\int_t^T r_s ds}$. $B(t,T)$ is price of zero coupon bond at t with maturity T. Note: $B(0,T)$ can be obersever for different maturies T.*
2. *$B_t = e^{\int_0^t r_s ds}$. B_t is the bank account.*

Definition 1.2 *Recall the definition of $B(t,T)$. Suppose it's smooth in the maturity varible T, then define:*

$$f_t(T) = \frac{\partial}{\partial T} log B(t,T)$$

Lemma 1.3 *For all $t \geq 0, f_t(t) = r_t$*

Sine $B_t = e^{\int_0^t r_s ds}$, $B(t,T)$ satisfies the following equation:

$$B(t,T) = \mathbb{E}_t \frac{B_t}{B_T}$$

because

$$f_t(T) = -\frac{\partial}{\partial T} logB(t,T)$$

which is equivalent as

$$B(t,T) = e^{-\int_t^T f_t(u)du}$$

Takeing derivative with respect to T, we can get

$$\begin{aligned}
\frac{\partial B(t,T)}{\partial T} &= \frac{\partial \mathbb{E}_t \frac{B_t}{B_T}}{\partial T} \\
&= \mathbb{E}_t \frac{\partial e^{-\int_t^T r_s ds}}{\partial T} \\
&= \mathbb{E}_t[-r_T \cdot e^{-\int_t^T r_s ds}]
\end{aligned}$$

On the other hand

$$\frac{\partial e^{-\int_t^T f_t(u)du}}{\partial T} = -f_t(T)e^{-\int_t^T f_t(u)du}$$

Therefore we can get:For $t \geq 0$,

$$f_t(t) = r_t$$

1.1.1 Forward Rate Model

Recall the relationship between B_t and r_t:

$$dB_t = r_t B_t dt \tag{1}$$

with initial value $B_0 = 1$.

Then the problem can be changed to:

$$dB_t = \begin{cases} r_t B_t dt, & B_0 = 1 \\ f_t(t) = r_t, & for \qquad t \geq 0 \\ df_t(u) = \alpha_t(u)dt + \beta_t(u)dW_t, & f_0(u) \end{cases} \tag{2}$$

In order to achieve the goal, model above should be built to satisfies:

1. The model should be arbitrage free.
2. Initial observation of the bond price $B(0,T)$ for all $T \geq 0$ from the market can be reproduced by the model. This could be called perfect calibration.

The second requirement can included in the initial value of $f_0(T)$ such that $f_0(T) = -\frac{\partial}{\partial T} log B(0,T)$ The first requirment will gives us the famous HJM drift condition. We explain below that enforcing this martingale property in a model leads to a constraint which is known under the name of drift condition. Since $B(t,T)$ is martingale under risk neutral measure P, this martingale property leads to a constraint which is known under name of drift condition.

Theorem 1.4 *Recall the definition of $\beta_t(u)$ and $\alpha_t(u)$.*

For all $0 \leq t \leq T$.

$$\alpha_t(T) = \beta_t(T) \cdot \int_t^T \beta_t(s)ds$$

Proof of this theorem will found [4].

The above formula shows that drift is completely determined by volatility.

The procedure to apply this HJM is: First we model the volatility of the forwrd rate. Second, we calculate the drift of this forward rate.

Example 1.5 *Suppose $\beta_t(T) = \sigma f_t(T)$, then according to the theorem above, we can have $\alpha_t(T) = \beta_t(T) \cdot \int_t^T \beta_t(s)ds = \sigma^2 f_t(T) f_t(u)du$. Heath, Jarrow and Morton [5] shows that this drift condition causes forward rates to explode.*

Example 1.6 *Shree[28] gives the following example:*

Suppose $\beta_t(T) = S_t\sigma(T-t)\min\{M, f_t(T)\}$, where S_t, $\sigma(T-t)$ are determin-

istic function and M is a constant number.

then they got:

$$\alpha_t(T) = \beta_t(T) \cdot \int_t^T \beta_t(u)du$$
$$= S_t^2 \sigma(T-t) \min\{M, f_t(T)\} \int_t^T \sigma(u-t) \min\{M, f_u(T)\} du$$

Given forward rate model $f_t(u)$, we can get $r_t = f_t(t)$. On the other hand, given spot rate model r_t, calculating the $f_t(T)$ is not very easy. Normally, we have to first calculate $B(0,T) = \mathbb{E}e^{-\int_0^T r_s ds}$ and then $f_0(T) = -\frac{\partial}{\partial T} logB(0,T)$. Therefore, we can have the analytic solution of $f_0(T)$ only if analytic solution of $B(0,T)$ is available.

Example 1.7 *Recall Vasicek modelL*

$$dr_t = (\alpha - \beta r_t)dt + \sigma dW_t$$

where α and σ are constants. For , they got:

$$B(0,T) = e^{A(T)+B(T)r_0} \tag{3}$$

where

$$A(T) = \frac{4\alpha\beta - 3\sigma^2}{4\beta^2} + \frac{\sigma^2 - 2\alpha\beta}{2\beta^2}T + \frac{\sigma^2 - \alpha\beta}{\beta^3}e^{-\beta T} - \frac{\sigma^2}{4\beta^3}e^{-2\beta T}$$

and

$$B(T) = -\frac{1}{\beta}(1 - e^{-\beta T})$$

In addition, we can get:

$$f_t(T) = r_t e^{-\beta(T-t)} + \frac{\alpha}{\beta}(1 - e^{-\beta(T-t)}) - (1 - e^{-\beta(T-t)})\frac{\sigma^2}{2\beta^2} \tag{4}$$

In practise, factor models are very popular. we discuss factor models in using Nelson and Siegel model as an example. Just liket HJM appraoch, local martinagle property is used to give no-arbitrage conditions.

A Factor model starts from a function G from $\Theta \times [0, \infty)$ into $[0, \infty)$ where Θ is an open set in R^d which we interpret as the set of possible values of a vector of parameters $\theta^1, \theta^2, ...\theta^d$. Then $G(\theta, .) : \tau \to G(\theta, \tau)$ can be viewed as a possible candidate for the forward curve. Nelson and Siegel has three parameters as

$$G(\theta, \tau) = \theta^1 + (\theta^2 + \theta^3\tau)e^{-\theta^4\tau}, \tau \geq 0$$

and

$$d\theta_t^i = b_t^i \cdot dt + \sum_{j=1}^{D} \sigma \cdot dW_t^j$$

with initial value θ_0^i.

Here θ_0^i is $\mathcal{F}_0-$measurable, and b and σ are progressively measurable process with values in R^4 and R^{4*D} respectively, such that $\int_0^t (|b_s| + |\sigma_s|)^2 ds < \infty$, $P-$almost surely for all finite t. Assuming further that G is twice continuously differentiable in the variables θ^j , we can use Ito's formula and derive the dynamics of $\overline{f}_t(\tau)$. The parameters $\theta_1$and θ_4 are assumed to be positive. θ_1 represents the asymptotic (long) forward rate, $\theta_1 + \theta_2$ gives the left end point of the curve, namely the shor rate, while θ_4 gives an asymptotic rate of decay. The set Θ of parameters is the subset of R^4 determined by $\theta_1 > 0, \theta_4 > 0$ and $\theta_1 + \theta_2 > 0$ since the short rate should not be negative. The parameter θ_3 is responsible for a hump when $\theta_3 > 0$ or a dip with $\theta_3 < 0$.

1.2 European Option Market

Given Probabiltiy space $(\Omega, (\mathcal{F})_{t\geq 0}, \mathcal{F}, P)$ where $(\mathcal{F}_{t\geq 0})$ satisfies the usual condition and P is the risk-netrual measure.

As well know, Black Scholes model is used to model the underlying asset to price the European option. However, volatility is assumed to be a constant number in the model which is totallty different from the observation from the market. In fact, implied volatility is a function of both time to maturity and strike price. Many approaches have been created to solve this problem for example implied volatility model and local volatility model. In this part, we summarize two famous model: implied forward volatility model by Schweizer, Wissel(2008) and local volatility dynamic model by Carmona, Sergey (2008)

1.2.1 Implied Forward Volatility

The spot volatility model is

$$dS_t = \mu_t S_t dt +_t S_t dW_t \tag{5}$$

where $\{\mu_t\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$ are adapted stochastic processes to be specified. In addition, $W = \{W_t\}_{t\geq 0}$ is $d-$dimensional Wiener process.

Schweizer, Wissel(2008) introduced forward implied variances $X(t,T)$ defined by

$$X(t,T) = \frac{\partial}{\partial T}((T-t)\Sigma_t(T)^2)$$

where $\Sigma_t(T)$ is the implied volatility at time t for maturity T which can be recoved from the Black Scholes forumula.

The implied forward volatility model is:

$$dS_t = \begin{cases} \mu_t S_t dt + \sigma_t S_t dW_t, & S_0 \\ X_t(t) = \sigma_t, & for \quad t \geq 0 \\ dX_t(u) = \alpha_t(u)dt + \beta_t(u)dW_t, & X_0(u) \end{cases} \quad (7)$$

Then they proved the spot consistency $X_t(t) = \sigma_t$ for $t \geq 0$ and Drift Condition in proposition 2.2 and theorem 2.1 in their paper.

1.2.2 Local Volatility Model

The spot volatility model is

$$dS_t = \sigma_t S_t dW_t \quad (8)$$

where $\{\mu_t\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$ are adapted stochastic processes to be specified. In addition, $W = \{W_t\}_{t\geq 0}$ is $d-$dimensional Wiener process.

Carmona, Sergey (2008) used local volatility which was introduced by Dupire (1994):

$$a_t^2(\tau, K) = \frac{2\partial C_t(\tau, K)}{K^2\partial^2_{KK}C_t(\tau, K)}$$

for $\tau > 0$ and $K > 0$. where $C_t(\tau, K)$ is the value of call option at time t for the maturity $t + \tau$.

The implied forward volatility model is:

$$dS_t = \begin{cases} \sigma_t S_t dW_t, & S_0 \\ a_t(0) = \sigma_t, & for \qquad t \geq 0 \\ da_t^2(\tau, x) = a_t^2(u)[\alpha_t(\tau, x)dt + \beta_t(\tau, x)dW_t], & a_0^2(\tau, x) \end{cases} \quad (9)$$

Carmona, Sergey (2008) gave the drift condition in theorem 4.1 in their paper.

Example 1.8 *Suppose $\beta_t(\tau, x) = 0$ for all $\tau > 0$ and $x > 0$. According to the drift condition, we can get: $a_t(\tau, x) = a_0(\tau + t, x)$. Therefore*

$$\sigma_t = a_0(t, logS_t)$$

Example 1.9

$$dS_t = \begin{cases} S_t r dt + S_t \sigma_t (\sqrt{1-\rho^2} dB_t^1 + \rho dB^{t)}, & S_0 \\ d\sigma_t = f(t,\sigma_t)dt + g(t,\sigma_t)dB_t^2, & \sigma_0 \end{cases}$$

where B_t^1 and B_t^2 are independent Brownian motions, $\rho \in [-1,1]$ $f(t,x)$ and $g(t,x)$ satisfy the usual conditions which guarantee the existence and uniqueness of a positive solution to the above system.

Carmona, Sergey (2008)the local volatility surface is given at time $t = 0$ by the formula

$$a^2(T,K) = \frac{[\sigma_T^2 \frac{\overline{S}_T}{\overline{\sigma}_T} e^{-\frac{d_1^2(T,K)}{2}}]}{E[\frac{\overline{S}_T}{\overline{\sigma}_T} e^{-\frac{d_1^2(T,K)}{2}}]}$$

Same as the fixed income market, factor models are very popular in practise. Brigo and Mercurio in [7,8] introduced the following factor model.

$$\Theta = (\sigma, \eta_1, \eta_2, \theta_1, \theta_2, p_1, p_2, s, u)$$

satisfying conditon: p_1, $p_2 > 0$,$p_1 + p_2 \leq 1$, $\theta_1, \theta_2 \geq 0$, $\sigma > 0$, $\mu \geq 0$ Let

$$v_i(\tau) = \sqrt{\theta_i + (\sigma^2 - \theta_i)\frac{1 - e^{-\eta_i \tau}}{\eta_i \tau}}$$

and

$$d_i(\tau, x) = \frac{s - x + (\mu + \frac{1}{2}v_i^2(\tau))\tau}{\sqrt{\tau}v_i(\tau)}$$

and $\eta_0 = 0$,$p_0 = 1 - p_1 - p_2$, $v_0(\tau) = \sigma$,$d_0(\tau, x) = \frac{s-x+(\mu+\frac{1}{2}\sigma^2)\tau}{\sqrt{\tau}\sigma}$ Then

$$a^2(\Theta, \tau, x) = \frac{\sum_i = 0^2 p_i(\theta_i + (\sigma^2 - \theta_i)e^{-\eta_i\tau})exp(-\frac{d_i^2(\tau,x)}{2})/v_i(\tau)}{\sum_i = 0^2 p_i exp(-\frac{d_i^2(\tau,x)}{2})/v_i(\tau)}$$

The meaning of each of the parameters is as follows: s is the logarithm of the current stock price.

σ is the spot volatility.

μ is the drift of the stock process (most likely, the difference between interest rate and the dividend payment rate).

$\{\eta_i, \theta_i\}_1^2$ define scenarios for the volatility process. p_i are the respective probabilities of these scenarios.

2 Chapter 2:Optimal Stopping Problem

Classic one optimal stopping problem is well-studied with nice results using martingale or markovian approach. We refer to Oksendal (2004),Peskir, Shiryaev(2006), Villenenve(2007) and Dayanik and Karatzas(2008) for classical accounts of the theory. For the classical problem, philosophy of backward induction is used to solve the optimal stopping time. For discrete case, Wald-Bellman equation is used to find optimal solution. For continuous case, Wald-Bellman equation changes to Snell Envelope.

There are a vast number of literature on application of optimal stopping problem: for example optimal stock selling time by Zhang(2001),Guo and Liu(2005); option pricing problem by Guo and Shepp(2001),Carmona and Touzi (2008); search problems by Nishimura and Ozaki (2004); optimal stopping problem with multiple priors by Riedel (2009).

2.1 Discrete Case

Let $G = (G_n)_{n \geq 0}$ be a sequence of random variables defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, P)$. G is adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$, in the sense that each G_n is $\mathcal{F}_n$-measurable. Recall that each $\mathcal{F}_n$ is σ akgebra if subsets of Ω such that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}$. Typically $(sF_n)_{n\geq 0}$ coincides with the natural filtration $(\mathcal{F}_n^G)_{n\geq 0}$

Definition 2.1 *A random variable $\tau : \Omega \to \{0, 1, ..., \infty\}$ is called Markov time if $\{\tau \leq n\} \in \mathcal{F}_n$ for all $0 \leq n \leq N$. A Markov time is called a stopping time if $\tau < \infty$ P.a.s.*

The family of all stopping times will be denoted by M.

Definition 2.2 $M_n^N = \{\tau \in M : n \leq \tau \leq N\}$

Assumption 2.3

$$\mathbb{E}(\sup_{0\leq k\leq N} |G_k|) < \infty$$

for all $N > 0$ with $G_N \equiv 0$ when $N = \infty$.

Consider the optimal stopping time:

$$V_N = \sup_{0\leq\tau\leq N} \mathbb{E}G_\tau$$

where τ is a stopping time.

Definition 2.4

$$S_n^N = \begin{cases} G_N, & for \qquad n = N \\ \max\left[G_n, \mathbb{E}\left[S_{n+1}^N \,|\mathcal{F}_n\right]\right], & for \quad n = N-1, ..., 0. \end{cases}$$

Definition 2.5

$$\tau_n^N = \inf\left\{n \leq k \leq N : S_k^N = G_k\right\}$$

for $0 \leq n \leq N$.

Note that the infimum above is always attained

Theorem 2.6 *Finite horizon*

Consider the optimal stopping problem [1] upon assuming that [1] holds. Then for $0 \leq n \leq N$ *we have:*

$$S_n^N \geq \mathbb{E}(G_\tau|\mathcal{F}_n) \tag{10}$$

for each $\tau \in m_n^N$.

$$S_n^N \geq \mathbb{E}(G_{\tau_n^N}|\mathcal{F}_n)$$

Moreover, we can have

1. *The stopping time* τ_0^N *is optimal in[1]*

2. *If* τ^* *is an optimal stopping time in [1], then* $\tau_0^N \leq \tau^*$ *P.a.s*

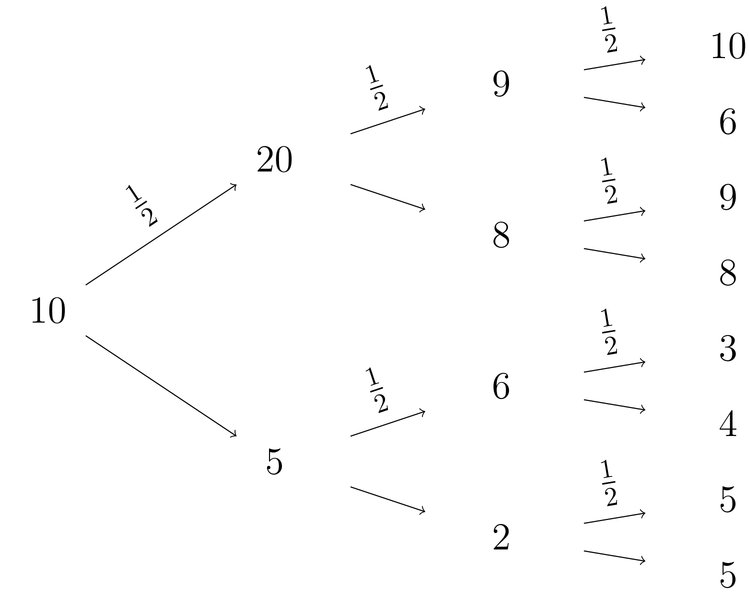
3. *The sequence* $(S_k^N)_{0 \leq k \leq N}$ *is the smallest supermartingale which dominates* $(G_k)_{n \leq k \leq N}$.

4. *The stopped sequence* $(S_{k \wedge \tau_n^N}^N)$ *is a martingale*

Detail proof can be found in [22].

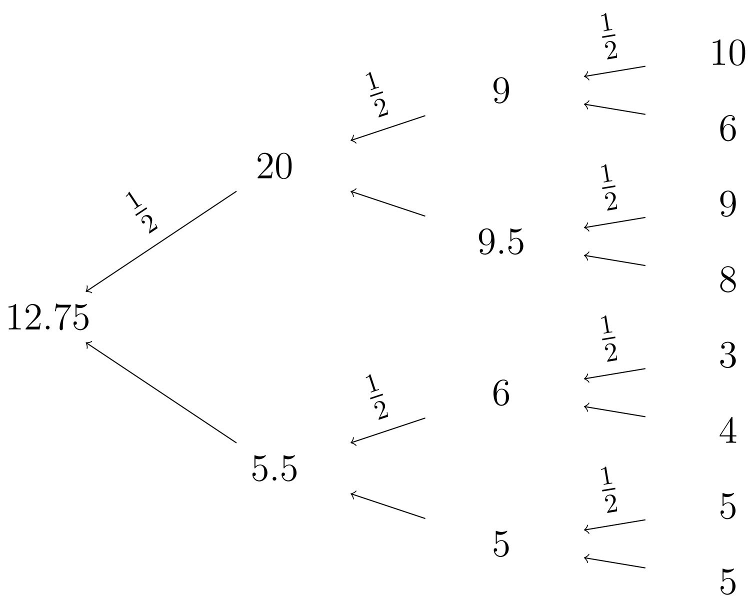
Binominal Tree Example:

Suppose we have a binominal tree for G_i for $0 \le i \le 3$



Then we can use the Bellman equation to get the value of V_i for $0 \le i \le 3$,

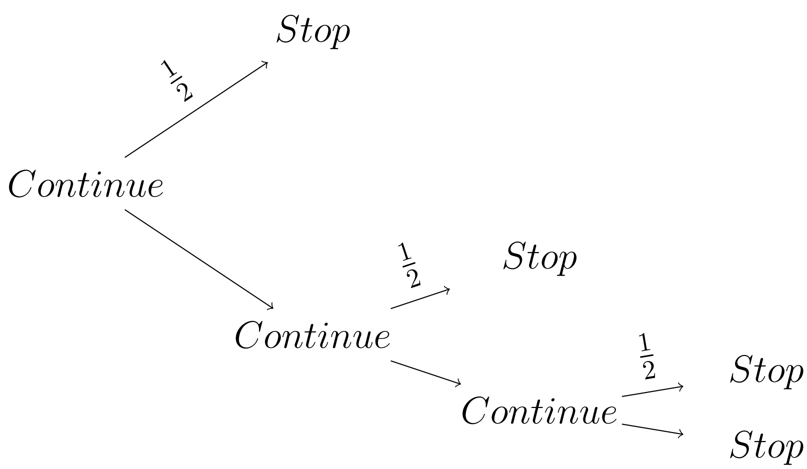
which starts $i = 3$ and let $V_3 = G_3$. For $0 \le i \le 2$, $V_i = \max\left[G_i, \mathbb{E}_i V_{i+1}\right]$.



After this, we can have the optimal stopping time:

$$\inf \{0 \leq i \leq 3 \,|V_i = G_i\}$$

The decision tree is as follows:



2.2 Continuous Case

Suppose $X = (X)_{0\leq t<\infty}$ is a strong Markov process with continuous paths in the probability space $(\Omega, (\mathcal{F})_{t\geq 0}, \mathcal{F}, P_x)$. Moreover, we assume X takes values in a measurable space $(R^d, \mathcal{B}(R^d))$ and $(\mathcal{F}_{t\geq 0})$ satisfies the usual condition.

Definition 2.7 *A random variable $\tau : \Omega \to [0, \infty]$ is called Markov time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. A Markov time is called a stopping time if $\tau < \infty$ $P.a.s.$*

The family of all stopping times will be denoted by M.

Definition 2.8 $M_t^T = \{\tau \in M : t \leq \tau \leq T\}$

Assumption 2.9 *Gain function $G : R^d \to R$ is Borel measurable function satisfying:*

$$E_x(\sup_{0\leq t\leq T} |G(X_t)|) < \infty \text{ and } G(X_\infty) = 0 \ P.a.s. \text{ for all } x \in R^d.$$

where, T is a fixed nubmer in $\overline{R^+}$.

Based on this assumption, we can get that $E_x |G(X_\tau)| < \infty$ and $\liminf_{t\to\infty} E_x I(\tau > t) |G(X_\tau)| < \infty$ for all $x \in R^d$ and stopping times τ. However, this assumption doesn't hold for some functions and processes. If this assumption doesn't hold, we can prove all thoeries are still true as long as the optimal stopping time in the set: $\Psi = \left\{\tau : \forall x \in R^d, E_x G(X_\tau) < \infty, \liminf_{t\to\infty} E_x I(\tau > t)\,|G(X_\tau)| < \infty\right\}$

Consider the (time independent)optimal stopping problem:

$$V(x) = \sup_{0\le\tau\le T} E_x G(X_\tau) \tag{12}$$

where τ is a stopping time with respect to $(\mathcal{F}_{t\ge0})$, $P_x(X_0 = x) = 1$ for $x \in R^d$ and $T \in \overline{R^+}$.

Consider the (time dependent)optimal stopping problem:

$$V(t,x) = ess \sup_{t\le\tau\le T} E_x G(X_\tau) \tag{13}$$

where τ is a stopping time with respect to $(\mathcal{F}_{t\ge0})$ and $T \in \overline{R^+}$.

It is well know that the above equation is called snell envelope. There are a few natural questions that arise at this point before we are going to solve the main problem:

1. Which decision we should make? Stop or continue?

2. If we choose to continue, how to find the optimal stopping time?

Let's try to solve the first question. If current value $G(x) \geq E_x G(X_\tau)$ for all stopping time τ, then we should choose to stop. Otherwise we will tend to lose value. On the other hand, if there exists a stopping time α such that $G(x) < E_x G(X_\alpha)$, then we should choose to continue because we can find at least one strategy to get more value.

Now, let's think of the second question. It's not difficult to get the following results. If X is time-homogeneous Markov process and optimal stopping problem is infinite case, then the continuation and stop region if exists doesn't change over time and is independent with the state variable. This is simply because we are actually facing a same question as time goes. Hence, we just need to find the optimal constant boundary in this case. For example, the boundary of the perpetual American put is constant if we assume the underlying asset follows geometric brownian motion. However, if X is time-inhomogeneous Markov process or optimal stopping problem is finite case, then the continuation and stop region change over time or depend on the state variable. For example, the boundary of the finite American put is a function of time if we assume the underlying asset follows geometric brownian motion. If the underlying asset doesn't follow geometric brownian motion, then the boundary may be a function of both time and state variablbe.

Following trivial cases are easy to get by using optional sampling theorem. If $\{G(X_t)\}_{0\leq t\leq T}$ is submartingale under P_x, then $\tau = T$. If $\{G(X_t)\}_{0\leq t\leq T}$ is supermartingale under P_x, then $\tau = 0$. However, what's the optimal stopping time if $\{G(X_t)\}_{0\leq t\leq T}$ is neither submartingale nor supermartingale? Generally, the optimal stopping time should be $\tau = inf\,\{t \geq 0 : X_t \in D\}$, where $D = \{x : V(x) = G(x)\}$. $V(x)$ represents the maximum possible value given time and state variable x. (Note: here x includes the time dimension.) Then the key thing is to find $V(x)$. Let me use the following example to show the importance of the assumption of uniformly integrability.

Example 2.10 *Consider the following optimal stopping problem:*

$$V(x) = \sup_{0\leq\tau\leq\infty} E_x(B_\tau - arctan(\tau))$$

It's easy to see the gain function is supermartingale, then optimal stopping time $\tau = 0$*. Therefore, we can get* $V(x) = x$*. However, define* $\tau^* = inf\left\{t \geq 0 : B_t = 2x + 1\right\}$*, then* $E_x(B_{\tau^*} - arctan(\tau^*)) \geq 2x$*. This means* $E_x(B_{\tau^*} - arctan(\tau^*)) > V(x)$*. Contradict with the theorem. The reason is this stopping doesn't satisfy uniformly integrability and is not our stopping time candidates.*

Theorem 2.11 *Suppose* $\widehat{V}$ *is the smallest superharmonic function which dominates the gain function* G *on* R^d*. In addition assumming that* $\widehat{V}$ *is lsc and* G *is usc. Set* $D = \left\{x \in R^d : \widehat{V} = G(x)\right\}$ *and* $\tau_D = inf\left\{t \geq 0 : X_t \in D\right\}$*. Then:*

If $P_x(\tau_D < \infty) = 1$ *for all* $x \in R^d$*, then* $\widehat{V} = V$ *and* τ_D *is optimal.*

If $P_x(\tau_D < \infty) < 1$ *for some* $x \in R^d$*, then there is no optimal stopping time.*

Detail proof can be found in[22].

Theorem 2.12 ***Value Function Independent on Time***

Suppose there exists a measurable function $\widehat{V}(x) : R^d \to R$ *satisfying*

1. $\widehat{V}(x) \geq G(x)$ *for all* $x \in R^d$*.*

2. $\widehat{V}(x)$ *is superharmonic function w.r.t* $(X)_{t\geq 0}$.

3. *There exists a stopping time* $\varsigma \in M_0^T$ *such that*

$$\widehat{V}(x) = E_x G(X_\varsigma)$$

for all $x \in R^d$.

Then we can have:

1. $\widehat{V}(x) = V(x)$ *for any* $x \in R^d$.

2. *If* $\widehat{V}$ *is lsc and* $G(x)$ *is usc, then* $\tau^* = inf\left\{t \geq 0 : \widehat{V}(X_t) = G(X_t)\right\}$ *is the smallest optimal stopping time.*

Proof: Because $\widehat{V}(x) \geq G(x)$ for all $x \in R^d$,it's easy to see that $E_x G(X_\tau) \leq E_x \widehat{V}(X_\tau)$. By the defiition of superharmonic function in the continuous time, we can conclude that $E_x G(X_\tau) \leq \widehat{V}(x)$ for any stopping time τ and $x \in R^d$. Therefore $\sup_{0\leq\tau<\infty} E_x G(X_\tau) \leq \widehat{V}(x)$ for any $x \in R^d$. Because of the third property, we can conclude that $\widehat{V}(x) = V(x)$ when $t = 0$. In order to prove τ^* is the smallest optimal time, we first claim that For any optimal stopping time τ $\widehat{V}(X_\tau) = G(X_\tau)$ $P.a.s$. This is true otherwise there exists an optimal stopping time such that $P(\widehat{V}(X_\tau) > G(X_\tau)) > 0$. Hence, $E_x G(X_\tau) <$

$E_x\widehat{V}(X_\tau) \leq \widehat{V}(x)$ which contradicts with the assumption that τ is optimal. Moreover, because $\widehat{V}(x)$ is lsc and $G(x)$ is usc, τ^* is a stopping. Hence, τ^* is the smallest optimal stopping time.

From the theorem above, we can get the following result: Suppose X_t be a d-dimensional process satisfying the setup and doesn't include time dimension.(For example, d-dimensional Ito diffusion process). In addition, let the gain function $G(x)$ satisfy assumption 1.2 w.r.t $X = (X_t)_{0\leq t<\infty}$ and x=L is the global maximum point of $G(x)$. Then $\widehat{V}(x) = G(L)$ and $\tau = inf\,\{t \geq 0 : X_t = L\}$ is an optimal stopping time if $\tau < \infty$ P.a.s.

Theorem 2.13 ***Value Function Dependent on Time***

Suppose there exists a measurable function $\widehat{V}(t,x) : R^+ \otimes R^d \to R$ satisfying

1. $\widehat{V}(t,x) \geq G(t,x)$ *for all* $(t,x) \in R^+ \otimes \in R^d$.

2. $\widehat{V}(t,x)$ *is superharmonic function w.r.t* $(t, X_t)_{t\geq 0}$.

3. *For any* $t \geq 0$, *there exists stopping times* $\varsigma_t \in M_t^T$ *such that*

$$\widehat{V}(t,x) = E_{(t,x)}G(\varsigma_t, X_{\varsigma_t})$$

for all $(t,x) \in R^+ \otimes R^d$.

Then we can have:

1. $\widehat{V}(t,x) = V(t,x)$ *for* $(t,x) \in R^+ \otimes R^d$.

2. *If* $\widehat{V}$ *is lsc and* $G(x)$ *is usc, then* $\tau^* = inf\left\{t \geq 0 : \widehat{V}(t,X_t) = G(t,X_t)\right\}$ *is the smallest optimal stopping time.*

Proof: Because $\widehat{V}(t,x) \geq G(t,x)$ for all $(t,x) \in R^+ \otimes \in R^d$, it's easy to see that $E_(t,x)G(\tau,X_\tau) \leq E_(t,x)\widehat{V}(\tau,X_\tau)$. By the defiition of superharmonic function in the continuous time, we can conclude that $E_(t,x)G(\tau,X_\tau) \leq \widehat{V}(t,x)$ for any stopping time $\tau \in M_t^T$ and $(t,x) \in R^+ \otimes \in R^d$. Therefore $\sup_{t\leq\tau\leq T} E_(t,x)G(\tau,X_\tau) \leq \widehat{V}(t,x)$ for any $(t,x) \in R^+ \otimes \in R^d$. Because of the third property, we can conclude that $\widehat{V}(t,x) = V(t,x)$ for all $(t,x) \in R^+ \otimes \in R^d$. In order to prove τ^* is the smallest optimal time, we first claim that $\widehat{V}(\tau,X_\tau) = G(\tau,X_\tau)$ $P.a.s$ for any optimal stopping time τ. If it's not ture, then $P(\widehat{V}(\tau,X_\tau) > G(\tau,X_\tau)) > 0$. Hence, $E_xG(\tau,X_\tau) < E_x\widehat{V}(\tau,X_\tau) \leq \widehat{V}(x)$ which contradicts with the assumption that τ is an optimal stopping time. Moreover, because $\widehat{V}(x)$ is lsc and $G(x)$ is usc, τ^* is a stopping. Hence, τ^* is the smallest optimal stopping time.

The procedure to apply this theorem is first to guess the stopping time ς_t (For example, the first hitting time to the constant bound). Then calculate $\widehat{V}(t,x) = E_(t,x)G(\varsigma_t, X_{\varsigma_t})$. If the function $\widehat{V}(t,x)$ satisfies the first two properties in the above theorem , then we can conclude that ς_t is optimal stopping times and $\widehat{V}(t,x) = V(t,x)$ for all $(t,x) \in R^+ \otimes R^d$. As we said before, bound is constant for infinite time horizon and time-homogeneous Markov process but it will depend on time for finite time horizon and time-homogeneous Markov process . Therefore, we will assume bound is $b(t,x)$ instead of constant b. The most difficult part for the above procedure is to calculate $\widehat{V}(x) = E_x G(X_\varsigma)$ for a very complexed function $G(x)$. In order to calculate the expected value, one way is to transform the problem to the boundary value problems. We give lots of examples in the appendix for drifted brownian motion with both one stopping time and two stopping times for both discounted function and integral functions.

2.3 Boundary Value Problem

For the boudary value problem, we refer Oksendal [23].Our goal in this subsection is to calculate $V(x) = E_x G(X_\tau)$ by using PDE method. We will first give the the PDE, which $V(x)$ should satisfy. Then we give the uniqueness theorems to prove the solution of the PDE $w(x)$ is also the solution of this expectation, i.e $w(x) = V(x)$. Now we assume D is a Borel set, τ_D is the first hitting time to D, i.e $\tau_D = inf\{t \geq 0 : X_t \in D\}$, $G : R^d \to R$ is a measurable function and $\lambda = (\lambda_t)_{t\geq 0}$ is given by $\lambda_t = \int_0^t \lambda(X_s)ds$ for a measurable continuous function $\lambda : R^d \to R$ and $L : R^d \to R$ is a continuous function. Then we have the following results:

1. Dirichlet Problem: If $V(x) = E_x G(X_{\tau_D})$ for $x \in R^d$, then the function V satisfies:

$$\mathcal{A}_X V = 0$$

for all $x \in C$.

2. Killed Dirichlet Problem: If V(x)=E$_x e^{-\lambda_{\tau_D}} G(X_{\tau_D})$ for $x \in R^d$, then the fuction V satisfies:

$$\mathcal{A}_X V = \lambda V$$

for all $x \in C$.

3. Poisson Problem: If $V(x) = E_x \int_0^{\tau_D} L(X_t)dt$ for $x \in R^d$, then the fuction V satisfies:

$$\mathcal{A}_X V = -L$$

for all $x \in C$.

4. Killed Poisson Problem: If $V(x) = E_x \int_0^{\tau_D} e^{-\lambda_t} L(X_t)dt$ for $x \in R^d$, then the fuction V satisfies:

$$\mathcal{A}_X V = \lambda V - L$$

for all $x \in C$.

Lemma 2.14 *Define:*

$$V(x) = E_x[\int_0^{\tau_D} e^{-rt} L(X_t)dt + e^{-r\tau_D} M(X_{\tau_D})]$$

where τ_D is the first hitting time to a Borel set D, X satisfies the setup. L is a continuous measurable function and M is measurable function. Then if characteristic operator exists for $V(x)$, then the following is true.

$$(\mathcal{A}_X - r)V(x) = -L(x)$$

for $x \in C$.

Proof will be found in the Appedix.

In fact, the Dirichlet problem already gives the PDE which the value function should satisfy if there exists an optimal stopping time. In general, if X is d dimension Markove process, we will get a d dimension PDE. However, if $G(x)$ has some speical form, we may have PDE with lower dimensions because of the property of the function. For example, if $Y_t = (t, X_t)$, the PDE should have form $L_Y = \frac{\partial}{\partial t} + L_X$. This is true for all $V(t,x)$ satisfying $V(0,x) = E_{(0,x)}G(\tau_D, X_{\tau_D})$. If we consider $G(t,x) = e^{-rt}G(x)$, then $L_Y = L_X - \lambda$. We will give the unique theorem below. Let C be a open connected set in R^d, $M \in C(\partial C)$ and $L \in C(C)$. L_X is the generator of Ito diffusion process. i.e $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$, where B_t is d dimensional Brownian motion. Moreover, we assume $\mu(x)$ and $\sigma(x)$ are continuous functions satisfying the existence of the SDE. Then the combined Dirichlet-Posisson problem is:

$$L_X w = -L \tag{14}$$

for $x \in C$ and

$$\lim_{x \to y, x \in C} w(x) = M(x) \tag{15}$$

for $y \in \partial C$. L_X is the generator of Ito diffusion process. i.e $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$, where B_t is d dimensional Brownian motion. Moreover, we assume $\mu(x)$ and $\sigma(x)$ are continuous functions satisfying the existence of the SDE.

Theorem 2.15 ***(Uniqueness theorem)***

Suppose the following statements are true:

1. *M is bouded.*

2. *L satisfies $E_x[\int_0^{\tau_D} |L(X_t)| \, dt] < \infty$.*

3. *$\tau_D < \infty$ P^x a.s. for all x.*

Then if $w \in C^2(C)$ is a bounded solution of the combined Dirichlet-Poisson problem above, we have

$$w(x) = E^x[M(X_{\tau_D})] + E^x[\int_0^{\tau_D} L(X_t)dt]$$

Proof can be found in [22].

3 Chapter 3: American Option Market

In this chapter, we will extend HJM approach to American option market by using theorems in the optimal stopping problems. As we will see later, there are a number of differences between European option and American option. First, there is no optimal stopping time in European option market but it is a very important concept for American option. Second, how to model volatility is the key issue for the European option. However, we will show how to model drift is the key issue for American option. Our focus will be about how to build a arbitrage free model for the drift.

In this part, we will give the HJM drift condition. In addition, as counterpart to the forward rates for bond market, the forward implied volatilities for European option market, here we introduce the forward drift for American option. Also, we introduce forward optimal stopping rule as counterpart to the classic stopping rule.

Given probability space $(\Omega, P, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F})$, where $\mathcal{F}_t$ is the natural filtration for Brownian motion W_t.

Let's consider the following problem. For a fix finite number T, we define G_t:

$$dG_t = \mu_t dt + \sigma_t dW_t$$

with initial value $G(0)$, where W is a multi-dimensional Brownian motion, μ_t and σ_t are adapted processes satisfying codintion that G_t has unique strong solution.

Assumption 3.1 $E(\sup_{0\leq\tau\leq T}|G_\tau|) < \infty$

Recall from the previous chapter, if we can find stopping times $\tau_t \in M_t^T$ for $t \geq 0$ such that $\widehat{V}(t,x) = \mathbb{E}_{(t,x)} G_{\tau_t}$ satisfying it's a superharmonic function dominating G_t. Then we proved that $\widehat{V}(t,x)$ is the snell envelop for G_t.

Since for any $\{\tau_t\}_{0\leq t\leq T}$ be stopping times such that $\tau_t \in M_t^T$. According to Hunt's stopping time theorem, we can get:

$$\begin{aligned}\mathbb{E}_t G_{\tau_t} &= G_t + \mathbb{E}_t \int_t^{\tau_t} \mu_u du + \mathbb{E}_t \int_t^{\tau_t} \sigma_u dW_u \\ &= G_t + \mathbb{E}_t \int_t^{\tau_t} \mu_u du \\ &= G_t + \mathbb{E}_t \int_t^{T} \mu_u 1(\tau_t \geq u) du \\ &= G_t + \int_t^{T} \mathbb{E}_t[\mu_u 1(\tau_t \geq u)] du\end{aligned}$$

Here, we can see the value $\mathbb{E}_t G_{\tau_t}$ doesn't depend on the volatility of the uderlying asset. In order to find $\mathbb{E}_t G_{\tau_t}$, we can assume the $\sigma_u = 0$ for $0 \leq u \leq T$, and $G(t)$ satisfies:

$$dG_t = \mu_t dt \tag{16}$$

with initial value G_0. As we can see from the result above, from the point of view of the optimal stopping problem, μ_t plays a very important role here.

3.1 Model Setup

Definition 3.2 *Recall the definition of G_t with no volatility part.*

Notation:

1. $V(0,T) = \sup_{0 \leq \tau \leq T} \mathbb{E} G_\tau$

2. $V(t,T) = ess\sup_{t\leq\tau\leq T}\mathbb{E}_t G_\tau$

3. $\tau_t = \inf\left\{t \leq s \leq T | V^G(s,T) = G(s)\right\}$

As counterpart to the forward rate or forward implied volatility, here we introduce forward drift:

$$f_t(u) = \mathbb{E}_t[\mu_u 1(\tau_t \geq u)] \tag{17}$$

then for all $0 \leq t \leq T$

$$f_t(t) = \mu_t \tag{18}$$

This is the spot condition for forward drift. Recall spot condition for forward rate is short rate for the fixed income market and for implied forward voltility is spot volatility for European option. Here we can see spot condition for forward drift is spot drift for American option market. In addition, recall the definition of τ_0, we have $\tau^{0=\inf\left\{0\leq t\leq T|\int_t^T{}_t f(u)du\leq 0\right\}}$. Here the problem for

American option market is:

$$dG_t = \begin{cases} \mu_t dt, & G_0 \\ \mu_t = f_t(t), & for \quad 0 \leq t \leq T \\ df_t(u) = \alpha_t(u)dt + \beta_t(u)dW_t, & f_0(u) \end{cases}$$

Then question is what is the drift condition given the initial observation $V(0,T)$, $G(0)$ and $f_0(T)$ for all T. We will show later that the Drift Condition here is described by admissible drift surface $\alpha_t^{\beta}(u)$ given the volatility surface $\beta_t(u)$.

3.2 Necessary Drift Condition

As we know $V(t,T)$ is martingale in the continuous region $t \le \tau_0$. Then we will use this property to derive the relation between α_t and β_t.

Theorem 3.3 *Recall $f_0(u)$ and the defintion of τ_0 we can prove: for $0 \le t \le \tau_0$,*

$$\int_t^T \alpha_t(u)du = 0$$

Proof:

Let $z(t,T) = \int_t^T f_t(u)du$

$$\begin{aligned} dz(t,T) &= \int_t^T df_t(u)du - f_t(t)dt \\ &= \int_t^T [\alpha_t(u)dt + \beta_t(u)dW_t]du - f_t(t)dt \\ &= [\int_t^T \alpha_t(u)du - f_t(t)]dt + \int_t^T \beta_t(u)dudW_t \end{aligned}$$

Therefore

$$\begin{aligned} dV(t,T) &= dG_t + dz(t,T) \\ &= [\mu_t + \int_t^T \alpha_t(u)du - f_t(t)]dt + \int_t^T \beta_t(u)dudW_t \\ &= \int_t^T \alpha_t(u)dudt + \int_t^T \beta_t(u)dudW_t \end{aligned}$$

Thus for $0 \leq t \leq \tau_0$, we have

$$\int_t^T \alpha_t(u) du = 0$$

◇

The theorem above give us the necessary condition for $\alpha_t(u)$ for $0 \leq t \leq \tau_0$ and $t \leq u \leq T$. There are two problem we haven't solved here. First τ_0 is still unknown so far. Second problem is we haven't given any condition for $\beta_t(u)$ for $t \leq u \leq T$. We will give the result in the follwing subsection.

3.3 Forward Stopping Rule

In this section, we will introduce a forward approach to solve the classic optimal stopping problem. We will use this approach to find the optimal stopping time τ_0 and the value $V(t,T)$ before the stopping time for American option. We won't focus on the value after the optimal stopping time.

Definition 3.4 *Recall* $f_0(u)$. *Given adapted stochastic process* $\alpha_t(u)$ *and* $\beta_t(u)$ *for* $t \leq u \leq T$, *define*

$$\overline{f}_t(u) = f_0(u) + \int_0^t \alpha_s(u)ds + \int_0^t \beta_s(u)dW_s$$

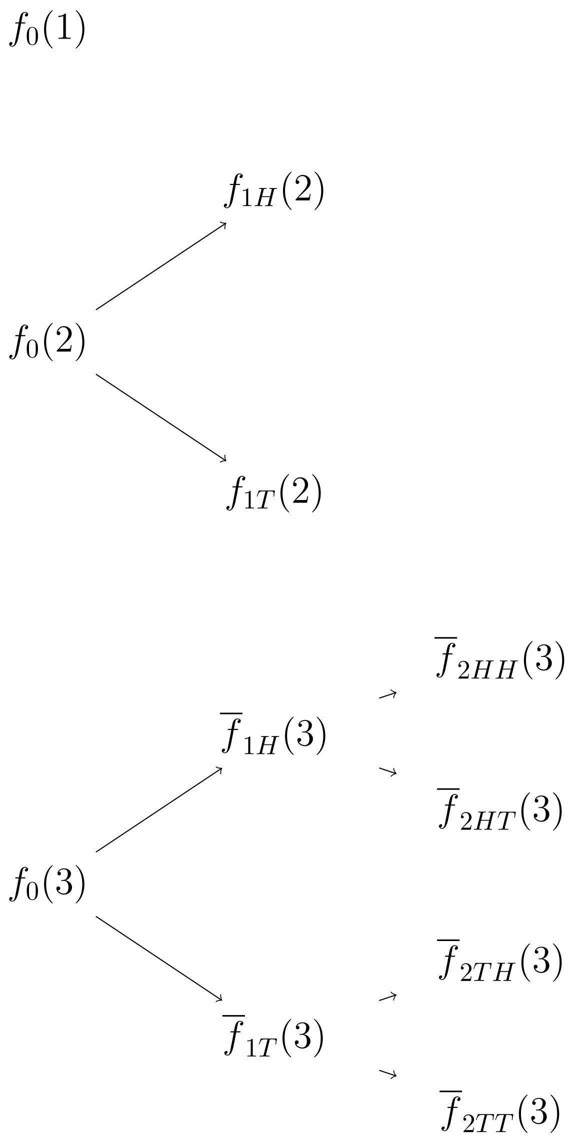
where $\alpha_t(u)$ *and* $\beta_t(u)$ *satisfies regular condition.*

Definition 3.5 $\widetilde{\tau}^* = \inf \left\{ 0 \leq t \leq T | \int_t^T f_t(u)du \leq 0 \right\}$

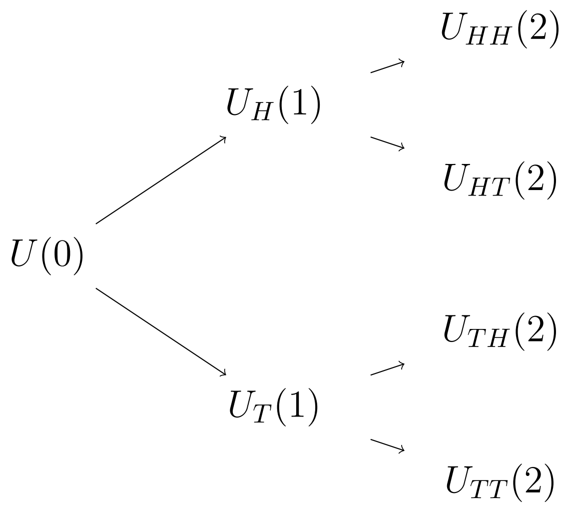
In order to explain forward stopping rule more clearly, we use following binominal tree as an example to compare with the classic approach backward induction.

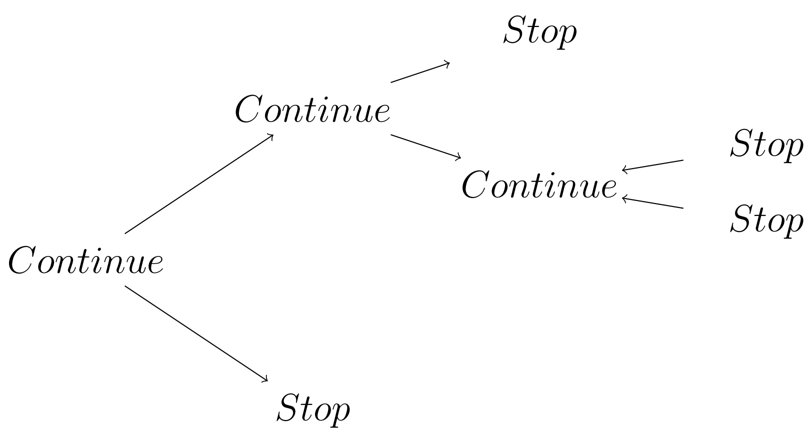
Forward Stopping Rule

In stead of Modelling G_t, we model $\overline{f}_t(u)$. Note $f_0(u)$ is observable in the market.



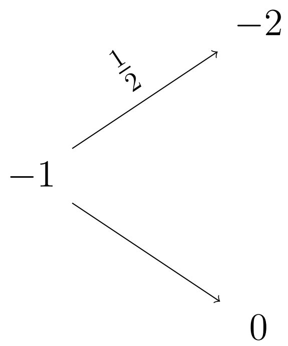
Then they will calculate the value of $U(i)$ for $0 \leq i \leq 3$, which starts $i = 0$ and let $U(0) = \sum_{i=1}^{3} f_0(i)$. If $U(0) \leq 0$, we stop. Otherwise, we will continuse and calculate $U(1) = \sum_{i=2}^{3} \overline{f}_1(i)$. If $U(1) \leq 0$, we stop. Otherwise, we will continuse and calculate $U(2) = \overline{f}_2(3)$. The will have the decision tree





Example 3.6 *In stead of Modelling* G_t*, we model* $\overline{f}_t(u)$*. Note* $f_0(u)$ *is observable in the market.*

2



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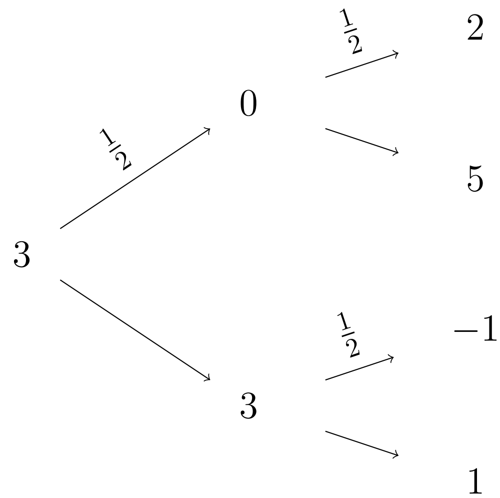
−1
0



2
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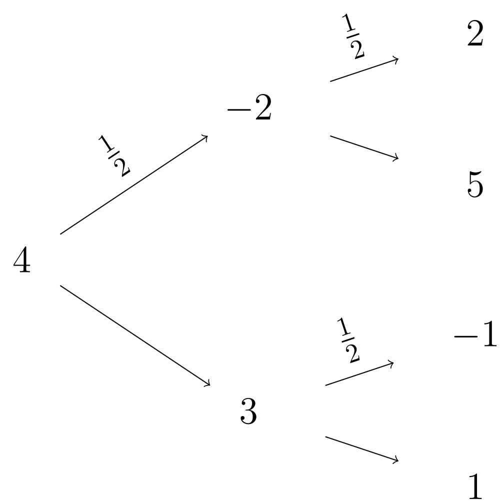
3
1/2
1



2
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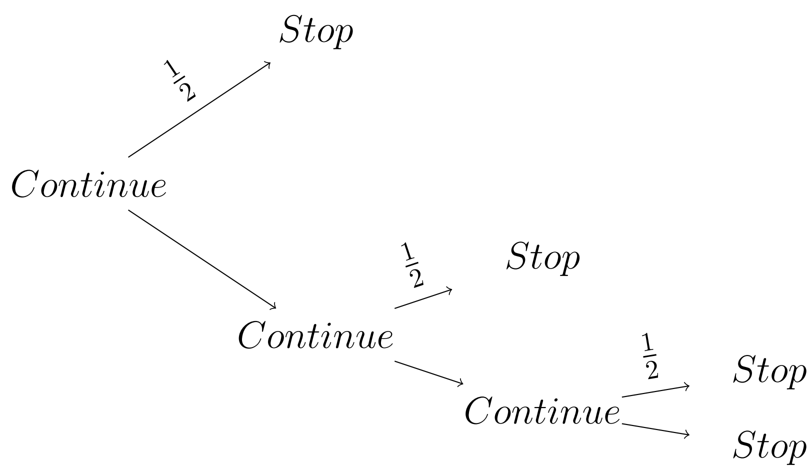

Using the backward induction approach, you have to calculate the whole tree starting from the end period. However, using forward decision approach, you don't need to calculate the whole tree. In addition, the decision you make will depend on the more recent data not the data far away.

Definition 3.7 *Recall the definition of* $\widetilde{\tau}^*$. *Given adapted process (volatility surface)* $\{\beta_t(u)\}_{0\leq t\leq u\leq T}$ *We call* $\{\alpha_t(u)\}_{0\leq t\leq u\leq T}$ *is addmisable drift surface if for* $0 \leq t \leq \widetilde{\tau}^*$

$$\int_t^T \alpha_t(u)du = 0$$

Then we use $\left\{\alpha_t^{\beta}(u)\right\}_{0\leq t\leq u\leq T}$ *to represent* $\{\beta_t(u)\}_{0\leq t\leq u\leq T}$ *addmissable drift surface.*

It's easy to see that if $\alpha_t(u) = 0$ for all $0 \le t \le u \le T$ is admissible drift surface for any volatility surface $\beta_t(u)$. For the discrete case, it's not difficult to check wether the drift surface is admissible given the volatility surface.

Example 3.8 *Given $\beta_t(u)$ is a constant σ, we want to check wether $\alpha_t(u) = \mu$ is $\beta_t(u)$ addmssible or not. First, we need to calculate $\overline{f}_t(u)$. According to the definition above, we know*

$$\overline{f}_t(u) = f_0(u) + \mu t + \sigma B_t$$

Then we have

$$\int_t^T \overline{f}_t(u)du = \int_t^T f_0(u)du + [\mu t + \sigma B_t](T-t)$$

and

$$\begin{aligned}
\widetilde{\tau}^* &= \inf\left\{0 \le t \le T | \int_t^T \overline{f}_t(u)du \le 0\right\} \\
&= \inf\left\{0 \le t \le T | \int_t^T f_0(u)du + [\mu t + \sigma B_t](T-t) \le 0\right\} \\
&= \inf\left\{0 \le t \le T | B_t \le -\frac{\int_t^T f_0(u)du}{\sigma(T-t)} - \frac{\mu t}{\sigma}\right\}
\end{aligned}$$

In this case it's very easy to chech that $\alpha_t(u) = \mu$ is not $\beta_t(u)$ admissible.

3.4 Sufficient Drift Condition

Definition 3.9 *Suppose* $dX_t = \mu_t^x dt + \sigma_t^x dW_t$, *if there exists a stopping time* $\overline{\tau} \geq 0$ *such that*

$$\mu_t^x \leq 0$$

for $t \geq \overline{\tau}$. *Then we call process* X_t *is a forward starting supermartingale and* $\overline{\tau}$ *is called the changing point for this process.*

For any initial obersvation $V(0,T)$, $G(0)$ and $f_0(T)$, we can always construct infinite many of forward starting supermartinagle such that it's consistent with the initial observations. These forward starting supermartingale will give us arbitrage free model.

Theorem 3.10 *Given* $V(0,T)$, $G(0)$*and* $f_0(u)$. *Recall the definition of (volatility surface)* $\beta_t(u)$, *its admissible (drift surface)* $\alpha_t^\beta(u)$, $\overline{f}_t(u)$ *and* $\widetilde{\tau}^*$. *Then choose (volatility surface)* $\beta_t(u)$, *Construct a forward supermartingale*

$$dX_t = \mu_t^x dt + \sigma_t^x dW_t$$

satisfying:

1. $X(0) = G(0)$.

2. $\overline{\tau}$ *is the changing point for* X_t *such that* $\overline{\tau} \le \widetilde{\tau}^*$.

3. $\mu_t^x = \overline{f}_t(t)$ *for* $0 \le t \le \overline{\tau}$.

Recall the definition of τ^*. *In this case, it's the optimal stopping time for* X_t *i.e* $\sup_{0\le\tau\le T} \mathbb{E}X_\tau = \mathbb{E}X_{\tau^*}$.

Then

1. $\tau^* = \widetilde{\tau}^*$.

2. *For* $0 \le t \le \widetilde{\tau}^*$, $V(t,T) = X_t + \int_t^T \overline{f}_t(u)du$.

3. $V(0,T) = \sup_{0\le\tau\le T} \mathbb{E}X_\tau = \mathbb{E}X_{\widetilde{\tau}^*}$.

According to the definition of X_t, we can get that X_t is supermartingale after $\widetilde{\tau}^*$, then

$$\tau^* \le \widetilde{\tau}^* \tag{19}$$

Define

$$\widehat{V}(t,T) = X_t + \int_t^T \overline{f}_t(u)du \tag{20}$$

and

$$U(t,T) = ess \sup_{t \le \tau \le T} \mathbb{E}_t X_\tau \tag{21}$$

Then we can get, for $0 \le t \le \widetilde{\tau}^*$:

$$d\widehat{V}(t,T) = [\sigma_t^x + \int_t^T \beta_t(u)du]dW_t \tag{22}$$

and also

$$\widehat{V}(t,T) \ge X_t \tag{23}$$

Then we can have for $0 \le t \le \tau^*$

$$\begin{aligned} U(t,T) &= \mathbb{E}_t X_{\tau^*} \\ &\le \mathbb{E}_t \widehat{V}(\tau^*, T) \\ &= \widehat{V}(t,T) \end{aligned} \tag{24}$$

On the other hand, since for $0 \le t \le \tau^*$

$$\widehat{V}(t,T) = \mathbb{E}_t X_{\widetilde{\tau}^*} \tag{25}$$

According to equation above, we will get for $0 \leq t \leq \tau^*$

$$U(t,T) = \widehat{V}(t,T) \tag{26}$$

Moreover, because X_t is continuous process, then we will have

$$U(\tau^*,T) = X_{\tau^*} \tag{27}$$

Then we have

$$\widehat{V}(\tau^*,T) = X_{\tau^*} \tag{28}$$

Thus

$$\tau^* = \widetilde{\tau}^* \tag{29}$$

and

$$\sup_{0\leq\tau\leq T} \mathbb{E}X_\tau = \mathbb{E}X_{\widetilde{\tau}^*} \tag{30}$$

According to equation (18), (23) and (28) we have:

$$\mathbb{E}X_{\widetilde{\tau}^*} = X(0) + \int_t^T f_0(u)du = V(0,T) \tag{31}$$

$\diamond$

As we already from the previous subsection, $\alpha_t(u) = 0$ for all $0 \leq t \leq u \leq T$ is admissible drift surface for any volaility surface $\beta_t(u)$, then we can have the following corollary.

Corollary 3.11 *Give $V(0,T)$, $G(0)$ and $f_0(u)$. Recall the definition of $f_0(u)$. Given adapted process (volatility surface) $\beta_t(u)$, define:*

$$\overline{f}_t(u) = f_0(u) + \int_0^t \beta_s(u)dW_s$$

Construct a forward supermartingale $dX_t = \mu_t^x dt + \sigma_t^x dW_t$ satisfying:

1. *$X(0) = G(0)$.*

2. *$\widetilde{\tau}^*$ is the changing point for X_t.*

3. *$\mu_t^x = \overline{f}_t(t)$ for $0 \leq t \leq \widetilde{\tau}^*$.*

Then

$$V(0,T) = \sup_{0 \le \tau \le T} \mathbb{E}X(\tau) = \mathbb{E}X(\widetilde{\tau}^*)$$

Example 3.12 *Given initial observation $V(0,T)$ and $G(0)$, Recall the definition of $f_0(u)$. According to theorem above, given any volatility surface $\beta_t(u)$ we can find a class of forward starting supermartingale X_t with $X_0 = G(0)$ which satisfies:*

$$V(0,T) = \sup_{0 \le \tau \le T} \mathbb{E}X(\tau)$$

Here we choose $\beta_t(u) = \sigma \cdot e^{-rt}$ and $\alpha_t(u) = 0$. Then we can get:

$$f_t(u) = f_0(u) + \sigma \cdot \int_0^t e^{-rs} dW_s$$

and

$$\tau^* = \inf \left\{ 0 \le t \le T | \int_t^T [f_0(u) + \sigma \cdot \int_0^t e^{-rs} dW_s] du \le 0 \right\}$$

which is equivalent as

$$\tau^* = \inf \left\{ 0 \le t \le T | \int_0^t e^{-rs} dW_s \le -\frac{\int_t^T f_0(u) du}{\sigma(T-t)} \right\}$$

Then we can construct $dX_t = \mu_t^x dt + \sigma_t^x dW_t$ *satisfies:*

$$\mu_t^x = \begin{cases} f_0(t) + \sigma \cdot \int_0^t e^{-rs} dW_s, & for \quad t \leq \tau^* \\ \leq 0, & for \quad t > \tau^* \end{cases}$$

In this case, the stopping time can be also written as:

$$\tau^* = \inf \left\{ 0 \leq t \leq T | \mu_t^x \leq f_0(t) - \frac{\int_t^T f_0(u)du}{T-t} \right\}$$

3.5 Spot Drift to Forward Drift

In this subsection, given $dG_t = \mu_t dt + \sigma_t dW_t$ with $G(0)$. let's consider the problem :

$$V(0) = \sup_{0 \le \tau \le T} \mathbb{E} G_\tau \tag{32}$$

For the bond market and European market, given spot rate model, they need to calculate $V(t, T)$ then can get the value of forward rate. For the American option market, it's difficult to compute the value of $V(t, T)$ given G_t for most cases. Other cases maybe trivial to calculate for example American Call option.

Example 3.13 *Let's consider the stock selling problem in this example. Stock process follows $dS_t = \rho S_t dt + \sigma S_t dW_t$ with initial value $S(0)$. Then the optimal stopping problem is*

$$V(0) = \sup_{\tau \ge 0} \mathbb{E} G_\tau$$

with

$$G_t = e^{-rt}(S_t - a)$$

where a, r, ρ and σ are constants.

Then we can get:

$$dG_t = e^{-rt}[(\rho - r)S_t + ar]dt + e^{-rt}\sigma S_t dW_t$$

For this infinite horizen problems, there usually exists constant boundary. In this case, the optimal time to sell stock is $\tau^* = \inf\{t \geq 0 | S(t) \geq b^*\}$. Then

$$f_t(t) = \mathbb{E}_t e^{-rt}[(\rho - r)S_t + ar] = e^{-rt}[(\rho - r)S_t + ar]$$

$$= e^{-rt}[(\rho - r)S(0)e^{(\rho - \frac{1}{2}\sigma^2)t + \sigma W_t} + ar]$$

For $u > t$,

$$f_t(u) = \mathbb{E}_t \left\{ e^{-ru}[(\rho - r)S_u + ar]1(\max_{t \leq s \leq u} S_s < b^*) \right\}$$

Normally there are more than one pair of volatility surface $\beta_t(u)$ and its admissible drift surface $\alpha_t^\beta(u)$ such that

$$\mu_t = f_t(t)$$

Definition 3.14

$$\Sigma = \left\{ \overline{f}^i_t(u) : 0 \leq t \leq u \leq T | \exists \beta_t(u), \alpha^\beta_t(u) s.t \mu_t = f^i_t(t) \right\}$$

Definition 3.15 *Recall the definition of $\widetilde{\tau}^*$, τ_i is the optimal time associated with $f^i_t(t)$.*

$$\Gamma = \left\{ \tau_i | \mu_t = f^i_t(t) \right\}$$

Theorem 3.16 *Recall the definition in the previous section.*

Suppose $|\mu_t| \leq B$ for all $0 \leq t \leq T$ and for some B.

Then $\overline{\tau}^* = ess \sup_{\tau_i \in \Gamma} \tau_i$ is the largest optimal stopping time of the problem:

$$\sup_{0 \leq \tau \leq T} \mathbb{E} G_\tau = \mathbb{E} G_{\overline{\tau}^*} \tag{33}$$

Proof:

Define:

$$Y_t = \mathbb{E}_t G_{\overline{\tau}^*} \tag{34}$$

and

$$\widehat{V}^i(t,T) = G_t + \int_t^T \overline{f}^i_t(u)du \tag{35}$$

Then according to the iterated condition, for $0 \le s \le t \le T$,

$$Y_s = \mathbb{E}_s G_{\overline{\tau}^*} = \mathbb{E}_s[\mathbb{E}_t G_{\overline{\tau}^*}] = \mathbb{E}_s Y_t \tag{36}$$

and

$$Y_{\overline{\tau}^*} = G_{\overline{\tau}^*} \tag{37}$$

If Γ is a infinite set, then there is a coutable sequence such that

$$\overline{\tau}^* = ess \sup_{i \ge 1} \tau_i$$

Define: $Z_n = \tau_1 \vee \tau_2 \vee ... \vee \tau_n$, then it's easy to see $Z_n \nearrow \overline{\tau}^*$.

Recall the property of $\widehat{V}^i(t,T)$, we have

$$\widehat{V}^i(t,T) = G_{\tau_i} \tag{38}$$

and for $0 \leq t \leq \tau_i$,

$$V(t,T) = \widehat{V}^i(t,T) \tag{39}$$

In the case Γ is a finite set with N elements, then we can get for $0 \leq t \leq \overline{\tau}^*$:

$$\begin{aligned}
Y_t &= \mathbb{E}_t[G_{\tau_1} 1(\overline{\tau}^* = Z_1) + G_{\tau_2} 1(Z_1 < \overline{\tau}^* = Z_2) + ... + G_{\tau_N} 1(Z_{N-1} < \overline{\tau}^* = Z_N)] \\
&= \mathbb{E}_t[\widehat{V}^1(\tau_1, T) 1(\overline{\tau}^* = Z_1) + \widehat{V}^2(\tau_2, T) 1(Z_1 < \overline{\tau}^* = Z_2) + ... + \widehat{V}^N(\tau_N, T) 1(Z_{N-1} < \overline{\tau}^* = \\
&\geq \mathbb{E}_t[\widehat{V}^1(t, T) 1(\overline{\tau}^* = Z_1) + \widehat{V}^2(t, T) 1(Z_1 < \overline{\tau}^* = Z_2) + ... + \widehat{V}^N(t, T) 1(Z_{N-1} < \overline{\tau}^* = Z \\
&\geq \mathbb{E}_t[G_t 1(\overline{\tau}^* = Z_1) + G_t 1(Z_1 < \overline{\tau}^* = Z_2) + ... + G_t 1(Z_{N-1} < \overline{\tau}^* = Z_N)] \\
&= G_t
\end{aligned}$$

In the case Γ is an infinite set, we can get for $0 \leq t \leq Z_m$,

$$\begin{aligned}
\mathbb{E}_t G_{\overline{\tau}^*} &= \mathbb{E}_t G_{Z_m} + \mathbb{E}_t \int_{Z_m}^{\overline{\tau}^*} \mu_u du \\
&\geq \mathbb{E}_t G_{Z_m} - B\mathbb{E}_t[\overline{\tau}^* - Z_m] \\
&\geq \mathbb{E}_t G_t - B\mathbb{E}_t[\overline{\tau}^* - Z_m]
\end{aligned}$$

Therefore we can get for $0 \leq t \leq \overline{\tau}^*$,

$$\mathbb{E}_t G_{\overline{\tau}^*} \geq G_t \tag{40}$$

Since $\tau^* \leq \overline{\tau}^*$, we will have

$$\sup_{0 \leq \tau \leq T} \mathbb{E}G_\tau = \mathbb{E}G_{\overline{\tau}^*} \tag{41}$$

$\diamond$

Example 3.17 *Let's back to the example in the last subsection.Suppose*

$$\mu_t = f_0(t) + \sigma \cdot \int_0^t e^{-rs} dW_s$$

Then according to theorem above, we can get: the optimal stopping for this process satisfies:

$$\tau^* \geq \inf \left\{ 0 \leq t \leq T | \mu_t \leq f_0(t) - \frac{\int_t^T f_0(u) du}{T - t} \right\}$$

4 Appendix

4.1 A1: Optimal Stopping Problems

Lemma 4.1 *Suppose* $V(t,T) = ess\sup_{t\leq\tau\leq T}\mathbb{E}_t^P G(\tau)$*, then we can prove*

$$V(t,T) = G(t) + \int_t^T \mathbb{E}_t^P[AG(u)1(G(u)\in C^*(u))]du$$

where $\{C^*(u)\}_{t\leq u\leq T}$ *is the optimal continous region. Inaddition, for arbitrary* $\{C(u)\}_{t\leq u\leq T}$*,*

$$V(t,T) \geq G(t) + \int_t^T \mathbb{E}_t^P[AG(u)1(G(u)\in C(u))]du$$

Proof:

$$
\begin{aligned}
V(t,T) &= ess \sup_{t\le\tau\le T} \mathbb{E}_t^P G(\tau) \\
&= \mathbb{E}_t^P V(T,T) - \mathbb{E}_t^P[\int_t^T AV(u,T)du] \\
&= \mathbb{E}_t^P G(T) - \mathbb{E}_t^P[\int_t^T AV(u,T)du] \\
&= G(t) + \mathbb{E}_t^P[\int_t^T AG(u)du] - \mathbb{E}_t^P[\int_t^T AV(u,T)du] \\
&= G(t) + \mathbb{E}_t^P[\int_t^T AG(u)du] - \mathbb{E}_t^P[\int_t^T AV(u,T)1(G(u)\in C^*(u))du] - \mathbb{E}_t^P[\int_t^T AV \\
&= G(t) + \mathbb{E}_t^P[\int_t^T AG(u)du] - \mathbb{E}_t^P[\int_t^T AV(u,T)1(G(u)\in D^*(u))du] \\
&= G(t) + \mathbb{E}_t^P[\int_t^T AG(u)du] - \mathbb{E}_t^P[\int_t^T AG(u)1(G(u)\in D^*(u))du] \\
&= G(t) + \mathbb{E}_t^P[\int_t^T AG(u)1(G(u)\in C^*(u))du] \\
&= G(t) + \int_t^T \mathbb{E}_t^P[AG(u)1(G(u)\in C^*(u))]du
\end{aligned}
$$

$\diamond$

4.2 A2: Boundary Value Problems

Lemma 4.2 *If the following equation holds.*

$$V(x) = E_x[\int_0^{\tau_D} e^{-rt}L(X_t)dt + e^{-r\tau_D}M(X_{\tau_D})]$$

where τ_D is the first hitting time to a Borel set D, X satisfies the setup. L is a continuous measurable function and M is measurable function. Then if *characteristic operator exists for $V(x)$, then the following is true.*

$$(\mathcal{A}_X - r)V(x) = -L(x)$$

for $x \in C$.

Proof: For any $x \in C$, which is the complement of set D, let U be an open set such that $x \in U \subset C$ and τ_{U^c} be the first hitting time to set U^c. Then it's easy to see that $\tau_{U^c} \le \tau_D$ P.a.s.

(42)

$$\begin{aligned}
E_x V(X_{\tau_{U^c}}) &= E_x E_{X_{\tau_{U^c}}}[\int_0^{\tau_D} e^{-rt} L(X_t)dt + e^{-r\tau_D} M(X_{\tau_D})] \\
&= E_x E_x[\int_0^{\tau_D} e^{-rt} L(X_t)dt + e^{-r\tau_D} M(X_{\tau_D})] \circ \theta_{\tau_{U^c}} |\mathcal{F}_{\tau_{U^c}} \\
&= E_x[\int_0^{\tau_D \circ \theta_{\tau_{U^c}}} e^{-rt} L(X_t \circ \theta_{\tau_{U^c}})dt + e^{-r\tau_D \circ \theta_{\tau_{U^c}}} M(X_{\tau_D} \circ \theta_{\tau_{U^c}})] \\
&= E_x[\int_0^{\tau_D - \tau_{U^c}} e^{-rt} L(X_{t+\tau_{U^c}})dt + e^{-r(\tau_D - \tau_{U^c})} M(X_{\tau_D})] \\
&= E_x[\int_{\tau_{U^c}}^{\tau_D} e^{-r(t-\tau_{U^c})} L(X_t)dt + e^{-r(\tau_D - \tau_{U^c})} M(X_{\tau_D})] \\
&= E_x[\int_0^{\tau_D} e^{-r(t-\tau_{U^c})} L(X_t)dt - \int_0^{\tau_{U^c}} e^{-r(t-\tau_{U^c})} L(X_t)dt + e^{-r(\tau_D - \tau_{U^c})} M(X_{\tau_D})]
\end{aligned}$$

According to the definition of characteristics operator, we can get:

$$\mathcal{A}_X V(x) = \lim_{U^c \downarrow x} \frac{E_x V(X_{\tau_{U^c}}) - V(x)}{E_x \tau_{U^c}} \tag{43}$$
$$= rV(x) - L(x)$$

Therefore, we can conclude that:

$$(\mathcal{A}_X - r)V(x) = -L(x) \tag{44}$$

for all $x \in C$*.Let* $V \in C^2$*, then we can prove* $\mathcal{A}_X V(x)$ *exists for Ito diffusion process and*

$$\mathcal{A}_X V(x) = \sum_{i=1}^{d} b_i \frac{\partial V}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} (\sigma \sigma^T)_i j \frac{\partial^2 V}{\partial x_i \partial x_j} \tag{45}$$

for d dimension Ito diffusion process.

In fact, the Dirichlet problem already gives the PDE which the value function should satisfy if there exists an optimal stopping time. In general, if X is d dimension Markove process, we will get a d dimension PDE. However, if $G(x)$ has some speical form, we may have PDE with lower dimensions because of the property of the function. For example, if $Y_t = (t, X_t)$, the PDE should have form $L_Y = \frac{\partial}{\partial t} + L_X$. This is true for all $V(t,x)$ satisfying $V(0,x) =$

$E_{(0,x)}G(\tau_D, X_{\tau_D})$. If we consider $G(t,x) = e^{-rt}G(x)$, then $L_Y = L_X - \lambda$.

4.3 A3: Drifted Brownian Motion

In this section, we assume the process $(X_t)_{0\le t<\infty}$ satisfying:

$$dX_t = \mu t + dB_t$$

where, $(B_t)_{0\le t<\infty}$ is 1-dimensional Brownian motion and μ is a constant.

Discounted Gain Function with One Stopping Time

The optimal stopping problem is:

$$V(x) = \sup_{0\le\tau<\infty} E_x e^{-r\tau}\tilde{G}(X_\tau)$$

where,$e^{-rt}\tilde{G}(x)$ satisfies assumption 1.1 w.r.t $(X_t)_{0\le t<\infty}$.

Let's first think of a special stopping time which is the first hitting time to a constant bound L, i.e $\tau_L = inf\left\{t \ge 0 : X_t = L\right\}$ and $\tau_L^t =$

$inf\,\{s \geq t : X_t = L\}$Then we have:

$$g(t,x) = E_{(t,x)} e^{-r(\tau_L^t)} \tilde{G}(X_{\tau_L^t})$$

$$(46) \qquad = \tilde{G}(L) E_{(t,x)} e^{-r\tau_L^t}$$

$$= \tilde{G}(L) e^{-rt} E_{(0,x)} e^{-r\tau_L}$$

By using the Laplace transform for the first passage time of drifted Brownian motion, we can get:

$$(47) \qquad g(t,x) = \begin{cases} \tilde{G}(L) e^{-rt} e^{(x-L)(-\mu+\sqrt{\mu^2+2r})}, & for \quad x < L \\ \tilde{G}(L) e^{-rt} e^{(L-x)(\mu+\sqrt{\mu^2+2r})}, & for \quad x \geq L \end{cases}$$

According to the theorem in the optimal stopping chaper, we can get the follwing propositions.

Proposition 4.3 *If $\tilde{G}$ has the following properties:*

1. $e^{-rt}\tilde{G}(x)$ *is superhamonic function w.r.t* (t, X_t)*, for all* $x \geq L$*.(Superharomic Propery)*
2. $\tilde{G}(L)e^{L(\mu-\sqrt{\mu^2+2r})} \geq \tilde{G}(x)e^{x(\mu-\sqrt{\mu^2+2r})}$*, for all* $x < L$*.(Dominating Property)*

In addition, $\mu \geq 0$, then

$$\widehat{V}(t,x) = \begin{cases} \tilde{G}(L)e^{-rt}e^{(x-L)(-\mu+\sqrt{\mu^2+2r})}, & for \quad x < L \\ e^{-rt}\tilde{G}(x), & for \quad x \geq L \end{cases}$$

and $\tau = inf\,\{t \geq 0 : X_t \in [L, \infty)\}$ is an optimal stopping time.

Proposition 4.4 *If G has continuous first derivative and there are at most finite points having no second derivative, then we can replace 1 in the above theorem by the following conditions:*

1. $(-r + \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2})\tilde{G}(x) \leq 0$ *for $x > L$.(Superharmonic Property)*
2. $\tilde{G}'(L_+) = \tilde{G}(L)(-\mu + \sqrt{\mu^2 + 2r})$.*(Smooth Pasting)*
3. $\tilde{G}(L)e^{L(\mu-\sqrt{\mu^2+2r})} \geq \tilde{G}(x)e^{x(\mu-\sqrt{\mu^2+2r})}$, *for all $x < L$.(Dominating Property)*

In addition, $\mu \geq 0$, then

$$\widehat{V}(t,x) = \begin{cases} \tilde{G}(L)e^{-rt}e^{(x-L)(-\mu+\sqrt{\mu^2+2r})}, & for \quad x < L \\ e^{-rt}\tilde{G}(x), & for \quad x \geq L \end{cases}$$

and $\tau = inf\,\{t \geq 0 : X_t \in [L, \infty)\}$ is an optimal stopping time.

Note: Smooth Pasting ensures that $\widehat{V}(t,x)$ is superharmonic function at $x = L$. Otherwise, there maybe local time at L.

Proposition 4.5 *If $\tilde{G}$ has the following properties:*

1. *$e^{-rt}\tilde{G}(x)$ is superhamonic function w.r.t (t, X_t), for all $x \leq L$.(Superharomic Propery)*
2. *$\tilde{G}(L)e^{L(\mu+\sqrt{\mu^2+2r})} \geq \tilde{G}(x)e^{x(\mu+\sqrt{\mu^2+2r})}$, for all $x > L$.(Dominating Property)*

In addition, $\mu \leq 0$, then

$$\widehat{V}(t,x) = \begin{cases} e^{-rt}\tilde{G}(x), & for \quad x \leq L \\ \tilde{G}(L)e^{-rt}e^{(L-x)(\mu+\sqrt{\mu^2+2r})}, & for \quad x > L \end{cases}$$

and $\tau = inf\,\{t \geq 0 : X_t \in (-\infty, L]\}$ is an optimal stopping time.

Proposition 4.6 *If G has continuous first derivative and there are at most finite points having no second derivative, then we can replace 1 in the above theorem by the following conditions:*

1. *$(-r + \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2})\tilde{G}(x) \leq 0$ for $x < L$.(Superharmonic Propery)*
2. *$\tilde{G}'(L_-) = \tilde{G}(L)(-\mu - \sqrt{\mu^2+2r})$. (Smooth Pasting)*

3. $\tilde{G}(L)e^{L(\mu+\sqrt{\mu^2+2r})} \geq \tilde{G}(x)e^{x(\mu+\sqrt{\mu^2+2r})}$, *for all* $x > L$*.(Dominating Property)*

In addition, $\mu \leq 0$*, then*

$$\widehat{V}(t,x) = \begin{cases} e^{-rt}\tilde{G}(x), & for \quad x \leq L \\ \tilde{G}(L)e^{-rt}e^{(L-x)(\mu+\sqrt{\mu^2+2r})}, & for \quad x > L \end{cases}$$

and $\tau = inf\,\{t \geq 0 : X_t \in (-\infty, L]\}$ *is an optimal stopping time.*

Example 4.7

$$V(x) = \sup_{0 \leq \tau < \infty} E_x[e^{-r\tau} bB_\tau]$$

where b *is a negative constant and* B_t *is Brownian motion.*

In this case, $\tilde{G}(x) = bx$ *and* $\mu = 0$*. Then it's easy to see that* $e^{-rt}bx$ *is superhamonic function for* $x < 0$*. Using the smooth pasting, we can get the boundary* $L = -\frac{1}{\sqrt{2r}}$*. After checking the third property in the corollary 1.4, we can conclude* $\tau = inf\left\{t \geq 0 : X_t \in (-\infty, -\frac{1}{\sqrt{2r}}]\right\}$ *is an optimal stopping* *time and*

$$\widehat{V}(t,x) = \begin{cases} e^{-rt}bx, & for \quad x \le -\frac{1}{\sqrt{2r}} \\ -e^{-rt}\frac{b}{\sqrt{2r}}e^{(-\frac{1}{\sqrt{2r}}-x)\sqrt{2r}}, & for \quad x > -\frac{1}{\sqrt{2r}} \end{cases}$$

Discounted Gain Function with Two Stopping Times

The optimal stopping problem is:

$$V(x) = \sup_{0 \le \tau \le \xi < \infty} E_x[e^{-r\xi}\tilde{K}(X_\xi) + e^{-r\tau}\tilde{G}(X_\tau)]$$

where,$e^{-rt}\tilde{K}(x)$ and $e^{-rt}\tilde{G}(x)$ satisfy assumption 1.1 w.r.t $(X_t)_{0 \le t < \infty}$.

Example 4.8

$$V(x) = \sup_{0 \le \tau \le \xi < \infty} E_x[-e^{-r\tau}cB_\tau + e^{-r\xi}bB_\xi]$$

where b and c are positive constants and B_t is Brownian motion.

According to the theorem in the first chapter and example 1.5, we can get:

$$\widehat{U}(t,x) = \begin{cases} e^{-rt}\frac{b}{\sqrt{2r}}e^{\sqrt{2r}x-1}, & for \quad x < \frac{1}{\sqrt{2r}} \\ e^{-rt}bx, & for \quad x \ge \frac{1}{\sqrt{2r}} \end{cases}$$

Then the problem changes to:

$$V(x) = \sup_{0 \le \tau < \infty} E_x[-e^{-r\tau} c B_\tau + U(\tau, B_\tau)]$$

The drift part of $-e^{-rt}cx + U(t,x)$ *is:*

$$\begin{cases} e^{-rt} crx dt, & for \quad x < \dfrac{1}{\sqrt{2r}} \\ e^{-rt}[crx - brx]dt, & for \quad x \ge \dfrac{1}{\sqrt{2r}} \end{cases}$$

The we can have the following conclusions:

Case 1: $b = c$. $-e^{-rt}cx + U(t,x)$ *is submartingale, there is no optimal stopping time.*

Case 2: $b > c$. $-e^{-rt}cx + U(t,x)$ *is supermartingale for* $x \ge 0$, *then stopping region* $D \in [0, \infty)$. *By using the theorems in the previous subsection, we can get* $D = [\frac{1}{\sqrt{2r}}, \infty)$.

Case 3: $b < c$. $-e^{-rt}cx + U(t,x)$ *is supermartingale for* $x \le 0$, *then stopping region* $D \in (-\infty, 0]$. *By using the theorems in the previous subsection, we can get* $D = (-\infty, -\frac{1}{\sqrt{2r}}]$.

Discounted and Integral Gain Function with One Stopping Time

The optimal stopping problem is:

$$V(x) = \sup_{0\leq\tau<\infty} E_x[\int_0^{\tau} e^{-rt}L(X_t)dt + e^{-r\tau}M(X_\tau)] \tag{48}$$

where X_t is 1-d drifted Brownian motion. In addition, L is cotinuous function and M is measurable function.

Let us first think of a special kind of stopping time. i.e. $\tau_a^b = inf\,\{t \geq 0 : X_t \in (-\infty, a] \cup [b, \infty)\}$ for some $a < b$. According to the theorem in the first chapter, we know $V(x) = E_x[\int_0^{\tau_a^b} e^{-rt}L(X_t)dt + e^{-r\tau_a^b}M(X_{\tau_a^b})]$ must satisfy the following PDE:

$$(\mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2} - r)V(x) = -L(x) \tag{49}$$

for $x \in (a, b)$. We assume the following boundary conditions:

$$\begin{aligned} V(a) &= M(a) \\ V(b) &= M(b) \end{aligned} \tag{50}$$

The general solution of this PDE is:

$$V(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + q(x) \tag{51}$$

where r_i is the solution of the following equation:

$$\frac{1}{2} r_i^2 + \mu r_i - r = 0 \tag{52}$$

and $r_1 < 0 < r_2$. And

$$C_1 = \frac{(M(a) - q(a))e^{r_2 b} - (M(b) - q(b))e^{r_2 a}}{e^{r_2 b + r_1 a} - e^{r_1 b + r_2 a}} \tag{53}$$

$$C_2 = \frac{(M(a) - q(a))e^{r_1 b} - (M(b) - q(b))e^{r_1 a}}{e^{r_2 a + r_1 b} - e^{r_2 b + r_1 a}} \tag{54}$$

Lemma 4.9 *Suppose the follwing statements are true:*

1. *$M(x)$ and $q(x)$ are finite for any $x \in R$.*

2. $\lim_{a \to -\infty} \frac{M(a) - q(a)}{e^{r_1 a}} = 0$.

3. $\lim_{b \to \infty} \frac{M(b) - q(b)}{e^{r_2 b}} = 0$.

Then if $a \to -\infty$

$$C_1 \to 0 \ and \ C_2 \to [M(b) - q(b)]e^{-r_2 b}$$

If $b \to \infty,$ *then*

$$C_1 \to [M(a) - q(a)]e^{-r_1 a} \ and \ C_2 \to 0$$

Lemma 4.10 *Let* $\tau_b = inf\,\{t \geq 0 : X_t \in [b, \infty)\}$, *consider the following problem:*

$$V_b(x) = E_x[\int_0^{\tau_b} e^{-rt}L(X_t)dt + e^{-r\tau_b}M(X_{\tau b})]$$

where X_t *is 1-d drifted Brownian motion with* $\mu > 0.L$ *is cotinuous function and* M *is measurable function. In addition* $M(x)$ *and* $q(x)$ *is finite for all* $x \in R$. *Moreover,*$\lim_{a \to -\infty} \frac{M(a)-q(a)}{e^{r_1 a}} = 0$. *Then*

$$V_b(x) = [M(b) - q(b)]e^{r_2(x-b)} + q(x)$$

for $x \in (-\infty, b)$.

Lemma 4.11 *Let* $\tau_a = inf\,\{t \geq 0 : X_t \in (-\infty, a]\}$, *consider the following problem:*

$$V_a(x) = E_x[\int_0^{\tau_a} e^{-rt}L(X_t)dt + e^{-r\tau_a}M(X_{\tau a})]$$

where X_t *is 1-d drifted Brownian motion with* $\mu < 0.L$ *is cotinuous function and* M *is measurable function. In addition* $M(x)$ *and* $q(x)$ *is finite for all* $x \in R$. *Moreover,* $\lim_{a \to -\infty} \frac{M(a)-q(a)}{e^{r_1 a}} = 0$. *Then*

$$V_a(x) = [M(a) - q(a)]e^{r_1(x-a)} + q(x)$$

for $x \in (a, \infty)$.

Now, let's define $\tau_a^t = inf\,\{s \geq t : X_s \in (-\infty, a]\}$ and

(55)
$$\begin{pmatrix} dZ_{1t} \\ \\ dZ_{2t} \end{pmatrix} = \begin{pmatrix} e^{-rt}L(X_t) \\ \\ \mu \end{pmatrix} dt + \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} dB_t$$

Moreover, if we define $V_a(t, z_1, z_2)$ as the following:

(56)
$$\begin{aligned} V_a(t, z_1, z_2) &= E_{(t,z_1,z_2)}[\textstyle\int_0^{\tau_a^t} e^{-rt}L(X_t)dt + e^{-r\tau_a^t}M(X_{\tau a}^t)] \\ &= E_{(t,z_1,z_2)}[\textstyle\int_0^t e^{-rs}L(X_s)ds + \int_t^{\tau_a^t} e^{-rs}L(X_s)ds + e^{-rt}e^{-r(\tau_a^t - t)}M(X_{\tau a}^t)] \\ &= z_1 + e^{-rt}[\textstyle\int_0^{\tau_a} e^{-rs}L(X_s)ds + e^{-r(\tau_a)}M(X_{\tau a})] \\ &= z_1 + e^{-rt}V_a(z_2) \end{aligned}$$

According to the sufficient theorem in the first chapter and lemma above, we can get the following theorems.

Proposition 4.12 *If the followings are true:*

1. $z_1 + e^{-rt}M(z_2)$ *is superhamonic function, for* $x \in [b, \infty)$*.(Superharomic Propery)*

2. $[M(b) - q(b)]e^{-r_2 b} \geq [M(x) - q(x)]e^{-r_2 x}$*, for all* $x \in (-\infty, b)$*.(Dominating Property)*

In addition, $\mu \geq 0$*, then*

$$\widehat{V}(t, z_1, z_2) = \begin{cases} z_1 + e^{-rt}[M(b) - q(b)]e^{r_2(z_2 - b)} + e^{-rt}q(z_2), & for \quad z_2 \in (-\infty, b) \\ z_1 + e^{-rt}M(z_2), & for \quad z_2 \in [b, \infty) \end{cases}$$

and $\tau = inf\,\{t \geq 0 : X_t \in [b, \infty)\}$ is an optimal stopping time.

Proposition 4.13 *If* M *has continuous first derivative and there are at most finite points having no second derivative, then we can replace 1 in the above theorem by the following conditions:*

1. $(-r + \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2})M(x) + L(x) \leq 0$ *for* $x \geq b$*. (Superharmoinic Property)*
2. $M^{'}(b_+) = [M(b) - q(b)]r_2 + q^{'}(b)$*. (Smooth Pasting)*
3. $[M(b) - q(b)]e^{-r_2 b} \geq [M(x) - q(x)]e^{-r_2 x}$*, for all* $x \in (-\infty, b)$*.(Dominating Property)*

In addition, $\mu \geq 0$*, then*

$$\widehat{V}(t, z_1, z_2) = \begin{cases} z_1(t) + e^{-rt}[M(b) - q(b)]e^{r_2(z_2(t)-b)} + e^{-rt}q(z_2(t)), & for \quad z_2(t) \in (-\infty, b) \\ z_1(t) + e^{-rt}M(z_2(t)), & for \quad z_2(t) \in [b, \infty) \end{cases}$$

and $\tau = inf\,\{t \geq 0 : X_t \in [b, \infty)\}$ is an optimal stopping time.
Note: Smooth Pasting determines the location of b.

Proposition 4.14 *If the followings are true:*

1. $z_1 + e^{-rt}M(z_2)$ *is superhamonic function for* $x \in (-\infty, a]$*.(Superharomic Propery)*
2. $[M(a) - q(a)]e^{-r_1 a} \geq [M(x) - q(x)]e^{-r_1 x}$*, for all* $x \in (a, \infty)$*.(Dominating Property)*

In addition, $\mu \leq 0$*, then*

$$\widehat{V}(t, z_1, z_2) = \begin{cases} z_1(t) + e^{-rt}M(z_2(t)), & for \quad z_2(t) \in (-\infty, a] \\ z_1 + e^{-rt}[M(a) - q(a)]e^{r_1(z_2(t)-a)} + e^{-rt}q(z_2(t)), & for \quad z_2(t) \in (a,) \end{cases}$$

and $\tau = inf\,\{t \geq 0 : X_t \in (-\infty, a]\}$ is an optimal stopping time.

Proposition 4.15 *If M has continuous first derivative and there are at most finite points having no second derivative, then we can replace 1 in the above theorem by the following conditions:*

1. $(-r + \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2})M(x) + L(x) \leq 0$ *for* $x \leq a$*. (Superharmonic Property)*
2. $M'(a_-) = [M(a) - q(a)]r_1 + q'(a)$*. (Smooth Pasting)*
3. $[M(a) - q(a)]e^{-r_1 a} \geq [M(x) - q(x)]e^{-r_1 x}$*, for all* $x \in (a, \infty)$*.(Dominating Property)*

In addition, $\mu \leq 0$*, then*

$$\widehat{V}(t, x) = \begin{cases} e^{-rt}M(x), & for \quad x \in (-\infty, a] \\ e^{-rt}[M(a) - q(a)]e^{r_1(x-a)} + e^{-rt}q(x), & for \quad x \in (a, \infty) \end{cases}$$

and $\tau = inf\,\{t \geq 0 : X_t \in (-\infty, a]\}$ is an optimal stopping time.

Example 4.16

$$V(x) = \sup_{0 \leq \tau < \infty} E_x[\int_0^\tau e^{-rt}aB_t dt + e^{-r\tau}bB_\tau]$$

where a is positive constant and b is negative constant.

In this example, $G(t, X_t) = \int_0^t e^{-rs}aB_s ds + e^{-rt}bB_t$*,* $L(x) = ax$*,* $M(x) = bx$*,* $q(x) = \frac{a}{r}x$ *and* $\mu = 0$*.*
Case 1: $a - rb > 0$
$G(t, x)$ *has a nonpositive drift for* $x \leq 0$*. Therefore, stop region* $D \in (-\infty, 0]$*. Let's guess*

$D = (-\infty, L]$. Here we use the smooth pasting to guess the value of L. From the corollary above, we get $L = -\frac{1}{\sqrt{2r}}$. According to the theorem above, we can conclude that D is $(-\infty, -\frac{1}{\sqrt{2r}}]$.

Case 2: $a - rb = 0$
$G(t, x)$ is a martingale. Therefore, we will always have the same value no matter which stopping we choose. Note: we assume the stopping time satisfies the assumption.

Case 3: $a - rb < 0$
$G(t, x)$ has a nonpositive drift for $x \geq 0$. Therefore, stop region $D \in [0, \infty)$ ... we can get D is $[\frac{1}{\sqrt{2r}}, \infty)$.

Discounted and Integral Gain Function with Two Stopping Times

The optimal stopping problem is:

$$V(x) = \sup_{0 \leq \xi \leq \tau < \infty} E_x[\int_{\xi}^{\tau} e^{-rt} L(X_t) dt + e^{-r\tau} M(X_\tau) + e^{-r\xi} N(X_\xi)] \tag{57}$$

where X_t is 1-d drifted Brownian motion. In addition, L is cotinuous function and M and N are measurable function.

(58)

$$\begin{aligned} V(x) = & \sup_{0 \leq \xi \leq \tau < \infty} E_x[\int_0^{\tau} e^{-rt} L(X_t) dt + e^{-r\tau} M(X_\tau) + \int_0^{\xi} -e^{-rt} L(X_t) dt + e^{-r\xi} N(X_\xi)] \\ \leq & \sup_{0 \leq \tau < \infty} E_x[\int_0^{\tau} e^{-rt} L(X_t) dt + e^{-r\tau} M(X_\tau)] + \sup_{0 \leq \xi < \infty} E_x[\int_0^{\xi} -e^{-rt} L(X_t) dt + e^{-r\xi} N(X_\xi)] \end{aligned}$$

The equality will hold if $P(\xi^* \leq \tau^*) = 1$.

Example 4.17 *Consider the following example:*

$$V(x) = \sup_{0 \leq \xi \leq \tau < \infty} E_x[\int_{\xi}^{\tau} e^{-rt} a B_t dt + e^{-r\tau} c B_\tau + e^{-r\xi} b B_\xi]$$

We assume a, b, c and $r > 0$ are constants. Moreover, we have $a + br > 0$, $a - cr > 0$ and $b + c > 0$.
Then it's easy to see that

$$V(x) \leq \sup_{0 \leq \tau < \infty} E_x[\textstyle\int_0^{\tau} e^{-rt} a B_t dt + e^{-r\tau} c B_\tau)] + \sup_{0 \leq \xi < \infty} E_x[\int_0^{\xi} -e^{-rt} a B_t dt + e^{-r\xi} b B_\xi)]$$

and equality holds if $P(\xi^* \leq \tau^*) = 1$.
From the result in the last subsection, we can get:

$$\tau^* = inf\left\{t \geq 0 : X_t \in (-\infty, -\frac{1}{\sqrt{2r}}]\right\}$$

and

$$\xi^* = inf\left\{t \geq 0 : X_t \in [\frac{1}{\sqrt{2r}}, \infty)\right\}$$

In addition, $P(\xi^* \leq \tau^*) = 1$ iff $x \geq \frac{1}{\sqrt{2r}}$.

However, what if $x \leq \frac{1}{\sqrt{2r}}$*.Define the following optimal stopping problems:*

$$U(s,y) = \sup_{\tau \geq s} E_y[\int_s^{\tau} e^{-rt} L(X_t)dt + e^{-r\tau} M(X_\tau)] \tag{59}$$

$$V(x) = \sup_{\xi \geq 0} E_x[U(\xi, X_\xi) + e^{-r\xi} L(X_\xi)] \tag{60}$$

It's easy to get that

$$U(s,y) = e^{-rs} U(0,y) = e^{-rs} U(y) \tag{61}$$

and

$$V(x) = \sup_{\xi \geq 0} E_x e^{-r\xi}(U(X_\xi) + L(X_\xi)) \tag{62}$$

Therefore, we can transfer the two optimal stopping problem to two one optimal stopping problems.

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