

GENERALIZED TCHEBYSHEV TRIANGULATIONS

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ABSTRACT. After fixing a triangulation L of a k -dimensional simplex that has no new vertices on the boundary, we introduce a triangulation operation on all simplicial complexes that replaces every k -face with a copy of L , via a sequence of induced subdivisions. The operation may be performed in many ways, but we show that the face numbers of the subdivided complex depend only on the face numbers of the original complex, in a linear fashion. We use this linear map to define a sequence of polynomials generalizing the Tchebyshev polynomials of the first kind and show, that in many cases, but not all, the resulting polynomials have only real roots, located in the interval $(-1, 1)$. Some analogous results are shown also for generalized Tchebyshev polynomials of the higher kind, defined using summing over links of all original faces of a given dimension in our generalized Tchebyshev triangulations. Generalized Tchebyshev triangulations of the boundary complex of a cross-polytope play a central role in our calculations, and for some of these we verify the validity of a generalized lower bound conjecture by the second author.

1. INTRODUCTION

This paper generalizes the following idea of a Tchebyshev triangulation introduced in [7]: given any simplicial complex K , subdivide each edge into two parts by adding a new midpoint, and triangulate K by performing a stellar subdivision at each of the newly added midpoints. The order in which these subdivisions had to be performed was subject to certain rules, and then the face numbers of the resulting complex K' were always the same. The effect of this triangulation operation on the face numbers f_j is most easily described in terms of the F -polynomial $\sum_{j \geq 0} f_{j-1}((x-1)/2)^j$ of these complexes: the operation taking the F -polynomial of K into the F -polynomial of K' is an instance of the linear map $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ that takes each x^n to $T_n(x)$, the n^{th} Tchebyshev polynomial of the first kind.

A key result of the present paper, Theorem 3.3, is a wide-reaching generalization of the idea presented above. It states that the stellar subdivision operations above may be performed in any order, and we always obtain the same face numbers. Furthermore, the statement may be generalized to the situation where instead of subdividing each edge into two parts, we subdivide each k -dimensional face in the same way, using a fixed triangulation L of the k -simplex that adds new vertices only in the interior. The resulting *generalized Tchebyshev triangulations* are the subject of study of our present paper.

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As we will see in Section 4, the face numbers in a generalized Tchebyshev triangulation can be easily computed knowing the number of faces of L with given numbers of vertices on the boundary and in the interior of the k -simplex. At the level of the F -polynomials, each fixed subdivision L induces a linear map $T^L : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$, giving rise to a natural generalization of Tchebyshev polynomials of the first kind, introduced in Section 5. These polynomials share many properties with the ordinary Tchebyshev polynomials: they satisfy a Fibonacci-type recurrence (whose degree depends on the dimension k), their multiset of zeros is symmetric to the origin, and all their real zeros belong to the interval $(-1, 1)$. The question naturally arises, whether these generalized Tchebyshev polynomials of the first kind also have only real roots. A first answer to this question is given in Section 6, where we will see that the answer is always “yes” for $k = 1$, and it is “no” for the simplest subdivision of a 3-simplex, obtained by adding one new vertex in the interior and performing a stellar subdivision. In Section 7 we prove that all roots are real also for generalized Tchebyshev polynomials of the first kind, induced by any valid subdivision of the two-dimensional simplex.

Generalizing the construction introduced in [7], in Section 8 we introduce analogues of Tchebyshev polynomials of the second kind, by considering summing over the links of all faces of a given dimension of the original complex, in the subdivided complex. A lot remains to be explored regarding these polynomials, but a few results indicate that we have found an “appropriate generalization”: our generalized Tchebyshev polynomials of the j^{th} kind (where $j \leq k + 1$) satisfy the same recurrence as our generalized Tchebyshev polynomials of the first kind, the multisets of their roots are also symmetric of the origin, and their real roots also belong to the interval $(-1, 1)$. We chose to postpone a deeper study of their real-rootedness to a future occasion, but we established the fact that, for $k = 1$, all generalized Tchebyshev polynomials of the second kind are real rooted.

Our results in Sections 5 and 8 underline the central importance of the generalized Tchebyshev triangulations of the boundary complex of a crosspolytope, as the coefficients of our generalized Tchebyshev polynomials can be directly read off the face count in these complexes, refined by distinguishing between original and newly added vertices. In the concluding Section 9 we prove the validity of a conjecture by the second author [11, Conjecture 1.5], on strong generalized lower bounds for the face numbers of some of these simplicial complexes.

Our generalized Tchebyshev triangulations offer infinitely many new ways to subdivide a simplicial complex in such a manner that the face numbers change in a predictable fashion. In this sense our triangulation operations generalize the notion of a barycentric subdivision. In fact, any barycentric subdivision arises by applying a sequence of generalized Tchebyshev triangulation operations as follows: for each k that is less than or equal to the dimension of the complex to be subdivided, we take the generalized Tchebyshev triangulation induced by the stellar subdivision of a k -simplex obtained after adding a single vertex in its interior (we perform these operation in decreasing order by k). Investigating whether some face counting polynomial associated to such a triangulation has only real roots is not a new concern: Brenti and Welker [4] showed that the h -polynomial of the barycentric subdivision of a simplicial complex with a nonnegative h -vector has only simple real zeros. In the future, it would be worth finding an exact description of all triangulations of a k -simplex that induces generalized Tchebyshev polynomials having only real roots. Another interesting question is to fix a specific generalized Tchebyshev triangulation operation, and to ask: to which simplicial complexes can we apply them and obtain real-rooted f -polynomials and/or h -polynomials? Finally, once we have a

better understanding of the generalized Tchebyshev polynomials of the higher kind, it will be worth finding out how they are interconnected.

2. PRELIMINARIES

First we recall some basic definitions and results related to simplicial complexes. For further background see, for instance, [2, 10]. Next we recall some basic facts on Tchebyshev polynomials. These polynomials play an important role in many areas of mathematics, including combinatorics, numerical analysis and orthogonal polynomials. However, we will only need facts on them that are discussed in any introductory work on orthogonal polynomials, see for instance [5]. Most important formulas on Tchebyshev polynomials are listed (without proof) in the work of Abramowitz and Stegun [1].

2.1. Simplicial complexes. A *simplicial complex* K on the *vertex set* V is a collection of subsets of V such that $\{v\} \in K$ for all $v \in V$, and if $G \subset F$ and $F \in K$, then $G \in K$. The elements of K are called *faces*. In particular, the empty set is a face of K . The *link* of a face σ is the subcomplex $\text{link}_K(\sigma) = \{\tau \in K : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in K\}$. The *join* of two simplicial complexes K and L on disjoint vertex sets is $K * L = \{\sigma \cup \tau : \sigma \in K, \tau \in L\}$. Thinking of the faces of K as simplices glued together gives K a topology, and the *geometric realization* of K , denoted $\|K\|$, stands for this topological space. We say that K is a *triangulation* of a topological space X if $\|K\|$ is homeomorphic to X .

The following well known result in piecewise linear topology will be needed later, see e.g. [8, Cor. 1.16– Lemma 1.18]:

Lemma 2.1. *Let L be a triangulation of a simplex such that the only vertices of L on the boundary $\partial(L)$ are the original vertices of the simplex, and let $\tau \in L$ be any face. Then $\text{link}_L(\tau)$ is homeomorphic to a sphere if and only if it contains at least one vertex in the interior of $\|L\|$, otherwise it is homeomorphic to a ball.*

Let $(A_i)_{i \in I}$ be a family of nonempty sets. Its *nerve* $\mathcal{N}((A_i)_I)$ is the simplicial complex with vertex set I and faces all $F \subseteq I$ such that $\cap_{i \in F} A_i \neq \emptyset$. A version of the Borsuk's nerve-theorem [3] that we will need is the following, see Björner [2, Theorem 10.6])

Theorem 2.2. *(Nerve theorem) Let $(A_i)_{i \in I}$ be a family of subcomplexes of a simplicial complex K such that $\cup_I A_i = K$ and for every $J \subseteq I$, $\cap_{i \in J} A_i$ is either empty or contractible. Then the nerve complex $\mathcal{N}((A_i)_I)$ is homotopy equivalent to K .*

The *dimension* of a face σ is defined by $\dim(\sigma) := |\sigma| - 1$; the dimension of a simplicial complex K is defined by $\dim(K) := \max\{\dim(\sigma) : \sigma \in K\}$. Let $f_i(K)$ be the number of i -dimensional faces (i -faces) of K , and let $f(K)$ be the f -vector of K , namely, $f(K) := (f_{-1}(K), f_0(K), \dots, f_{\dim(K)}(K))$. In polynomial form, the f -polynomial of K is $f(K, x) := \sum_{0 \leq i \leq \dim(K)+1} f_{i-1}(K)x^i$. This information can also be encoded in the h -polynomial of K , $h(K, x) := \sum_{0 \leq i \leq \dim(K)+1} h_i(K)x^i$, given by $h_i =$

$\sum_{j=0}^i (-1)^{i-j} \binom{n-j}{i-j} f_{j-1}$ where $n = \dim(K) + 1$. In particular, $f_{i-1} = \sum_{j=0}^i \binom{n-j}{i-j} h_j$. Given a simplicial complex K and a map (called *coloring*) $a : V(K) \rightarrow \{x_1, x_2, \dots, x_s\}$, the *flag f -polynomial* of (K, a) is

$$f_a(K; x_1, \dots, x_s) := \sum_{F \in K} \prod_{v \in F} a(v) \in \mathbb{Z}[x_1, \dots, x_s].$$

A set F is called a *missing face* of a simplicial complex K if $F \notin K$ and its boundary complex $\partial(F) = 2^F \setminus \{F\}$ is a subcomplex of K . For $F \in K$, the *stellar subdivision* of K at F is the simplicial complex $K(F) = \text{Stellar}_K(F) = (K \setminus \{T \in K : F \subseteq T\}) \cup \{v_F\} * \partial(F) * \text{link}_K(F)$, where v_F is not a vertex of K . The *j -skeleton* of K , denoted $K_{\leq j}$, is the subcomplex of K consisting of all faces in K of dimension $\leq j$.

2.2. Tchebyshev polynomials. The Tchebyshev polynomials $T_n(x)$ of the first kind and the Tchebyshev polynomials $U_n(x)$ of the second kind are usually defined by the formulas

$$(2.1) \quad T_n(\cos(\alpha)) = \cos(n \cdot \alpha) \quad \text{and} \quad U_n(\cos(\alpha)) = \frac{\sin((n+1)\alpha)}{\cos(\alpha)},$$

see [1, (22.3.15) and (22.3.16)]. Equivalently, they may be defined recursively as follows. Both polynomial sequences satisfy the same recurrence $P_n(x) = 2xP_{n-1}(x) - P_{n-2}(x)$ (all occurrences of the letter P need to be replaced by T resp. U), see [1, (22.7.4) and (22.7.5)]. Only the initial conditions are different: we have $T_0(x) = 1$, $T_1(x) = x$; $U_0(x) = 1$ and $U_1(x) = 2x$.

They share the following properties, which will be explored for our generalized Tchebyshev polynomials.

Theorem 2.3. *For all $n \geq 0$, the polynomials $T_n(x)$ and $U_n(x)$ satisfy:*

- (1) *their degree is n ,*
- (2) *symmetry: $(-1)^n P_n(-x) = P_n(x)$ for $P_n = T_n, U_n$, and*
- (3) *all their roots are real, simple, and belong to the interval $(-1, 1)$.*

The first two statements are immediate consequences of the recursive definition, the third statement may be shown in at least two different ways: by direct calculation of the roots from (2.1), or by invoking general results from the theory of orthogonal polynomials. We refer the reader to [5] for details which we will not review here as most of our generalized Tchebyshev polynomials will *not* be sequences of orthogonal polynomials, see Remark 5.6.

3. GENERALIZED TCHEBYSHEV TRIANGULATIONS

In this section we fix a triangulation L of the k -dimensional simplex such that the only vertices of L on the boundary $\partial(L)$ are the original vertices of the simplex. We will use the notation $\partial(L)$ for the subcomplex of boundary faces (this is also the boundary complex of the original k -simplex) and the notation $\text{Int}(L)$ for the family of (closed) faces contained in the interior of L . Given any family of faces C we will use $V(C)$ to denote the set of vertices of the faces in the family C . Any face $\sigma \in L$ may be uniquely written as the disjoint union of $\sigma \cap V(\partial(L))$ and $\sigma \cap V(\text{Int}(L))$.

Definition 3.1. Let L be a triangulation of the k simplex, containing new vertices in the interior only. Given any simplicial complex K , a simplicial complex K' is a generalized Tchebyshev triangulation of K induced by L if there is an ordered list $\sigma_1, \dots, \sigma_m$ of the k -dimensional faces of K , listing each k -dimensional face exactly once, and an ordered list K_0, K_1, \dots, K_m of simplicial complexes such that $K_0 = K$, $K_m = K'$ and, for each $i \geq 1$, the complex K_i is obtained from K_{i-1} by replacing the face σ_i with an isomorphic copy L_i of L and the family of faces $\{\sigma_i \cup \tau : \tau \in \text{link}(\sigma_i)\}$ containing σ_i with the subdivided complex $\{\sigma' \cup \tau : \sigma' \in L_i, \tau \in \text{link}(\sigma_i)\}$.

Obviously, given any ordered list $\sigma_1, \dots, \sigma_m$ of the k -dimensional faces of K , and a list of bijections $\phi_{\sigma_i} : V(\partial(L)) \rightarrow V(\sigma_i)$ for $1 \leq i \leq m$, there is exactly one list of simplicial complexes K_0, K_1, \dots, K_m satisfying the condition given in Definition 3.1. Using a different list may result in a non-isomorphic triangulation, as shown in the following example.

Example 3.2. Let L be the path with 2 edges (triangulating the 1-simplex), K be the union of the two triangles $\{v_1, v_2, v_3\}$ and $\{v_1, v_2, v_4\}$ sharing the edge $\{v_1, v_2\}$. Let K' be the generalized Tchebyshev triangulation of K defined by the ordering of edges $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_4\}$ and let K'' be the generalized Tchebyshev triangulation of K defined by the ordering $\{v_1, v_3\}$, $\{v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_1, v_4\}$, $\{v_1, v_2\}$. Then K' is a cone (over an 8-cycle) and K'' is not, see Fig. 1. (Here specifying the bijection ϕ_{σ_i} is not important as we obtain isomorphic complexes for both choices.)

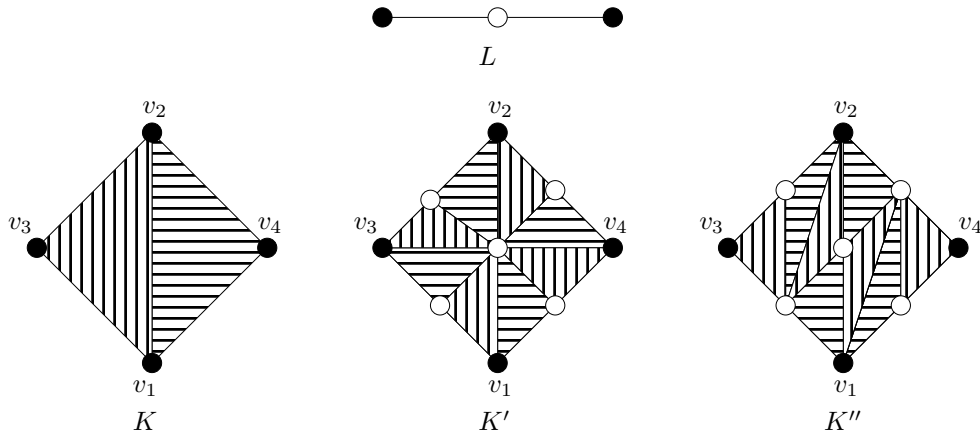


FIGURE 1. Illustration to Example 3.2

However, K' and K'' have the same f -vector. This is not a coincidence as the following result shows.

Theorem 3.3. Given a triangulation L of the k -dimensional simplex and an arbitrary simplicial complex K , all generalized Tchebyshev triangulations of K , induced by L , have the same f -vector.

Theorem 3.3 follows from setting $y = x$ in the following, more general statement.

Let $c = c_K : V(K') \rightarrow \{x, y\}$ be the coloring $c(v) = x$ if $v \in V(K)$ and $c(v) = y$ if $v \in V(K') - V(K)$.

Theorem 3.4. *Given a triangulation L of the k -dimensional simplex, let K' be any generalized Tchebyshev triangulation of K induced by L . Then the flag f -polynomial of (K', c) , namely*

$$f_c(K'; x, y) = \sum_{\sigma \in K'} x^{|\sigma \cap V(K)|} y^{|\sigma \cap (V(K') \setminus V(K))|},$$

depends only on the f -vector of K in a linear fashion and is independent of the particular choice of K' . Thus, we will denote it by $f(K; x, y)$.

Proof. We need to show that given L , there exist linear functionals $l_{i,j} : \mathbb{R}[z] \rightarrow \mathbb{R}$ such that for any simplicial complex K and any generalized Tchebyshev triangulation K' of K induced by L , one has

$$f_c(K'; x, y) = \sum_{i,j} l_{i,j}(f(K, z)) x^i y^j.$$

First we reduce to the case where K is a simplex. Do to so, we make the following easy observation:

(*) If H is a subcomplex of K then the induced order on the k -simplices in H (keeping the same bijections $\phi_\sigma : V(\partial(L)) \rightarrow V(\sigma)$ for all k -faces $\sigma \in H$) gives H' which equals to the restriction of K' to the subspace $\|H\|$ of $\|K\|$.

In particular, by restriction c_K induces a coloring of $V(H')$ which is the same as $c_H : V(H') \rightarrow \{x, y\}$. Thus, if F is a top dimensional face of K then

$$f_c(K'; x, y) = f_c((K - \{F\})'; x, y) + f_c(F'; x, y) - f_c((\partial F)'; x, y),$$

hence, by repeating for all top dimensional faces of K ,

$$f_c(K'; x, y) = f_c((K_{\leq \dim(K)-1})'; x, y) + f_{\dim(K)}(K)(f_c(F'; x, y) - f_c((\partial F)'; x, y)).$$

By induction on dimension, we already know that there exist linear functionals $l_{i,j}^{(\dim(K))} : \mathbb{R}[z]_{\leq \dim(K)} \rightarrow \mathbb{R}$ such that for all complexes T of dimension $< \dim(K)$, $f_c(T'; x, y) = \sum_{i,j} l_{i,j}^{(\dim(K))}(f(T), z) x^i y^j$. If we show the assertion of the theorem for the $(\dim(K))$ -simplex, then $l_{i,j}^{(\dim(K))}$ can be extended to $l_{i,j}^{(\dim(K)+1)} : \mathbb{R}[z]_{\leq \dim(K)+1} \rightarrow \mathbb{R}$ by setting $l_{i,j}^{(\dim(K)+1)}(z^{\dim(K)+1})$ to be the coefficient of $x^i y^j$ in $f_c(F'; x, y) - f_c((\partial F)'; x, y)$.

Thus, assume K is an $(n-1)$ -dimensional simplex. We will show now by induction on n that $f_c(K'; x, y)$ does not depend on the choice of K' , only on n , in which case we use $f_n(x, y)$ to stand for $f_c(K'; x, y)$. We claim that $f_n(x, y)$ is given by $f_n(x, y) = (1+x)^n$ for $n \leq k$ and by the recurrence formula (which shows the independence of the choice of K')

$$(3.1) \quad f_n(x, y) = \sum_{\emptyset \neq \mathcal{I} \subseteq \mathcal{F}(L)} (-1)^{|\mathcal{I}|-1} (1+y)^{|\bigcap_{F \in \mathcal{I}} F \cap V(\text{Int}(L))|} f_{n-k-1+|\bigcap_{F \in \mathcal{I}} F \cap V(\partial(L))|}(x, y) \quad \text{for } n \geq k+1.$$

Here $\mathcal{F}(L)$ is the set of facets of L and the summation runs over all nonempty subfamilies \mathcal{I} of $\mathcal{F}(L)$. For $n > k$, we argue by induction: the subdivision of σ_1 induces a bijection ι from $\mathcal{F}(\sigma'_1)$ to $\mathcal{F}(K_1)$ by $\iota(F) = F \cup (V(K) - V(\sigma_1))$. Denote by $\bar{U} = 2^U$ the simplex on the finite vertex set U and all its faces, and for a family \mathcal{U} of finite sets let $\bar{\mathcal{U}}$ be the simplicial complex $\cup_{U \in \mathcal{U}} \bar{U}$. For a face $\sigma \in K'$

let $N(\sigma) := \sigma \cap (V(K') \setminus V(K))$ and $O(\sigma) = \sigma \cap V(K)$. (The letters N and O in $N(\sigma)$ and $O(\sigma)$, respectively, are meant to refer to “old”, respectively, “new” vertices.) Observe that the restriction of K' to $|\overline{\iota(F)}|$ is the subcomplex $(\overline{\iota(F)})' = \overline{N(F)} * (\overline{V(K) - V(\sigma)} * \overline{O(F)})'$, and by the induction hypothesis its contribution to $f_c(K'; x, y)$ is $(1+y)^{|N(F)|} f_{n-|N(F)|}(x, y)$. (Note that indeed $|N(F)| > 0$ as all facets of σ_1 contain a vertex in $\text{Int}(\sigma_1)$.) As $K' = \cup_{F \in \mathcal{F}(\sigma'_1)} \overline{\iota(F)}'$, inclusion-exclusion gives (3.1). Here are the details:

$$f_n(x, y) = \sum_{\emptyset \neq S \subseteq \mathcal{F}(\sigma'_1)} (-1)^{|S|-1} f_c(\cap_{F \in S} \overline{\iota(F)}'; x, y) = \sum_{\emptyset \neq S \subseteq \mathcal{F}(\sigma'_1)} (-1)^{|S|-1} f_c(\cap_{F \in S} \overline{N(F)} * (\cap_{F \in S} \overline{O(F)} * (\overline{V(K) - V(\sigma_1)}))'; x, y).$$

Now, $f_c(\cap_{F \in S} \overline{N(F)}; x, y) = (1+y)^{|\cap_{F \in S} N(F)|}$ and, by induction,

$$f_c(\cap_{F \in S} (\overline{O(F)} * (\overline{V(K) - V(\sigma_1)}))'; x, y) = f_{n-(k+1)+|\cap_{F \in S} O(F)|}(x, y).$$

The bijection ι finishes the proof. \square

Example 3.5. Let $k = 1$ and let L be the triangulation of the 1-dimensional simplex obtained by adding the midpoint of the 1-simplex as a new vertex, as in Example 3.2. Certain generalized Tchebyshev triangulations induced by this complex L were considered in [7], where it was shown that the face numbers in these triangulations are independent of the numbering of the vertices. Theorem 3.3 generalizes these results even for this particular choice of L .

Using Lemma 2.1 we may rephrase (3.1) as follows.

Proposition 3.6. *The polynomials $f_n(x, y)$ are also given by $f_n(x, y) = (1+x)^n$ for $n \leq k$ and by the recurrence*

$$f_n(x, y) = \sum_{\sigma \in L \setminus \partial(L)} (-1)^{k+1-|\sigma|} (1+y)^{|\sigma \cap V(\text{Int}(L))|} \cdot f_{n-k-1+|\sigma \cap V(\partial(L))|}(x, y) \quad \text{for } n \geq k+1.$$

Proof. Let us fix a face $\sigma \in L$ and consider only those subsets $\mathcal{I} \subseteq \mathcal{F}(L)$ for which we have

$$(3.2) \quad \sigma = \bigcap_{F \in \mathcal{I}} F.$$

Note that $\mathcal{I} \neq \emptyset$ is equivalent to $\sigma \in L$. Each $\mathcal{I} \subseteq \mathcal{F}(L)$ satisfying (9.4) contributes a term of the form $(-1)^{|\mathcal{I}|-1} (1+y)^{|\sigma \cap V(\text{Int}(L))|} \cdot f_{n-k-1+|\sigma \cap V(\partial(L))|}(x, y)$ to the right hand side of (3.1).

Assume first that σ is not a facet of L . Then we may identify each \mathcal{I} with the collection of facets $\mathcal{I}' := \{F \setminus \sigma : F \in \mathcal{I}\}$ of $\text{link}(\sigma)$. Condition (9.4) is then equivalent to requiring that the intersection of the facets of $\text{link}(\sigma)$ listed in \mathcal{I}' is empty, equivalently the vertex set \mathcal{I}' is not a face of the nerve complex $\mathcal{N}(\text{link}(\sigma))$ of $\text{link}(\sigma)$. In this case the total contribution of all families $\mathcal{I} \subseteq \mathcal{F}(L)$ satisfying (9.4) to the right hand side of (3.1) is

$$\left(\sum_{\mathcal{I}' \subseteq V(\mathcal{N}(\text{link}(\sigma)))} (-1)^{|\mathcal{I}'|-1} - \tilde{\chi}(\mathcal{N}(\text{link}(\sigma))) \right) \cdot (1+y)^{|\sigma \cap V(\text{Int}(L))|} \cdot f_{n-k-1+|\sigma \cap V(\partial(L))|}(x, y).$$

Here $\tilde{\chi}(\mathcal{N}(\text{link}(\sigma)))$ is the reduced Euler characteristic of $\mathcal{N}(\text{link}(\sigma))$ which is the same as $\tilde{\chi}(\text{link}(\sigma))$ by Borsuk's nerve theorem, Theorem 2.2. By Lemma 2.1, $\tilde{\chi}(\text{link}(\sigma))$ is nonzero exactly when $\sigma \cap V(\text{Int}(L)) \neq \emptyset$ and then it is $(-1)^{k-|\sigma|}$, the reduced Euler characteristic of a $(k - |\sigma|)$ -dimensional sphere. Since σ is not a facet of L , we have $|V(\mathcal{N}(\text{link}(\sigma)))| \geq 1$ and the sum $\sum_{\mathcal{I}' \subseteq V(\mathcal{N}(\text{link}(\sigma)))} (-1)^{|\mathcal{I}'|-1}$ is zero.

Finally, consider the case when σ is a facet of L . Then (9.4) holds for exactly one $\mathcal{I} \subseteq \mathcal{F}(L)$, namely the family $\mathcal{I} = \{\sigma\}$. The contribution of this \mathcal{I} to the right hand side of (3.1) is

$$(-1)^{|\mathcal{I}|-1} \cdot (1+y)^{|\sigma \cap V(\text{Int}(L))|} \cdot f_{n-k-1+|\sigma \cap V(\partial(L))|}(x, y)$$

which equals

$$(-1)^{k+1-|\sigma|} \cdot (1+y)^{|\sigma \cap V(\text{Int}(L))|} \cdot f_{n-k-1+|\sigma \cap V(\partial(L))|}(x, y),$$

since $|\mathcal{I}| - 1 = k + 1 - |\sigma| = 0$. □

4. A GENERATING FUNCTION FOR THE POLYNOMIALS $f_n(x, y)$

The recurrence given in Proposition 3.6 allows to write a generating function formula for the polynomials $f_n(x, y)$. To state it in a more concise fashion we introduce the *magic polynomial* $r_L(u, v)$ of the simplicial complex L given by

$$(4.1) \quad r_L(u, v) = \sum_{\sigma \in L \setminus \partial L} u^{|\sigma \cap V(\partial L)|} v^{|\sigma \cap V(\text{Int}(L))|} - u^{k+1}.$$

Proposition 4.1. *The generating function $f(x, y, t) := \sum_{n=0}^{\infty} f_n(x, y) t^n$ is given by*

$$f(x, y, t) = \frac{1}{1-t(x+1)} \left(1 - \frac{r_L(-1-x, -1-y)}{r_L(-1/t, -1-y)} \right).$$

Proof. Proposition 3.6 may be rewritten as

$$\begin{aligned} f(x, y, t) &= \sum_{n=0}^k (1+x)^n \cdot t^n \\ &+ \sum_{\sigma \in L \setminus \partial L} (-1)^{k+1-|\sigma|} (1+y)^{|\sigma \cap V(\text{Int}(L))|} t^{k+1-|\sigma \cap V(\partial(L))|} \sum_{n=k+1}^{\infty} f_{n-k-1+|\sigma \cap V(\partial(L))|}(x, y) t^{n-k-1+|\sigma \cap V(\partial(L))|} \\ &= \frac{1 - (1+x)^{k+1} t^{k+1}}{1 - (1+x)t} \\ &+ \sum_{\sigma \in L \setminus \partial L} (-1)^{k+1-|\sigma|} (1+y)^{|\sigma \cap V(\text{Int}(L))|} t^{k+1-|\sigma \cap V(\partial(L))|} \left(f(x, y, t) - \sum_{n=0}^{|\sigma \cap V(\partial(L))-1} (1+x)^n t^n \right). \end{aligned}$$

After subtracting $\sum_{\sigma \in L \setminus \partial L} (-1)^{k+1-|\sigma|} (1+y)^{|\sigma \cap V(\text{Int}(L))|} t^{k+1-|\sigma \cap V(\partial L)|} f(x, y, t)$ on both sides, the left hand side becomes

$$\left(1 - \sum_{\sigma \in L \setminus \partial L} (-1)^{k+1-|\sigma|} (1+y)^{|\sigma \cap V(\text{Int}(L))|} t^{k+1-|\sigma \cap V(\partial L)|} \right) f(x, y, t) = -(-t)^{k+1} r_L \left(-\frac{1}{t}, -1-y \right) f(x, y, t),$$

and the right hand side becomes

$$\frac{1 - (1+x)^{k+1} t^{k+1}}{1 - (1+x)t} - t^{k+1} \sum_{\sigma \in L \setminus \partial L} (-1)^{k+1-|\sigma|} (1+y)^{|\sigma \cap V(\text{Int}(L))|} \left(\frac{1}{t} \right)^{|\sigma \cap V(\partial L)|} \frac{1 - ((1+x)t)^{|\sigma \cap V(\partial L)|}}{1 - (1+x)t}$$

which is easily seen to be equal to

$$\frac{-(-t)^{k+1} r_L(-1/t, -1-y) + (-t)^{k+1} r_L(-1-x, -1-y)}{1 - (1+x)t}$$

Dividing both sides by $-(-t)^{k+1} r_L(-1/t, -1-y)$ yields the stated equality. \square

Let $f_n^o(x, y)$ denote the contribution to $f(K; x, y)$ of adding a single facet of dimension $(n-1)$. Knowing $f_n^o(x, y)$ allows to express $f(K; x, y)$ directly since we have

$$(4.2) \quad f(K; x, y) = \sum_{j=0}^{\dim(K)+1} f_{j-1}(K) f_j^o(x, y).$$

Applying (4.2) to the case when K is the $(n-1)$ -dimensional simplex yields

$$(4.3) \quad f_n(x, y) = \sum_{j=0}^n \binom{n}{j} f_j^o(x, y).$$

As an immediate consequence we obtain the generating function identity

$$\sum_{n=0}^{\infty} f_n(x, y) t^n = \sum_{j=0}^{\infty} f_j^o(x, y) t^j \sum_{k=0}^{\infty} \binom{k+j}{j} t^k = \sum_{j=0}^{\infty} f_j^o(x, y) \frac{t^j}{(1-t)^{j+1}}.$$

Substituting $t := u/(1+u)$ in the previous equation and rearranging yields

$$\sum_{j=0}^{\infty} f_j^o(x, y) u^j = \frac{1}{1+u} \sum_{n=0}^{\infty} f_n(x, y) \left(\frac{u}{1+u} \right)^n = \frac{1}{1+u} f \left(x, y, \frac{u}{1+u} \right).$$

This equation and Proposition 4.1 have the following consequence.

Corollary 4.2. *The generating function $f^o(x, y, t) := \sum_{n=0}^{\infty} f_n^o(x, y) t^n$ is given by*

$$f^o(x, y, t) = \frac{1}{1-tx} \left(1 - \frac{r_L(-1-x, -1-y)}{r_L\left(-\frac{1-t}{t}, -1-y\right)} \right).$$

5. GENERALIZED TCHEBYSHEV POLYNOMIALS OF THE FIRST KIND

Following [7] we define the F -polynomial of a simplicial complex K as the polynomial

$$F(K, x) := \sum_{j=0}^{\dim K+1} f_{j-1}(K) \left(\frac{x-1}{2} \right)^j.$$

Let L be the simplicial complex considered in Example 3.5. As an immediate generalization of [7, Proposition 3.3], Theorem 3.4 gives the following.

Proposition 5.1. *Let L be the path with two edges. Let K be any simplicial complex and K' be any generalized Tchebyshev triangulation of K induced by L . Let $T_n(x)$ be the n -th Tchebyshev polynomial of the first kind. Then the linear map $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ given by $T(x^n) := T_n(x)$ satisfies*

$$T(F(K, x)) = F(K', x).$$

Inspired by Proposition 5.1 we make the following definition.

Definition 5.2. *Let L be a triangulation of the k -dimensional simplex such that the only vertices of L on the boundary $\partial(L)$ are the original vertices of the simplex. We define the generalized Tchebyshev polynomial $T_n^L(x)$ of the first kind as the image of x^n under the unique linear map $T^L : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ that has the following property: given any simplicial complex K and any generalized Tchebyshev triangulation K' of K , induced by L , we have*

$$(5.1) \quad T^L(F(K, x)) = F(K', x).$$

The linear map T^L in Definition 5.2 above is well-defined: let $T_1 : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ be the invertible linear map satisfying $T_1(f(K, x)) = F(K, x)$ for all simplicial complexes K , and $T_2 : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ be the linear map from Theorem 3.4 satisfying $T_2(f(K, x)) = f(K', x)$ (plugging $y = x$). Then $T^L = T_1 T_2 T_1^{-1}$. We now compute $T_n^L(x)$ explicitly. When K is an $(n-1)$ -dimensional simplex, we have

$$F(K, x) = \sum_{j=0}^n \binom{n}{j} \left(\frac{x-1}{2} \right)^j = \left(\frac{x+1}{2} \right)^n \quad \text{and} \quad F(K', x) = f_n \left(\frac{x-1}{2}, \frac{x-1}{2} \right).$$

As a consequence T^L is given by

$$(5.2) \quad T^L \left(\left(\frac{x+1}{2} \right)^n \right) = f_n \left(\frac{x-1}{2}, \frac{x-1}{2} \right).$$

Since

$$x^n = \left(2 \cdot \frac{x+1}{2} - 1 \right)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} 2^k \left(\frac{x+1}{2} \right)^k,$$

equation (5.2) is equivalent to

$$(5.3) \quad T_n^L(x) = T^L(x^n) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} 2^k f_k \left(\frac{x-1}{2}, \frac{x-1}{2} \right).$$

Using (5.3) we obtain the following generating function formula for the polynomials $T_n^L(x)$.

$$\begin{aligned} \sum_{n=0}^{\infty} T_n^L(x)t^n &= \sum_{k=0}^{\infty} 2^k f_k \left(\frac{x-1}{2}, \frac{x-1}{2} \right) t^k \sum_{m=0}^{\infty} \binom{m+k}{k} (-t)^m \\ &= \sum_{k=0}^{\infty} f_k \left(\frac{x-1}{2}, \frac{x-1}{2} \right) \cdot \frac{(2t)^k}{(1+t)^{k+1}}, \quad \text{i.e.,} \end{aligned}$$

$$(5.4) \quad \sum_{n=0}^{\infty} T_n^L(x)t^n = \frac{1}{1+t} f \left(\frac{x-1}{2}, \frac{x-1}{2}, \frac{2t}{1+t} \right).$$

This equation and Proposition 4.1 yield

$$(5.5) \quad \sum_{n=0}^{\infty} T_n^L(x)t^n = \frac{1}{1-xt} \cdot \frac{r_L \left(-\frac{1+t}{2t}, -\frac{1+x}{2} \right) - r_L \left(-\frac{1+x}{2}, -\frac{1+x}{2} \right)}{r_L \left(-\frac{1+t}{2t}, -\frac{1+x}{2} \right)}.$$

Combining Equation (5.3) with (4.3) yields

$$\begin{aligned} T_n^L(x) &= \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} 2^k \sum_{j=0}^k \binom{k}{j} f_j^o \left(\frac{x-1}{2}, \frac{x-1}{2} \right) \\ &= \sum_{j=0}^n f_j^o \left(\frac{x-1}{2}, \frac{x-1}{2} \right) 2^j \binom{n}{j} \sum_{k=j}^n \binom{n-j}{k-j} (-1)^{n-k} 2^{k-j}. \end{aligned}$$

The inside sum is $(2-1)^{n-j} = 1$ and we obtain

$$(5.6) \quad T_n^L(x) = \sum_{j=0}^n f_j^o \left(\frac{x-1}{2}, \frac{x-1}{2} \right) 2^j \binom{n}{j},$$

thus by (4.2) we observe that:

Corollary 5.3. $T_n^L(x)$ is the F -polynomial of the generalized Tchebyshev triangulation of the boundary complex of an n -dimensional cross-polytope, induced by L .

Corollary 5.3 allows us to prove several properties of the generalized Tchebyshev polynomials of the first kind.

Theorem 5.4. For all $n \geq 0$, the polynomials $T_n^L(x)$ have the following properties:

- (1) $T_n^L(x)$ is a polynomial of degree n ;
- (2) $T_n^L(1) = 1$;
- (3) $(-1)^n T_n^L(-x) = T_n^L(x)$;
- (4) all real roots of $T_n^L(x)$ belong to the interval $(-1, 1)$.

Proof. Let (f_{-1}, \dots, f_{n-1}) , respectively (h_0, \dots, h_n) be the f -vector and h -vector, respectively, of the generalized Tchebyshev triangulation of the boundary complex of an n -dimensional cross-polytope,

induced by L . By Corollary 5.3 we have

$$T_n^L(x) = \sum_{j=0}^n f_{j-1} \left(\frac{x-1}{2} \right)^j.$$

Clearly $T_n^L(x)$ has degree n . Substituting $f_{j-1} = \sum_{i=0}^j \binom{n-i}{n-j} h_i$ for each j , the previous equation may be rewritten as

$$T_n^L(x) = \sum_{j=0}^n \left(\frac{x-1}{2} \right)^j \sum_{i=0}^j \binom{n-i}{n-j} h_i = \sum_{i=0}^n h_i \sum_{j=i}^n \binom{n-i}{n-j} \left(\frac{x-1}{2} \right)^j.$$

By the binomial theorem we obtain

$$(5.7) \quad T_n^L(x) = \frac{1}{2^n} \sum_{i=0}^n h_i (x-1)^i (x+1)^{n-i}.$$

Substituting $x = 1$ into (5.7) yields $T_n^L(1) = h_0 = 1$. The third statement follows from the Dehn-Sommerville equations $h_i = h_{n-i}$.

As a consequence, the set of real zeros of $T_n^L(x)$ is symmetric to the origin. Thus, to prove the last statement, we only need to show that $T_n^L(x)$ has no real zero that is larger than 1. This is an immediate consequence of (5.7) and the fact that the h -vector of a simplicial sphere has only nonnegative entries, with $h_0 = h_n = 1$ being strictly positive. \square

We remark that the above proof shows that the statements in Theorem 5.4 are valid for the F -polynomial of any homology sphere.

We conclude this section with a recursive description of the polynomials $T_n^L(x)$.

Theorem 5.5. *The polynomials $T_n^L(x)$ satisfy $T_n^L(x) = x^n$ for $n \leq k$. For all $n \geq k+1$, the polynomial $T_n^L(x)$ satisfies a recurrence of the form*

$$T_n^L(x) = \sum_{j=1}^{k+1} p_j^L(x) T_{n-j}^L(x).$$

Here each $p_j^L(x)$ is a polynomial of x and it equals to the coefficient of t^j in $(-2t)^{k+1} r_L \left(-\frac{1+t}{2t}, -\frac{1+x}{2} \right)$.

Proof. For $n \leq k$, the generalized Tchebyshev triangulation of the boundary complex of an n -dimensional cross-polytope is the boundary complex itself whose F -polynomial is x^n .

To prove the second part of the statement, let us rewrite (5.5) as

$$\sum_{n=0}^{\infty} T_n^L(x) t^n = \frac{1}{1-xt} \cdot \frac{(-2t)^{k+1} r_L \left(-\frac{1+t}{2t}, -\frac{1+x}{2} \right) - (-2t)^{k+1} r_L \left(-\frac{1+x}{2}, -\frac{1+x}{2} \right)}{(-2t)^{k+1} r_L \left(-\frac{1+t}{2t}, -\frac{1+x}{2} \right)}.$$

Since the total degree in u and v of each term of $r_L(u, v)$ is at most $k+1$, the denominator and the numerator of the second factor on the right hand side are polynomials of x and t . Substituting $x = 1/t$ into the numerator on the right hand side makes it vanish. As a consequence, we may always simplify

by $(1 - tx)$ on the right hand side, yielding a numerator of degree at most k in t . The degree of the denominator $(-2t)^{k+1} r_L\left(-\frac{1+t}{2t}, -\frac{1+x}{2}\right)$, as a polynomial of t is exactly $k + 1$, and the coefficient of t^0 is (-1) since t^{k+1} comes only from the term $-u^{k+1}$ of $r_L(u, v)$. Multiplying both sides with the denominator on the right hand side and comparing coefficients of t^n on both sides yields a recurrence of the stated form. \square

Remark 5.6. Theorem 5.5 implies that $\{T_n^L(x)\}_{n \geq 0}$ is not a sequence of orthogonal polynomials if the dimension of L is more than one. Indeed, every sequence $\{P_n(x)\}_{n \geq 0}$ of monic orthogonal polynomials satisfies a recurrence of the form

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x) \quad \text{for } n \geq 1,$$

where $P_{-1}(x) = 0$, $P_0(x) = 1$, the numbers c_n and λ_n are constants, $\lambda_n \neq 0$ for $n \geq 2$, and λ_1 is arbitrary (see [5, Ch. I, Theorem 4.1]). If the dimension of L is greater than one, we have $T_n^L(x) = x^n$ for $n \leq 2$, forcing $c_1 = 0$, $c_2 = 0$ and $\lambda_2 = 0$; in contradiction with the requirement of $\lambda_n \neq 0$ for $n \geq 2$.

Theorem 5.5 may be used to find an explicit formula for $T_n^L(x)$, whenever the characteristic equation associated to the linear recurrence can be solved. Note that, by Theorem 5.5, this characteristic equation is obtained by replacing each t^j by q^{k+1-j} in $(-2t)^{k+1} r_L\left(-\frac{1+t}{2t}, -\frac{1+x}{2}\right)$ and finding the zeros of the resulting polynomial of q . This transformation is the same as substituting $t = 1/q$ and multiplying by q^{k+1} , thus the characteristic equation of the linear recurrence is

$$(5.8) \quad (-2)^{k+1} r_L\left(-\frac{1+q}{2}, -\frac{1+x}{2}\right) = 0.$$

If we find $k + 1$ linearly independent solutions $q_0(x), q_1(x), \dots, q_k(x)$ of Equation (5.8) above then we may look for a general formula of the form

$$T_n^L(x) = \alpha_0(x)q_0(x)^n + \dots + \alpha_k(x)q_k(x)^n.$$

Since $T_n^L(x) = x^n$ holds for $n \leq k$, the array of functions $(\alpha_0(x), \dots, \alpha_k(x))$ may be found as the solution of the system of equations

$$(5.9) \quad \begin{pmatrix} 1 & 1 & \dots & 1 \\ q_0(x) & q_1(x) & \dots & q_k(x) \\ \vdots & \vdots & \ddots & \vdots \\ q_0(x)^k & q_1(x)^k & \dots & q_k(x)^k \end{pmatrix} \begin{pmatrix} \alpha_0(x) \\ \alpha_1(x) \\ \vdots \\ \alpha_k(x) \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^k \end{pmatrix}.$$

Such a system of equations may be solved using Cramer's rule and the formula for the Vandermonde determinant. Explicit examples will be worked out in Section 6.

6. GENERALIZED TCHEBYSHEV POLYNOMIALS OF THE FIRST KIND AND REAL ROOTEDNESS

By Theorem 5.4 the generalized Tchebyshev polynomials of the first kind $T_n^L(x)$ possess many important properties of the ordinary Tchebyshev polynomials of the first kind $T_n(x)$. An important property of the polynomials $T_n(x)$ is that all their roots are distinct and real. Since $T_n^L(x) = x^n$ holds

for $n \leq k$, for $k \geq 2$ the roots of $T_n^L(x)$ are not distinct for all n any more. The question remains whether all roots of all polynomials $T_n^L(x)$ could still be real. In this section we explore this question.

We begin with a complete description of the case $k = 1$. The only way to subdivide a 1-dimensional simplex is to select $s \geq 1$ distinct vertices in its interior, thus creating a path of length $s + 1$. The magic polynomial $r_L(u, v)$ is given by

$$r_L(u, v) = sv + 2uv + (s - 1)v^2 - u^2.$$

To use Theorem 5.5, we observe that

$$(-2t)^2 r_L\left(-\frac{1+t}{2t}, -\frac{1+x}{2}\right) = t^2((x^2 - 1)(s - 1) - 1) + 2xt - 1,$$

yielding the recurrence

$$T_n^L(x) = 2x \cdot T_{n-1}^L(x) + ((x^2 - 1)(s - 1) - 1)T_{n-2}^L(x) \quad \text{for } n \geq 2.$$

Note that for $s = 1$ the above recurrence degenerates into the well-known recurrence of the Tchebyshev polynomials $T_n(x)$. Taking into account the initial conditions $T_0^L(x) = 1$ and $T_1^L(x) = x$ it is not hard to derive (after solving a quadratic characteristic equation) the following explicit formula:

$$(6.1) \quad T_n^L(x) = \frac{(x - \sqrt{s(x^2 - 1)})^n + (x + \sqrt{s(x^2 - 1)})^n}{2} \quad \text{for } n \geq 0.$$

Proposition 6.1. *Let $s \geq 1$ be an integer and L be the subdivision of the 1-simplex by s interior vertices. Then the polynomial $T_n^L(x)$ has n distinct real roots in the open interval $(-1, 1)$.*

Proof. Consider the function

$$\phi(x) = \frac{x}{\sqrt{x^2 + s(1 - x^2)}}$$

on the interval $[-1, 1]$. Its derivative, $\phi'(x) = s/(s(1 - x^2) + x^2)^{3/2}$, is positive on $(-1, 1)$, thus $\phi(x)$ is strictly increasing on $(-1, 1)$. Obviously we also have $\lim_{x \rightarrow -1} \phi(x) = -1$ and $\lim_{x \rightarrow 1} \phi(x) = 1$. Therefore $\phi(x)$ is a bijection from $[-1, 1]$ to itself. The function $\alpha(x) := \arccos(\phi(x))$ is well-defined and maps the interval $[-1, 1]$ bijectively onto the interval $[0, \pi]$. Using (6.1), it is not difficult to show that we have

$$(6.2) \quad T_n^L(x) = \left(\sqrt{x^2 + s(1 - x^2)}\right)^n \cos(n\alpha(x)),$$

which, for $s = 1$, is equivalent to the first half of (2.1). Now the statement follows from the fact that there are n different values of α in $(0, \pi)$ for which $\cos(n\alpha) = 0$. \square

Another interesting special case is when L is obtained by adding just one vertex to the interior of a k -dimensional simplex and we subdivide the simplex into $k + 1$ facets by connecting this new vertex to all other vertices of the simplex. The resulting magic polynomial is

$$r_L(u, v) = (1 + u)^{k+1}v - (1 + v)u^{k+1}.$$

Example 6.2. For $k = 3$, direct calculation shows that

$$T_6^L(x) = 6 - 9x^2 - 60x^4 + 64x^6.$$

By Descartes' rule of signs, the polynomial $6 - 9x - 60x^2 + 64x^3$ has at most two positive roots. As a consequence $T_6^L(x)$ has at most 4 real roots (Maple finds 4 real roots indeed, but this is unimportant). None of these roots can be a double root, because the derivative of $T_6^L(x)$ is relative prime to $T_6^L(x)$. Therefore not all roots of $T_6^L(x)$ are real.

As a special case of Theorem 7.14, in the next section we will see that, for $k = 2$, all roots of $T_n^L(x)$ are real.

7. TWO-DIMENSIONAL GENERALIZED TCHEBYSHEV TRIANGULATIONS

Let L be an arbitrary triangulation of the two-dimensional simplex that has new vertices only in its interior. In this section we will show that the generalized Tchebyshev polynomials $T_n^L(x)$ have only real roots.

Let m be the number of interior vertices in L and let e be the number of edges in L with one end on the boundary and the other end in the interior of L . Thus the total number of vertices in L is

$$f_0(L) = m + 3.$$

Each edge, except for the three edges on the boundary, is included in exactly two faces, yielding $2f_1(L) = 3(f_2(L) + 1)$, whereas Euler's formula gives $f_0(L) + f_2(L) = f_1(L) + 1$. Solving these equations for $f_1(L)$ and $f_2(L)$ yields

$$f_1(L) = 3(m + 1) \quad \text{and} \quad f_2(L) = 2m + 1.$$

In order to compute the magic polynomial, we need to refine the above face count. Let us say that a face has type (i, j) if it has i vertices on the boundary and j vertices in the interior. Of the $3m + 3$ edges, 3 edges have type $(2, 0)$, e edges have type $(1, 1)$, and the remaining $3m - e$ edges have type $(0, 2)$. Of the $2m + 1$ 2-faces, three have type $(2, 1)$. To count the number of faces of type $(1, 2)$, observe that each face of type $(1, 2)$ or $(2, 1)$ contains exactly two edges of type $(1, 1)$ and, conversely, each edge of type $(1, 1)$ belongs to exactly two faces of type $(1, 2)$ or $(2, 1)$. Thus the total number of faces of types $(1, 2)$ or $(2, 1)$ is the same as the number of type $(1, 1)$ edges, that is, e . Since the number of type $(2, 1)$ faces is 3, there are $e - 3$ faces of type $(1, 2)$. Finally, the remaining $2m + 1 - e$ faces must have type $(0, 3)$. Therefore the magic polynomial associated to L is

$$(7.1) \quad r_L(u, v) = mv + euv + (3m - e)v^2 + 3u^2v + (e - 3)uv^2 + (2m + 1 - e)v^3 - u^3.$$

A closer look at the face-counting argument above also implies the following statement.

Lemma 7.1. *The parameters e and m above satisfy $e \leq \min(2m + 1, 3m)$. Furthermore, $e = 3m$ is only possible when $e = 3$ and $m = 1$.*

Indeed, as observed above, $3m - e$ is the number of edges of type $(0, 2)$, and $2m + 1 - e$ is the number of faces of type $(0, 3)$. The number of edges of type $(0, 2)$ can be zero only if there is one vertex in the interior of L , in which case we have $e = 3$.

By (7.1) we have

$$(-2t)^3 r_L \left(-\frac{1+t}{2t}, -\frac{1+x}{2} \right) = ((2m+1-e) \cdot x^3 + (e-2m)x) \cdot t^3 + ((e-3)x^2 - e) \cdot t^2 + 3x \cdot t - 1.$$

By Theorem 5.5, the polynomials $T_n^L(x)$ satisfy the initial conditions $T_0^L(x) = 1$, $T_1^L(x) = x$, $T_2^L(x) = x^2$ and the recurrence

$$T_n^L(x) = 3xT_{n-1}^L(x) + ((e-3)x^2 - e)T_{n-2}^L(x) + ((2m+1-e) \cdot x^3 + (e-2m)x) \cdot T_{n-3}^L(x) \quad \text{for } n \geq 3.$$

The characteristic equation associated to the above recurrence is

$$(7.2) \quad (q-x)^3 + e(1-x^2)(q-x) + 2mx(1-x^2) = 0.$$

According to Cardano's formula, this characteristic equation has the following three solutions:

$$(7.3) \quad q_j(x) = x + \omega^j u(x) + \omega^{2j} v(x)$$

where $j \in \{0, 1, 2\}$, $\omega = e^{i2\pi/3}$,

$$(7.4) \quad u(x) = \sqrt[3]{1-x^2} \cdot \sqrt[3]{-mx + \sqrt{m^2x^2 + \frac{e^3(1-x^2)}{27}}} \quad \text{and}$$

$$(7.5) \quad v(x) = \sqrt[3]{1-x^2} \cdot \sqrt[3]{-mx - \sqrt{m^2x^2 + \frac{e^3(1-x^2)}{27}}}.$$

We restrict the domain of the functions $q_j(x)$ to real values of x in the interval $[-1, 1]$. Note that $q_0(x)$ is a real-valued function, whereas $q_1(x)$ and $q_2(x)$ are complex valued functions such that $q_2(x)$ is the complex conjugate of $q_1(x)$. The common length of $q_1(x)$ and $q_2(x)$ is given by

$$\|q_1(x)\|^2 = \|q_2(x)\|^2 = q_1(x) \cdot q_2(x) = (x + \omega u(x) + \omega^2 v(x))(x + \omega^2 u(x) + \omega v(x)), \quad \text{that is,}$$

$$(7.6) \quad \|q_1(x)\|^2 = \|q_2(x)\|^2 = x^2 - (u(x) + v(x)) \cdot x + u(x)^2 + v(x)^2 - u(x)v(x).$$

Similarly, for $j = 0$, (7.3) yields $|q_0(x)|^2 = (x + u(x) + v(x))^2$, that is,

$$(7.7) \quad |q_0(x)|^2 = x^2 + u(x)^2 + v(x)^2 + 2x(u(x) + v(x)) + 2u(x)v(x).$$

For future reference we note that

$$(7.8) \quad u(0) = \sqrt{e/3}, \quad v(0) = -\sqrt{e/3}, \quad \text{implying}$$

$$(7.9) \quad q_0(0) = 0, \quad q_1(0) = \sqrt{e}i \quad \text{and} \quad q_2(0) = -\sqrt{e}i.$$

Similarly

$$(7.10) \quad u(1) = u(-1) = v(1) = v(-1) = 0 \quad \text{implies}$$

$$(7.11) \quad q_j(-1) = -1 \quad \text{and} \quad q_j(1) = 1 \quad \text{for } j = 0, 1, 2.$$

As a part of the derivation of Cardano's formula, $u(x)$ and $v(x)$ are known to satisfy the following equalities:

$$(7.12) \quad u(x) \cdot v(x) = -\frac{e}{3} \cdot (1-x^2), \quad \text{and}$$

$$(7.13) \quad u(x)^3 + v(x)^3 = 2mx(x^2 - 1).$$

Besides these classical identities, we will use the following two inequalities about $u(x)$ and $v(x)$.

Lemma 7.2. *The function $u(x)^3 - v(x)^3$ is nonnegative on the interval $[-1, 1]$. Equality to zero holds only when $x = \pm 1$.*

This lemma is a direct consequence of

$$u(x)^3 - v(x)^3 = 2(1 - x^2) \sqrt{m^2 x^2 + \frac{e^3(1 - x^2)}{27}}.$$

Lemma 7.3. *The functions $u(x)$ and $v(x)$ satisfy*

$$x(u(x) + v(x)) \leq 0$$

for all real $x \in [-1, 1]$. Equality holds exactly when $x = \pm 1$ or $x = 0$.

Proof. Consider the function $w : [-1, 1] \rightarrow \mathbb{R}$, given by

$$w(x) = \sqrt[3]{-mx + \sqrt{m^2 x^2 + \frac{e^3(1 - x^2)}{27}}} + \sqrt[3]{-mx - \sqrt{m^2 x^2 + \frac{e^3(1 - x^2)}{27}}}$$

This is a continuous function on $[-1, 1]$, satisfying $w(1) = \sqrt[3]{-2m} < 0$ and $w(-1) = \sqrt[3]{2m} > 0$. Furthermore, the only solution of $w(x) = 0$ on the interval $[-1, 1]$ is $x = 0$. We obtain that $w(x)$ is positive on $[-1, 0)$ and negative on $(0, 1]$. Since, by (7.4) and (7.5), $w(x)$ satisfies $u(x) + v(x) = \sqrt[3]{1 - x^2} \cdot w(x)$, the function $u(x) + v(x)$ has the same signature on the interval $(-1, 1)$, and the statement follows directly. \square

As indicated at the end of Section 5, we may look for $T_n^L(x)$ in the form

$$(7.14) \quad T_n^L(x) = \alpha_0(x)q_0(x)^n + \alpha_1(x)q_1(x)^n + \alpha_2(x)q_2(x)^n,$$

where the functions $\alpha_0(x)$, $\alpha_1(x)$ and $\alpha_2(x)$ may be found by solving (5.9).

Lemma 7.4. *On the interval $(-1, 1)$, the functions $\alpha_0(x)$, $\alpha_1(x)$ and $\alpha_2(x)$ are given by*

$$\alpha_0(x) = \frac{(u(x)^2 - u(x)v(x) + v(x)^2)(u(x) - v(x))}{3(u(x)^3 - v(x)^3)}, \quad \text{and}$$

$$\alpha_j(x) = \frac{(u(x)^2 + (-1)^j i \sqrt{3} u(x)v(x) - v(x)^2)(u(x) + v(x))}{3(u(x)^3 - v(x)^3)} \quad \text{for } j = 1, 2.$$

Proof. We use Cramer's formula to solve (5.9). For all $\alpha_j(x)$, the denominator in this formula is the Vandermonde determinant

$$\det \begin{pmatrix} 1 & 1 & 1 \\ q_0(x) & q_1(x) & q_2(x) \\ q_0(x)^2 & q_1(x)^2 & q_2(x)^2 \end{pmatrix} = (q_1(x) - q_0(x))(q_2(x) - q_0(x))(q_2(x) - q_1(x)),$$

which, by (7.3), equals

$$((\omega - 1)u(x) + (\omega^2 - 1)v(x))((\omega^2 - 1)u(x) + (\omega - 1)v(x))((\omega^2 - \omega)u(x) + (\omega - \omega^2)v(x)).$$

After taking out a $(\omega - 1)$ from the first factor, $(\omega^2 - 1)$ from the second factor and $(\omega^2 - \omega)$ from the third factor, and after noting that

$$(\omega - 1)(\omega^2 - 1)(\omega^2 - \omega) = -3\sqrt{3}i,$$

we obtain that the common denominator in Cramer's formula is

$$-3\sqrt{3}i(u(x) - \omega^2 v(x))(u(x) - \omega v(x))(u(x) - v(x)) = -3\sqrt{3}i(u(x)^3 - v(x)^3).$$

The numerators in Cramer's formula are also Vandermonde determinants and may be computed in a completely analogous way. The stated equalities follow after simplifying by $-\sqrt{3}i$. \square

By Lemma 7.2, $u(x)^3 - v(x)^3$ is real and strictly positive on the interval $(-1, 1)$, hence the formulas stated in Lemma 7.4 above are well-defined. In order to extend the definition of $\alpha_j(x)$ to $x = \pm 1$ in a continuous fashion, we state the following, equivalent formulas for $\alpha_j(x)$.

Lemma 7.5. *On the set $(-1, 1) \setminus \{0\}$, the functions $\alpha_j(x)$ are equivalently given by*

$$(7.15) \quad \alpha_j(x) = \frac{mx}{e(q_j(x) - x) + 3mx} \quad \text{for } j = 0, 1, 2.$$

These formulas may be continuously extended to $[-1, 1]$ by setting $\alpha_j(1) = 1/3$, $\alpha_j(-1) = 1/3$ for $j = 0, 1, 2$, $\alpha_j(0) = 0$ for $j = 1, 2$ and $\alpha_0(0) = 1$.

Proof. Observe first that, by (7.13), the sum $u(x)^3 + v(x)^3$ is nonzero on the set $(-1, 1) \setminus \{0\}$ thus the same holds for $u(x) + v(x)$ by $u(x)^3 + v(x)^3 = (u(x) + v(x))(u(x)^2 - u(x)v(x) + v(x)^2)$. Using these observations, we may rewrite $\alpha_0(x)$ as

$$\alpha_0(x) = \frac{\frac{u(x)^3 + v(x)^3}{u(x) + v(x)}(u(x) - v(x))}{3(u(x)^3 - v(x)^3)} = \frac{2mx(x^2 - 1)}{3(u(x)^2 + u(x)v(x) + v(x)^2)(u(x) + v(x))}.$$

Here $u(x) + v(x)$ may be replaced by $q_0(x) - x$. Furthermore, by (7.12), the factor $u(x)^2 + u(x)v(x) + v(x)^2$ in the denominator above may be rewritten as

$$u(x)^2 + u(x)v(x) + v(x)^2 = (u(x) + v(x))^2 - u(x)v(x) = (q_0(x) - x)^2 - \frac{e(x^2 - 1)}{3}.$$

Thus we obtain

$$\alpha_0(x) = \frac{2mx(x^2 - 1)}{3 \left((q_0(x) - x)^2 - \frac{e(x^2 - 1)}{3} \right) (q_0(x) - x)} = \frac{2mx(x^2 - 1)}{3((q_0(x) - x)^3 - e(x^2 - 1)(q_0(x) - x))}.$$

After expanding $(q_0(x) - x)^3$ and using (7.2) to replace $q_0(x)^3$ with a linear expression of $q_0(x)$, we obtain

$$\alpha_0(x) = \frac{2mx(x^2 - 1)}{6mx(x^2 - 1) + 2e(x^2 - 1)(q_0(x) - x)}.$$

Simplifying by $2(x^2 - 1)$ yields the stated equation for $\alpha_0(x)$. The calculations for $\alpha_1(x)$ and $\alpha_2(x)$ are completely analogous, therefore omitted.

Substituting $x = 1$, respectively $x = -1$, in the stated formulas for $\alpha_j(x)$ yields $\alpha_j(1) = 1/3$ and $\alpha_j(-1) = 1/3$, as we have $q_j(1) = 1$ and $q_j(-1) = -1$ for $j = 0, 1, 2$. These are obviously continuous

extensions of the functions $\alpha_j(x)$. By (7.9), for $j \in \{1, 2\}$ the denominator $e(q_j(x) - x) + 3mx$ is nonzero at $x = 0$ and $\alpha_j(0) = 0$ is a continuous extension of the given formula. Finally, to find the limit of $\alpha_0(x)$ at $x = 0$, observe that using (7.13) we may rewrite

$$q_0(x) = x + u(x) + v(x) = x + \frac{u(x)^3 + v(x)^3}{u(x)^2 - u(x)v(x) + v(x)^2}$$

as

$$q_0(x) = x \left(1 + \frac{2m(x^2 - 1)}{u(x)^2 - u(x)v(x) + v(x)^2} \right).$$

Using (7.8), the last equation yields

$$(7.16) \quad \lim_{x \rightarrow 0} \frac{q_0(x)}{x} = \frac{e - 2m}{e}.$$

Equation (7.16) implies

$$\lim_{x \rightarrow 0} \alpha_0(x) = \lim_{x \rightarrow 0} \frac{m}{e(q_0(x)/x - 1) + 3m} = 1.$$

□

Definition 7.6. For $j = 0, 1, 2$, we define the functions $\alpha_j(x)$ on the interval $[-1, 1]$ by the formulas stated in Lemma 7.5.

Note that the functions $\alpha_j(x)$ are also given by the equation (5.9) on the interval $(-1, 1)$, and for such values of x our definition is equivalent to the solution given in Lemma 7.4. Our definition extends these functions to $x = \pm 1$ in a continuous way, such that they are still solutions of the system (5.9) which is degenerate for these values of x .

Corollary 7.7. The function $\frac{\alpha_1(x)}{x}$ is well-defined and nowhere zero on $[-1, 1]$.

Indeed, by Lemma 7.5 we may write

$$(7.17) \quad \frac{\alpha_1(x)}{x} = \frac{m}{e(q_1(x) - x) + 3mx}.$$

For a real x , the denominator can only be zero when $q_1(x) - x = \omega u(x) + \omega^2 v(x)$ is a real number, i.e., when $u(x) = v(x)$. The only solutions of $u(x) = v(x)$ are $x = \pm 1$, however, by (7.11), the denominator is nonzero at $x = \pm 1$.

Next we make an analogous observation for $q_1(x)$.

Proposition 7.8. The function $q_1(x)$ is nowhere zero on the interval $[-1, 1]$.

Proof. If $q_1(x) = 0$, then (7.6) gives

$$x^2 - (u(x) + v(x)) \cdot x + u(x)^2 + v(x)^2 - u(x)v(x) = 0.$$

Consider this as a quadratic equation for x , with real coefficients. It can only have a real solution when its discriminant

$$D = (u(x) + v(x))^2 - 4(u(x)^2 + v(x)^2 - u(x)v(x))$$

is not negative. Using (7.12), the discriminant may be rewritten as

$$D = -3(u(x)^2 + v(x)^2) + 2e(x^2 - 1).$$

Here $-3(u(x)^2 + v(x)^2)$ is at most zero, and, for $x \in [-1, 1]$, we also have $2e(x^2 - 1) \leq 0$. Thus $D \geq 0$ is only possible when $x = \pm 1$. However, $q_1(x)$ is not zero at $x = \pm 1$, as we have $q_1(1) = 1$ and $q_1(-1) = -1$. \square

The proof of the main result of this section depends on two key inequalities, stated in the next two propositions.

Proposition 7.9. *We have $\|q_1(x)\| \geq |q_0(x)|$ for all $x \in [-1, 1]$. Equality holds exactly when $x = \pm 1$.*

Proof. The difference of (7.6) and (7.7) is

$$\|q_1(x)\|^2 - |q_0(x)|^2 = -3x(u(x) + v(x)) - 3u(x)v(x).$$

Here, for any $x \in [-1, 1]$, the summand $-3x(u(x) + v(x))$ is nonnegative by Lemma 7.3 and the summand $-3u(x)v(x)$ is nonnegative by Equation (7.12). The sum is zero only when both summands are zero, which is only possible when $x = \pm 1$. \square

Proposition 7.10. *The functions $\alpha_j(x)$ and $q_j(x)$ satisfy*

$$2\|\alpha_1(x)q_1(x)\| \geq |\alpha_0(x)q_0(x)|$$

on the interval $[-1, 1]$. Equality is only possible when $x = 0$.

Proof. Assume, by way of contradiction, that

$$|\alpha_0(x)q_0(x)| \geq 2\|\alpha_1(x)q_1(x)\| = \|\alpha_1(x)q_1(x)\| + \|\alpha_2(x)q_2(x)\|$$

holds for some $x \in [-1, 1] \setminus \{0\}$. Then, by the triangle inequality, we also have

$$|\alpha_0(x)q_0(x)| \geq \|\alpha_1(x)q_1(x) + \alpha_2(x)q_2(x)\|.$$

Using (7.14) with $n = 1$ yields

$$|\alpha_0(x)q_0(x)| \geq |x - \alpha_0(x)q_0(x)|.$$

Since we excluded the possibility of $x = 0$, we obtain that the sign of $\alpha_0(x)q_0(x)$ must equal the sign of x . Using (7.15) and the Viète formulas associated to the characteristic equation (7.2) it is easy to derive the following formula:

$$\alpha_0(x)\alpha_1(x)\alpha_2(x) = \frac{m^2x^2}{27m^2x^2 + e^3(1-x^2)}$$

On the left hand side, $\alpha_1(x)\alpha_2(x) = \|\alpha_1(x)\|^2$ is positive by Corollary 7.7. The right hand side is also positive. We obtain that $\alpha_0(x)$ must be positive and thus the sign of x must also equal the sign of $q_0(x)$. Since we also have $3m - e \geq 0$ (by Lemma 7.1), using (7.15) we may write

$$|\alpha_0(x)q_0(x)| = |mx| \left| \frac{q_0(x)}{eq_0(x) + (3m - e)x} \right| = |mx| \frac{|q_0(x)|}{e|q_0(x)| + (3m - e)|x|}.$$

The rightmost expression can only increase if we replace $|q_0(x)|$ with a larger number. Thus, Proposition 7.9 yields

$$|\alpha_0(x)q_0(x)| \leq |mx| \frac{\|q_1(x)\|}{e\|q_1(x)\| + (3m - e)|x|}.$$

Applying the triangle inequality to the denominator on the right hand side yields

$$|\alpha_0(x)q_0(x)| \leq \left\| \frac{mxq_1(x)}{eq_1(x) + (3m - e)x} \right\| = \|\alpha_1(x)q_1(x)\|,$$

which contradicts our assumptions unless $\alpha_1(x)q_1(x) = \alpha_0(x)q_0(x) = 0$, impossible for $x \neq 0$ by Corollary 7.7 and Proposition 7.8. \square

As a consequence of Corollary 7.7 and Proposition 7.8, for $n \geq 1$ we may rewrite (7.14) as
(7.18)

$$\frac{T_n^L(x)}{x} = \frac{\|\alpha_1(x)q_1(x)^n\|}{|x|} \left(\frac{\alpha_0(x) \frac{q_0(x)}{x}}{\left\| \frac{\alpha_1(x)}{x} q_1(x) \right\|} \left(\frac{q_0(x)}{\|q_1(x)\|} \right)^{n-1} + \sum_{j=1}^2 \frac{\frac{\alpha_j(x)}{x} q_j(x)}{\left\| \frac{\alpha_1(x)}{x} q_1(x) \right\|} \left(\frac{q_j(x)}{\|q_1(x)\|} \right)^{n-1} \right).$$

Introducing the functions

$$g_n(x) = \frac{\alpha_0(x) \frac{q_0(x)}{x}}{\left\| \frac{\alpha_1(x)}{x} q_1(x) \right\|} \left(\frac{q_0(x)}{\|q_1(x)\|} \right)^{n-1}, \quad \varepsilon(x) = \frac{\frac{\alpha_1(x)}{x} q_1(x)}{\left\| \frac{\alpha_1(x)}{x} q_1(x) \right\|} \quad \text{and} \quad \rho(x) = \frac{q_1(x)}{\|q_1(x)\|},$$

we may rewrite (7.18) as

$$(7.19) \quad \frac{T_n^L(x)}{x} = \frac{\|\alpha_1(x)q_1(x)^n\|}{|x|} \left(g_n(x) + \varepsilon(x) \cdot \rho(x)^{n-1} + \overline{\varepsilon(x)} \cdot \overline{\rho(x)}^{n-1} \right).$$

The next three lemmas gather properties of the functions $g_n(x), \varepsilon(x), \rho(x)$ that will be needed later for the proof of real-rootedness.

Lemma 7.11. *For $n > 1$, the function $g_n : [-1, 1] \rightarrow \mathbb{R}$ is a real-valued function satisfying $g_n(-1) = (-1)^{n-1}$, $g_n(0) = 0$ and $g_n(1) = 1$. Furthermore, there exists a positive constant $c < 2$ such that $|g_n(x)| \leq c$ holds for all $x \in [-1, 1]$.*

Proof. The function $g_n(x)$ is continuous and real-valued, because the same holds for the functions $\alpha_0(x)$ and $q_0(x)/x$; see (7.3) and Lemma 7.5. Direct substitution (in Equations (7.9) and (7.11), using Lemma 7.5 and Equations (7.16) and (7.17)) yields $g_n(0) = 0$, $g_n(1) = 1$ and $g_n(-1) = (-1)^{n-1}$. For $x \neq 0$, we have

$$|g_n(x)| = \frac{\alpha_0(x)q_0(x)}{\|\alpha_1(x)q_1(x)\|} \cdot \left(\frac{|q_0(x)|}{\|q_1(x)\|} \right)^{n-1}$$

and the inequality is a direct consequence of Propositions 7.9 and 7.10 as $|g_n(0)| < 2$ (using compactness of $[-1, 1]$). \square

Lemma 7.12. *The function $\rho : [-1, 1] \rightarrow \mathbb{C}$ is a continuous function whose range is the upper half of the unit circle, centered at the origin. $\rho(x)$ is real if and only if $x = \pm 1$, where we have $\rho(-1) = -1$ and $\rho(1) = 1$.*

Proof. Clearly ρ is continuous and we must have $\|\rho(x)\| = 1$ for all $x \in [-1, 1]$. The imaginary part of $q_1(x)$ is $\sqrt{3} \cdot (u(x) - v(x)) \cdot i$ and $u(x) - v(x)$ is strictly positive on $(-1, 1)$, see (7.4) and (7.5). \square

Lemma 7.13. *The function $\varepsilon : [-1, 1] \rightarrow \mathbb{C}$ is continuous and its range is a proper subset of the unit circle, centered at the origin. The real number -1 is not part of the range. If $e = 3m$ then $\varepsilon(x) = 1$ for all $x \in [-1, 1]$. If $e \neq 3m$ then $\varepsilon(x)$ is real only when $x \in \{-1, 0, 1\}$ and, for all other values of x , the sign of the imaginary part of $\varepsilon(x)$ is the same as the sign of x .*

Proof. Clearly ε is continuous and satisfies $\|\varepsilon(x)\| = 1$. Direct substitution (into (7.9), (7.11) and (7.17)) yields $\varepsilon(-1) = 1$, $\varepsilon(1) = 1$ and $\varepsilon(0) = 1$. In the case when $e = 3m$, we have

$$\frac{\alpha_1(x)}{x} \cdot q_1(x) = \frac{m}{eq_1(x)} \cdot q_1(x) = \frac{1}{3}$$

and ε is identically 1. Assume from now on that $e \neq 3m$. Assume also that $x \notin \{-1, 0, 1\}$ and $\varepsilon(x)$ is real. Substituting (7.15) into the definition of $\varepsilon(x)$ we obtain

$$\frac{mq_1(x)}{e(q_1(x) - x) + 3mx} = r$$

for some $r \in \mathbb{R}$, which may be rearranged as

$$(m - er)q_1(x) = r(3m - e)x.$$

On the right hand side we have a real number, whereas on the left hand side $m - er$ is real but $q_1(x)$ is not real for $x \in (-1, 1) \setminus \{0\}$. The two sides can only be equal, if $m - er = 0$ but then x must be zero, in contradiction with our assumptions.

Assume $x \in (0, 1)$. We have seen in the proof of Lemma 7.12 that the imaginary part of $q_1(x)$ is positive. Since $3m - e$ is positive, the argument of $e \cdot q_1(x) + (3m - e)x$ is smaller than the argument of $q_1(x)$, but the imaginary part of $e \cdot q_1(x) + (3m - e)x$ is also positive. We obtain that the argument of the quotient

$$\frac{\alpha_1(x)}{x} \cdot q_1(x) = \frac{mq_1(x)}{e \cdot q_1(x) + (3m - e)x}$$

belongs to the interval $(0, \pi)$ and the imaginary part of $\varepsilon(x)$ is positive. A completely analogous reasoning may be used to prove that the imaginary part $\varepsilon(x)$ is negative for negative x . \square

Theorem 7.14. *Let L be any subdivision of the triangle, with no new vertices added to the boundary. Then the polynomials $T_n^L(x)$ have only real roots.*

Proof. We only need to show the statement for $n \geq 3$. Since we have $T_n^L(0) = 0$, it suffices to show that the polynomial $T_n^L(x)/x$ has $n - 1$ distinct roots in the interval $[-1, 1]$. Consider the expression of $T_n^L(x)/x$ given in (7.19). It suffices to show that the function

$$g_n(x) + \varepsilon(x) \cdot \rho(x)^{n-1} + \overline{\varepsilon(x)} \cdot \overline{\rho(x)}^{n-1}$$

has at least $n - 1$ zeroes in the interval $[-1, 1]$. By Lemma 7.11, the graph of the continuous function $-g_n(x)$ is in between the horizontal lines $y = -c$ and $y = c$ for some $0 < c < 2$. As $\varepsilon(x) \cdot \rho(x)^{n-1}$ is a unit complex number,

$$f_n(x) := \varepsilon(x) \cdot \rho(x)^{n-1} + \overline{\varepsilon(x)} \cdot \overline{\rho(x)}^{n-1}$$

equals twice the cosine of the argument of $\varepsilon(x) \cdot \rho(x)^{n-1}$. As a consequence, the graph of the continuous real-valued function $f_n(x)$ is between the horizontal lines $y = -2$ and $y = 2$. At the endpoints of the interval $[-1, 1]$ we have $f(-1) = 2 \cdot (-1)^{n-1}$ and $f(1) = 2$. It suffices to prove that there are $n - 2$ real numbers x_1, x_2, \dots, x_{n-2} satisfying $-1 < x_1 < \dots < x_{n-2} < 1$ and $f(x_j) = 2 \cdot (-1)^{n-1-j}$ for $j = 1, \dots, n - 2$. Introducing $x_0 = -1$ and $x_{n-1} = 1$ we can then say that, for each $j \in \{1, \dots, n - 1\}$, in each interval (x_{j-1}, x_j) , the graph of $f_n(x)$ enters and leaves the region between $y = -c$ and $y = c$, and crosses the graph of $-g_n(x)$ at least once, where we have a root of $f_n(x) + g_n(x)$.

Consider first the special case when $e = 3m$. By Lemma 7.13 $\varepsilon(x)$ is identically 1 and $\varepsilon(x)\rho(x)^{n-1} = \rho(x)^{n-1}$. By Lemma 7.12, as x moves from -1 to 1 , the argument of ρ continuously changes from π to 0 . We may select x_j as the least real number for which the argument of $\rho(x_j)$ is $\frac{n-1-j}{n-1}\pi$. Then the argument of $\rho(x)^{n-1}$ is $(n - 1 - j)\pi$ and we have $f(x_j) = 2 \cdot (-1)^{n-1-j}$. Because of the continuity of ρ we must also have $-1 < x_1 < \dots < x_{n-2} < 1$.

Consider finally the case when $e \neq 3m$. For $j = 0, \dots, n - 1$, let z_j be the least real number such that the argument of $\rho(z_j)$ is $\frac{n-1-j}{n-1}\pi$. Clearly we have $-1 = z_0 < z_1 < \dots < z_{n-1} = 1$. Let us set $x_0 = -1$ and $x_{n-1} = 1$. Let us denote by k the index for which we have $z_k < 0 \leq z_{k+1}$. For $j = 1, \dots, k$ we will show that we may select x_j as an element of the interval (z_{j-1}, z_j) and for $j = k + 1, \dots, n - 2$ we will show that we may select x_j as an element of the interval (z_j, z_{j+1}) . This selection automatically guarantees $-1 = x_0 < x_1 < \dots < x_{n-2} < x_{n-1} = 1$, we only need to show that the argument of $\varepsilon(x_j)\rho(x_j)^{n-1}$ is $(n - 1 - j)\pi$ for the x_j we selected.

Case 1: $1 \leq j \leq k$, implying $z_j < 0$. By Lemma 7.13, the imaginary part of $\varepsilon(x)$ is negative for all $x \in (z_{j-1}, z_j)$, in other words, the argument of $\varepsilon(x)$ belongs to the interval $(-\pi, 0)$ and the argument of $\varepsilon(x)^{-1}$ belongs to the interval $(0, \pi)$. The graph of the function $(n - 1 - j)\pi + \arg(\varepsilon(x)^{-1})$ stays strictly between the horizontal lines $y = (n - 1 - j)\pi$ and $y = (n - j)\pi$. As x moves from z_{j-1} to z_j , the argument of $\rho(x)^{n-1}$ moves from $(n - 1 - j + 1)\pi$ down to $(n - 1 - j)\pi$, in a continuous fashion. Thus the graph of $\arg(\rho(x)^{n-1})$ crosses the graph of $(n - 1 - j)\pi + \arg(\varepsilon(x)^{-1})$ at some $x_j \in (z_{j-1}, z_j)$. For this x_j , the argument of $\varepsilon(x_j)\rho(x_j)^{n-1}$ is $(n - 1 - j)\pi$.

Case 2: $k + 1 \leq j \leq n - 2$, implying $z_j \geq 0$. By Lemma 7.13, the imaginary part of $\varepsilon(x)$ is positive for all $x \in (z_j, z_{j+1})$. The handling of this case is left to the reader as it is completely analogous to the previous case.

□

8. GENERALIZED TCHEBYSHEV POLYNOMIALS OF THE HIGHER KIND

As a direct generalization of the construction introduced in [7], we may introduced generalized Tchebyshev polynomials of the higher (second, third, \dots , $(k + 1)^{\text{st}}$) kind as follows.

Definition 8.1. *Let L be a triangulation of the k -dimensional simplex such that the only vertices of L on the boundary $\partial(L)$ are the original vertices of the simplex, and let $j \in \{2, \dots, k + 1\}$. Let us define $U^{L,j} : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ as the unique linear map satisfying $U^{L,j}(x^n) = 0$ for $n \leq j - 2$ and having*

the following property: given any simplicial complex K and any generalized Tchebyshev triangulation K' of K , induced by L , we have

$$(8.1) \quad U^{L,j}(F(K, x)) = \sum_{\sigma \in K, |\sigma|=j-1} F(\text{link}_{K'}(\sigma), x).$$

We define the generalized Tchebyshev polynomial $U_n^{L,j}(x)$ of the j th kind by

$$U_n^{L,j}(x) = 2^{1-j} \cdot (j-1)! U^{L,j}(x^{n+j-1}).$$

Similarly to the map T^L , the linear maps $U^{L,j}$ are well-defined, as a consequence of Theorem 3.4. To see this, it is enough to show that $\sum_{\sigma \in K, |\sigma|=j-1} f(\text{link}_{K'}(\sigma), x)$ depends linearly on $f(K, x)$. By Theorem 3.4, there are linear functionals $l_{i,p}$ such that $f_c(K'; x, y) = \sum_{i,p} l_{i,p}(f(K)) x^i y^p$. Now, any face $\tau \in K'$ with $|V(K) \cap \tau| = i + j - 1$ and $|(V(K') \setminus V(K)) \cap \tau| = p$ contributes $\binom{i+j-1}{j-1}$ to the coefficient of $x^i y^p$ in the polynomial $\sum_{\sigma \in K, |\sigma|=j-1} f_c(\text{link}_{K'}(\sigma); x, y)$. Thus,

$$\sum_{\sigma \in K, |\sigma|=j-1} f(\text{link}_{K'}(\sigma), z) = \frac{1}{(j-1)!} \frac{\partial^{j-1}}{\partial x^{j-1}} \left(\sum_{i,p} l_{i,p}(f(K)) x^i y^p \right) \Big|_{x=y=z}.$$

Example 8.2. Let L be the path with two edges, considered in Examples 3.2 and 3.5. Using Theorem 3.4, as an immediate generalization of [7, Proposition 4.4] we obtain that the polynomials $U_n^{L,2}(x)$ are the ordinary Tchebyshev polynomials of the second kind.

In general, to compute $U^{L,j}$, by linearity it suffices to find its value when K is an $(n-1)$ -dimensional simplex, where $n \geq j-1$. When K is an $(n-1)$ -dimensional simplex, we have

$$F(K, z) = \left(\frac{z+1}{2} \right)^n \quad \text{and}$$

$$\sum_{\sigma \in K, |\sigma|=j-1} F(\text{link}_{K'}(\sigma), z) = \frac{1}{(j-1)!} \frac{\partial^{j-1}}{\partial x^{j-1}} f_n(x, y) \Big|_{\substack{x=(z-1)/2 \\ y=(z-1)/2}}.$$

As a consequence, $2^{1-j}(j-1)!U^{L,j}$ is given by

$$(8.2) \quad 2^{1-j}(j-1)!U^{L,j} \left(\left(\frac{z+1}{2} \right)^n \right) = 2^{1-j} \frac{\partial^{j-1}}{\partial x^{j-1}} f_n(x, y) \Big|_{\substack{x=(z-1)/2 \\ y=(z-1)/2}}.$$

Since

$$z^{n+j-1} = \left(2 \cdot \frac{z+1}{2} - 1 \right)^{n+j-1} = \sum_{k=0}^{n+j-1} \binom{n+j-1}{k} (-1)^{n+j-1-k} 2^k \left(\frac{z+1}{2} \right)^k,$$

Equation (8.2) is equivalent to

$$(8.3) \quad U_n^{L,j}(z) = \sum_{k=j-1}^{n+j-1} \binom{n+j-1}{k} (-1)^{n+j-1-k} 2^{1-j+k} \frac{\partial^{j-1}}{\partial x^{j-1}} f_k(x, y) \Big|_{\substack{x=(z-1)/2 \\ y=(z-1)/2}}.$$

In analogy to the derivation of (5.4), we may use (8.3) to obtain the following generating function formula for the polynomials $U_n^{L,j}(x)$.

$$(8.4) \quad \sum_{n=0}^{\infty} U_n^{L,j}(z)t^n = \frac{2^{1-j}}{1+t} \frac{\partial^{j-1}}{\partial x^{j-1}} f \left(x, y, \frac{2t}{1+t} \right) \Bigg|_{\substack{x=(z-1)/2 \\ y=(z-1)/2}}.$$

In analogy to Corollary 5.3, a completely analogous computation has the following consequence.

Corollary 8.3. *Let $\diamond(n+j-1)$ be the boundary complex of an $(n+j-1)$ -dimensional cross-polytope, and let $\diamond(n+j-1)'$ be a Tchebyshev triangulation of it, induced by L . Then we have*

$$U_n^{L,j}(x) = 2^{1-j} \cdot (j-1)! \sum_{\sigma \in \diamond(n+j-1)', |\sigma|=j-1} F(\text{link}_{\diamond(n+j-1)' }(\sigma), x).$$

Using this corollary, it is easy to prove the following analogue of Theorem 5.4.

Theorem 8.4. *For all $n \geq 0$, the polynomials $U_n^{L,j}(x)$ have the following properties:*

- (1) $U_n^{L,j}(x)$ is a polynomial of degree n ;
- (2) $(-1)^n U_n^{L,j}(-x) = U_n^{L,j}(x)$;
- (3) all real roots of $U_n^{L,j}(x)$ belong to the interval $[-1, 1]$.

Theorem 8.4 naturally inspires the question: which triangulations L induce Tchebyshev polynomials of the higher kind having only real roots? We postpone the study of this question to a future occasion. Here we only wish to highlight one important observation that may help handle this problem in complete analogy of the same question for the generalized Tchebyshev polynomials of the first kind: as it is the case for the ordinary Tchebyshev polynomials, the polynomials $U_n^{L,j}(x)$ satisfy the same recurrence as the polynomials $T_n^L(x)$.

Theorem 8.5. *For all $n \geq k+1$, the polynomials $U_n^{L,j}(x)$ satisfy a recurrence of the form*

$$U_n^{L,j}(x) = \sum_{\ell=1}^{k+1} p_{\ell}^L(x) U_{n-\ell}^{L,j}(x).$$

Here each $p_{\ell}^L(x)$ is a polynomial of x and it equals to the coefficient of t^{ℓ} in $(-2t)^{k+1} r_L \left(-\frac{1+t}{2t}, -\frac{1+x}{2} \right)$.

Proof. To obtain a proof of this statement, observe that the proof of Theorem 5.5 depends on (5.5), which follows from (5.4) and from Proposition 4.1. In the proof of Theorem 5.5 we observed that on the right hand side of (5.5) we may simplify by $(1-tx)$. Note that we can make an analogous observation “one step earlier” about the right hand side of Proposition 4.1: using $r_L(-1/t, -1-y)$ as the common denominator on the right hand side, we may simplify the numerator $r_L(-1/t, -1-y) - r_L(-1-x, -1-y)$ by $1-t(x+1)$ and obtain a formula of the form

$$f(x, y, t) = \frac{\tilde{r}_L(x, y, t)}{r_L(-1/t, -1-y)}$$

for some function $\tilde{r}_L(x, y, t)$ that is a polynomial of x, y and $1/t$. The denominator $r_L(-1/t, -1 - y)$ is independent of x , and remains unchanged when we take the partial derivative with respect to x , even repeatedly. We conclude our proof by referring to (8.4) instead of (5.4). \square

Using Corollary 8.3 and Theorem 8.5 it is easy to answer the question on real roots when the dimension of L is 1.

Proposition 8.6. *Let $s \geq 1$ be an integer and L be the subdivision of the 1-simplex by s interior vertices. Then the polynomial $U_n^{L,2}(x)$ has n distinct real roots in the open interval $(-1, 1)$.*

Proof. Using Corollary 8.3 we obtain that $U_0^{L,2}(x) = 2^{1-2} \cdot 2 = 1$ and

$$U_1^{L,2}(x) = 2^{1-2} \cdot 4 \cdot (1 + 2 \cdot (x - 1)/2) = 2x.$$

In analogy of (6.1) it is easy to derive

$$(8.5) \quad U_n^{L,2}(x) = \frac{(x + \sqrt{s(x^2 - 1)})^{n+1} - (x - \sqrt{s(x^2 - 1)})^{n+1}}{2\sqrt{s(x^2 - 1)}} \quad \text{for } n \geq 0.$$

The statement now follows from the fact that, in analogy to (6.2), we have

$$(8.6) \quad U_n^{L,2}(x) = \left(\sqrt{x^2 + s(1 - x^2)} \right)^n \frac{\sin((n+1)\alpha(x))}{\sin(\alpha(x))},$$

where $\alpha(x)$ is the function introduced in the proof of Proposition 6.1, and from the observation that there are n different values of α in $(0, \pi)$ for which $\sin((n+1)\alpha) = 0$. Note that, for $s = 1$, (8.6) is equivalent to the second half of (2.1). \square

9. GENERALIZED LOWER BOUNDS ON FACE NUMBERS

We follow [11], with notational change that dimension d there is replaced by $d - 1$ here. For $d, i \geq 1$ integers, let $\mathcal{HS}(i, d)$ be the family of $(d - 1)$ -dimensional homology spheres without missing faces of dimension $> i$. For $\Delta \in \mathcal{HS}(i, d)$ let $g^{(i)}(\Delta) := g^{(d,i)}(h(\Delta, t))$ be the vector of coefficients when expressing the h -polynomial $h(\Delta, t)$ in the basis $B_{d,i} := (P_{d,i}(t), tP_{d-2,i}(t), t^2P_{d-4,i}(t), \dots, t^{\lfloor \frac{d}{2} \rfloor} P_{d-2\lfloor \frac{d}{2} \rfloor, i}(t))$, where $P_{d,i}(t) := (1 + t + \dots + t^i)^q (1 + t + \dots + t^r)$, and $q \geq 0, 1 \leq r \leq i$ are the unique integers such that $d = qi + r$.

Conjecture 9.1. [11, Conjecture 1.5] If $\Delta \in \mathcal{HS}(i, d)$ then $g^{(i)}(\Delta) \geq 0$ (componentwise).

The case $i \geq d$ gives the usual g -vector and the well known g -conjecture, see e.g. [12] for more details on the latter, and the case $i = 1$ gives Gal's γ -vector and conjecture [6]. Generalizing the usual g -polynomial and Gal's γ -polynomial we introduce the *generalized g -polynomial*

$$(9.1) \quad g^{(i)}(\Delta, t) = \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} g^{(i)}(\Delta)_j t^j$$

Conjecture 9.1 is obviously equivalent to stating that, for any $\Delta \in \mathcal{HS}(i, d)$, all coefficients in $g^{(i)}(\Delta, t)$ are nonnegative.

The following related results [11, Propositions 1.6 and 4.1] will be needed.

Lemma 9.2. *Let $\Delta \in \mathcal{HS}(i, d)$ and $\Delta' \in \mathcal{HS}(i, d')$.*

(1) *If $g^{(i)}(\Delta) \geq 0$ then $g^{(i+1)}(\Delta) \geq 0$.*

(2) *$\Delta * \Delta' \in \mathcal{HS}(i, d + d')$ and if $g^{(i)}(\Delta) \geq 0$ and $g^{(i)}(\Delta') \geq 0$ then $g^{(i)}(\Delta * \Delta') \geq 0$.*

We will verify Conjecture 9.1 for simplicial spheres arising as generalized Tchebyshev triangulations of $\diamond(d)$, the boundary complex of the d -cross polytope, induced by L , for certain triangulations L considered in previous sections. Denote any simplicial sphere obtained in this way by $\diamond(d, L)$. We have seen in Theorem 3.3 that, although the $\diamond(d, L)$ have different combinatorial types, they all have the same f -vector, and hence the same generalized g -polynomial.

Theorem 9.3. *Let $\Delta \in \mathcal{HS}(i, d)$ and $F \in \Delta$ of dimension $\leq i$. Note that $\text{link}_\Delta(F) \in \mathcal{HS}(i, d - |F|)$ and the stellar subdivision $\Delta(F) := \text{Stellar}_\Delta(F) \in \mathcal{HS}(i, d)$. Assume $g^{(i)}(\Delta) \geq 0$ and $g^{(i)}(\text{link}_\Delta(F)) \geq 0$. Then $g^{(i)}(\Delta(F)) \geq 0$.*

Proof. Indeed $\text{link}_\Delta(F) \in \mathcal{HS}(i, d - |F|)$, see e.g. [11, Lemma 2.3]. To see that $\Delta(F) \in \mathcal{HS}(i, d)$ note that the missing faces of $\Delta(F)$ and not of Δ are F and some missing edges containing the new vertex v_F of $\Delta(F)$.

Let $u \in F$, then the *link condition* $\text{link}_{\Delta(F)}(uv_F) = \text{link}_{\Delta(F)}(v_F) \cap \text{link}_{\Delta(F)}(u)$ holds. Moreover, this complex is in $\mathcal{HS}(i, d - 2)$ and equals the join $\partial(F \setminus u) * \text{link}_\Delta(F)$ of two complexes, where $\text{link}_\Delta(F) \in \mathcal{HS}(i, d - |F|)$ and $\partial(F \setminus u) \in \mathcal{HS}(i, |F| - 3)$. Note that $g^{(|F|-2)}(\partial(F \setminus u), t) = g(\partial(F \setminus u), t) = 1$, thus by Lemma 9.2(1) $g^{(i)}(\partial(F \setminus u)) \geq 0$. By assumption, $g^{(i)}(\text{link}_\Delta(F)) \geq 0$, so by Lemma 9.2(2), $g^{(i)}(\text{link}_{\Delta(F)}(uv_F)) \geq 0$.

Now, the contraction $v_F \mapsto u$ in $\Delta(F)$ results in Δ . An easy computation shows $h(\Delta(F), t) = h(\Delta, t) + th(\text{link}_{\Delta(F)}(uv_F), t)$. Thus the generalized g -polynomial satisfies

$$g^{(i)}(\Delta(F), t) = g^{(i)}(\Delta, t) + tg^{(i)}(\text{link}_{\Delta(F)}(uv_F), t).$$

By our assumption, both summands on the right hand side have nonnegative coefficients, therefore the same holds for the left hand side. \square

Corollary 9.4. *Let L be the subdivision of the j -simplex with one interior vertex, namely the one obtained by starring. Assume $1 \leq j \leq i$. Then $g^{(i)}(\diamond(d, L)) \geq 0$.*

Proof. Note that $g^{(1)}(\diamond(d), t) = \gamma(\diamond(d), t) = 1$, thus by Lemma 9.2(1) $g^{(i)}(\diamond(d)) \geq 0$. Now $\diamond(d, L)$ is obtained from $\diamond(d)$ by a sequence of stellar subdivisions at faces of dimension j in $\diamond(d)$. As $j \leq i$, thanks to Theorem 9.3, it is enough to verify that when subdividing the $(k + 1)$ 'th j -face F_{k+1} of $\Delta = \Delta_0 = \diamond(d)$, as a face in the k th complex Δ_k , then $g^{(i)}(\text{link}_{\Delta_k}(F_{k+1})) \geq 0$.

The key observation here is that for any simplicial complex K , the operations link and stellar subdivision commute, more precisely, if $T \not\subseteq F$ are sets, then

$$\text{link}_{K(T)}(F) \cong (\text{link}_K(F))(T \setminus F).$$

(Here, if $F' \notin K'$ for a complex K' and a set F' then define $K'(F') := K'$. For $|T \setminus F| = 1$ and $F \cup T \in K$ the isomorphism is given by mapping v_T to the vertex of $T \setminus F$ and the other vertices to themselves.) Let the order of the j -faces in Δ be F_1, F_2, \dots . We will prove the following stronger assertion by double induction on d and k (for fixed $1 \leq j \leq i$):

(**) If $F \in \Delta$ is of dimension $\geq j$, and does not contain any of F_1, \dots, F_k , then $g^{(i)}(\text{link}_{\Delta_k}(F)) \geq 0$.

The base case $d < j$ is trivial and the base case $k = 0$ follows for any d as $\text{link}_{\diamond(d)}(F) \cong \diamond(d - |F|)$, so it has $g^{(1)}(\text{link}_{\diamond(d)}(F), t) = 1$, hence $g^{(i)}(\text{link}_{\diamond(d)}(F)) \geq 0$. For $k > 0$, $\text{link}_{\Delta_k}(F) \cong (\text{link}_{\Delta_{k-1}}(F))(F_k \setminus F)$. If $F_k \cup F \notin \Delta_{k-1}$ then $\text{link}_{\Delta_k}(F) = \text{link}_{\Delta_{k-1}}(F)$ and we are done by induction on k . Else, as $F_k \cup F \in \Delta_{k-1}$ and $F_k, F \in \Delta$ we conclude that $F_k \cup F \in \Delta$. By induction on k , $g^{(i)}(\text{link}_{\Delta_{k-1}}(F)) \geq 0$. Also, $\text{link}_{\text{link}_{\Delta_{k-1}}(F)}(F_k \setminus F) = \text{link}_{\Delta_{k-1}}(F \cup F_k)$. By construction of Δ_{k-1} , $F \cup F_k$ does not contain any of F_1, \dots, F_{k-1} (as $F \cup F_k \in \Delta_{k-1}$), hence the induction on k says $g^{(i)}(\text{link}_{\Delta_{k-1}}(F \cup F_k)) \geq 0$. Thus, by Theorem 9.3 we conclude that $g^{(i)}(\text{link}_{\Delta_k}(F)) \geq 0$. \square

Remark 9.5. For any subdivision L of the 1-simplex (say with k interior points), $g^{(1)}(\diamond(d, L)) = \gamma(\diamond(d, L)) \geq 0$. This is known, and also follows from Theorem 9.3, as L is obtained by a sequence of k stellar subdivisions at an edge.

We now turn to arbitrary subdivisions L of the 2-simplex.

Theorem 9.6. *Let L be a subdivision of the 2-simplex, where all the new vertices are in the interior. Then $\diamond(d, L) \in \mathcal{HS}(2, d)$ and satisfies $g^{(2)}(\diamond(d, L)) \geq 0$.*

It is clear that $\diamond(d, L) \in \mathcal{HS}(2, d)$. Below we state and prove two generalizations of the second statement.

Theorem 9.7. *Let L be a subdivision of the 2-simplex, where all the new vertices are in the interior. Then complexes $\Delta_k = \diamond(d)_k$, arising in the definition of a $\diamond(d, L)$, satisfy*

- (i) $g^{(2)}(\Delta_k) \geq 0$ and
- (ii) $g^{(2)}(\text{link}_{\Delta_k}(T_{k+1})) \geq 0$ where T_{k+1} is the $(k+1)$ th 2-simplex of $\Delta_0 = \diamond(d)$ that is subdivided.

Proof. We proceed by induction on d and k and instead of (ii) we will prove the following stronger assertion:

(iii) If $F \in \Delta_0$ is of dimension ≥ 2 , and does not contain any of T_1, \dots, T_k , then $g^{(2)}(\text{link}_{\Delta_k}(F)) \geq 0$.

The base case $d < 2$ is trivial, and the case $k = 0$ is clear as both Δ_0 and $\text{link}_{\Delta_0}(T_1)$ are boundary complexes of cross polytopes. Let m be the number of interior vertices in L , and T be the 2-simplex

L subdivides. By Euler's formula applied to the 2-sphere $S = L \cup \{T\}$, one gets that the polynomial $f(L^0, t)$ counting faces of $L^0 := L \setminus \partial L$ satisfies $f(L^0, t) - t^3 = mt + 3mt^2 + 2mt^3$.

Then

$$(9.2) \quad \begin{aligned} f(\Delta_{k+1}, t) &= f(\Delta_k, t) - t^3 f(\text{link}_{\Delta_k}(T_{k+1}), t) + f(L^0, t) f(\text{link}_{\Delta_k}(T_{k+1}), t) \\ &= f(\Delta_k, t) + f(\text{link}_{\Delta_k}(T_{k+1}), t)(mt + 3mt^2 + 2mt^3). \end{aligned}$$

For any $(d-1)$ -dimensional homology sphere Δ , $h(\Delta, t) = (t-1)^d f(\Delta, \frac{1}{t-1})$, and combined with equation (9.2) we get

$$h(\Delta_{k+1}, t) = h(\Delta_k, t) + mt(t+1)h(\text{link}_{\Delta_k}(T_{k+1}), t).$$

Note that the suspension $\Sigma\Delta$, i.e. the join of Δ with the two points complex, has h -vector $(1+t)h(\Delta, t)$. Thus

$$h(\Delta_{k+1}, t) = h(\Delta_k, t) + mt \cdot h(\Sigma \text{link}_{\Delta_k}(T_{k+1}), t).$$

By induction, $g^{(2)}(\Delta_k) \geq 0$ and $g^{(2)}(\text{link}_{\Delta_k}(T_{k+1})) \geq 0$, so by Lemma 9.2 also $g^{(2)}(\Sigma \text{link}_{\Delta_k}(T_{k+1})) \geq 0$. Thus,

$$g^{(2)}(\Delta_{k+1}, t) = g^{(2)}(\Delta_k, t) + mt \cdot g^{(2)}(\Sigma \text{link}_{\Delta_k}(T_{k+1}), t)$$

has only nonnegative coefficients, proving (i).

To prove (iii), if $F \cup T_k \notin \Delta_k$ then $\text{link}_{\Delta_k}(F) = \text{link}_{\Delta_{k-1}}(F)$ and we are done. Else, we treat different cases according to the cardinality of $F \cap T_k$:

Case $|F \cap T_k| = 0$: Then $\text{link}_{\Delta_k}(F) = (\text{link}_{\Delta_{k-1}}(F))(T_k)$. By induction on k , $g^{(2)}(\text{link}_{\Delta_{k-1}}(F)) \geq 0$, and $g^{(2)}(\text{link}_{\text{link}_{\Delta_{k-1}}(F)}(T_k)) = g^{(2)}(\text{link}_{\Delta_{k-1}}(F \cup T_k)) \geq 0$. Thus, by Theorem 9.3 we are done.

Case $|F \cap T_k| = 2$: Then $\text{link}_{\Delta_k}(F) \cong \text{link}_{\Delta_{k-1}}(F)$, via the isomorphism mapping the vertex $v \in \text{Int}(T_k)$ adjacent to the edge $T_k \cap F$ to the vertex $T_k \setminus F$, and the other vertices to themselves. We are done by induction on k .

Case $|F \cap T_k| = 1$: Let v be the common vertex of F and T_k , and let P be the link of v in the subdivision of T_k induced by L and the bijection $\phi : V(\partial L) \rightarrow T_k$. Then P is a path, say with s interior points (then $s \geq 1$).

Then $\text{link}_{\Delta_k}(F)$ equals the subdivision of $\text{link}_{\Delta_{k-1}}(F)$ induced by subdividing the edge $T_k \setminus F$ by s interior points. Thus,

$$f(\text{link}_{\Delta_k}(F), t) = f(\text{link}_{\Delta_{k-1}}(F), t) + st(1+t)f(\text{link}_{\text{link}_{\Delta_{k-1}}(F)}(T_k \setminus F), t),$$

equivalently,

$$h(\text{link}_{\Delta_k}(F), t) = h(\text{link}_{\Delta_{k-1}}(F), t) + st \cdot h(\text{link}_{\text{link}_{\Delta_{k-1}}(F)}(T_k \setminus F), t),$$

equivalently,

$$g^{(2)}(\text{link}_{\Delta_k}(F), t) = g^{(2)}(\text{link}_{\Delta_{k-1}}(F), t) + st \cdot g^{(2)}(\text{link}_{\text{link}_{\Delta_{k-1}}(F)}(T_k \setminus F), t).$$

By induction, both summands on the right hand side have nonnegative coefficients (for the rightmost summand consider $\text{link}_{\Delta_{k-1}}(T_k \cup F)$), hence the left hand side has also only nonnegative coefficients. \square

Theorem 9.6 is a special case of Theorem 9.7 above, since $\diamond(d, L)$ is the last complex in the sequence of complexes $\Delta_1, \Delta_2, \dots$. Before proving the second generalization, let us make the following observation. Obviously, for any homology sphere $\Delta \in \mathcal{HS}(i, d)$, we have $g_0^{(i)}(\Delta) = 1$, since we have $h_0(\Delta) = 1$, the constant term of $P_{d,i}(t)$ is 1 and all other polynomials in the basis $B_{d,i}$ have zero constant term. Therefore $g^{(i)}(\Delta) \geq 0$ holds (componentwise) whenever the generalized g -polynomial given in (9.1) has only real negative roots. Theorem 9.6 is thus also a consequence of the already shown Corollary 5.3, Theorems 5.4 and 7.14, and of Theorem 9.8 below.

Theorem 9.8. *Let Δ be a homology sphere. Then the following are equivalent:*

- (i) *the roots of $F(\Delta, t)$ are all real numbers in the interval $(-1, 1)$;*
- (ii) *the roots of $h(\Delta, t)$ are all real and negative;*
- (iii) *the roots of $g^{(2)}(\Delta, t)$ are all real numbers in the interval $[-1, 0)$.*

Proof. The equivalence of the first two statements may be shown by refining the argument presented in [7, Section 6]. It was noted there that the F -polynomial and the h -polynomial of Δ are connected by the formula

$$(1-t)^d \cdot F\left(\Delta, \frac{1+t}{1-t}\right) = h(\Delta, t).$$

The Preliminaries of [7] remind of the well-known fact that the map $\mu : x \mapsto t = (x-1)/(x+1)$ establishes a bijection between the unit disk $|x| < 1$ and the open left t -halfplane. Using this bijection it is easy to show that the *Schur-stability* of $F(\Delta, x)$, defined as having all its roots inside the unit disk $|x| < 1$, implies the *Hurwitz-stability* of $h(\Delta, t)$, defined as having all its zeros in the open left t -halfplane. As noted in [7, Proposition 6.4], the converse is also true when the reduced Euler characteristic of Δ is not zero, which is the case for homology spheres. To arrive at the presently stated equivalence we only need to observe that the restriction of μ to the interval $(-1, 1)$ establishes a bijection between this interval and the set of all negative real numbers.

We are left to show the equivalence of the second and the third statement. Directly from the definitions we have

$$P_{k,2}(t) = \begin{cases} (1+t+t^2)^{k/2} & \text{for even } k; \\ (1+t+t^2)^{(k-1)/2}(1+t) & \text{for odd } k. \end{cases}$$

Using this formula it is easy to show

$$h(\Delta, t) = \begin{cases} (1+t+t^2)^{d/2} g^{(2)}(\Delta, t/(1+t+t^2)) & \text{for even } d; \\ (1+t+t^2)^{(d-1)/2} (1+t) g^{(2)}(\Delta, t/(1+t+t^2)) & \text{for odd } d. \end{cases}$$

Without loss of generality we may assume d is odd, the case of even d being similar but simpler. Assume first all roots of $g^2(\Delta, t)$ are real from $[-1, 0)$, i.e., we have

$$g^{(2)}(\Delta, t) = r(t-r_1)(t-r_2) \cdots (t-r_{(d-1)/2})$$

for some positive real number r and some negative real numbers $r_1, \dots, r_{(d-1)/2} \in [-1, 0)$. (The fact that r is real and positive follows from $g^{(2)}(\Delta)_0 = 1$.) Then we have

$$(9.3) \quad h(\Delta, t) = r(1+t)(t-r_1(1+t+t^2)) \cdots (t-r_{(d-1)/2}(1+t+t^2)).$$

The roots of $h(\Delta, t)$ are -1 and the roots of all quadratic equations of the form $t - r_k(1 + t + t^2) = 0$, that is numbers of the form

$$(9.4) \quad s_k = \frac{(r_k - 1) - \sqrt{(r_k - 1)^2 - 4r_k^2}}{(-2r_k)} \quad \text{and of the form} \quad t_k = \frac{(r_k - 1) + \sqrt{(r_k - 1)^2 - 4r_k^2}}{(-2r_k)}.$$

Here $r_k - 1$ is negative, the summand $\sqrt{(r_k - 1)^2 - 4r_k^2}$ is real but strictly less than $|r_k - 1|$, and the denominator $-2r_k$ is positive. We obtain that each s_k and t_k is a negative real number. To prove the converse, observe that Equation (9.3) holds in general for any homology sphere Δ , with some complex roots $r_1, \dots, r_{(d-1)/2}$ and complex leading coefficient r . The roots of $h(\Delta, t)$ are still -1 and the complex numbers s_k and t_k given by (9.4). (Recall that taking the square root of a complex number is unique up to sign, thus the pair $\{s_k, t_k\}$ is well-defined.) Assuming that each s_k and t_k is a negative real number, we obtain that each

$$\frac{1 - r_k}{r_k} = s_k + t_k$$

is a negative real number and so each r_k is a real number, belonging to the set $(-\infty, 0) \cup (1, \infty)$. Thus

$$\sqrt{(r_k - 1)^2 - 4r_k^2} = (-r_k)(t_k - s_k)$$

is also a real number, and we must have $(r_k - 1)^2 - 4r_k^2 \geq 0$. This is equivalent to $r_k \in [-1, 1/3]$. The intersection of $[-1, 1/3]$ with $(-\infty, 0) \cup (1, \infty)$ is the set $[-1, 0)$. \square

Remark 9.9. An analogous statement for $i = 1$ was shown by Gal [6, Remark 3.1.1] who proved that, for a homology sphere Δ , the polynomial $h(\Delta, t)$ has only negative real roots if and only if the same holds for $g^{(1)}(\Delta, t)$.

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