THE SPECTRAL ANALYSIS OF SCHRÖDINGER OPERATOR ON GENERAL GRAPHS

by

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ABSTRACT

LUKUN ZHENG. The spectral analysis of Schrödinger operator on general graphs.
(Under the direction of DR. STANISLAV MOLCHANOV)

The goal of this dissertation is to give the sufficient conditions for the absence of a.c.spectrum or existence of the pure point (p.p.) spectrum for the deterministic or random Schrödinger operators on the general graphs. For the particular situations of "non-percolating" graphs like Sierpiński lattice and Quasi-1 dimensional tree, we'll prove the Simon-Spencer type results and the localization theorem for Anderson Hamiltonians. Technical tools here are the extensions of the real-analytic methods presented for the 1D lattice Z^1 and corresponding Schrödinger operators in [10]. The central moment is the cluster expansion of the resolvent with respect to appropriate partition of Γ.

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CHAPTER 1: INTRODUCTION

1.1 A General Summary

The idea on the possible connection between the spectral phase transition for the Anderson Hamiltonian on the lattices Z^d, $d \geq 3$ and the percolation was always popular in the physical literature. In one direction it is definitely correct. Consider on the lattice Z^1 the random Schrödinger operator

$$H\phi(x) = \Delta\phi(x) + \sigma V(s, \omega),\ \Delta\phi(x) = \phi(x+1) + \phi(x-1)$$

where σ is a coupling constant and $V(x, \omega)$ are unbounded i.i.d., say, $N(0, 1)$ random variables. Then for arbitrary small σ and for arbitrary large M there are infinitely many points x : $|V(x, \omega)| > M$. I.e. the set $\{x : |V(x, \omega)| \leq M\}$ contains only finite connected components, or they doesn't percolate, though its concentration can be arbitrary close to 1. We call Z^1 non-percolating graph. More formally, let Γ be an abstract graph and $X(x, \omega), x \in \Gamma$ is the Bernoulli field such that $P\{X(\cdot) = 1\} = \epsilon$, $P\{X(\cdot) = 0\} = 1 - \epsilon$. We call Γ a non-percolating graph if for arbitrary $\epsilon > 0$ the set $\{x : X(x) = 0\}$ contains $P - a.s.$ only bounded connected components. The class of non-percolating graphs is very rich. It includes, for instance, all nested fractal lattices, among them, the Sierpiński lattice.

Let's consider for such graphs Anderson type operators

$$H\phi(x) = \sum_{x':x\sim x'} \phi(x') + \sigma V(x,\omega)$$

where $\{x' : x' \sim x\}$ is the set of nearest neighbours of x and $V(x,\omega)$ are unbounded i.i.d. random variables. At the level of the physical intuition we have the following picture: the realization of $\sigma V(\cdot)$ contains the sequence of the higher and higher "walls" and the quantum particle can't avoid the interaction with such walls in its attempts to reach infinite. Since the tunnelling through higher and higher walls has smaller and smaller probability one can expect here some kind of localization phenomena, say, the absence of the a.c. spectrum. In the case of 1D lattice Z^1 these a bit fuzzy arguments were transformed in the mathematical theorem in the famous Simon-Spencer theorem [3]. Let

$$H\phi(x) = \phi(x+1) + \phi(x-1) + V(x)\phi(x), x \in Z^1.$$

If the potential $V(x)$ is unbounded near $\pm\infty$ (i.e. $\limsup_{x\to+\infty} |V(x)| = \limsup_{x\to-\infty} |V(x)| = +\infty$), then $\Sigma_{a.c.} = \emptyset$.

Of course, for the random i.i.d. unbounded potentials this results provides the absence of a.c. spectrum $P - a.s.$. The Simon-Spencer theorem can be extended on some class of the potentials even in $R^d, d \geq 2$, see [8]. All results of such kind include the very strong assumptions of the existence of the infinite system of the "rings" or belts b_n around origin, where potential V is higher and higher (i.e. $\min_{x\in b_n} V(x) = h_n \to +\infty,\ n \to \infty$). They also require the additional conditions on fast increasing of the "heights" h_n. B.Simon [3] constructed (for $\Gamma = Z^2$) such Schrödinger operator, where

$V(x) \geq 0$ contains the system of the higher and higher walls (i.e. set $\{x : V(x) \leq M\}$ doesn't percolate for any M), but spectrum of H contains the a.c. components.

Unfortunately the lattice Z^d, $d > 1$ “percolate”, there exist the critical thresholds h_{cr}, $\tilde{h}_{cr}$: if $h > h_{cr}$ then the set $\{x : V(x,\omega) > h\}$ doesn't percolate, but for $h < \tilde{h}_{cr}$ it contains the infinite component.

The goal of this paper is to give the sufficient conditions for the absence of a.c.spectrum or existence of the pure point (p.p.) spectrum for the deterministic or random Schrödinger operators on the general graphs. For the particular situations of “non-percolating” graphs we'll prove the Simon-Spencer type results and the localization theorem for Anderson Hamiltonians. Technical tools here are the extensions of the real-analytic methods presented for the 1D lattice Z^1 and corresponding Schrödinger operators in [10]. The central moment is the cluster expansion of the resolvent with respect to appropriate partition of Γ.

1.2 Two Particular Examples

The general theory provide the following particular results.

Theorem (A). *Consider the Sierpiński lattice S^∞ (see figure 1) with Laplacian $\Delta\psi(x) = \sum_{x' \sim x} \psi(x')$ and boundary condition $\psi(0) = 0$. Let $H = \Delta + V(x,\omega)$ is the Anderson Hamiltonian and $V(x,\omega)$ are i.i.d. unbounded random variables. Then $P-a.s.$ $H = H(\omega)$ has no absolutely continuous spectrum.*

This theorem is a particular case of the much more general result, see Corollary 16.

Theorem (B). *Consider the Quasi-1 Dimensional Tree T (see figure 2) with Laplacian*

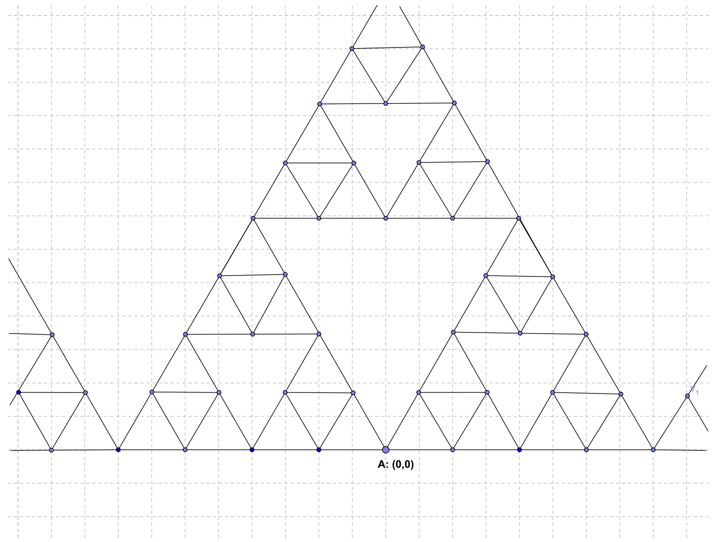

Figure 1: Sierpiński lattice S^{∞}.

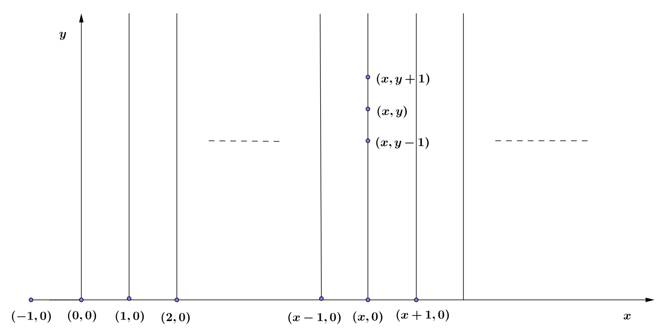

Figure 2: Quasi-1 dimensional tree T

$\Delta\psi(x) = \sum_{x' \sim x} \psi(x')$, $x \neq (-1,0)$ *and boundary condition* $\psi(-1,0) = 0$. *Let* $H = \Delta + V(x,\omega)$ *be the Anderson Hamiltonian and* $V(x,\omega)$ *are i.i.d. random variables with bounded distribution density* $f(t)(|f(t)| < M,\ t \in R,$ *for some* $M < \infty)$ *and* $\int_{|t|>A} f(t)dt > 0$ *for any* $A > 0$. *Then the spectrum of* H *is pure point with probability 1.*

CHAPTER 2: BASIC DEFINITIONS AND THE MODEL

Let (Γ, A) be a undirected infinite graph with symmetric adjacency matrix $A(\Gamma) = A^*(\Gamma) = [a(x,y)]$. If $a(x,y) = 1$, we will call x and y are the nearest neighbors, denoted by $x \sim y$. Suppose that for each $x \in \Gamma$, the number $\nu(x)$ of nearest neighbors x': $x' \sim x$ is bounded: $\nu(x) \leq K$ for some fixed $K : 2 \leq K \leq \infty$. For the homogeneous graph (with appropriate group of the symmetries) $\nu(x) \equiv K$. In this case, K is called the index of branching of the graph (Γ, A). Typical examples of homogeneous graphs are: lattices $\mathbb{Z}^d$ with index of branching equal to $2d$, groups with finite number of generators, homogeneous trees, etc.

The path $[\gamma]$ of length $|[\gamma]| = n$, from point x to point y on Γ, denoted by $[\gamma] : x \to y$ is defined to be a sequence of the points

$$[\gamma] = \{x_i\}_{i=0}^{n}$$

such that $x_i \sim x_{i-1}$ for $i = 1, 2, \cdots, n$ and $x_0 = x,\ x_n = y$. Here $x_0 = x$ is called the start point, $x_n = y$ is called the end point. All the other points are called the internal points of the path. We denote $(\gamma) : x \to y$ the internal part of $[\gamma] : x \to y$, that is,

$$(\gamma) = \{x_i\}_{i=1}^{n-1}$$

with $x_1 \sim x_0 = x$ and $x_{n-1} \sim x_n = y$. If a path contains infinitely many points, then it is called an infinite path. Any non-empty set $B \subset V$ with more than one points is called connected or 1-connected if $\forall x, y \in B,\ \exists[\gamma] : x \to y$ and $[\gamma] \subset B$. The boundary of B is defined as:

$$\partial B = \{y : y \notin B, \text{ and } y \sim x, \text{ for some } x \in B\}.$$

In this paper, we assume that graph Γ is 1-connected.

A metric $d(x, y)$ on Γ and distances $d(x, B),\ d(B_1,\ B_2)$ are defined in the standard way. The volume of the ball, centered at point $x_0 \in \Gamma$, can not increases faster than an exponential:

$$|B_R(x_0)| = |\{x : d(x, x_0) \leq R\}| \leq K^R + 1.$$

Recall that $K = \max_{x \in \Gamma} \nu(x)$.

Let $\ell_2(\Gamma)$ be the Hilbert space of square-summable functions $f(x) : \Gamma \to \mathbb{C}$ with the inner product and norm

$$(f, g) = \sum_{x \in \Gamma} f(x)\overline{g}(x),\ ||f||^2 = \sum_{x \in \Gamma} |f(x)|^2. \tag{1}$$

for $f, g \in \ell_2(\Gamma)$.

The lattice Laplacian in the space $\ell_2(\Gamma)$ is given by a usual formula

$$\Delta f(x) = \sum_{x' : d(x', x) = 1} f(x') \tag{2}$$

As easy to see,

$$||\Delta|| = \sup_{f : ||f|| = 1} ||\Delta f|| \leq K.$$

i.e. the lattice Laplacian is bounded.

The Schrödinger operator(Hamiltonian), by the definition, has a form

$$H = \Delta + V(x) \tag{3}$$

where $V(x)$ is an arbitrary real-valued potential. In the most interesting case, $V(x) = \sigma\xi(x,\omega)$ and $\xi(x,\omega)$ will be the family of i.i.d.r.v. with bounded continuous density $p_\xi(\cdot)$. Here $x \in \Gamma$, $\omega \in (\Omega, \mathcal{F}, P)$(a basic probability space), and $\sigma > 0$ is the coupling constant(the measure of disorder). In this case, we will call H the Anderson Hamiltonian on $l^2(\Gamma)$. The fundamental and still unsolved problem is to determine the spectral type of H for the general graphs (or at least for lattices $\mathbb{Z}^d, d \geq 2$). It is known ([6])that for arbitrary graph Γ and the very general symmetric bounded operators $\mathcal{L}$, the spectral measure is pure point(p.p) for the large disorder, $\sigma > \sigma_0$, where σ_0 can be effectively determined by the geometry of the graph Γ. If $\Gamma = \mathbb{Z}^1$ (or $\Gamma = \mathbb{Z}^1 \times A, Card(A) < \infty$) the spectrum of H is p.p. P-a.s. and the corresponding eigenfunctions are exponentially decaying for arbitrary small coupling parameter σ (see details in [9]). The second case where the spectral picture is well understood is the homogeneous tree T^N, see [5]. The last case is the only one which demonstrate the Anderson type phase transition from the p.p. spectrum for large σ to the mixed spectrum (a.c. component plus p.p. component) for small σ.

CHAPTER 3: EXPANSION THEORY OF THE RESOLVENT KERNEL

3.1 Path Expansion

The resolvent kernel of the operator H

$$R_\lambda(x,y) = (H - \lambda I)^{-1}(x,y)$$

is well defined at least for complex λ. We will use the following "exact" formula for $R_\lambda(x,y)$(so-called path expansion) see [10].

Proposition 1. Let $V(x)$ be the potential of the Schrödinger operator on Γ with $Range(V) = \{V(x) : x \in \Gamma\} \subset R$. Then

$$R_\lambda(x,y) = \frac{\delta_y(x)}{\lambda - V(x)} + \sum_{[\gamma]:x\to y} \left(\prod_{z\in\gamma} \frac{1}{\lambda - V(z)} \right). \qquad (4)$$

where $\delta_y(x) = 1$ if $x = y$ and 0 otherwise. This formula holds, at least for λ's such that $d(\lambda, \overline{Range(V)}) \geq K + \delta$, for some $\delta > 0$.

Proof. Since $H\psi(x) - \lambda\psi(x) = \delta_y(x)$, we have

$$\sum_{x':x'\sim x} \psi(x') + V(x)\psi(x) - \lambda\psi(x) = \delta_y(x).$$

Thus,

$$\psi(x) = \frac{\delta_y(x)}{\lambda - V(x)} + \sum_{x':x'\sim x} \frac{\psi(x')}{\lambda - V(x)}.$$

For each x', one can use the same formula, and continue these iterations, to get

(4).

The number of paths $[\gamma]$ with the fixed start point x and the length n is at most K^n and $\left|\prod_{z\in[\gamma]}\frac{1}{\lambda-V(z)}\right|\leq\frac{1}{(K+\delta)^{|[\gamma]|+1}}$. These facts lead to the convergence of the series (4). □

Similar construction works for the restriction of H on subsets B of Γ. Consider $H_B=\triangle+V(x)$ with Dirichlet boundary condition:

$$\phi(x)=0$$

for $x\in\Gamma-B$.

Proposition 2. Let $R_\lambda^{(B)}(x,y) = (H_B-\lambda I)^{-1}(x,y)$ be the resolvent kernel of H_B (which is well defined at least for $|Im\lambda|\geq K+\delta$, $\delta>0$). Then for $x,y\in B$,

$$R_\lambda^{(B)}(x,y)=\frac{\delta_y(x)}{\lambda-V(x)}+\sum_{\substack{[\gamma]:x\to y\\ [\gamma]\subset B}}\left(\prod_{z\in[\gamma]}\frac{1}{\lambda-V(z)}\right). \tag{5}$$

Note that for cardinality of the set B: $|B|<\infty$ the spectrum of H_B contains $|B|$ real eigenvalues (which can be multiple), i.e. $R_\lambda^{(B)}(x,y)$ is real and finite for all real λ, except the finite set of eigenvalues of H_B.

3.2 Cluster Expansion

Let us fix some point $x_0\in\Gamma$, called *reference point*. A set $b\subset\Gamma$ is called a *belt* with respect to x_0 if any path that starts from point x_0 and goes to infinite intersects with set b. Define the set of all the points trapped by the belt b the *enclosure* E_b of the belt b: $E_b=\{x:\exists[\gamma]:x_0\to x,\text{ and }[\gamma]\cap b=\emptyset\}$. Thus E_b is 1-connected, that is, $\forall x,y\in E_C,\exists[\gamma]:x\to y,[\gamma]\subset E_b$. The *inner boundary* of the belt b is defined to be the set $\partial b^- = \partial b \cap E_b$ and the *outer boundary* of the belt b is $\partial b^+ = \partial b - \partial b^-$. A *counter* is defined to be a belt c for which every point in c is irreducible. In other word, c is not going to be a belt if any point of it is taken away.

We can define now the following quantity (similar to resolvent of b): for $x \in \partial b^-,\ y \in \partial b^+$, put

$$\beta^b_\lambda(x,y) = \sum_{\substack{[\gamma]:x\to y \\ (\gamma)\subset b}} \prod_{z\in(\gamma)} \frac{1}{\lambda - V(z)}. \tag{6}$$

(and similar expression if $x \in \partial b^+,\ y \in \partial b^-$). Note that path (γ) stays inside belt b. In the future the potential $V(\cdot)$ will be large on b, i.e. for $\lambda \in I$ (I is a fixed interval) the quantity $\beta^b_\lambda(x,y)$ will be small. Physically it means that the tunnelling of the quantum particle through the belt is very unlikely.

Now assume that $\{b_i : i \geq 1\}$ is a sequence of disjoint belts with respect to the reference point x_0 and the corresponding sequence of the enclosures is $\{E_{b_i}\}_{i=1}^{\infty}$. Assume that $E_{b_i} \subset E_{b_{i+1}}$, for any $i = 1, 2, 3, \cdots$. Let $S_0, S_1, \cdots$ be subsets of Γ between successive belts(S_0 contains x_0). That is, $S_i = E_{b_i} - E_{b_{i-1}} - b_{i-1}$ for $i = 1, 2, \cdots$. We define the main blocks $(0), (1), \cdots$ as:

$(0) = S_0 \cup b_1, \partial(0) = \partial b_1^+,$

$(1) = b_1 \cup S_1 \cup b_2, \partial(1) = \partial b_1^- \cup \partial b_2^+$

and so on. See figure (3)

We call ∂b_i^- the inner boundary of b_i and ∂b_i^+ the outer boundary of b_i for $i \geq 1$.

Now we will study the cluster expansion of $R_\lambda(x_0, x)$. Consider the path expansion of H:

$$R_\lambda(x_0,x) = \frac{\delta_{x_0}(x)}{\lambda - V(x_0)} + \sum_{[\gamma]:x_0\to x} \left(\prod_{z\in[\gamma]} \frac{1}{\lambda - V(z)} \right)$$

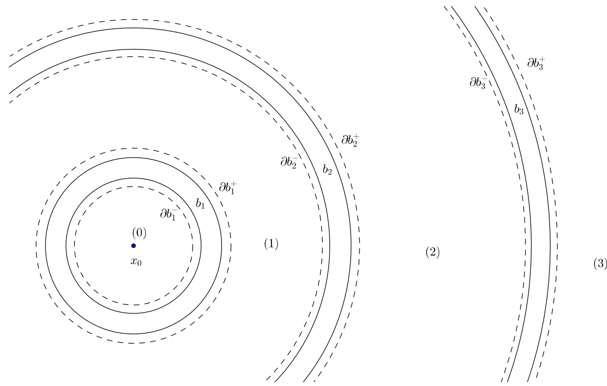

Figure 3: Main blocks and belts

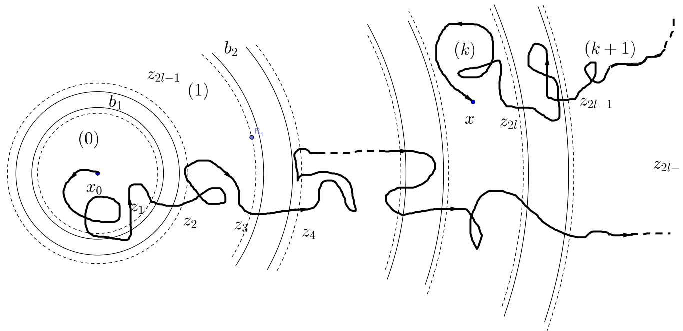

Figure 4: The path $[\gamma]$

and introduce the procedure of the union of the paths into the special groups(clusters).

Consider a particle following some path $[\gamma] : x_0 \to x$ (Figure 4) . It will stay in the main block (0) for certain amount of time. And then at some moment it will reach ∂b_1^+. By definition, it is the moment of transition from main block (0) to main block (1). The particles again can stay for certain amount of time in $(0) \cup (1)$ doing transitions between (0) and (1) until at some moment it reaches ∂b_2^+, and this moment, by definition, is the transition from (1) to (2), etc.

Therefore, for any $[\gamma] : x_0 \to x$, one can define the sequence of the main blocks with which $[\gamma]$ successively intersects. Let's introduce the graph $\tilde{\Gamma}$ with the vertices being the main blocks: (0), (1), ... and possible transitions $(n) \to (n \pm 1), n > 0$ and $(0) \to 1$. For each path $[\tilde{\gamma}]$ on $\tilde{\Gamma}$, one can consider the cluster of the elementary paths $[\gamma]$ following $[\tilde{\gamma}]$ between main blocks. If so, we say that $[\gamma] \in [\tilde{\gamma}]$. The class of all the paths which stays in (0) forever corresponds to the $[\tilde{\gamma}] = (0)$.

Put

$$R_\lambda^{[\tilde{\gamma}]}(x_0, x) = \sum_{[\gamma]\in[\tilde{\gamma}]} \prod_{z\in[\gamma]} \left(\frac{1}{\lambda - V(z)} \right).$$

Here the $[\gamma]$s under summation are from x_0 to x in the kth main block (k).

By definition, we have

$$R_\lambda(x_0, x) = \frac{\delta_{x_0}(x)}{\lambda - V(x_0)} + \sum_{[\tilde{\gamma}]:(0)\to(k)} R_\lambda^{[\tilde{\gamma}]}(x_0, x).$$

Let's provide a deeper analysis of any path $[\gamma] \in [\tilde{\gamma}]$. Assume that $[\gamma]$ contains, say, transitions $(0) \to (1) \to (2)$. Then at the moment τ_1^+ the path enters to some point z_2 on ∂b_1^+ for the first time (the first entrance to (1)). Let τ_1^- is the last moment before τ_1^+ when the path enters to the belt b_1 through ∂b_1^- and stays in b_1 until the moment τ_1^+. After moment τ_1^+, the trajectory $[\gamma]$ is moving inside (1) and at the moment τ_2^+ enters to ∂b_2^+ (the first entrance to (2)). τ_2^- is similarly defined as τ_1^- so that in the time interval $[\tau_2^-, \tau_2^+]$ it stays inside b_2, etc.

This classification leads to the simple combinatorial representation of $R_\lambda^{[\tilde{\gamma}]}(x_0, x)$. For instance, consider the special case when $x = x_0$. If $[\tilde{\gamma}] \equiv (0)$, $R_\lambda^{[\tilde{\gamma}]}(x_0, x_0) = R_\lambda^{(0)}(x_0, x_0)$.

For other paths $[\tilde{\gamma}]$, say, $[\tilde{\gamma}] = (0) \to (1) \to \cdots \to (2) \to (1) \to (0)$, their contribution to $R_\lambda(x_0, x_0)$ equals

$$R_\lambda^{[\tilde{\gamma}]}(x_0,x_0)=\sum R_\lambda^{(0)}(x_0,z_1)\beta_\lambda^{b_1}(z_1,z_2)R_\lambda^{(1)}(z_2,\cdot)\times\cdots$$

$$\times\, R_\lambda^{(2)}(\cdot,z_{2l-3})\beta_\lambda^{b_2}(z_{2l-3},z_{2l-2})R_\lambda^{(1)}(z_{2l-2},z_{2l-1})\beta_\lambda^{b_1}(z_{2l-1},z_{2l})R_\lambda^{(0)}(z_{2l},x_0) \qquad (\star\star)$$

where the summation is over $z_1\in\partial b_1^-,\ z_2\in\partial b_1^+,\dots,z_{2l-3}\in\partial b_2^+,\ z_{2l-2}\in\partial b_2^-,\ z_{2l-1}\in \partial b_1^+,\ z_{2l}\in\partial b_1^-$ with l being the number of times the path go through the belts until it returns back to x_0.

For each term $R_\lambda^{(k)}(z_i,z_j), k=1,2,\cdots$, there are four possible versions according to the different transitions through the kth belt, i.e.

$$R_\lambda^{(k)}(z_i,z_j), z_i\in\partial b_k^+, z_j\in\partial b_{k+1}^-;\ \ R_\lambda^{(k)}(z_i,z_j), z_i\in\partial b_k^+, z_j\in\partial b_k^+;$$

$$R_\lambda^{(k)}(z_i,z_j), z_i\in\partial b_{k+1}^-, z_j\in\partial b_{k+1}^-;\ \ R_\lambda^{(k)}(z_i,z_j), z_i\in\partial b_{k+1}^-, z_j\in\partial b_k^+.$$

For each term $R_\lambda^{(0)}(z_i,z_j)$, there are two possible versions:

$$R_\lambda^{(0)}(x_0,z_j), z_j\in\partial b_1^-;\ \ R_\lambda^{(0)}(z_j,x_0), z_j\in\partial b_1^-.$$

The similar formulas have place for $R^{[\tilde{\gamma}]}(x_0,x),\ x\in(k)$. Say, $[\tilde{\gamma}]=(0)\to(1)\to\cdots\to(k+1)\to(k)$,

$$R_\lambda^{[\tilde{\gamma}]}(x_0,x)=\sum R_\lambda^{(0)}(x_0,z_1)\beta_\lambda^{b_1}(z_1,z_2)R_\lambda^{(1)}(z_2,z_3)\beta_\lambda^{b_2}(z_3,z_4)R_\lambda^{(2)}(z_4,\cdot)\times\cdots$$

$$\times R_\lambda^{(k+1)}(\cdot,z_{2l-1})\beta_\lambda^{b_{k+1}}(z_{2l-1},z_{2l})R_\lambda^{(k)}(z_{2l},x)$$

where the summation is over all $z_1\in\partial b_1^-,\ z_2\in\partial b_1^+, z_3\in\partial b_2^-,\ z_4\in\partial b_2^+,\dots,\ z_{2l-1}\in \partial b_{k+1}^+,\ z_{2l}\in\partial b_{k+1}^-$ with l being the number of times the path go through the belts until it reaches to x.

For convenience of the reference, let's formulate the main result of this section as

Theorem 3 (The cluster expansion theorem)*.* Let x_0 be the reference point and $x \in (k)$, then

$$R_\lambda(x_0, x) = \frac{\delta_{x_0}(x)}{\lambda - V(x_0)} + \sum_{[\tilde{\gamma}]:(0)\to(k)} R_\lambda^{[\tilde{\gamma}]}(x_0, x). \tag{7}$$

Here

$$R_\lambda^{[\tilde{\gamma}]}(x_0, x) = \sum R_\lambda^{(0)}(x_0, z_1)\beta_\lambda^{b_1}(z_1, z_2)R_\lambda^{(1)}(z_2, z_3)\beta_\lambda^{b_2}(z_3, z_4)R_\lambda^{(2)}(z_4, \cdot) \times \cdots$$

$$\times R_\lambda^{(k+1)}(\cdot, z_{2l-1})\beta_\lambda^{b_{k+1}}(z_{2l-1}, z_{2l})R_\lambda^{(k)}(z_{2l}, x) \tag{8}$$

where the summations are described above.

CHAPTER 4: TECHNICAL LEMMAS BASED ON THE RESOLVENT KERNEL $R_\lambda(X_0, X)$ WHERE $IM(\lambda) = \epsilon > 0$

In this chapter we will formulate and prove several criteria for the absence of the a.c. spectrum, localization, etc. All results here will be based on the complex analysis. See real analytic approach in Chapter 5.

Due to general theory, $H = \int_{SpH} \lambda E(d\lambda)$. If $f \in l^2(\Gamma)$, then $\mu_f(d\lambda) = (E(d\lambda)f, f)$ is the spectral measure of the element f and $\int_{Sp(H)} \mu_f(d\lambda) = \int_{R^1} \mu_f(d\lambda) = ||f||_2^2$. The resolvent $R_\lambda = (H - \lambda I)^{-1}$ is the bounded operator for $\lambda \notin Sp(H)$, for instance, for $\lambda = \lambda_0 + i\epsilon, \lambda_0 \in R^1$, $(R_\lambda f, f) = \int_{Sp(H)} \frac{\mu_f(dz)}{z - \lambda}$. The function $f_0(x) \in l^2(\Gamma)$ has the *maximal spectral type* if $\mu_{f_0}(d\lambda) \gg \mu_g(d\lambda)$ for any $g \in l^2(\Gamma)$. The dense set of functions $f_0 \in l^2(\Gamma)$ have the maximal spectral type.

The measure $\mu_f(d\lambda)$ has the Lebesgue decomposition into a.c. component $\mu_{f,a.c.}$, singular continuous component $\mu_{f,s.c.}$, and finally point component $\mu_{f,p}$.

The a.c. components is responsible for the transport of the quantum particles (electric conductivity, scattering, etc). The point component is related to the localization phenomena.

Theorem 4. The limit

$$\pi \lim_{\epsilon \to 0} Im(R_{\lambda+\epsilon i} f,\ f)$$

exists for a.e. $\lambda \in R$ and equals to $\rho_f(\lambda)$. Here $\rho_f(\lambda)$ is the density of the a.c. part of the spectral measure $\mu_f(d\lambda)$. See details in [2].

Corollary 5. For given energy interval $I \subset R$, the a.c. part of the spectral measure $\mu_{f,a.c.}(d\lambda)$ is equal to 0 if

$$\lim_{\epsilon\to 0} Im(R_{\lambda+\epsilon i}f, f) = 0$$

a.e. on $\lambda \subset I$.

Assume that $\mu_{a.c.}(d\lambda) = 0$ (operator H has no a.c. spectrum). How to separate μ_p and $\mu_{s.c.}$? The following theorem (see [4]) gives the simple criteria for (p.p) spectrum.

Theorem 6 (Simon-Wolff). Assume that for real λ on the resolvent kernel, $R_{\lambda+\epsilon i}(x_0, y) \overset{\epsilon\to 0}{\to} R_{\lambda+0i}(x_0, y)$ and for any $x_0 \in \Gamma$

$$\lim_{\epsilon\to 0} \sum_{y\in\Gamma} |R_{\lambda+i\epsilon}(x_0, y)|^2 = \sum_{y\in\Gamma} |R_{\lambda+0i}(x_0, y)|^2 < \infty.$$

Then the operator H "typically" has p.p. spectrum.

Remark 1. It is not difficult to prove the existence of the limit above.

The last sentence "typically" means that, in the subspace $\ell_2(\delta_{x_0}) = Span(R_\lambda \delta_{x_0})$, the perturbed operator $H_a = H + a\delta_{x_0}(\cdot),\ x_0 \in \Gamma$ (rank-one perturbation) has p.p. spectrum for a.e. $a \in R$. At the same time(in the wide class of situations), the operator H_a has the pure singular spectrum for some a from the appropriate G_δ set (with measure 0), see [1].

This result is especially convenient for the random Schrödinger operators (Anderson Hamiltonians):

$$H(\omega) = \Delta + \sigma\xi(x, \omega),\ x \in \Gamma,\ \omega \in (\Omega, \mathcal{F}, P),$$

where σ is a coupling constant (the measure of disorder) and the random variables $\xi(x, \omega)$ are i.i.d. with bounded continuous distribution density.

As we already mentioned, for large σ (large disorder) the operator $H(\omega)$ has p.p. spectrum P-a.s., see [6].

Let's formulate and prove the result, closely related to Simon-Spencer approach to the theorems on the absence of the a.c. spectrum.

Theorem 7. Let Γ be a graph with estimate $\nu(x) \leq K$, Δ is the Laplacian, $H = \Delta + V(x)$ is the Schorödinger operator with potential $V(\cdot)$. Assume that for some sequence of points $\mathcal{D} = \{x_n : n = 1, 2, \dots\}$

$$\sum_{n=1}^{\infty} \frac{1}{|V(x_n)|} < \infty.$$

Consider the new operator $\tilde{H} = \Delta + V(x)$ with boundary condition $\phi(x_n) = 0,\ n = 1, 2, \dots$, i.e. the restriction of H on $\Gamma - \mathcal{D}$ with Dirichlet boundary condition on $\mathcal{D}$. Then

$$Sp_{a.c.}(H) = Sp_{a.c.}(\tilde{H})$$

and for any $f \in L^2(\Gamma)$, the a.c. components of the spectral measures of f for H and $\tilde{H}$ are mutually a.c.. In particular, $Sp_{a.c.}(\tilde{H}) = \emptyset \Longleftrightarrow Sp_{a.c.}(H) = \emptyset$.

Proof. The proof is based on Kato-Birman criterion [11]. See also [2, 3]. If for some fixed $\lambda_0 = Ci$, the difference between the resolvents $R_{\lambda_0} = (H - \lambda_0 I)^{-1}$ and $\tilde{R}_{\lambda_0} = (\tilde{H} - \lambda_0 I)^{-1}$ belongs to the trace class, i.e.

$$\sum_{x\in\Gamma} \left| R_{\lambda_0}(x,x) - \tilde{R}_{\lambda_0}(x,x) \right| < \infty,$$

then $\mu_{a.c.}(H) = \mu_{a.c.}(\tilde{H})$

Now we will use the path expansion to calculate $R_{\lambda_0}(x,x)$ and $\tilde{R}_{\lambda_0}(x,x)$, $x \in \Gamma$ for

$\lambda_0 = K(K+1)i$.

We have

$$R_{\lambda_0}(x,x) = \frac{1}{\lambda_0 - V(x)} + \sum_{[\gamma]:x\to x} \prod_{z\in[\gamma]} \left(\frac{1}{\lambda_0 - V(z)}\right)$$

for $x \in \Gamma$, and

$$\tilde{R}_{\lambda_0}(x,x) = \frac{1}{\lambda_0 - V(x)} + \sum_{\substack{[\gamma]:x\to x\\ [\gamma]\cap\mathcal{D}=\emptyset}} \prod_{z\in[\gamma]} \left(\frac{1}{\lambda_0 - V(z)}\right),\ \ x \in \Gamma - \mathcal{D}$$

and $\tilde{R}_{\lambda_0}(x,x) = 0,\ \ x \in \mathcal{D}$.

Note that $\left|\frac{1}{\lambda_0 - V(z)}\right| \le \frac{1}{K(K+1)}$. If $x_n \in \mathcal{D}$, then

$$\begin{aligned}
&\left|R_{\lambda_0}(x,x) - \tilde{R}_{\lambda_0}(x,x)\right| = |R_{\lambda_0}(x,x)| \\
&\le \left|\frac{1}{\lambda_0 - V(x_n)}\right| \left(1 + \frac{K}{K(K+1)} + \frac{K^2}{(K(K+1))^2} + \dots\right) \\
&\le 2\left|\frac{1}{\lambda_0 - V(x_n)}\right| \le \frac{2}{|V(x_n)|}.
\end{aligned}$$

Therefore,

$$\sum_{x\in\mathcal{D}} \left|R_{\lambda_0}(x,x) - \tilde{R}_{\lambda_0}(x,x)\right| \le 2\sum_{n=1}^{\infty} \frac{1}{|V(x_n)|} < \infty. \tag{9}$$

But if $x \notin \mathcal{D}$, we have

$$\left|R_{\lambda_0}(x,x) - \tilde{R}_{\lambda_0}(x,x)\right| = \left|\sum_{\substack{[\gamma]:x\to x\\ [\gamma]\cap\mathcal{D}\neq\emptyset}} \prod_{z\in[\gamma]} \left(\frac{1}{\lambda_0 - V(z)}\right)\right|.$$

Since $[\gamma]\cap\mathcal{D} \neq \emptyset$, the path $[\gamma]$ must intersect $\mathcal{D}$ for the first time at one of the points $x_1, x_2, \dots$. Let $A_j = \{[\gamma] : x \to x$ and γ and $\mathcal{D}$ intersect for the first time at $x_j\}$, $j = 1, 2, \dots$.

Then

$$\begin{aligned}
&\left|\sum_{\gamma\in A_j}\prod_{z\in[\gamma]}\left(\frac{1}{\lambda_0-V(z)}\right)\right| \\
&\leq \sum_{\gamma\in A_j}\frac{1}{(K(K+1))^{|\gamma|-1}}\frac{1}{|V(x_j)|} \\
&\leq \frac{1}{|V(x_j)|}\left(\frac{K^{d(x,x_j)}}{(K(K+1))^{d(x,x_j)}}+\frac{K^{d(x,x_j)+1}}{(K(K+1))^{d(x,x_j)+1}}+\dots\right) \\
&\leq \frac{1}{|V(x_j)(K+1)^{d(x,x_j)}}\left(1+\frac{1}{K}\right).
\end{aligned}$$

Therefore, for $x\notin\mathcal{D}$,

$$\left|R_{\lambda_0}(x,x)-\tilde{R}_{\lambda_0}(x,x)\right| \leq \left(1+\frac{1}{K}\right)\sum_{j=1}^{\infty}\frac{1}{|V(x_j)(K+1)^{d(x,x_j)}}$$

$$\sum_{x\in\Gamma-\mathcal{D}}\left|R_{\lambda_0}(x,x)-\tilde{R}_{\lambda_0}(x,x)\right| \leq \left(1+\frac{1}{K}\right)\sum_{j=1}^{\infty}\frac{1}{|V(x_j)|}\sum_{x\in\Gamma-\mathcal{D}}\frac{1}{(K+1)^{d(x,x_j)}}$$

But

$$\sum_{x\in\Gamma-\mathcal{D}}\frac{1}{(K+1)^{d(x,x_j)}} \leq 1+\frac{K}{K+1}+\frac{K^2}{(K+1)^2}+\cdots=K+1<\infty.$$

.

Hence, we have

$$\sum_{x\in\Gamma-\mathcal{D}}\left|R_{\lambda_0}(x,x)-\tilde{R}_{\lambda_0}(x,x)\right| \leq \frac{(K+1)^2}{K}\sum_{j=1}^{\infty}\frac{1}{|V(x_j)|}<\infty. \tag{10}$$

Due to equation (9) and (10), we have

$$\sum_{x\in\Gamma}\left|R_{\lambda_0}(x,x)-\tilde{R}_{\lambda_0}(x,x)\right|<\infty.$$

.

Since the difference belongs to the trace class, operators H and $\tilde{H}$ have the same a.c. spectrum. □

Corollary 8. Assume that for the Schrödinger operator $H = \Delta + V(x)$ one can find the set $\mathcal{D} = \{x_n; n = 1, 2, \dots\}$ such that $\Gamma - \mathcal{D}$ is the union of the disjoint finite sets (in our terminology, $\mathcal{D}$ is the union of the belts). Then under the assumption of the above theorem ($\sum_n \frac{1}{|V(x_n)|} < \infty$), H has no *a.c.* spectrum.

CHAPTER 5: THE REAL ANALYTIC APPROACH TO THE ABSENCE OF A.C. SPECTRUM AND LOCALIZATION

In all future constructions, the belts $\{b_l,\ l = 1, 2, \dots\}$ will be selected from the assumption that for the fixed energy interval I on λ-axis:

$$\max_{z_1 \in \partial b_l^-, z \in \partial b_1^+} \left| \beta_\lambda^{b_l}(z_1, z_2) \right| \leq \delta_l,\ \delta_l \to 0,\ l \to \infty \tag{11}$$

for all $\lambda \in I$.

There are different options to guarantee the small values of $\beta_\lambda^{b_l}(\cdot, \cdot)$. For instance, assume that $|V(x)| \geq h_l,\ x \in b_l$ and h_l is sufficiently large, then, for fixed energy interval I,

$$\left| \frac{1}{\lambda - V(z)} \right| \leq \frac{2}{h_l},\ \lambda \in I, z \in b_l,$$

i.e.

$$\beta_\lambda^{b_l}(\cdot, \cdot) \leq C \left(\frac{2K}{h_l} \right)^{t_l} = \delta_l,$$

with $t_l = d(\partial b_l^-, \partial b_l^+)$, for some constant C.

The each product in the cluster expansion contains factors $\beta_\lambda^{b_l}(\cdot, \cdot)$ which will be small and the resolvents of the main blocks $R_\lambda^{(l)}(\cdot, \cdot)$, where we have no information about the potentials. We will use here the Kolmogorov's lemma which states that $R_\lambda^{(l)}(\cdot, \cdot)$ for typical λ is not "too large" (see the proof of this result in [10] in the convenient form).

Proposition 9 (Kolmogorov Lemma). Let $M > 0$, then

$$m\left(\lambda : |F(z)| = \left|\sum_{l=1}^{N} \frac{\alpha_l}{\lambda - \lambda_l}\right| \geq M\right) \leq \frac{4\sum_l |\alpha_l|}{M}. \tag{12}$$

Here λ_l are different real numbers, and α_l are also real.

If $\alpha_l > 0,\ l = 1, 2, \ldots, N$, then elementary algebraic calculations give

$$m(\lambda : |F(z)| \geq M) = \frac{2\sum_l \alpha_l}{M}.$$

If $\alpha_l,\ l = 1, 2, \ldots, N$ have positive terms α_l^+ and negative terms α_l^-, then

$$m(\lambda : |F(z)| \geq M) \leq m\left(\lambda : \left|\sum_i \frac{\alpha_i^+}{\lambda - \lambda_i}\right| \geq \frac{M}{2}\right) + m\left(\lambda : \left|\sum_i \frac{\alpha_j^-}{\lambda - \lambda_j}\right| \geq \frac{M}{2}\right) \leq \frac{4\sum_l |\alpha_l|}{M}.$$

The following proposition is the central point of our approach.

Proposition 10. Let $H_B = \Delta + V(x)$ with Dirichlet boundary condition: $\phi(x) = 0,\ x \in \Gamma - B$, where $B \subset \Gamma$. Consider $\left(R_\lambda^{(B)} f, g\right),\ f, g \in l^2(B)$, then in the obvious notations,

$$\left(R_\lambda^{(B)} f, g\right) = \sum_{k=1}^{|B|} \frac{(f, \psi_k)(g, \psi_k)}{\lambda - \lambda_k^B} = \tag{13}$$

$$||f||_2 ||g||_2 \sum_k \frac{[(f, \psi_k)/||f||_2]\,[(g, \psi_k)/||g||_2]}{\lambda - \lambda_k^B}. \tag{14}$$

where $\lambda_k^B, k = 1, 2, \ldots, |B|$ are the eigenvalues of H_B and ψ_k are the corresponding orthonormal eigenfunctions. Due to Cauchy-Shwartz inequality and Kolmogorov's lemma, we have

$$m\left(\lambda \in R : \left|\left(R_\lambda^{(B)} f, g\right)\right| > M\right) \leq \frac{4||f||_2 ||g||_2}{M}. \tag{15}$$

Corollary 11. In particular, let $f = g = I_{\partial b_l^+}$ be the indicator function on ∂b_l^+ and $B = (l)$, we have

$$m\left(\lambda \in R : \left|\sum_{z_1, z_2 \in \partial b_l^+} R_\lambda^{(l)}(z_1, z_2)\right| = \left|\left(R_\lambda^{(l)} I_{\partial b_l^+}, I_{\partial b_l^+}\right)\right| > M_l^+\right) \leq \frac{4||I_{\partial b_l^+}||^2}{M_l^+} = \frac{4|\partial b_l^+|}{M_l^+}.$$

Similarly,

$$m\left(\lambda \in R : \left|\sum_{z_1 \in \partial b_l^+, z_2 \in \partial b_{l+1}^-} R_\lambda^{(l)}(z_1, z_2)\right| > \sqrt{M_l^+ M_{l+1}^-}\right) \leq \frac{4\sqrt{|\partial b_l^+||\partial b_{l+1}^-|}}{\sqrt{M_l^+ M_{l+1}^-}}; \quad (16)$$

$$m\left(\lambda \in R : \left|\sum_{z_1, z_2 \in \partial b_{l+1}^-} R_\lambda^{(l)}(z_1, z_2)\right| > M_{l+1}^-\right) \leq \frac{4|\partial b_{l+1}^-|}{M_{l+1}^-}; \quad (17)$$

$$m\left(\lambda \in R : \left|\sum_{z_1 \in \partial b_{l+1}^-, z_2 \in \partial b_l^+} R_\lambda^{(l)}(z_1, z_2)\right| > \sqrt{M_l^+ M_{l+1}^-}\right) \leq \frac{4\sqrt{|\partial b_l^+||\partial b_{l+1}^-|}}{\sqrt{M_l^+ M_{l+1}^-}}. \quad (18)$$

where $M_l > 0$ is a constant.

Let's return now to the theorems 1, 2 giving the criteria for localization (p.p. spectrum) or absence of a.c. spectrum. Both of them contains $R_{\lambda+i\epsilon}(\cdot, \cdot)$. Unfortunately, Kolmogorov's lemmma is not applicable for the complex-valued resolvent kernal $R_{\lambda+i\epsilon}(x, y)$, $\epsilon > 0$. We need something else.

In the case of localization, the real analytic result is simple, see [2] or [10].

Theorem 12. Assume that $Q_n \uparrow \Gamma$ is the increasing family of the connected sets and $R_{n,\lambda}(x, y)$, $n = 1, 2, \ldots$ are the resolvents of the operators H_n: the restrictions of H on Q_n with Dirichlet boundary condition ($\psi \equiv 0, x \notin Q_n$). Assume also that for any $x_0 \in \Gamma$ and for a.e. $\lambda \in \mathbb{R}^1$,

$$\limsup_{n \to \infty} \sum_{y \in Q_n} (R_{n,\lambda}(x_0, y))^2 \leq c(\lambda) < \infty.$$

Then

$$\lim_{\epsilon\to 0}\sum_{y\in\Gamma}|R_{\lambda+i\epsilon}(x_0,y)|^2=\sum_{y\in\Gamma}(R_{\lambda+i0}(x_0,y))^2\le c(\lambda)<\infty,$$

i.e. one can apply the theorem 2 on the localization (with appropriate randomization).

The desirable result on the absence of a.c. spectrum (in real analytic terms) gives the following

Theorem 13 (**A.Gordon**). If the operator $H=\Delta+V$ in $L^2(\Gamma)$ has a.c. spectrum (say in the subspace generated by $\delta_{x_0}(\cdot)$), then

$$\limsup_{n\to\infty}|R_{n,\lambda}(x_0,x_0)|=\infty$$

on the essential support of the a.c. component of the spectral measure of the element $\delta_{x_0}(\cdot)$.

This result will be published in the parallel article.

Corollary 14. If, for any $x_0\in\Gamma$,

$$\limsup_{n\to\infty}|R_{n,\lambda}(x_0,x_0)|<\infty$$

a.e. on some interval I, then the spectral measure of the operator H is singular on I (i.e. its a.c. component equal to 0 in each subspace, generated by $\delta_{x_0}(\cdot)$).

CHAPTER 6: THEOREMS ON THE ABSENCE OF A.C. SPECTRUM AND LOCALIZATION

In this chapter, we'll prove the major results of the paper. They will be illustrated by the examples in sections **??** and **??**.

Theorem (I)**.** *Consider the graph Γ introduced in section 1 and the Hamiltonian $H = \Delta + V(x)$. Assume that one can find the system of the counters $b_n,\ n = 1, 2, \dots$ (the belts of thickness 1) such that for*

$$h_n = \max_{x \in b_n} |V(x)|, \ \text{and } |b_n|/h_n \to 0, \ n \to \infty.$$

Then $\mu_{a.c.}(H, f) \equiv 0$.

This result is the trivial corollary of the Theorem 4. In fact if $h_n^{-1}|b_n| \to 0$, then one can find the sequence $\{n'\}$: $\sum_{n'} h_{n'}^{-1}|b_{n'}| < \infty$ but $\sum_{z \in \cup b_{n'}} \frac{1}{|V(x)|} \le \sum_{n'} \frac{|b_{n'}|}{h_{n'}} < \infty$.

The following example illustrates the above theorem.

Example 15. Let $\Gamma = S^\infty$ be the Sierpiński lattice and $H = \Delta + V(x, \omega),\ x \in S^\infty$, where $V(x, \omega)$ is a system of unbounded i.i.d. random variables. Then P-a.s. $\mu_{a.c.}(H) = 0$.

Proof. Consider the following events $A_n = \{|V(2^n \vec{i}, \omega)| > h_n,\ |V(2^n \vec{w}, \omega)| > h_n\}$, where $\vec{i} = (1, 0),\ \vec{w} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Since events A_n are independent and $P(A_n) = P^2(|V| > h_n) > 0$ for appropriate $h_n \to \infty$, $\sum_n P(A_n) = \infty$. The second Borel-Cantelli lemma provides the existence P-a.s. of the infinitely many events $A_{n'}$. One can apply now the above Theorem (I) to the counters containing only two points: $b_n = \{2^n\vec{i}, 2^n\vec{w}\}$. □

In fact, we didn't use here the specific structure of S^∞ and proved the following result.

Corollary 16. If for the general graph Γ with conditions $\nu(x) \le K$, one can find infinite system of the belts b_n : $|b_n| \le C_0 < \infty$ for appropriate C_0 and the i.i.d. potential $V(x,\omega)$ is unbounded($P(|V| > A) = P(A) > 0$ for any A). Then the Anderson Hamiltonian $H = \Delta + V(x,\omega)$ has no a.c. spectrum, P-a.s..

We'll return to the spectral theory of the self-similar fractal graphs(like Sierpiński lattice or snow flacks) in another paper to cover cases with the bounded random potentials, where again $\mu_{a.c.}(H) = 0$.

The following result is more general than Theorem (I).

Theorem (II)**.** *The condition*

$$\delta_l \sqrt{|\partial b_l^+||\partial b_l^-|} \to 0,\ l \to \infty \tag{19}$$

implies that $Sp_{a.c.}(H) = \emptyset$. *Here* $\delta_l = \max_{z_1 \in \partial b_l^-, z_2 \in \partial b_1^+} \left|\beta_\lambda^{b_l}(z_1, z_2)\right|$.

The proof of this result is the significant part of the future localization theorem.

Proof. Selecting appropriate subsequence of the belts $\{b_{l'}\}$, one can assume without loss of generality that

$$\sum_{l'} \delta_{l'} \sqrt{|\partial b_{l'}^+||\partial b_{l'}^-|} < \infty.$$

Let's now apply the Borel-Cantelli lemma, which will show that the resolvent kernals $R_\lambda^{(l)}(\cdot,\cdot)$ are not "too large".

Put $M_l^- = \dfrac{|\partial b_l^-|}{\alpha_l}$, $M_l^+ = \dfrac{|\partial b_l^-|}{\alpha_l}$, $l = 1, 2, \ldots$, where $\alpha_l > 0$ is any sequence such that $\sum_l \alpha_l < \infty$. Then formulas in Corollary 11 will have a form

$$m\left(\lambda \in R : |\cdot| > \cdot\right) \leq \alpha_l.$$

And Borel-Cantelli lemma gives that for a.e. $\lambda \in R$ and $l \geq L_0(\lambda)$,

$$\left| \sum_{z_1, z_2 \in \partial b_l^+} R_\lambda^{(l)}(z_1, z_2) \right| \leq \frac{|\partial b_l^+|}{\alpha_l}.$$

Similarly,

$$\left| \sum_{z_1, z_2 \in \partial b_{l+1}^-} R_\lambda^{(l)}(z_1, z_2) \right| \leq \frac{|\partial b|_{l+1} -}{\alpha_{l+1}}; \tag{20}$$

$$\left| \sum_{z_1 \in \partial b_l^+, z_2 \in \partial b_{l+1}^-} R_\lambda^{(l)}(z_1, z_2) \right| \leq \frac{\sqrt{|\partial b_l^+||\partial b_{l+1}^-|}}{\sqrt{\alpha_l \alpha_{l+1}}}; \tag{21}$$

$$\left| \sum_{z_1 \in \partial b_{l+1}^-, z_2 \in \partial b_l^+} R_\lambda^{(l)}(z_1, z_2) \right| \leq \frac{\sqrt{|\partial b_l^+||\partial b_{l+1}^-|}}{\sqrt{\alpha_l \alpha_{l+1}}}. \tag{22}$$

In the case $\sum_{z_1 \in \partial b_1^-} R_\lambda^{(0)}(x_0, z_1)$ one can put

$$\left| \sum_{z_1 \in \partial b_1^-} R_\lambda^{(0)}(x_0, z_1) \right| \leq \frac{\sqrt{|\partial b_1^-|}\sqrt{1}}{\sqrt{\alpha_1}}.$$

Firstly, assume $L_0(\lambda) = 0$ and choose $\alpha_l = 3\delta_l\sqrt{|\partial b_l^-||\partial b_l^+}$. Then, due to Theorem 3, using the fact that the number of of paths $[\tilde{\gamma}]$ of length n is no more than 2^n,

$$\left|R_\lambda^{[\tilde\gamma]}(x_0,x_0)\right| \le \left|\sum_{z_1} R_\lambda^{(0)}(x_0,z_1)\right| \beta_\lambda^{b_1}(z_1,z_2) \left|\sum_{z_2,z_3} R_\lambda^{(1)}(z_2,z_3)\right| \times \cdots$$
$$\times \left|\sum_{z_{2l-4},z_{2l-3}} R_\lambda^{(2)}(z_{2l-4},z_{2l-3})\right| \beta_\lambda^{b_2}(z_{2l-3},z_{2l-2}) \left|\sum_{z_{2l-2},z_{2l-1}} R_\lambda^{(1)}(z_{2l-2},z_{2l-1})\right| \beta_\lambda^{b_1}(z_{2l-1},z_{2l}) \left|\sum_{z_{2l}} R_\lambda^{(0)}(z\right.$$
$$\le \left(\frac{\sqrt{|\partial b_1^-|}}{\sqrt{\alpha_1}}\delta_1 \frac{\sqrt{|\partial b_1^+|}}{\sqrt{\alpha_1}}\right)\left(\frac{\sqrt{|\partial b_2^-|}}{\sqrt{\alpha_2}}\delta_2 \frac{\sqrt{|\partial b_2^+|}}{\sqrt{\alpha_2}}\right)\cdots$$

where the summations are over $z_1 \in \partial b_1^-,\ z_2 \in \partial b_1^+, \ldots, z_{2l-3} \in \partial b_2^+,\ z_{2l-2} \in \partial b_2^-,\ z_{2l-1} \in \partial b_1^+,\ z_{2l} \in \partial b_1^-$ with l being the number of times the path go through the belts until it returns back to x_0.

Therefore

$$|R_\lambda(x_0,x_0)| \le \sum_{n=0}^{\infty} \frac{2^n}{3^n} = 3.$$

Let $S_k = \{\lambda \in R : L_0(\lambda) = k\},\ k \ge 0$. We proved that one the set S_0:

$$R_\lambda(x_0,x_0) = \lim_{N\to\infty} R_\lambda^{(N)}(x_0,x_0)$$

and A. Gordon's Theorem 5.5 states that the a.c. component of the spectral measure $\rho_f(d\lambda),\ f = \delta_{x_0}(x)$ equals to 0 on S_0.

Assumen that $L_0(\lambda) > 0$. If $L_0(\lambda) = k \ge 1$, we introduce the new operator

$$\tilde H_N = \Delta + \tilde V$$

where

$$\tilde V = \begin{cases} V(x) & \text{if } x \in (m) : m \ge k \\ A_k & \text{if } x \in (m) : m < k \end{cases}$$

We select constants A_k so large that on $b_m : m < k$ such that $\dfrac{\sqrt{|\partial b_m^-|}\sqrt{|\partial b_m^+|}\delta_m}{\alpha_m} < \dfrac{1}{3}$.

Then for the resolvent kernel $\tilde{R}_{\lambda,k}(x,x_0) = \left(\left(\tilde{H}_k - \lambda I\right)^{-1}\delta_{x_0}(x), \delta_{x_0}\right)$, we can repeat for any $k \geq 1$ the previous consideration, which gives that on the set $S_k\{\lambda \in R : L_0(\lambda) = k\}$, there is no a.c. spectrum of $\tilde{H}_k$ for $f = \delta_{x_0}(x)$. Due to Kato-Birman theorem [11], the same is true for H (transition from H to $\tilde{H}_k$ is the finite rank perturbation of H). It proves the theorem. □

The last theorem in this section gives the sufficient conditions for the localization.

Theorem (III)**.** *Assume that there exists the sequence of the belts* $\{b_l,\ l \geq 1\}$ *with boundaries* $\partial b_l^{\pm}$ *such that for given energy interval* I*, the quantities* $\{\delta_l,\ l \geq 1\}$ *introduced above satisfy the central condition of the Theorem (II):*

$$\delta_l \sqrt{|\partial b_l^+||\partial b_l^-|} \to 0,\ \ l \to \infty$$

and the following series converges for a.e. $\lambda \in I$*:*

$$\sum_{n=1}^{\infty} \left(\prod_{l=1}^{n} \delta_l^2 |\partial b_l^+||\partial b_l^-| \right) \sqrt{|(n)|} (\sqrt{|\partial b_n^+|} + \sqrt{|\partial b_{n+1}^-|}) n^{1+\delta}$$

for some $\delta > 0$ *where* $|(n)|$ *is the cardinality of the nth main block. Then for any* $x_0 \in \Gamma$

$$|R_\lambda(x_0, \cdot)| \in L^2(\Gamma),\ \ \textit{for a.e. } \lambda \in I.$$

Corollary 17. If one can construct such belts for the random Schrödinger operator (and arbitrary $x_0 \in \Gamma$):

$$H = \Delta + V(x,\omega),\ \omega \in (\Omega, \mathcal{F}, P)$$

with i.i.d. random variable $V(x,\cdot)$, $x \in \Gamma$ which have bounded distribution density

$f(\nu)$, then $P-a.s.$ the spectrum of $H=H(\omega)$ is pure point.

We will prove this theorem using the following strategy. Let's fix point $x_0 \in \Gamma$ and consider the resolvent kernel $R_\lambda(x_0,x), \lambda \in I$. Using the convergence of the series $\sum_{l=1}^{\infty} \delta_l \sqrt{|\partial b_l^+||\partial b_l^-|}$ and the technical Lemma 18 (similar to Kolmogorov's lemma) we'll prove that

$$\| R_\lambda(x_0,\cdot) \|^2 = \sum_{x\in\Omega} R_\lambda^2(x_0,x) < \infty$$

P-a.s. and a.e. for $\lambda \in I$.

In this proof we will not use the existence of the distribution density $p(x)$ for the random variables $V(x,\omega)$, but only the assumptions on δ_l, and $|(l)|$.

In fact we'll prove that $\| R_\lambda(x_0,\cdot) \|^2 < \infty$ for fixed value of the potential at the reference point $x_0 \in \Gamma$. Using rank-one perturbation (see details in [2] or [10]), one can present $R_\lambda(x_0,x)$ as the rational function of $V(x_0)$ and $\tilde{R}_\lambda(x_0,x)$, where $\tilde{R}_\lambda(x_0,x)$ is the corresponding resolvent kernel of the operator

$$\tilde{H} = \Delta + \tilde{V}(x)$$

where $\tilde{V}(x) = V(x)$ for $x \neq x_0$ and $\tilde{V}(x_0) = 0$. Since $V(x_0)$ has a.c. distribution (density), the Simon-Wolf criterion [4] ($H = \tilde{H} + V(x_0,\omega)\delta_{x_0}$) gives now that spectrum of H is pure point P-a.s. on the interval I in the cyclic subspace $l^2(\Gamma,\delta_{x_0}) \subset l^2(\Gamma)$ generated by the function $f(x) = \delta_{x_0}(x)$.

But the point x_0 is arbitrary and $l^2(\Gamma)$ is generated by the δ-functions $\delta_{x_0},\ x_0 \in \Gamma$. It completes the proof.

Lemma 18. Let $h_2^{(n)}(\lambda) = \sum_{a\in\partial b_n^+} \sum_{x\in(n)} R_\lambda^2(a,x)$ for $n \geq 1$. then

$$m\left\{\lambda : h_2^{(n)}(\lambda) > M\right\} \leq \frac{4\sqrt{|(n)||\partial b_n^+|}}{M}. \tag{23}$$

This Lemma is a generalization of the similar result in [10], where ∂b_n^+ contains a single point. In fact, for $\lambda \notin \{\lambda_{n,k}\}$

$$\begin{aligned} h_2^{(n)}(\lambda) &= \sum_{a\in\partial b_n^+} \sum_{x\in(n)} \left(\sum_{k=1}^{|(n)|} \frac{\psi_{n,k}(a)\psi_{n,k}(x)}{\lambda - \lambda_{n,k}}\right)^2 \\ &= \sum_{a\in\partial b_n^+} \sum_{k=1}^{|(n)|} \frac{\psi_{n,k}^2(a)}{(\lambda-\lambda_{n,k})^2} = \sum_{k=1}^{|(n)|} \frac{\zeta_{n,k}}{(\lambda-\lambda_{n,k})^2}. \end{aligned}$$

Here $\zeta_{n,k} = \sum_{a\in\partial b_n^+} \psi_{n,k}^2(a) = \|I_{\partial b_n^+}\|^2$ and $\{\psi_{n,k}\}$, $\{\lambda_{n,k}\}$, $k = 1, \ldots, |(n)|$ are the eigenfuntions and eigenvalues of the restriction of H onto the main block (n).

Now one can repeat the corresponding arguments in [10]. The similar arguments work for summations over $a \in \partial b_{n+1}^-$.

Corollary 19. Using the Borel-Cantelli lemma, one can prove that, for any $\delta > 0$, P-a.s.

$$\sum_{a\in\partial b_n^+\cup\partial b_n^-} \left(\sum_{x\in(n)} R_\lambda^2(a,x)\right) \leq 4\sqrt{|(n)|}\left(\sqrt{|\partial b_n^+|} + \sqrt{|\partial b_n^-|}\right) n^{1+\delta} \tag{24}$$

for $n \geq n_0(\omega)$.

Now we can repeat the same calculation as in [10] (Theorem 2.4) or in Theorem (II) above: we assume first that we have desirable estimations for $R_\lambda^{(l)}(a,b)$, $a \in \partial b_l^+, b \in \partial b_l^+$ or $a \in \partial b_l^+, b \in \partial b_{l+1}^-$, etc. and for $\| R_\lambda^{(l)}(a,\cdot) \|_2^2$ (which are true for all $l \geq l_0(\omega)$ P-a.s. and a.e. in $\lambda \in I$ due to Borel-Cantelli lemma). I.e. we assume that estimates (21) – (23) and (24) have place for $n \geq 1$. Consider some path $[\tilde{\gamma}]$ on the set of main blocks $(l), l = 1, 2, \ldots$ which end at (n).

We'll estimate first the contribution of $[\tilde{\gamma}]$ into $\| R_\lambda^{(n)}(a, \cdot) \|_2^2$. The main contribution is given by the shortest path $(0) \to (1) \to \cdots \to (n)$ of length n and it can be estimated as

$$|R_\lambda(x_0, x)|^2 \leq \prod_{l=1}^{n} \left(\delta_l \sqrt{|\partial b_l^+||\partial b_l^-|} \right)^2 \sum_{a \in \partial b_n^+} \left[R_\lambda^{(n)}(a, x) \right]^2 .$$

Since $\delta_l \sqrt{|\partial b_l^+||\partial b_l^-|} \leq \frac{1}{3}$, the contributions of the paths of length $n+1$, $n+2$, $\ldots$, $n+k, \ldots$ are exponentially decaying in k. Summations over all paths $[\tilde{\gamma}]$ of the length $n, n+1, \ldots$ and after over n complete the proof for $l_0(\omega) = 1$.

If $l_0(\omega) \geq 1$, we can consider new Hamiltonian $\tilde{H}$ which has "very large potentials" inside the first l_0 main blocks and our initial potential $V(x, \omega)$ inside blocks (n) with $n > l_0$.

The operator $\tilde{H}$ is the finite rank perturbation of H, which preserves the square integrability of the resolvent, see [2]. But for $\tilde{H}$ we have desirable estimations for all $l \geq 1$.

The idea of transition using finite rank perturbation from the general l_0 to $l_0 = 1$ is the same as in the proof of the absence of the a.c. spectrum (Theorem (I), (II)).

CHAPTER 7: EXAMPLES

We will illustrate the general Theorem (III) by several examples. In all these examples, the belts will be "relatively short". The belt factors in these examples will be compensated by large values of the potential on the belts.

Example 20 (Localization on Sierpiński lattice S^∞). Let's start from the fractal (nested) lattice and consider as a typical example the Sierpiński Lattice S^∞. Let S^n be the part of S^∞ with vertices $\vec{0}$, $2^n\vec{i}$, and $2^n\vec{w}$. The volume of S^n is given as $|S^n| = \dfrac{3^{n+1}+3}{2}$. See fig 1.

Consider the Anderson Hamiltonian $H = \Delta + V(\vec{x}, \omega)$, where $\vec{x} \in S^\infty$ and $V(\cdot, \omega)$ are unbounded i.i.d. random variables with bounded density function $f(x)$ on R. Consider for fixed A the sequence of independent events $B_{A,n} = \{|V(2^n\vec{i}, \omega)| > A,\, |V(2^n\vec{w},\, \omega)| > A\}$, where $\vec{i} = (1,0)$ and $\vec{w} = (1/2,\, \sqrt{3}/2)$. Then $P(B_{A,n}) = p^2(A) = \left(\int_{|x|>A} f(x)dx\right)^2$. Let τ_A be the moment of the first occurrence of $B_{A,n}$ in the sequence $B_{A,0}$, $B_{A,1}$, ... Then τ_A has geometric distribution: $P(\tau_A = k) = (1 - p^2(A))^{k-1}p^2(A)$ for $k = 1, 2, \ldots$, with $E\tau_A = \dfrac{1}{p^2(A)}$. It is easy to see that $p^2(A)\tau_A \overset{law}{\to} Exp(1)$ as $A \to \infty$.

Consider the increasing sequence $\{A_n = n\} : n \to \infty$ and the moments τ_n. Since $\sum_n P\left(p^2(n)\tau_n > (1+\epsilon)\ln n\right) \leq \sum_n \dfrac{c_0}{n^{1+\epsilon}} < \infty$ for some constant c_0 and any $\epsilon > 0$, we have (due to Borel- Cantelli lemma) P-a.s. $\tau_n \leq \dfrac{(1+\epsilon)\ln n}{p^2(n)}$, for $n \geq n_0(\omega)$.

The successive belts $b_n, n \geq 1$ contains the pairs of the points $\{2^{\tau_{A_n}}\vec{i},\, 2^{\tau_{A_n}}\vec{w}\}$. Of course, we have $|(n)| \leq c_1 3^{\tau_n} \leq c_1 \exp\left([(1+\epsilon)\ln 3 \dfrac{\ln n}{p^2(A)}\right) < \infty$ and for fixed energy interval I, $\beta_\lambda^{b_k} \leq \dfrac{c_2(I)}{n}$, where c_1, c_2 are some constants.

Assume that $P\{|V(\cdot)| > A\} = p(A) \geq \frac{c_3}{A^\theta}$ for any $A > 0$, where c_3 is a constant. Then Theorem (III) provides the P-a.s. localization certenly if

$$\sum_n \left(\prod_{k=1}^n \beta_\lambda^{b_k}\right)^2 \sqrt{|(n)|} \leq \sum_n \exp\left(c_4 n - 2n\ln n + c_5 n^{2\theta} \ln n\right)$$

(since $\sqrt{|(n)|}\left(\sqrt{|\partial b_n^+|} + \sqrt{|\partial b_{n+1}^-|}\right) \leq 4|(n)|$). The last series converges if $\theta < \frac{1}{2}$.

Theorem 21. Condition $P\{|V(\cdot)| > A\} = p(A) \geq \dfrac{c}{A^\theta}$, $\theta < \dfrac{1}{2}$ is sufficient for P-a.s. localization on S^∞.

The same proof works for all nested fractal lattices.

Let's stress that we didn't use here the fundamental properties of the self-similarity of S^∞. The spectral analysis of the Laplacian on S^∞ can provide much better localization results and cover the case of the "light" tails of $V(\cdot,\omega)$. We'll return to this subject in other publications and prove the localization for cases when $p(A) \geq \dfrac{C}{A^\theta}$ for any $\theta > 0$.

Example 22. Consider the Quasi-1 dimensional tree as shown in figure 2, denoted by T. The set of vertices is $\{\vec{x} = (x_1, x_2) : x_1, x_2$ are nonnegative integers$\} \cup \{(-1, 0)\}$. Consider the Anderson Hamiltonian $H = \Delta + V(\vec{x}, \omega)$ on T, where $V(\cdot,\, \omega)$ is are i.i.d. random variables with density $f(x)$ such that $P(V(\vec{x}; w) > A) = \displaystyle\int_A^\infty f(x)dx = p(A) > 0$ for all $A \in R$.

For fixed energy interval I, let's select a constant A such that $\left|\dfrac{\lambda}{A}\right| \leq \dfrac{1}{2}$, $\lambda \in I$ and introduce the following points on $x - axis$ and on the vertical lines $\{(x, y) : y > 0\}$ for positive integers x. Put

$$\tau_1 = \min\{x_1 > 0 : |V(x_1, 0, \omega)| > A, \}$$

$$\tau_2 = \min\{x_1 : |V(x_1, 0, \omega)| > A, |V(x_1 + 1, 0, \omega)| > A\}$$

$$\dots$$

$$\tau_n = \min\{x_1 > \tau_{n-1} : |V(x_1, 0, \omega)| > A, \dots, |V(x_1 + n - 1, 0, \omega)| > A\}.$$

$$\dots$$

Similarly, for fixed x, on the vertical line $\{(x, y) : y > 0\}$, we define

$$\tau_{x,1} = \min\{y > 0 : |V(x, y, \omega)| > A, \}$$

$$\tau_{x,2} = \min\{y > \tau_{x,1} : |V(x, y, \omega)| > A, |V(x, y + 1, \omega)| > A\}$$

$$\dots$$

$$\tau_{x,n} = \min\{y > \tau_{x,n-1} : |V(x, y, \omega)| > A, \dots, |V(x, y + n - 1, \omega)| > A\}$$

$$\dots$$

The random variables $\tau_1, \tau_2 - \tau_1, \dots, \tau_n - \tau_{n-1}, \dots; \tau_{x,1}, \tau_{x,2} - \tau_{x,1}, \dots, \tau_{x,n} - \tau_{x,n-1}, \dots, x = 0, 1, 2, \dots$ are independent. As easy to see, $\tau_n - \tau_{n-1}$ or $\tau_{x,n} - \tau_{x,n-1}$ are majorated by the random variable $n\theta_n^*$ or $n\theta_{x,n}^*$, where θ_n^* and $\theta_{x,n}^*$ are geometrically distributed with parameter $p^n(A)$. Like in the previous example $p^n(A)\theta_n^* \to Exp(1)$. and Borel-Cantelli lemma gives P-a.s.

$$\theta_n^* \leq \frac{(1 + \epsilon) \ln n}{p^n(A)}, n \geq n_0(\omega)$$

i.e.

$$\tau_n - \tau_{n-1} \leq \frac{(1+\epsilon)n\ln n}{p^n(A)}, n \geq n_0(\omega)$$

The same calculations show that $p^n(A)\theta^*_{x,n} \leq (1+\epsilon)(\ln n + \ln x)$ except finitely many pairs (x, n), i.e.

$$\tau_{x,n} \leq \frac{(1+\epsilon)\ln(nx)}{p^n(A)}.x + n \geq n_1(\omega).$$

The belt b_n consists of the points $\{(x_1, 0), \tau_n \leq x_1 \leq \tau_n + n - 1\}$ on the $x - axis$ and for any fixed x the points $\{(x, y), \tau_{x,n} \leq y \leq \tau_{x,n} + n - 1\}$. As a result,

$$|(n)| \leq \frac{(1+\epsilon)n\ln n}{p^n(A)} \cdot \frac{(1+\epsilon)n\ln\left(n \cdot \frac{(1+\epsilon)n\ln n}{p^n(A)}\right)}{p^n(A)} \leq \frac{c(A)n^3\ln n}{p^{2n}(A)} \leq c(A)n^3\ln n e^{\vartheta(A)n}$$

for some $c(A), \vartheta(A) > 0$. Also we have $\beta_\lambda^{b_n} \leq \left(\frac{1}{2}\right)^n$ and $\left(\prod_{k=1}^{n}\beta_\lambda^{b_k}\right)^2 \leq e^{-\vartheta n^2}$, for some $\vartheta > 0$.

Applying the general result of Theorem (III), we'll get

Theorem 23. For the Anderson Hamiltonian on the graph T(see figure 2)with boundary condition $\psi(-1, 0) = 0$, where $V(\vec{x}, \omega)$ are i.i.d. random variables with bounded distribution density $f(\nu)$ such that $\int_{|\nu|>A} f(\nu)d\nu = p(A) > 0$ for any $A > 0$, the spectrum of H is pure point with probability 1.

Remark 2. One can prove that the spectrum of the pure Laplacian Δ on the graph T is a.c..

Remark 3. The Hausdorff dimension of the graph T equals 2: it is simply the lattice Z^2 after removing some edges.

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