

**INTERMITTENCY FOR BRANCHING RANDOM WALKS WITH  
HEAVY TAILS**

by

**Asmaa Getan**

A dissertation submitted to the faculty of  
The University of North Carolina at Charlotte  
in partial fulfillment of the requirements  
for the degree of Doctor of Philosophy in  
Applied Mathematics

Charlotte

2016

Approved by:

---

**Dr. Boris Vainberg**

---

**Dr. Stanislav Molchanov**

---

**Dr. Yuri Godin**

---

**Dr. Heather Coffey**



## ABSTRACT

**ASMAA GETAN. Intermittency for Branching Random Walks with Heavy Tails.** (Under the direction of **DR. BORIS VAINBERG**)

Branching Markov process models populations in which each individual in generation  $n$  produces some random number of individuals in the next generation,  $n + 1$ , according to a certain probability distribution.

Branching processes play important role in the study of the evolution of various population plants, where members of the population may die or produce offspring independently of the rest. They can be used to model reproduction of bacteria where each bacteria generates several offspring with some probability in a single time unit. And they can be used to model other systems with similar dynamics, e.g., the spread of surnames in genealogy or the propagation of neutrons in a nuclear reactor.

In our dissertation, we consider a long time behavior for a model of branching random walk problem of a population of particles on the  $d$ - dimensional lattice  $Z^d$ .

In this model, the number of particles increases exponentially by duplicating, with a constant rate of birth, (each particle can split into two particles), and the particles spread everywhere by jumping to not necessary a neighbor place, (it could be a faraway distance), under probability of jumps that is described to be, a heavy tailed probability. Branching or jumping of each particle occurs independently of the other particles.

Under these two conditions, (constant rate of birth and heavy tailed probability of jumps), the front of propagation (where local growth occurs) has been found to be

moving exponentially fast.

A well developed non-uniformity concept called intermittency, is used to investigate the uniformity of the distribution of the particles, on, inside and outside of the front. A random field is called intermittent, if it is distributed very non-uniformly, where huge values can appear with a very small probability. For instance, the magnetic field of the sun is highly intermittent, as almost all of its energy is concentrated in black spots which covers only small parts of the surface of the sun.

In our work, we found that particles on, and outside the front exhibit intermittent behavior. We proved that, the same is true for some region inside the front. Despite that the front of propagation itself moves exponentially fast, the front of intermittency moves with a small power rate,  $|x| > t^\gamma$ , inside the first front. In the area between those two fronts, the particles are concentrated in very sparse spots with clustered density. This means that, the zone of non-intermittency extends with that rate too. This rate has been found exactly.

## ACKNOWLEDGMENTS

I would like to I would like to express my deepest thanks and gratitude to my advisor Dr. Boris Vainberg for supervising me with his wide knowledge and guidance. I appreciate his persistent help and encouragement from the initial classes I took with him until I finished my dissertation.

I am grateful to Dr. Stanislav Molchanov, the professor who inspired me to move forward and from whom I learned a lot.

I am so thankful to Dr. Joel Avrin, the former graduate coordinator in the Math department who made the process easier for me to join this PhD program.

I would like to thank all the members in my dissertation committee for accepting to serve on it, Dr. Youri Godin, Dr. Heather Coffy and Dr. Stanislav Molchanov.

Many thanks to all of my professors who always pushed me to be better, and to all the staff at UNCC.

I would like to thank my beloved family, my husband Layth, my daughter, Huda and my sons, Ahmed, Zaid and Ibrahim. They were always patient, supportive, proud and encouraging throughout my studies.

## TABLE OF CONTENTS

LIST OF FIGURES	viii
LIST OF TABLES	ix
CHAPTER 1: INTRODUCTION	1
1.1. A General Summary	1
1.2. Description of the main problem	1
1.3. Previous Studies	3
1.4. <i>KPP</i> problem on $R^d$	3
1.5. <i>KPP</i> problem on the lattice $Z^d$	5
1.6. The difference between <i>KPP</i> model and our model	6
1.7. The first and second moments in our model ( $m_1$ and $m_2$ )	6
1.7.1. The first moment $m_1$	6
1.7.2. The second moment $m_2$	7
1.8. Front of propagation	7
1.9. Intermittency	8
1.10. Our goal	9
CHAPTER 2: THE FIRST AND SECOND MOMENTS	10
2.1. Derivation of the equation for $m_1$	10
2.2. Integral representations of $p$ and $m_1$	11
2.3. Derivation of the equation for $m_2$	14
2.4. Integral representation of $m_2$	14
2.5. Starting from the origin	15

CHAPTER 3: ASYMPTOTIC BEHAVIOR OF THE FIRST MOMENT	17
3.1. Asymptotic behavior of $\hat{a}(\sigma)$ at 0	17
3.2. Asymptotic behavior of $m_1(t, x)$	18
3.3. Simplified version theorem of the asymptotics of $p(t, x)$	20
3.4. Finding the front	23
CHAPTER 4: INTERMITTENCY	25
4.1. Intermittency on and outside the front:	25
4.2. Intermittency inside the Front	28
4.3. No Intermittency in the region $ x  \leq Bt^\gamma$	30
4.4. Contribution to $\frac{m_2}{m_1}$ from the region $D_{12}$	32
4.4.1. Case 1 : region $D_{12}$ when $2t^{\frac{1}{\alpha}} <  x  \leq Bt^\gamma$	32
4.4.2. Case 2 : region $D_{12}$ , when $ x  \leq 2t^{\frac{1}{\alpha}}$	34
4.5. Contribution to $\frac{m_2}{m_1}$ from the region $D_{11}$ (inside of both paraboloids)	35
4.6. Contribution to $\frac{m_2}{m_1}$ from region $D_{22}$ (outside of both paraboloids)	37
4.6.1. Case 1: region $D_{22}$ when $\frac{1}{2}t^{1/\alpha} \leq  x  \leq Bt^\gamma$	37
4.6.2. Case 2: Region $D_{22}$ when $ x  < \frac{1}{2}t^{1/\alpha}$	40
4.7. Contribution to $\frac{m_2}{m_1}$ from the region $D_{21}$	42
4.7.1. Case 1: region $D_{21}$ when $2t^{\frac{1}{\alpha}} <  x  \leq Bt^\gamma$ .	42
4.7.2. Case 2: region $D_{21}$ when $ x  \leq 2t^{\frac{1}{\alpha}}$	44
REFERENCES	46

## LIST OF FIGURES

FIGURE 1: This graph is for $d=1$ .	31
-------------------------------------	----



## LIST OF TABLES

## CHAPTER 1: INTRODUCTION

### 1.1 A General Summary

We consider a population of particles on a  $d$ -dimensional lattice  $Z^d$ , where each particle sits in its place for a while before it either jumps randomly to another place, or it splits in the same place into two particles. After that, each of those particles (the original particle after it jumps to the new place, the original particle after producing offspring particle or the offspring itself ) starts again to either jump to another new place or split into two particles. This process is continued forever, under the condition that each particle jumps and splits independently of the others. For this process, we impose another two conditions. First, we consider constant rate branching, which means that the number of particles is exponentially increasing in time. And second, we assume that the particles can jump not only to neighbor points but also for distant points, with not necessarily small probability. The limit structure of the particle population inside the propagating front (where global or local growth occur) will be investigated. This structure is described in terms of intermittency.

### 1.2 Description of the main problem

Let us provide a little more detailed description of our branching random walk model.

We assume that each particle in this population, at any position  $x \in Z^d$ , at a given

time  $t$ , stays at this point a randomly exponentially distributed period of time which is  $\Delta t$ , with a parameter 1, before it either splits into two particles in the same place, or it jumps with a probability  $a(z)$ , to a new position  $x + z \in Z^d$ . In addition, if  $\nu$  is a constant that represents the rate of particle splitting, then at any time interval  $(t, t + \Delta t)$ , this particle which is located at any point  $x \in Z^d$ , may split with the rate  $\nu \Delta t$ , into two particles located at the same point  $x$ . The process of splitting or jumping of any particle is independent of any other particle. Later on, each of those particles, (parental one and the offspring), evolve independently of each other by the same law as the initial particle.

Obviously, the probability of the jump  $a(z)$ , satisfies the two conditions that,

$a(z) \geq 0$  and  $\sum_{z \in Z^d} a(z) = 1$ . Here we will assume that the distribution of the jump is symmetric, which means that,  $a(z) = a(-z)$  for all  $z \in Z^d$ .

In our work, a main assumption is included in our model. We assume that the probability for any particle to make a long jump is not necessarily small. Thus we are assuming the following behavior of  $a(z)$  at infinity:

$$a(z) = \frac{a_0(\dot{z})}{|z|^{d+\alpha}}(1 + o(1)), \quad |z| \rightarrow \infty, \quad \dot{z} = \frac{z}{|z|}, \quad (1)$$

with

$$a_0(\dot{z}) > \delta > 0, \quad 0 < \alpha < 2.$$

Such a probability distribution is called heavy tailed. Under this heavy tailed probability assumption, the second moments  $\sum z^2 a(z)$  are not defined when  $\alpha < 2$ . They would exist only when  $\alpha > 2$ .

In the case when the particle jumps without being split, the initial particle would perform the symmetric  $d$ -dimensional random walk  $X(t)$  with the generator

$$(\mathcal{L}f)(x) = \sum_{z \in Z^d} [f(x+z) - f(x)] a(z), \quad (2)$$

which is a bounded operator on the space  $\ell^2(Z^d)$ .

The symmetry of the jumps (namely  $a(z) = a(-z)$ ), implies that  $\mathcal{L} = \mathcal{L}^*$  is self adjointed operator in  $\ell^2(Z^d)$ .

### 1.3 Previous Studies

The mathematical study of branching processes goes back to the work of Galton and Watson [22] who were interested in the probabilities of long-term survival of family names. Later similar mathematical models were used to describe the evolution of a variety of biological populations, in genetics [11, 12, 13, 14], and in the study of certain chemical and nuclear reactions [20, 15]. The branching processes (in particular, branching diffusions) play important role in the study of the evolution of various populations such as bacteria, cancer cells, carriers of a particular gene, etc., where each member of the population may die or produce offspring independently of the rest.

### 1.4 $KPP$ problem on $R^d$

The original classical model in branching diffusion processes belongs to Kolmogorov, Petrovski and Piskunov. It is published in their famous paper in 1937, see [16]. Their model is called  $KPP$  model. The  $KPP$  model was applied to biological problems

and underlines processes in the case of Brownian motion in  $R^d$ . This diffusion process can be described by Poisson kernel  $p(t, x, y)$ , namely the probability density to find particles at the point  $x \in R^d$  at a time  $t$  when the starting point at  $t = 0$  is  $y \in R^d$ , is given by

$$p(t, x, y) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{(x-y)^2}{4t}}. \quad (3)$$

The generating operator in *KPP* model is the pure Laplacian  $\Delta\psi$  in the Euclidean space  $R^d$ , and the governing equation of this model is the heat equation

$$\frac{\partial p}{\partial t} = \Delta p \quad (4)$$

$$p(0, x, y) = \delta(x - y) \quad \text{on } R^d.$$

In the case that *KPP* model splitting with constant rate  $\nu$ , the expectation density to find particles at  $x$  is denoted by  $m_1$ . This function satisfies the equation

$$\frac{\partial m_1}{\partial t} = \Delta m_1 + \nu m_1 \quad (5)$$

$$m_1(0, x, y) = \delta(x - y) \quad \text{on } R^d.$$

In this case we have

$$m_1(t, x, y) = p(t, x, y)e^{\nu t}. \quad (6)$$

In this model, it can be easily found that the front (where  $m_1 \approx 1$ ) is growing linearly in time, since equations (3) and (6) with  $m_1 \approx 1$  imply that

$$-\frac{|x-y|^2}{4t} + \nu t \approx 0. \quad (7)$$

So by starting from the origin  $y = 0$ , the front is the region defined by

$$|x| \approx 2\sqrt{\nu t}$$

The regions where  $\frac{m_2}{m_1^2} \rightarrow \infty$  are considered in our thesis. These regions are called the regions of intermittent behavior. Here  $m_1$  and  $m_2$  are called the first moment and the second moment respectively.

For *KPP* model on  $R^d$ , Koralov and Molchanov in ([18]) studied different levels of intermittency. They are the regions where,  $\frac{m_j}{m_1^j} \rightarrow \infty$  as  $t \rightarrow \infty$  and  $m_j$  is the  $j$ th moment in this model.

### 1.5 *KPP* problem on the lattice $Z^d$

In the direct analog of *KPP* model on the lattice, the right hand side of the heat equation (4) will be replaced by the Laplacian operator, which is defined on lattice  $Z^d$  and given by

$$(\Delta\psi)(z) = \frac{1}{2d} \sum_{|e|=1} [\psi(z+e) - \psi(z)]. \quad (8)$$

This operator is somewhat simpler than (4) since  $\psi$  is defined on the lattice instead of  $R^d$ . On the other hand it is much more difficult to handle since it is not spherically symmetric.

## 1.6 The difference between *KPP* model and our model

**1-** The most important difference between our problem and the *KPP* problem is related to the fact that the generating operator in our model, given in (2), allows particles to jump not only in neighbor points, but also to make long jumps on the lattice  $Z^d$  with relatively not small probability. Any particle  $x$  after it splits, can jump to the point  $x + z$  with a probability  $a(z)$ . The point  $x + z$  can be found as faraway from  $x$  as we please. Note that in the *KPP* model, the probability to find a particle at the time  $t = 1$ , with a distance  $d$  from the parent particle is exponentially small, it is of order  $e^{-d^2/4}$ .

**2-** The front propagates linearly in *KPP* model, while in our model, the front will be proved to be propagated exponentially fast, due to the assumption of a long jump probability.

## 1.7 The first and second moments in our model ( $m_1$ and $m_2$ )

### 1.7.1 The first moment $m_1$

Let  $n(t, x, y)$  be the random variable of the number of particles at a point  $x \in Z^d$ , at the time  $t \geq 0$ , under the condition that  $n(0, x, y) = \delta(x - y)$ , which means that the process starts at the initial time  $t = 0$ , with a single particle located at the point  $y \in Z^d$ .

For each arbitrary constant rate of splitting,  $\nu > 0$ , let  $m_1 = m_1(t, x, y) = E(n(t, x, y))$ , be the expected value of  $n(t, x, y)$ . It is called the first moment of

the random variable  $n(t, x, y)$ . Let  $p = p(t, x, y)$  be the expected value of the random variable  $n(t, x, y)$  in the case when  $\nu = 0$ . Then  $m_1$  and  $p$  satisfy the relations (9) and (10) given below respectively:

$$\frac{\partial m_1}{\partial t}(t, x, y) = \mathcal{L}m_1(t, x, y) + \nu m_1(t, x, y) \quad , \quad t \geq 0 ; \quad (9)$$

$$m_1(0, x, y) = \delta(x - y).$$

$$\frac{\partial p}{\partial t}(t, x, y) = \mathcal{L}p(t, x, y) \quad , \quad t \geq 0 ; \quad (10)$$

$$p(0, x, y) = \delta(x - y).$$

It is clear that if  $p(t, x, y)$  is a fundamental solution of (10), then

$m_1(t, x, y) = p(t, x, y)e^{\nu t}$  is a fundamental solution of (9).

### 1.7.2 The second moment $m_2$

The second moment  $m_2(t, x, y)$  of the random variable  $n(t, x, y)$  is defined by

$m_2(t, x, y) = E(n^2(t, x, y))$ . It satisfies the following differential equation

$$\frac{\partial m_2}{\partial t}(t, x, y) = (\mathcal{L} + \nu)m_2(t, x, y) + 2\nu m_1^2(t, x, y) \quad (11)$$

$$m_2(0, x, y) = \delta(x - y).$$

The Derivations of (9), (10) and (11) will be given in the next chapter.

### 1.8 Front of propagation

The concept of front of propagation is essential in our work. The region in  $Z^d$  which separates the large and small values of  $m_1(t, x, y)$  is called the **front**. In our



work we define the front as the boundary of the set where  $m_1 < 1$ .

## 1.9 Intermittency

The notion of intermittency (or intermittent random fields), is popular in natural sciences, (astrophysics, biology, etc). From the qualitative point of view, intermittent random fields are distinguished by the formation of sparse spatial structures such as high peaks, clumps, patches, etc., giving the main contribution to the process in the medium. For instance, the magnetic field of the Sun is highly intermittent as almost all its energy is concentrated in the black spots, which cover only a very small part of the surface of the Sun. Many bio-populations also exhibit strong clumping (clustering). A random variable is called intermittent if it is distributed very non-uniformly. It means that huge values can appear with a very small probability.

Intermittency is a well developed non-uniformity concept. For physicists, the magnetic field of the Sun is intermittent since, say, 99% of its magnetic energy is concentrated on less than 1% of the surface. For mathematicians, 0.1, 0.01 or  $10^{-6}$  are not necessarily small numbers, and a limiting process must be considered instead. The definition of intermittency based on the progressive growth of the statistical moments was proposed in the review [24], a more formal presentation can be found in [5]. In the simplest form, a field  $n(t, x)$ ,  $x \in \mathbb{Z}^d$ , is intermittent as  $t \rightarrow \infty$  on a non-decreasing family of sets,  $\omega(t)$ , if

$$\lim_{t \rightarrow \infty} \frac{En^2(t, x)}{(En(t, x))^2} = \infty$$

uniformly in  $x \in \omega(t)$ .

Let us provide an example to illustrate the meaning of intermittency. Consider a

random variable  $X$  with expectation equal to one.

Let this random variable take the values  $\alpha_n$ ,  $1 \leq n \leq 100$ , with probabilities  $p_n$ ,  $\sum p_n = 1$ . Then  $m_1 = E(X) = \sum \alpha_n p_n$ , and  $m_2 = E(X^2) = \sum \alpha_n^2 p_n$ .

Let  $\alpha_1 = \frac{c}{\epsilon}$ , and  $p_1 = \epsilon$ , for  $0 < c < 1$  and  $\epsilon$  is a very small number. Let  $\alpha_n \approx 1 - c$  for all  $n > 1$  with  $\sum_2^\infty p_n = 1 - \epsilon$ . Then it is clear that  $m_1 \approx 1$  and  $\frac{m_2}{m_1^2} \approx c/\epsilon \rightarrow \infty$  as  $\epsilon \rightarrow 0$ . Thus  $X$  is an intermittent random variable, as one can see that the  $X$  is distributed non-uniformly with large values  $\alpha_1 = \frac{c}{\epsilon}$  having a small probability.

### 1.10 Our goal

In this thesis, we consider the random variable  $n(t, x, 0)$ , which is the number of particles at a point  $x \in Z^d$  at the time  $t$  when the process started from a single particle located at the origin 0.

We will discuss the intermittency of  $n(t, x, 0)$  on and outside the propagating front.

Our first goal is to find the asymptotic of  $m_1(t, x)$ , as  $t, x \rightarrow \infty$ , in order to find the front of propagation of the particles in our problem.

The second goal is to study if intermittent regions exist on, outside and inside of that front.

In fact we will show that the front of propagation grows exponentially, and our random variable exhibits intermittent behavior on the front and outside of it. For the region inside the front, we are going to prove that the intermittent behavior occurs when  $|x| > t^\gamma$ , with a specific values of  $\gamma > 0$  that will be found.

The estimation of  $p(t, x)$  given in ([1]) will play an important role in our study.

## CHAPTER 2: THE FIRST AND SECOND MOMENTS

### 2.1 Derivation of the equation for $m_1$

In order to derive the equations (9) that governs  $m_1(t, x, y)$ , we need to evaluate  $m_1(t + \Delta t, x, y)$ .

For this reason, the time interval  $(0, t + \Delta t)$  can be split up in the backward form into two successive intervals which are  $(0, \Delta t)$  and  $(\Delta t, t + \Delta t)$ , of the lengths  $\Delta t$  and  $t$  respectively. Then  $m_1(t + \Delta t, x, y)$  can be represented in the form

$$\begin{aligned} m_1(t + \Delta t, x, y) \sim & \sum_{z \in Z^d} a(z) \Delta t m_1(t, x, y + z) + 2\nu \Delta t m_1(t, x, y) \\ & + (1 - \Delta t - \nu \Delta t) m_1(t, x, y) \end{aligned} \quad (12)$$

The terms on the right hand side of the relation (12) can be explained as follows: The first term is the probability for the particle to jump from the point  $y$  to the point  $y + z$  during the time  $\Delta t$ , which is  $a(z)\Delta t$ , multiplied by the expectation of the number of particles at the point  $x$  when the walk starts at a single point  $y + z \in Z^d$ , which is  $m_1(t, x, y + z)$ . The second term is the probability  $\nu\Delta t$  of branching during the time  $\Delta t$ , multiplied by the expected value of the number of particles at  $x$  that are descendant of both the original and the new born particle at  $y$ , which is  $2m_1(t, x, y)$ . The last term is the expectation of the number of particles at the point  $x$  under the condition that the initial particle stays at  $y$  without splitting and without jumping

during the time interval  $\Delta t$ .

We subtract  $m_1(t, x, y)$  from both sides of the equation (12) above.

Using that  $\sum a(z) = 1$ , divide by  $\Delta t$ , and pass to the limit as  $\Delta t \rightarrow 0$ . This implies the following equation

$$\frac{\partial m_1}{\partial t}(t, x, y) = \sum_{z \in Z^d} [m_1(t, x, y + z) - m_1(t, x, y)]a(z) + \nu m_1(t, x, y), \quad t \geq 0 \quad (13)$$

such that

$$m_1(0, x, y) = \delta(x - y). \quad (14)$$

By using the definition of  $(\mathcal{L}f)(x)$  given in (2) in the equation (13) above, the derivation of the equations (9) is complete.

The solution of (13) gives the integral representation of  $m_1(t, x, y)$ . The details are given in the section 2.2 below.

Note that the term,  $\sum_{z \in Z^d} m_1(t, x, y + z)a(z)$  in (13), can be considered as a convolution in  $y$  of the two functions  $a(y)$  and  $m_1(t, x, y)$ .

## 2.2 Integral representations of $p$ and $m_1$

The integral representations of  $p$  and  $m_1$  are obtained by solving the equation (13) together with the initial condition (14) for the two cases, when  $\nu = 0$  and when  $\nu \neq 0$  respectively.

One can easily show that  $m_1$  and  $p$  depend, not on  $x$  and  $y$ , but on their difference, i.e.,  $m_1(t, x, y) = m_1(t, x - y, 0)$  and  $p(t, x, y) = p(t, x - y, 0)$ .

Thus we can drop the argument  $y$  in both  $m_1(t, x, y)$  and  $p(t, x, y)$ .

In this case  $p(t, x)$  denotes the solution of (13) with  $\nu = 0$  and  $y = 0$ , and  $m_1(t, x)$

denotes the solution of (13) with  $\nu \neq 0$  and  $y = 0$ .

For the case when  $\nu = 0$ , we find  $p(t, x)$  as follows:

Consider  $p(t, x)$  and  $a(x)$  as Fourier coefficients, (defined at an integer points  $x \in Z^d$ ), of the two functions  $\widehat{p}(t, \sigma)$  and  $\widehat{a}(\sigma)$  respectively, with  $\sigma \in [-\pi, \pi]^d$ . In this case

$$\widehat{p}(t, \sigma) = \sum_{x \in Z^d} p(t, x) e^{-i(\sigma, x)} \quad \text{and} \quad \widehat{a}(\sigma) = \sum_{x \in Z^d} a(x) e^{-i(\sigma, x)}. \quad (15)$$

We use that  $\sum_{x \in Z^d} a(x) = 1$ , and pass to the Fourier series in (13), we get

$$\begin{aligned} \frac{\partial \widehat{p}}{\partial t}(t, \sigma) &= \sum_{x \in Z^d} \frac{\partial p}{\partial t}(t, x) e^{-i(\sigma, x)} \\ &= \sum_{x \in Z^d} \left[ \sum_{z \in Z^d} p(t, x - z) a(z) \right] e^{-i(\sigma, x)} - \sum_{x \in Z^d} p(t, x) e^{-i(\sigma, x)} \\ &= \sum_{z \in Z^d} \left[ \sum_{x \in Z^d} p(t, x - z) e^{-i(\sigma, (x-z))} \right] a(z) e^{-i(\sigma, z)} - \widehat{p}(t, \sigma), \end{aligned}$$

i.e.,  $\widehat{p}(t, \sigma)$  is the solution of the following ordinary differential equation

$$\frac{\partial \widehat{p}}{\partial t}(t, \sigma) = \widehat{p}(t, \sigma) [\widehat{a}(\sigma) - 1]. \quad (16)$$

With the condition that,  $\widehat{p}(0, \sigma) = 1$ .

This condition is true since we have  $p(0, x) = \delta(x)$ .

It is clear that the solution of the problem (16) above is:

$$\widehat{p}(t, \sigma) = e^{[\widehat{a}(\sigma) - 1]t}.$$

The Fourier coefficients  $p(t, x)$  of the function  $\widehat{p}(t, \sigma)$  are given by

$$p(t, x) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \widehat{p}(t, \sigma) e^{i(\sigma, x)} d\sigma$$

Thus

$$p(x, t) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{[\widehat{a}(\sigma)-1]t+i(\sigma, x)} d\sigma. \quad (17)$$

The equation (17) gives the integral representation of the function  $p(t, x)$ .

For the case when  $\nu \neq 0$ , then, by passing to the Fourier series in (13), we obtain the following ordinary differential equation and initial condition

$$\frac{\partial \widehat{m}_1}{\partial t}(t, \sigma) = [\widehat{a}(\sigma) - 1] \widehat{m}_1(t, \sigma) + \nu \widehat{m}_1(t, \sigma)$$

$$\widehat{m}_1(0, \sigma) = 1.$$

The solution of this problem is

$$\widehat{m}_1(t, \sigma) = e^{[\widehat{a}(\sigma)-1+\nu]t} = e^{\nu t} e^{[\widehat{a}(\sigma)-1]t}.$$

Hence the integral representation of  $m_1(t, x)$ , is given by

$$m_1(t, x) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{\nu t} e^{[\widehat{a}(\sigma)-1]t+i(\sigma, x)} d\sigma \quad (18)$$

It is clear that

$$m_1(t, x) = e^{\nu t} p(t, x).$$

### 2.3 Derivation of the equation for $m_2$

For the derivation of equation (11) that is satisfied by  $m_2$ , we again consider  $m_2(t + \Delta t, x, y)$  on the time interval  $(0, t + \Delta t)$ . We split this interval in the backward way into two successive intervals,  $(0, \Delta t)$  and  $(\Delta t, t + \Delta t)$ . Then  $m_2(t + \Delta t, x, y)$  can be represented on those two intervals by the following form

$$m_2(t + \Delta t, x, y) \sim \sum_{z \in Z^d} a(z) \Delta t m_2(t, x, y + z) + \nu \Delta t E(n_1 + n_2)^2 + (1 - \Delta t - \nu \Delta t) m_2(t, x, y). \quad (19)$$

Here the terms on the right hand side of (19), are similar to the terms on the right hand side of (12). In this case,  $n_1 = n_1(t, x, y)$  is the number of particles in  $x$  that are descendant of the original particle, and  $n_2 = n_2(t, x, y)$  is the number of particles in  $x$  that are descendant of the new born particle.

In (19) we use the fact that

$$E(n_1 + n_2)^2 = E(n_1)^2 + E(n_2)^2 + 2E(n_1)E(n_2) = 2m_2(t, x, y) + 2m_1^2(t, x, y).$$

We subtract  $m_2(t, x, y)$  from both sides of (19), divide by  $\Delta t$ , and pass in both sides to the limit as  $\Delta t \rightarrow 0$ . After doing all these steps, the derivation of (11) is obtained.

### 2.4 Integral representation of $m_2$

The solution of (11) gives the integral representation of  $m_2$ . This solution can be found by using the following Duhamel formula:

$$m_2(t, x, y) = m_1(t, x, y) + \int_0^t u(t-s, x, y) ds. \quad (20)$$

This formula is used under the condition that,  $u(t, x)$  is the solution of the following problem:

$$\frac{\partial u}{\partial t}(t, x, y) = (\mathcal{L} + \nu)u, \quad t > s \quad (21)$$

Such that

$$u(s, x, y) = 2\nu m_1^2(s, x, y).$$

Now since  $m_1(t, x, y)$  is a fundamental solution of the problem (21) above, the solution of (21) on the lattice will be given by

$$u(t, x, y) = 2\nu \sum_{z \in Z^d} m_1(t, x-z, y) m_1^2(s, z, y). \quad (22)$$

Hence the substitution of (22) in the equation (20), implies the solution of (11).

Thus the integral representation of  $m_2(t, x, y)$  is the following

$$m_2(t, x, y) = m_1(t, x, y) + 2\nu \int_0^t ds \sum_{z \in Z^d} m_1(t-s, x-z, y) m_1^2(s, z, y). \quad (23)$$

## 2.5 Starting from the origin

Finally we note that, the first and second moments,  $m_1$  and  $m_2$ , depend only on the difference  $x - y$ , (but not on  $x$  and  $y$  separately) . Thus we can replace  $y$  in (20) and (22) by  $y = 0$ , and consider the two functions  $m_1 = m_1(t, x)$  and  $m_2 = m_2(t, x)$



as the first and second moments respectively of the random variable  $n(t, x)$  of the number of particles at each point  $x \in Z^d$ , under the condition that the process starts with one particle located at the origin.

Now we can consider the main equation for  $m_1(t, x)$ ,  $p(t, x)$  and  $m_2(t, x)$  to be the following:

$m_1(t, x)$  is the solution of the equation

$$\frac{\partial m_1}{\partial t}(t, x) = (\mathcal{L} + \nu) m_1(t, x), \quad t \geq 0, \quad (24)$$

such that

$$m_1(0, x) = \delta(x).$$

Function  $p(t, x)$  coincides with  $m_1$  when  $\nu = 0$ . It is the solution of the equation

$$\frac{\partial p}{\partial t}(t, x) = \mathcal{L}p(t, x), \quad t \geq 0, \quad (25)$$

such that

$$p(0, x) = \delta(x).$$

Also from (23), the main integral representation equation for  $m_2(t, x)$  has the form

$$m_2(t, x) = m_1(t, x) + 2\nu \int_0^t ds \sum_{z \in Z^d} m_1(t-s, x-z) m_1^2(s, z). \quad (26)$$

## CHAPTER 3: ASYMPTOTIC BEHAVIOR OF THE FIRST MOMENT

### 3.1 Asymptotic behavior of $\widehat{a}(\sigma)$ at 0

For  $\widehat{a}(\sigma)$  defined in (15), the following Lemma 1, has been proved in [1]. It provides the asymptotic behavior of  $\widehat{a}(\sigma)$  at 0. Namely, we assume that:

$$a(z) = \sum_{j=0}^{d+\epsilon} \frac{a_j(\dot{z})}{|z|^{d+\alpha+j}} + O\left(\frac{1}{|z|^{2d+\alpha+1+\epsilon}}\right), \quad |z| \rightarrow \infty, \quad \alpha \in (0, 2), \quad (27)$$

where

$$a_j \in C^{d+1-j+\epsilon}(S^{d-1}), \quad a_0(\dot{z}) > \delta > 0$$

and  $\epsilon = 1$  if  $\alpha = 1$ ,  $\epsilon = 0$  otherwise.

**Lemma 1.** If (27) holds, then

$$\widehat{a}(\sigma) = 1 - \sum_{j=0}^d b_j(\dot{\sigma}) |\sigma|^{\alpha+j} + f(\sigma), \quad \sigma \in T^d = [-\pi, \pi]^d, \quad f(0) = 0, \quad (28)$$

where  $b_j \in C^{d+[\alpha]+1}(S^{d-1})$  and function  $f$ , being extended periodically on  $R^d$ , belongs to  $C^{d+[\alpha]+1}(R^d)$ . Moreover, the homogeneous function  $b_0(\dot{\sigma})|\sigma|^\alpha$  in  $R^d$  is the Fourier transformation of the homogeneous (of order  $-d - \alpha$ ) distribution that is equal to  $a_0(\dot{x})|x|^{-d-\alpha}$  when  $0 \neq x \in R^d$ , and

$$b_0(\dot{\sigma}) = -\Gamma(-\alpha) \cos \frac{\alpha\pi}{2} \int_{S^{d-1}} a_0(\dot{x}) |(\dot{x}, \dot{\sigma})|^\alpha dS_{\dot{x}} > 0, \quad (29)$$

where  $\Gamma$  is the gamma-function.

### Remarks

1) Note that  $f$  can not be omitted, since a change in the values of  $a(z)$  at several points does not perturb its asymptotic behavior at infinity, but changes  $\widehat{a}(\sigma)$  by an analytic function.

2) The next two properties of  $\widehat{a}(\sigma)$  follow immediately from the properties of  $a_0(\dot{x})$ :

$$\widehat{a}(-\sigma) = \widehat{a}(\sigma) \quad \text{and} \quad -1 < \widehat{a}(\sigma) < 1, \quad \text{when } 0 \neq \sigma \in T^d. \quad (30)$$

The second property in (30), follows from (15) and the relation  $\sum_{x \in Z^d} a(x) = 1$ , under the condition that, for each  $\sigma \in T^d$ ,  $\sigma \neq 0$ , there is a point  $z \in Z^d$ , where  $e^{-i(z,\sigma)} \neq 1$  and  $a(z) \neq 0$ . Such points  $z$  exist due to (27).

### 3.2 Asymptotic behavior of $m_1(t, x)$

The asymptotic behavior of  $m_1(t, x) = e^{\nu t} p(t, x)$  is a direct result of the uniform asymptotics of the function  $p(t, x)$  given in the following theorem which is one of the main results of [1].

**Theorem 2.** Let the conditions (28)-(30) hold.

Then, the the following asymptotis holds for  $p(t, x)$

$$(i) \quad p(t, x) = \frac{1}{t^{d/\alpha}} S\left(\frac{x}{t^{1/\alpha}}\right) (1 + o(1)), \quad \text{when } x \in Z^d, \quad |x| + t \rightarrow \infty. \quad (31)$$

Here,  $S(y) = \frac{1}{(2\pi)^d} \int_{R^d} e^{i(\sigma,y) - b_0(\hat{\sigma})|\sigma|^\alpha} d\sigma > 0$ , is the stable density  $S = S_{\alpha, a_0}(y)$ , which depends on  $\alpha \in (0, 2)$ , and coefficient  $a_0$ , is defined in (27).

(ii) If  $\frac{|x|}{t^{1/\alpha}} \rightarrow \infty$ ,  $|x| \geq 1$ , then the previous statement can be specified as follows:

$$p(t, x) = \frac{a_0(\dot{x})}{t^{d/\alpha}} \left( \frac{t^{1/\alpha}}{|x|} \right)^{d+\alpha} (1 + o(1)) = \frac{a_0(\dot{x})t}{|x|^{d+\alpha}} (1 + o(1)). \quad (32)$$

In the next section, a simplified version of this theorem will be proved. For this reason, we need to prove the following lemma:

**Lemma 3.** Function  $p(t, x)$  given in (17), is strictly positive for all  $x \in Z^d$ ,  $t > 0$ .

**Proof.** Denote by  $a_n(x)$  the convolutions of  $n$  copies of  $a(x)$ :

$$a_n(x) := a(x) * a(x) * \dots * a(x), \quad (33)$$

where  $a(x) * b(x) = \sum_{z \in Z^d} a(x - z) * b(z)$ .

We multiply both sides of (33) by  $e^{(-i\sigma x)}$ , then we take the summation for all  $x \in Z^d$ . This implies that  $\widehat{a}_n(\sigma) = [\widehat{a}(\sigma)]^n$ . Further, since  $\widehat{a}(0) = 1$ , the second property in (30) implies that,  $|\widehat{a}(\sigma)| \leq 1$ , and therefore,  $|\widehat{a}_n(\sigma)| \leq 1$ . This allows us to write  $p(t, x)$  as follows:

$$\begin{aligned} p(t, x) &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{[\widehat{a}(\sigma)-1]t+i(\sigma, x)} d\sigma \\ &= \frac{e^{-t}}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left[ 1 + \sum_{n=1}^{\infty} \frac{[\widehat{a}(\sigma)]^n}{n!} t^n \right] e^{i(\sigma, x)} d\sigma. \end{aligned}$$

Thus

$$p(t, x) = e^{-t} [\delta(x) + \sum_{n=1}^{\infty} \frac{a_n(x)}{n!} t^n]. \quad (34)$$

Since  $a(x) \geq 0$ , all the convolutions  $a_n(x)$  are non-negative. Hence (34) will imply

the statement of the lemma if we show that  $a_2(x)$  is strictly positive for all  $x \in Z^d$ .

We have:

$$a_2(x) = \sum_{z \in Z^d} a(x-z)a(z). \quad (35)$$

Here  $a(\cdot) \geq 0$ , and from (27) it follows that the terms in (35) are positive for each fixed  $x$  if  $z$  is large enough. Thus  $a_2(x) > 0$ , and the proof of the lemma is completed.

### 3.3 Simplified version theorem of the asymptotics of $p(t,x)$

As a consequence of the results above, the following theorem 4 has been proved, it can be considered as a simplified version of Theorem 2. First we need to introduce the notion of the equivalency between two functions.

**Definition 1.** Functions  $a$  and  $b$  are called equivalent, and it will be denoted by  $a \sim b$ , if two constants  $c_1$  and  $c_2$  exist, such that  $c_1 b < a < c_2 b$ .

**Theorem 4.** Let the condition given in (1) be hold.

For arbitrary  $a_2 \geq a_1 > 0$ , the following relations hold

$$(i) \quad |p(t, x)| \sim \frac{t}{|x|^{d+\alpha}} \quad \text{when } |x| > a_1 t^{\frac{1}{\alpha}}. \quad (36)$$

$$(ii) \quad |p(t, x)| \sim t^{-\frac{d}{\alpha}} \quad \text{when } 0 \neq |x| \leq a_2 t^{\frac{1}{\alpha}}, \quad t > \epsilon > 0. \quad (37)$$

$$(iii) \quad |p(t, x)| \sim C \quad \text{when } x = 0 \text{ and } t < 1, \quad C \neq 0 \text{ is a constant.} \quad (38)$$

**Proof.** The last statement is obvious since the boundedness of  $p(t, x)$  follows immediately from (17).

In order to prove the first two statements, we need to split the lattice into three

regions,  $U_1 = U_1(t)$ ,  $U_2 = U_2(t)$ , and  $U_3 = U_3(t)$ . Where  $U_1$  is the region defined by the inequality  $|x| > At^{\frac{1}{\alpha}}$  with  $A$  so large that the remainder term in (32) is less than  $1/2$ . If needed we increase the value of  $A$  to be sure that it is not smaller than the constant  $a_2$  chosen in (37).

Note that  $|x| > At^{\frac{1}{\alpha}}$  implies that  $x \neq 0$ . Thus,  $|x| \geq 1$  in  $U_1$  since  $x$  is a point on the lattice.

From (32) it follows that:

$$\left| p(t, x) - \frac{a_0(\dot{x})t}{|x|^{d+\alpha}} \right| < \frac{1}{2} \frac{a_0(\dot{x})t}{|x|^{d+\alpha}}, \text{ for all } x \in U_1.$$

and therefore,

$$\frac{1}{2} \frac{a_0(\dot{x})t}{|x|^{d+\alpha}} < p(t, x) < \frac{3}{2} \frac{a_0(\dot{x})t}{|x|^{d+\alpha}}, \text{ for all } x \in U_1.$$

Since  $a_0(\dot{x})$  is continuous and does not vanish on the unite sphere  $|\dot{x}| = 1$ , it follows that:

$$|p(t, x)| \sim \frac{t}{|x|^{d+\alpha}} \text{ when } |x| > At^{\frac{1}{\alpha}}. \quad (39)$$

This justifies (36) for  $a_1 = A$ , but not for any arbitrary  $a_1$ .

Now one can find a large enough constant  $B$ , such that the remainder term in (31) is less than  $1/2$ , (i.e. ,  $|o(1)| < 1/2$ ), whenever  $|x| + t > B$ .

Let  $U_2$  be defined by the inequalities  $|x| \leq At^{\frac{1}{\alpha}}$  and  $|x| + t \geq B$ , with the same  $A$  used in the previous step.

Since  $\frac{|x|}{t^{\frac{1}{\alpha}}} \leq A$  is bounded in  $U_2$ , and function  $S(y)$  is positive and continuous, it follows that  $S\left(\frac{|x|}{t^{\frac{1}{\alpha}}}\right)$  in (31) has upper and lower positive bounds in  $U_2$ . Thus (31)

implies that  $p(t, x) \sim \frac{1}{t^{\frac{d}{\alpha}}}$  on the region  $U_2$ .

Now consider the region  $U_3$  defined as follows

$$U_3 = \{|x| \leq At^{\frac{1}{\alpha}}, \epsilon < |x| + t \leq B\} \setminus \{(t, x) : x = 0, t \leq 1\}.$$

We take into account that region  $U_3$  is bounded with  $t \geq \epsilon > 0$ . Since  $p(t, x) > 0$  for  $t > 0$ , (see Lemma 3), and  $p$  is continuous on  $U_3$ , it follows that  $p$  has lower and upper bounds on  $U_3$ , i.e., there are two constants  $C_1$  and  $C_2$  such that

$$0 < C_1 < p(t, x) < C_2 \text{ on } U_3.$$

Similar estimates are valid for the function  $\frac{1}{t^{\frac{d}{\alpha}}}$ .

Thus,  $p(t, x) \sim \frac{1}{t^{\frac{d}{\alpha}}}$  on  $U_2 \cup U_3$ , and the second statement of the theorem is proved with  $a_2 = A$ . Therefore it is valid for arbitrary  $a_2$ , since  $A$  was chosen to be such that  $A \geq a_2$ .

To complete the proof of the theorem, it remains to show that the equivalency relation (39) can be extended to the region  $At^{\frac{1}{\alpha}} \geq |x| > a_1t^{\frac{1}{\alpha}}$ . Thus the proof of the theorem will be completed as soon as the following lemma is proved.

**Lemma 5.** In the intermediate region  $\epsilon < a_1t^{\frac{1}{\alpha}} < |x| \leq At^{\frac{1}{\alpha}}$ , the following relations hold

$$p(t, x) \sim \frac{1}{t^{\frac{d}{\alpha}}} \sim \frac{t}{|x|^{d+\alpha}}.$$

**Proof.** Relation (37) holds in the region  $a_1t^{\frac{1}{\alpha}} < |x| \leq At^{\frac{1}{\alpha}}$ , described in the lemma. So it is enough to show that  $\frac{1}{t^{\frac{d}{\alpha}}} \sim \frac{t}{|x|^{d+\alpha}}$  in this region.

The ratio of the latter two functions is equal to

$$\frac{\frac{1}{t^{\frac{d}{\alpha}}}}{\frac{t}{|x|^{d+\alpha}}} = \frac{|x|^{d+\alpha}}{t^{1+\frac{d}{\alpha}}} = \frac{|x|^{d+\alpha}}{t^{\frac{d+\alpha}{\alpha}}} = \left(\frac{|x|}{t^{\frac{1}{\alpha}}}\right)^{d+\alpha}.$$

This ratio is bounded from below by  $a_1^{d+\alpha}$  and bounded from above by  $A^{d+\alpha}$ . Hence the proofs of Lemma 5 and Theorem 4 are complete.

Lemma 5 allows us to use any of the equivalency results (36) or (37) for  $p(t, x)$  on the region where  $t^{\frac{d}{\alpha}} < |x| < At^{\frac{1}{\alpha}}$ .

We also can combine the estimates (37) and (38) in the region  $|x| \leq t^{\frac{1}{\alpha}}$  by the following form, given in the next lemma, with no need to mention that  $t \geq 1$ .

**Lemma 6.** The relation,  $|p(t, x)| \sim \frac{1}{(t+1)^{\frac{d}{\alpha}}}$ , holds in the region where  $|x| \leq a_2 t^{\frac{1}{\alpha}}$ .

### 3.4 Finding the front

Recall that the front of propagation is the region where,  $m_1(t, x) \sim 1$  as  $t \rightarrow \infty$ .

The following theorem states that, the front propagates exponentially in time.

**Theorem 7.** The front of propagation is located in the region where,

$$|x| = (a_0(\dot{x})t)^{\frac{1}{d+\alpha}} e^{\frac{\nu t}{d+\alpha}} (1 + o(1)) , \quad t \rightarrow \infty.$$

**Proof.** There is no front in the region where  $|x| \leq t^{\frac{1}{\alpha}}$ . This follows from (37) since in this region, we have

$$m_1(t, x) = p(t, x) e^{\nu t} \sim \frac{1}{t^{\frac{d}{\alpha}}} e^{\nu t} \rightarrow \infty. \quad (40)$$

From (40), it is clear that  $m_1(t, x)$  is unbounded and grows exponentially as  $t \rightarrow \infty$ .



Now consider the region  $|x| > t^{\frac{1}{\alpha}}$ , as  $t \rightarrow \infty$ .

From (18) and (32), we have:

$$m_1(t, x) = p(t, x)e^{\nu t} = \frac{a_0(\dot{x})t}{|x|^{d+\alpha}} e^{\nu t}(1 + o(1)). \quad (41)$$

The statement of the theorem follows immediately from here.

## CHAPTER 4: INTERMITTENCY

### 4.1 Intermittency on and outside the front:

Recall that our goal is to investigate the intermittent regions in the domain of our problem. They are the regions in which

$$\frac{m_2(t, x)}{m_1^2(t, x)} \rightarrow \infty, \text{ as } t \rightarrow \infty.$$

For this reason, we will estimate  $\frac{m_2}{m_1^2}$  separately, on, inside and outside of the front of propagation (see Theorem 7 for the definition of the front).

In the following theorem, we are going to prove that the regions on and outside the front are intermittent. The intermittency inside the front will be discussed in much detail in the following sections.

**Theorem 8.** The region on and outside the front is intermittent, i.e.,

$$\frac{m_2(t, x)}{m_1^2(t, x)} \rightarrow \infty, \text{ in the region where } |x| \geq t^{\frac{1}{d+\alpha}} e^{\frac{\nu t}{d+\alpha}}, \text{ as } t \rightarrow \infty.$$

**Proof.** For the proof we are going to directly estimate  $\frac{m_2(t, x)}{m_1^2(t, x)}$  in the region given above.

Since always  $m_1(t, x) \geq 0$ , the equality (26) implies that

$$m_2(t, x) > \int_0^t \sum_{z \in Z} m_1(t-s, x-z) m_1^2(s, z) ds. \tag{42}$$

We need to find an estimation for the integral on the right hand side of (42). For

this reason, we use that  $m_1(s, z) = p(s, z) \cdot e^{\nu s}$ , and replace  $m_1(t-s, x-z)$  and  $m_1(s, z)$  by their corresponding formulas from (36) and (37).

For estimating this integral, we divide the set  $\{(s, z) : s > 0, z \in Z^d\}$  into several sub-regions by the following two parbloids, each of them in  $R^1 \times Z^d$ , namely:

$$|z| = s^{\frac{1}{\alpha}} \text{ and } |x - z| = (t - s)^{\frac{1}{\alpha}} .$$

It is clear that this division depends on the values of  $t$  and  $x$ .

We will estimate  $m_2(t, x)$  in each of those sub-regions. In fact it is enough to find a small sub-region,  $\omega$ , in  $R^1 \times Z^d$ , where the expression in the right hand side of (42) restricted to  $\omega$  is unbounded as  $t \rightarrow \infty$ , because in this case, the contribution to the right side of (42) from the other parts of  $R^1 \times Z^d$  will only increase the low bound to the value of  $m_2(t, x)$ .

Let us consider the small sub-region,  $\omega : z = 0, t - 1 \leq s \leq t$ . It is located inside the region  $|z| \leq s^{\frac{1}{\alpha}}$ . In this case (42) implies that

$$m_2(t, x) > \int_0^t \sum_{z \in Z} m_1(t-s, x-z) m_1^2(s, z) ds \geq \int_{t-1}^t m_1(t-s, x-0) m_1^2(s, 0) ds. \quad (43)$$

Function  $m_1(t-s, x-0)$  in (43), can be estimated from below using (36) as follows:

since we have

$$m_1(t-s, x-z) \geq \frac{C(t-s)}{|x-z|^{\alpha+d}} e^{\nu(t-s)} \text{ as } \frac{|x-z|}{(t-s)^{\frac{1}{\alpha}}} \rightarrow \infty .$$

So when,  $z = 0, t - 1 \leq s \leq t$ , then

$$m_1(t-s, x-0) \geq \frac{C(t-s)}{|x|^{d+\alpha}} e^{\nu(t-s)} \text{ as } \frac{|x|}{(t-s)^{\frac{1}{\alpha}}} \rightarrow \infty .$$

Also,  $m_1(s, 0)$  in (43), can be estimated from below using (37) by

$$m_1(s, 0) \geq \frac{C}{s^{\frac{d}{\alpha}}} e^{\nu s}.$$

Hence  $m_2(t, x)$  in (43) is estimated by the following

$$m_2(t, x) \geq C \int_{t-1}^t \frac{(t-s)}{|x|^{d+\alpha}} e^{\nu(t-s)} \frac{1}{s^{\frac{2d}{\alpha}}} e^{2\nu s} ds. \quad (44)$$

For further estimation of  $m_2(t, x)$  in (44), we can use that

$$e^{\nu(t-s)} \cdot e^{2\nu s} = e^{\nu(t+s)} \geq ce^{2\nu t} \text{ since } s \geq t-1 \text{ and } \frac{1}{s^{\frac{2d}{\alpha}}} > \frac{1}{t^{\frac{2d}{\alpha}}} \text{ (as we have } s < t \text{).}$$

We use that  $\int_{t-1}^t (t-s)ds = 1/2$ . Hence (44) can be read as follows

$$m_2(t, x) \geq \frac{Ce^{2\nu t}}{|x|^{d+\alpha}} \cdot \frac{1}{t^{\frac{2d}{\alpha}}}. \quad (45)$$

So if  $x$  is located on the front (see Theorem 7, where  $|x| \sim t^{\frac{1}{d+\alpha}} e^{\frac{\nu t}{d+\alpha}}$ ,  $t$  is large, and  $|x|$  is exponentially large) or outside of it, then (41) and (45), together with the relation  $m_1 < C$ , imply that

$$\frac{m_2(t, x)}{m_1^2(t, x)} \Big|_{|x| \geq t^{\frac{1}{d+\alpha}} e^{\frac{\nu t}{d+\alpha}}} \geq \frac{Ce^{\nu t}}{t^{1+2d/\alpha}} \rightarrow \infty.$$

In the above result,  $\frac{m_2}{m_1^2}$  is unbounded since it has an exponential growth as  $t \rightarrow \infty$ .

Hence the theorem is proved.

In fact, in Theorem 9, given in the next section, we will show that intermittency takes place inside the front for some extent of the form  $|x| = t^\gamma$ .

## 4.2 Intermittency inside the Front

Now we consider the region where  $|x| \gg t^{\frac{1}{\alpha}}$  but still inside the front, i.e., we consider the region where,

$$t^{\frac{1}{\alpha}} < |x| < t^{\frac{1}{d+\alpha}} e^{\frac{vt}{d+\alpha}}.$$

The following theorem is essential in our work. In this theorem we are going to prove that, there are intermittent regions inside the front to the extent where  $|x| > t^{\gamma+\epsilon}$ , such that  $\gamma = \frac{2\alpha + d}{\alpha(\alpha + d)}$ .

**Theorem 9.** Let the heavy tail condition (1) holds. Then

- 1) The ratio  $\frac{m_2(t, x)}{m_1^2(t, x)}$  is uniformly bounded in each ball  $|x| < Bt^\gamma$ , such that  $\gamma = \frac{2\alpha+d}{\alpha(\alpha+d)}$ , i.e., the random variable  $n$  is non-intermittent in this ball.
- 2) For each domain  $\Omega_\varepsilon(t) = \{x : |x| > t^{\gamma+\varepsilon}\}$ ,  $\varepsilon > 0$ , we have  $\frac{m_2(t, x)}{m_1^2(t, x)} \rightarrow \infty$  uniformly in  $x \in \Omega_\varepsilon(t)$ , as  $t \rightarrow \infty$ , i.e.,  $n$  is intermittent in  $\Omega_\varepsilon(t)$ .

**Proof:** The proof of the first part of this theorem will be given in the following sections of this chapter starting from the next one.

The proof of the second part is given here.

From the estimation of  $m_1(t, x)$  given in (41), when  $t \rightarrow \infty$ , we have:

$$m_1^2(t, x) \leq \frac{ct^2}{|x|^{2(d+\alpha)}} e^{2vt}. \quad (46)$$

We will consider the contribution to the lower bound of  $m_2(t, x)$  given in (42) from the region where,  $t - 1 < s < t$ ,  $|z| \leq t^{\frac{1}{\alpha}}/2$  (this region is clear in Figure 1 when  $d=1$ ).

We replace  $m_1(t-s, x-z)$  and  $m_1(s, z)$  in (42) by their estimates given in (36) and (37) respectively. Then

$$m_2(t, x) \geq C \int_{t-1}^t \sum_{\substack{|z| \leq \frac{t^{\frac{1}{\alpha}}}{2}, \\ z \in Z^d}} \frac{(t-s)}{|x-z|^{d+\alpha}} e^{\nu(t-s)} \frac{1}{s^{\frac{2d}{\alpha}}} e^{2\nu s} ds. \quad (47)$$

We are going to estimate the right hand side of (47) from below as follows:

Since  $|x| \geq t^{\gamma+\epsilon}$ , and since  $|z| \leq \frac{t^{\frac{1}{\alpha}}}{2} < \frac{t^{\gamma+\epsilon}}{2} < \frac{|x|}{2}$ , these imply that  $|x-z| < \frac{|x|}{2}$ . Hence we can replace  $\frac{1}{|x-z|}$  in (47) by  $\frac{C_1}{|x|}$ . We also use that  $\frac{1}{s^{\frac{2d}{\alpha}}} \geq \frac{1}{t^{\frac{2d}{\alpha}}}$  since  $t-1 \leq s \leq t$ .

Hence, (47) can be read as follows:

$$m_2(t, x) \geq \frac{ce^{\nu t}}{t^{\frac{2d}{\alpha}}} \int_{t-1}^t (t-s) e^{\nu s} ds \sum_{\substack{|z| \leq \frac{t^{\frac{1}{\alpha}}}{2}, \\ z \in Z^d}} \frac{1}{|x|^{\alpha+d}}. \quad (48)$$

The Summation in (48) is taken over all the integer points  $z$  in the ball

$|z| \leq t^{\frac{1}{\alpha}}/2$ . The number of these points has an order of  $O(t^{\frac{1}{\alpha}})^d$ , as  $t \rightarrow \infty$ . Hence this summation can be replaced by  $t^{\frac{d}{\alpha}}$ .

After performing the integration with respect to  $s$  in (48),  $m_2(t, x)$  is estimated for  $x \in \Omega_\epsilon(t)$  as  $t \rightarrow \infty$ , by

$$m_2(t, x) \geq \frac{ce^{2\nu t}}{t^{\frac{d}{\alpha}}} \frac{1}{|x|^{d+\alpha}}, \quad (49)$$

Hence if  $|x| > t^\gamma$ , for any  $\gamma \geq 0$ , then from (49) and (46), we have

$$\frac{m_2(t, x)}{m_1^2(t, x)} \geq \frac{C|x|^{(d+\alpha)}}{t^{2+d/\alpha}} = \frac{C|x|^{(d+\alpha)}}{t^{\frac{2\alpha+d}{\alpha}}} = C\left(\frac{|x|}{t^{\frac{2\alpha+d}{\alpha}}}\right)^{d+\alpha} = C\left(\frac{|x|}{t^\gamma}\right)^{d+\alpha}.$$

The expression above is unbounded when  $|x| \geq t^{\gamma+\epsilon}$ , which means that there is an intermittency in the region  $\Omega_\epsilon$ , where  $|x| \geq t^{\gamma+\epsilon}$ , with  $\gamma = \frac{2\alpha+d}{\alpha(d+\alpha)}$ . This proves the second part of the theorem.

Notice that for  $\gamma = \frac{2\alpha+d}{\alpha(d+\alpha)}$  then  $t^\gamma < t^{\frac{1}{d+\alpha}} e^{\frac{\nu t}{d+\alpha}}$ . This means that the region  $\Omega_\epsilon$  is located inside the front where  $|x|$  is given as in Theorem 7.

#### 4.3 No Intermittency in the region $|x| \leq Bt^\gamma$

For the proof of the first part of the theorem, we need to prove that, for any  $B > 0$ ,  $\frac{m_2}{m_1^2}$  is bounded in the ball where  $|x| \leq Bt^\gamma$ ,  $\gamma = \frac{2\alpha+d}{\alpha(\alpha+d)}$ .

Using (26) and the fact that  $m_1 \geq 1$  inside the front,  $\frac{m_2}{m_1^2}$  satisfies the following relation

$$\frac{m_2(t, x)}{m_1^2(t, x)} \leq 1 + \frac{2\nu}{m_1^2(t, x)} \int_0^t \sum_{z \in \mathbb{Z}^d} m_1(t-s, x-z) m_1^2(s, z) ds, \quad |x| \leq Bt^\gamma, \quad t \rightarrow \infty. \quad (50)$$

We are going to find an estimation from above, for the right hand-side of (50). It is as follows :

For fixed  $s$ , such that  $0 \leq s \leq t$ , we split the region  $\mathbb{Z}^d$  in (50) into four regions, separated by the two spheres which are,  $|z| < s^{\frac{1}{\alpha}}$  and  $|x-z| < (t-s)^{\frac{1}{\alpha}}$ .

Let  $P_1 = P_1(s)$ , and  $P_2 = P_2(x, t-s)$ , be two (bounded) sets of points  $z \in \mathbb{Z}^d$ .  $P_1$  is located inside or at the boundary of the first sphere and  $P_2$  is located inside or at the boundary of the second sphere, i.e.,

$$P_1 = \{z \in \mathbb{Z}^d : |z| \leq s^{\frac{1}{\alpha}}\}, \quad \text{and} \quad P_2 = \{z \in \mathbb{Z}^d : |x-z| \leq (t-s)^{\frac{1}{\alpha}}\}.$$

For  $i, j = 1, 2$ , we denote by each  $D_{i,j} = D_{i,j}(s, t, x)$ , the set of points  $z \in \mathbb{Z}^d$  such

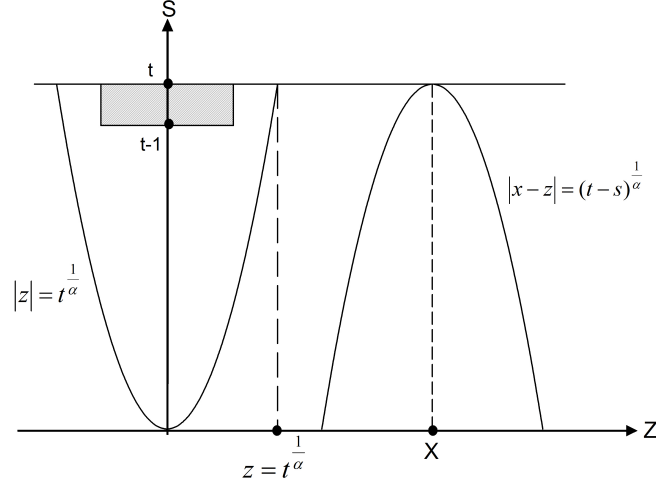


Figure 1: This graph is for  $d=1$ .

that  $0 \leq s \leq t$ ,  $x \in \mathbb{Z}^d$  and satisfy the following

$D_{11}$  is the the set of points  $z \in \mathbb{Z}^d$  located inside or at the boundary of both spheres, i.e.,  $D_{11} = P_1 \cap P_2$ .

$D_{22}$  is the the set of points  $z \in \mathbb{Z}^d$ , located outside of both spheres, i.e.,

$$D_{22} = \mathbb{Z}^d \setminus (P_1 \cup P_2).$$

$D_{12}$  is the set of points  $z \in \mathbb{Z}^d$  located outside of the first sphere, but inside of the second one or on its boundary, i.e.,  $D_{12} = (\mathbb{Z}^d \setminus P_1) \cap P_2$ .

$D_{21}$  is the the set of points  $z \in \mathbb{Z}^d$  located inside of the first sphere or on its boundary, but outside of the second one, i.e.,  $D_{21} = (P_1 \setminus P_2) \cup P_2$ .

It is convenient to visualize the domains  $D_{ij}$  by drawing the two paraboloids,  $|z| < s^{\frac{1}{\alpha}}$  and  $|x-z| < (t-s)^{\frac{1}{\alpha}}$ , in the  $(1+d)$  dimensional space  $R^1 \times Z^d$ . The graph given above is for  $d = 1$ . Then  $D_{ij}$  is defined by the intersection of the region inside or outside of the corresponding paraboloid with the space  $s = \text{constant}$ .

Now under the condition that  $|x| \leq Bt^\gamma$ ,  $t \rightarrow \infty$ , we are going to estimate the



contribution to the lower bound of  $\frac{m_2}{m_1^2}$ , given in the right hand side of (50), that is made by integration/summation over each of the above regions  $D_{ij}$  instead of the integral for all  $Z^d$ .

Thus

$$\frac{m_2(t, x)}{m_1^2(t, x)} \leq 1 + 2\nu \sum_{2 \geq i, j \geq 1} I_{ij}, \quad (51)$$

such that for  $i, j = 1, 2$ ,  $I_{ij}$ , is given as follows

$$I_{ij} := \frac{1}{m_1^2(t, x)} \int_0^t \sum_{z \in Z^d \subset D_{ij}} m_1(t-s, x-z) \cdot m_1^2(s, z) ds. \quad (52)$$

In the following sections, we are going to estimate the upper bound of the right hand side of (52), for each  $I_{ij}$ ,  $i, j = 1, 2$ . We will prove that they are bounded as  $t \rightarrow \infty$ .

#### 4.4 Contribution to $\frac{m_2}{m_1^2}$ from the region $D_{12}$

Let us start with the region  $D_{12}$ , and discuss two cases: first when  $2t^{\frac{1}{\alpha}} < |x| \leq \beta t^\gamma$ , and second when  $|x| \leq 2t^{\frac{1}{\alpha}}$ . Note that, from the definition of the region  $D_{12}$ , we have  $|z| > s^{\frac{1}{\alpha}}$  and  $|x-z| \leq (t-s)^{\frac{1}{\alpha}}$ , so the two balls  $P_1$  and  $P_2$ , are separated in the first case (when  $0 \leq s \leq t$ ) (see figure 1), and they may intersect each other in the second case.

##### 4.4.1 Case 1 : region $D_{12}$ when $2t^{\frac{1}{\alpha}} < |x| \leq \beta t^\gamma$

In this case, since  $|x| > 2t^{\frac{1}{\alpha}}$ , estimate (36) implies that

$$|m_1^2(t, x)| \sim \frac{t^2}{|x|^{2(d+\alpha)}} e^{2tv}.$$

In  $D_{12}$ , because we have  $|z| > s^{\frac{1}{\alpha}}$  and  $|x-z| \leq (t-s)^{\frac{1}{\alpha}}$ , estimate (36) and Lemma

6 imply that

$$m_1(s, z) \sim \frac{s}{|z|^{(d+\alpha)}} e^{sv}, \text{ and } m_1(t-s, x-z) \sim \frac{1}{(t-s+1)^{\frac{1}{\alpha}}} e^{(t-s)v}.$$

Hence for  $I_{12}$ , defined in (52), we have

$$I_{12} < \frac{C|x|^{2(d+\alpha)}}{t^2 e^{2tv}} \int_0^t \sum_{z \in D_{12}} \frac{s^2 e^{nu(t+s)}}{(t-s+1)^{\frac{d}{\alpha}} |z|^{2(d+\alpha)}} ds, \quad 2t^{\frac{1}{\alpha}} < |x| \leq \beta t^\gamma. \quad (53)$$

For the estimation of the right hand side of (53), we can replace  $z$  by  $x$ . This is true because, in this region we have,  $|x| > 2t^{\frac{1}{\alpha}}$ , hence,  $|x-z| \leq (t-s)^{\frac{1}{\alpha}} \leq t^{\frac{1}{\alpha}} \leq \frac{1}{2}|x|$ . This implies that  $|z| \geq \frac{1}{2}|x|$  in  $D_{12}$ , and accordingly,  $\frac{1}{|z|^{2(d+\alpha)}} \leq \frac{C}{|x|^{2(d+\alpha)}}$ .

The summation in (53), is applied to the  $z$ -independent terms and therefore the summation sign can be replaced by the factor,  $K$ , that estimates the number of terms in this sum from above. This factor  $K$  can be estimated by the volume of the ball  $P_2$  that results from the second paraboloid  $|x-z| \leq (t-s)^{\frac{1}{\alpha}}$  (when  $s$  is fixed). It is clear that

$$K \leq C[(t-s)^{\frac{1}{\alpha}} + 1]^d \sim C[(t-s)^{\frac{d}{\alpha}} + 1].$$

Thus

$$I_{12} < \frac{C}{t^2 e^{2vt}} \int_0^t \frac{[(t-s)^{\frac{d}{\alpha}} + 1] s^2}{(t-s+1)^{\frac{d}{\alpha}}} e^{\nu(t+s)} ds.$$

By using that,  $\frac{[(t-s)^{\frac{d}{\alpha}} + 1]}{(t-s+1)^{\frac{d}{\alpha}}} \leq 1$ , we have

$$I_{21} < \frac{C_1}{t^2 e^{vt}} \int_0^t s^2 e^{\nu s} ds. \quad (54)$$

After performing the integral above, it is clear that the right side of (54), is bounded when  $t \rightarrow \infty$ .

Thus we have proved that,  $I_{12}$  is bounded in the first case when,  $2t^{\frac{1}{\alpha}} < |x| \leq Bt^\gamma$ , as  $t \rightarrow \infty$ .

#### 4.4.2 Case 2 : region $D_{12}$ , when $|x| \leq 2t^{\frac{1}{\alpha}}$

In case  $|x| \leq 2t^{\frac{1}{\alpha}}$ , due to (37), then,  $|m_1^2(t, x)| \sim \frac{1}{t^{\frac{2d}{\alpha}}} e^{2t\nu}$ , for  $t \geq 1$ .

Further, due to (36) and Lemma 4, relation (52) implies that

$$I_{12} < \frac{Ct^{\frac{2d}{\alpha}}}{e^{2\nu t}} \int_0^t \sum_{z \in D_{12}} \frac{s^2 e^{\nu(t+s)}}{[(t-s+1)^{\frac{d}{\alpha}}] |z|^{2(d+\alpha)}} ds, \quad |x| \leq 2t^{\frac{1}{\alpha}}, \quad t \rightarrow \infty. \quad (55)$$

For the estimation of the right hand side of (55), we split the integral above so that

$$\int_0^t = \int_0^{t/2} + \int_{t/2}^t.$$

For estimating the integral  $\int_0^{t/2}$ , we replace  $|z|^{2(d+\alpha)}$  by 1 since by the definition of  $D_{12}$  we have  $|z| \geq 1$  on the lattice.

Similar to the previous case, the summation along  $z$  can be replaced by the factor  $K \leq C[(t-s)^{\frac{d}{\alpha}} + 1]$ .

Hence

$$\frac{Ct^{\frac{2d}{\alpha}}}{e^{2\nu t}} \int_0^{t/2} \frac{[(t-s)^{\frac{d}{\alpha}} + 1] s^2 e^{\nu(t+s)}}{(t-s+1)^{\frac{d}{\alpha}}} ds \leq \frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} \int_0^{t/2} s^2 e^{\nu s} ds \leq C < \infty. \quad (56)$$

The right hand side of (56) is bounded since it decays exponentially as  $t \rightarrow \infty$ .

Now consider the case when the integral in (55) is  $\int_{t/2}^t$ . In this case we can replace  $|z|$  by  $(t/2)^{\frac{1}{\alpha}}$ . The reason for that is,  $t/2 < s < t$  implies that  $|z| \geq s^{\frac{1}{\alpha}} \geq (t/2)^{\frac{1}{\alpha}}$ .

After that, we replace the summation in  $z$  by the factor  $K \leq C[(t-s)^{\frac{d}{\alpha}} + 1]$  as before.

This leads to the following estimate,

$$\frac{Ct^{\frac{2d}{\alpha}}}{e^{2\nu t}t^{\frac{2(d+\alpha)}{\alpha}}} \int_{t/2}^t s^2 e^{\nu(t+s)} ds \leq C < \infty, \quad t \rightarrow \infty. \quad (57)$$

After implementing the integral, the expression (57) above is bounded as  $t \rightarrow \infty$ .

Now (56) and (57) imply that,  $I_{12}$  is bounded in the second case when  $|x| \leq 2t^{\frac{1}{\alpha}}$ .

The two estimates of  $I_{12}$ , obtained in the previous two cases, prove that  $I_{12}$ , given in (55), is bounded in the region  $D_{12}$ , under the condition that  $|x| < t^\gamma$ .

#### 4.5 Contribution to $\frac{m_2}{m_1^2}$ from the region $D_{11}$ (inside of both paraboloids)

From the definition of  $D_{11}$ , given in section 4.3, we have

$$|z| < s^{\frac{1}{\alpha}} \text{ and } |x - z| < (t - s)^{\frac{2}{\alpha}}.$$

Hence, the two balls  $P_1$  and  $P_2$ , defined in Section 4.3, do not intersect each other when  $|x| > 2t^{\frac{1}{\alpha}}$  and  $(0 \leq s \leq t)$ .

For this reason, we assume, in this case, that  $|x| \leq 2t^{\frac{1}{\alpha}}$ .

For  $I_{11}$ , given in (52), we use that,  $m_1^2(t, x) \sim \frac{C}{t^{\frac{2d}{\alpha}}} e^{2\nu t}$  for  $t \rightarrow \infty$  and replace  $m_1(t - s, x - z)$  and  $m_1(t, x)$  by their corresponding formulas in  $D_{11}$  using (37) and lemma 6.

Now the following inequality is obtained

$$I_{11} < \frac{Ct^{\frac{2d}{\alpha}}}{e^{2\nu t}} \int_0^t \sum_{z \in D_{11}} \frac{1}{(t - s + 1)^{\frac{d}{\alpha}} (s + 1)^{\frac{2d}{\alpha}}} e^{\nu(t+s)} ds, \quad \text{as } t \rightarrow \infty. \quad (58)$$

The estimation of the right hand side of (58), can be done as follows:

The summation,  $\sum_{z \in D_{11}}$ , can be replaced by the factor  $K$  which has been defined

in subsection 4.4.1. And by the same reason, we use that

$$\sum_{z \in D_{11}} \leq C[(t-s)^{\frac{d}{\alpha}} + 1]$$

. After that, we use in (58) the fact that  $\frac{C[(t-s)^{\frac{d}{\alpha}} + 1]}{(t-s+1)^{\frac{d}{\alpha}}} < C_1$ .

Hence, the following relation is true:

$$I_{11} < \frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} \int_0^t \frac{e^{\nu s}}{(s+1)^{\frac{2d}{\alpha}}} ds, \text{ as } t \rightarrow \infty. \quad (59)$$

For further estimation of  $I_{11}$ , given in (59), the integral in the right hand side of (59) is split so that

$$\int_0^t = \int_0^{t/2} + \int_{t/2}^t .$$

When the integral is  $\int_0^{t/2}$ , we drop the bottom of the integrand in (59) since

$(s+1)^{\frac{2d}{\alpha}} > 1$ . Hence

$$\frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} \int_0^{t/2} e^{\nu s} ds \leq \frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} [e^{\nu t/2} - e^{\nu}]. \quad (60)$$

The right side of (60) is bounded, as it decays exponentially when  $t \rightarrow \infty$ .

For the second integral  $\int_{t/2}^t$ , we drop 1 from the bottom of the integrand of (59), and make the substitution  $s = t\tau$ , hence

$$\int_{t/2}^t \frac{e^{\nu s}}{s^{\frac{2d}{\alpha}}} ds = \int_{1/2}^1 \frac{e^{\nu t\tau}}{(t\tau)^{\frac{2d}{\alpha}}} t d\tau = \int_{1/2}^1 \frac{e^{\nu t\tau}}{t^{\frac{2d}{\alpha}-1} \tau^{\frac{2d}{\alpha}}} d\tau \quad (61)$$

The integral in the right hand side of (61) can be estimated by using the Laplace method, as  $t \rightarrow \infty$ . It is less than,  $\frac{1}{t^{\frac{2d}{\alpha}-1}} \frac{e^{\nu t}}{t\nu}$ .

Hence,

$$\frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} \int_{t/2}^t \frac{e^{\nu s}}{s^{\frac{2d}{\alpha}}} ds < \frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} \frac{1}{t^{\frac{2d}{\alpha}-1}} \frac{e^{\nu t}}{t\nu}. \quad (62)$$

It is clear that the right hand side of (62) is bounded by a constant as  $t \rightarrow \infty$ .

The boundedness of both (60) and (62), prove that, (59) is bounded as  $t \rightarrow \infty$ .

Hence  $I_{11}$  is bounded in  $D_{11}$ .

4.6 Contribution to  $\frac{m_2}{m_1}$  from region  $D_{22}$  (outside of both paraboloids)

From the definition of the region  $D_{22}$ , we have  $|z| > s^{\frac{1}{\alpha}}$  and  $|x - z| > (t - s)^{\frac{1}{\alpha}}$ .

Then due to (36), we can replace the factors under the summation in (52) by

$$m_1(s, z) \sim \frac{se^{\nu s}}{|z|^{d+\alpha}} \quad \text{and} \quad m_1(t - s, x - z) \sim \frac{(t - s)e^{\nu(t-s)}}{|x - z|^{d+\alpha}}.$$

For this region, we will consider the two cases, first when  $\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma$ , and second when  $|x| < \frac{1}{2}t^{1/\alpha}$ . These two cases will be given in following two subsections.

4.6.1 Case 1: region  $D_{22}$  when  $\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma$

In this case, from (36), we have  $m_1^2(t, x) \sim \frac{t^2}{|x|^{2(d+\alpha)}} e^{2\nu t}$ .

Hence  $I_{22}$ , defined in (52) satisfies the following relation

$$I_{22} < \frac{C|x|^{2(d+\alpha)}}{t^2 e^{2\nu t}} \int_0^t \sum_{z \in D_{22}} \frac{(t - s)s^2}{|x - z|^{d+\alpha} |z|^{2(d+\alpha)}} e^{\nu(t+s)} ds. \quad (63)$$

For an estimation of the right hand side of (63), we divide the region  $D_{22}$  into two sub-regions, namely,

$$D_{22}^{(1)} = D_{22} \cap \{z : |z| > \frac{x}{2}\} \quad \text{and} \quad D_{22}^{(2)} = D_{22} \cap \{z : |z| \leq \frac{x}{2}\}.$$

Let  $I_1$  and  $I_2$  be the right-hand side of (63), with  $D_{22}$  is replaced by  $D_{22}^{(1)}$  and  $D_{22}^{(2)}$

respectively. It is clear that  $I_{22} \leq I_1 + I_2$ .

In order to find an estimation for  $I_1$  from above, we use in (63) the following:

$|x - z|$  is strictly greater than  $(t - s)^{\frac{1}{\alpha}}$  on the lattice in  $D_{22}$ . This implies that,

$|x - z| \geq 1$ , for all  $z \in D_{22}$ .

Now since we have,  $\sum_{z \in D_{22}} \frac{1}{|x - z|^{d+\alpha}} < \sum_{z \in \mathbb{Z}^d \setminus \{x\}} \frac{1}{|x - z|^{d+\alpha}}$ , and the later series converges and does not depend on  $x$ . This implies that,

$\sum_{z \in D_{22}} \frac{1}{|x - z|^{d+\alpha}}$  is bounded by a constant.

We also use that  $\frac{1}{|z|^{2(d+\alpha)}} < \frac{1}{(|x|/2)^{2(d+\alpha)}}$ . This is true since  $|z| > |x|/2$ . Because

those two reasons, the following relation is true

$$\sum_{z \in D_{22}^{(1)}} \frac{1}{|x - z|^{d+\alpha} |z|^{2(d+\alpha)}} \leq \left(\frac{2}{|x|}\right)^{2(d+\alpha)} \sum_{z: |x-z| \geq 1} \frac{1}{|x - z|^{d+\alpha}} \leq \frac{C}{|x|^{2(d+\alpha)}}. \quad (64)$$

Using (64) in (63), we find that

$$I_1 \leq \frac{c}{t^2 e^{\nu t}} \int_0^t (t - s) s^2 e^{\nu s} ds. \quad (65)$$

For further estimation from above of  $I_1$ , given in (65), as  $t \rightarrow \infty$ , in the case when

$\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma$ , we use the following Lemma, which is a consequence of the

Laplace method, see ([10]). This Lemma will be used to show that  $I_1$  is bounded

from above as  $t \rightarrow \infty$ .

**Lemma 10.** The asymptotics of the integral  $I = \int_0^1 (1 - s)^\alpha g(s) e^{\phi(s)t} ds$ , for  $\alpha > -1$ ,

when  $\phi(s)$  has a maximum at  $s = 1$  and  $\phi'(1) \neq 0$ , is

$$I = C_o \frac{g(1)}{[\phi'(1)]^{\alpha+1}} \cdot \frac{e^{\phi(1)t}}{t^{\alpha+1}} (1 + o(1)), \quad t \rightarrow \infty. \quad (66)$$

In order to use the lemma 10 above, we change the variable  $s$  in (65) by,  $s = \tau t$ .

This implies that

$$I_1 \leq \frac{Ct^2}{e^{\nu t}} \int_0^1 (1 - \tau)\tau^2 e^{\nu\tau t} d\tau. \quad (67)$$

Using Lemma 10, the integral in (67) is estimated by  $\frac{e^{\nu t}}{\nu^2 t^2}$  as  $t \rightarrow \infty$ . Hence it is easy to see that,  $I_1$  is bounded from above by a constant when  $t \rightarrow \infty$ .

Now let us estimate  $I_2$ .

The inequality  $|z| \leq \frac{|x|}{2}$ , implies that  $|x - z| \geq \frac{|x|}{2}$ . Hence  $\frac{1}{|x - z|^{(d+\alpha)}} \leq \left(\frac{2}{|x|}\right)^{(d+\alpha)}$ .

From here and (63), it follows that, for  $\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma$ ,  $t \rightarrow \infty$ , we have

$$I_2 \leq \frac{C|x|^{(d+\alpha)}}{t^2 e^{\nu t}} \int_0^t (t - s)s^2 e^{\nu s} \left( \sum_{z \in D_{22}^{(2)}} \frac{1}{|z|^{2(d+\alpha)}} \right) ds. \quad (68)$$

Since we have,  $|z| > s^{\frac{1}{\alpha}}$  in  $D_{22}$ , and  $|z| \geq 1$  on the lattice when  $z \neq 0$ . This implies that the summation in (68) does not exceed  $\frac{C}{(1 + s)^{\frac{2\alpha+d}{\alpha}}}$ .

Thus

$$I_2 \leq \frac{C|x|^{d+\alpha}}{t^2 e^{\nu t}} \int_0^t \frac{(t - s)s^2}{(1 + s)^{\frac{2\alpha+d}{\alpha}}} e^{\nu s} ds. \quad (69)$$

In (69), we can also use that  $\frac{|x|^{(d+\alpha)}}{t^2} < C_1 t^{d/\alpha}$ . This is due to reason that  $|x| \leq Bt^\gamma$ , with  $\gamma = \frac{2\alpha+d}{\alpha(\alpha+d)}$ . Hence, for  $\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma$ ,  $t \rightarrow \infty$ , we have

$$I_2 \leq \frac{Ct^{d/\alpha}}{e^{\nu t}} \int_0^t \frac{(t - s)s^2}{(1 + s)^{\frac{2\alpha+d}{\alpha}}} e^{\nu s} ds, \quad (70)$$

For further estimation of  $I_2$ , given in (70), we split the integral above as follows:

$$\int_0^t = \int_0^{t/2} + \int_{t/2}^t .$$



In the case when the integral in (70) is  $\int_0^{t/2}$ , we replace  $s^2$  in the integrand by  $t^2$  since  $s \leq t/2$ , we replace  $(t-s)$  by  $t$  since  $(t-s) \leq t$ , and we drop the term  $\frac{1}{(1+s)^{\frac{2\alpha+d}{\alpha}}}$  since it is less than 1. Then

$$\frac{Ct^{d/\alpha}}{e^{\nu t}} \int_0^{t/2} \frac{(t-s)s^2}{(1+s)^{\frac{2\alpha+d}{\alpha}}} e^{\nu s} ds < \frac{Ct^{3+\frac{d}{\alpha}}}{e^{\nu t/2}}. \quad (71)$$

The right hand side of (71) is bounded, since it decays exponentially when,

$$\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma, \text{ and } t \rightarrow \infty.$$

Now Consider (70) when the integral is  $\int_{t/2}^t$ . In this case we drop 1 from the denominator of the term  $\frac{1}{(1+s)^{\frac{2\alpha+d}{\alpha}}}$ , and make the change of variable  $s = \tau t$ .

Now the right side of (70) satisfies that

$$\frac{Ct^{d/\alpha}}{e^{\nu t}} \int_{t/2}^t \frac{(t-s)}{s^{\frac{d}{\alpha}}} e^{\nu s} ds \leq \frac{Ct^2}{e^{\nu t}} \int_{1/2}^1 \frac{(1-\tau)}{\tau^{\frac{d}{\alpha}}} e^{\nu \tau t} d\tau. \quad (72)$$

Using Lemma 10 to estimate the integral in the right hand side of (72), it is easy to see that this side is bounded, as  $t \rightarrow \infty$ .

The boundedness of each of (71) and (72) implies that,  $I_2$  given in (70) is bounded when,  $|x| \leq Bt^\gamma$ , with  $\gamma = \frac{2\alpha+d}{\alpha(\alpha+d)}$ .

The boundedness of both  $I_1$  and  $I_2$  prove that,  $I_{22}$  is bounded in the first case when  $\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma$ ,  $t \rightarrow \infty$ .

#### 4.6.2 Case 2: Region $D_{22}$ when $|x| < \frac{1}{2}t^{1/\alpha}$

In this case,  $|m_1^2(t, x)| \sim \frac{1}{t^{\frac{2d}{\alpha}}} e^{2\nu t}$  for  $t \rightarrow \infty$ . Hence,  $I_{22}$  defined in (52), satisfies that

$$I_{22} < \frac{Ct^{2d/\alpha}}{e^{2\nu t}} \int_0^t \sum_{z \in D_{22}} \frac{(t-s)s^2}{|x-z|^{d+\alpha}|z|^{2(d+\alpha)}} e^{\nu(t+s)} ds. \quad (73)$$

From the definition of  $D_{22}$ , we have

$$|z| > s^{\frac{1}{\alpha}}, \quad |x-z| > (t-s)^{\frac{1}{\alpha}}, \quad 0 < s < t, \quad x, z \in \mathbb{R}^d. \quad (74)$$

This implies that, there exists  $\beta > 0$ , such that,  $|z| \geq \beta > 0$ , when  $t = 1$ .

Indeed, if we assume that  $|z|$  can be as small as we pleased, i.e.,  $|z| = \epsilon$ , then the first inequality in (74) implies that,  $|s| < \epsilon^\alpha$ , and when  $t = 1$ , the second inequality in (74) implies that  $|x| > 1/2$ . The latter inequality contradicts our assumption that  $|x| < \frac{1}{2}t^{\frac{1}{\alpha}}$ ,  $t = 1$ .

Now we can use the homogeneity argument as follows:

Assume that  $s_1 = \frac{s}{t^{1/\alpha}}$ ,  $z_1 = \frac{z}{t^{1/\alpha}}$ ,  $x_1 = \frac{x}{t^{1/\alpha}}$ ,  $t_1 = \frac{t}{t^{1/\alpha}}$ . Then one can easily see that the inequalities in (74) imply that,  $|z_1| > \frac{1}{2}$  for  $t_1 = 1$ . Thus for an arbitrary  $t > 0$ , we have

$$|z| \geq \beta t^{1/\alpha}.$$

Now replacing  $|z|$  in (73) by  $\beta t^{1/\alpha}$ , we find that

$$I_{22} \leq \frac{Ct^{2d/\alpha}}{e^{2\nu t} t^{2(d+\alpha)/\alpha}} \int_0^t \sum_{z \in D_{22}} \frac{(t-s)s^2 e^{\nu(t+s)} ds}{|x-z|^{d+\alpha}}, \quad \text{when } |x| < \frac{1}{2}t^{1/\alpha}, \quad t \rightarrow \infty.$$

Note that for all  $z \in D_{22}$ , then  $|x-z| \geq 1$  on the lattice, and consequently,  $\sum_{z \in D_{22}} \frac{1}{|x-z|^{d+\alpha}}$  is a convergent series since it can be estimated from above by the convergent series  $\sum_{|x-z| \geq 1} \frac{1}{|x-z|^{d+\alpha}}$ . The latter series does not depend on  $x$ . Hence

$$I_{22} \leq \frac{C}{t^2 e^{\nu t}} \int_0^t (t-s)s^2 e^{\nu s} ds. \quad (75)$$

By changing the variable  $s = \tau t$  in (75), then

$$I_{22} \leq \frac{Ct^2}{e^{\nu t}} \int_0^1 (1-\tau)\tau^2 e^{\nu\tau t} d\tau. \quad (76)$$

Estimating the integral in (76) by using Lemma 10, it can easily be seen that  $I_{22}$  is bounded when  $t \rightarrow \infty$ . Thus we have proved that  $I_{22}$  is bounded in the second case when  $|x| < \frac{1}{2}t^{1/\alpha}$ ,  $t \rightarrow \infty$ .

Together with the boundedness of  $I_{22}$  in the first case given in section 4.6.1, when  $2t^{\frac{1}{\alpha}} \leq |x| \leq Bt^\gamma$ , we have proved that  $I_{22}$  is bounded.

#### 4.7 Contribution to $\frac{m_2}{m_1}$ from the region $D_{21}$

In the region  $D_{21}$ , we have,  $|z| \leq s^{\frac{1}{\alpha}}$  and  $|x-z| > (t-s)^{\frac{1}{\alpha}}$ . We will consider the two cases: first, when  $2t^{\frac{1}{\alpha}} < |x| \leq \beta t^\gamma$  and second, when  $|x| \leq 2t^{\frac{1}{\alpha}}$ . These two cases has been considered in section 4.4 when we discussed the contribution to  $\frac{m_2}{m_1}$  from the region  $D_{12}$ . The reason is that, the two balls  $P_1$  and  $P_2$ , defined in section 4.3, are separated in the first case, and they may be intersect each other in the second case.

##### 4.7.1 Case 1: region $D_{21}$ when $2t^{\frac{1}{\alpha}} < |x| \leq Bt^\gamma$ .

From (36), it follows that  $m_1^2(t, x) \sim \frac{t^2}{|x|^{2(d+\alpha)}} e^{2\nu t}$ , for  $t > 1$ .

Formula (52) implies that

$$I_{21} \leq \frac{C|x|^{2(d+\alpha)}}{t^2 e^{2\nu t}} \int_0^t \sum_{z \in D_{21}^{(s)}} \frac{(t-s)}{|x-z|^{d+\alpha} (s+1)^{\frac{2d}{\alpha}}} e^{\nu(t+s)} ds. \quad (77)$$

Since,  $|z| \leq s^{\frac{1}{\alpha}} \leq t^{\frac{1}{\alpha}} < |x|/2$ , it follows that  $|x-z| \geq |x|/2$ .

So in order to estimate the right side of (77), we replace  $|x-z|$  in (77) by  $|x|/2$ .

After that, the summation sign,  $\sum_{z \in D_{21}^{(s)}}$ , in (77) can be replaced by the number of terms in this sum which is  $\kappa_1$ . Obviously,  $\kappa_1 \leq C(A_1^d + 1)$ , where  $A_1$  is the radius of the first ball  $P_1$ , i.e.,  $\kappa_1 \leq C[s^{\frac{d}{\alpha}} + 1]$ .

Then

$$I_{21} \leq \frac{C|x|^{d+\alpha}}{t^2 e^{\nu t}} \int_0^t \frac{(t-s)}{(s+1)^{\frac{d}{\alpha}}} e^{\nu s} ds.$$

Since we have  $|x| \leq Bt^\gamma$ ,  $\gamma = \frac{2\alpha + d}{\alpha(\alpha + d)}$ , we can replace  $|x|^{d+\alpha}$  by  $Bt^{\gamma(d+\alpha)}$ .

Hence

$$I_{21} \leq \frac{Ct^{d/\alpha}}{e^{\nu t}} \int_0^t \frac{(t-s)}{(s+1)^{\frac{d}{\alpha}}} e^{\nu s} ds. \quad (78)$$

For the estimation of (78), we split the integral in its right hand side so that:

$$\int_0^t = \int_0^{t/2} + \int_{t/2}^t.$$

Let us denote by,  $I_{21}^{(1)}$  and  $I_{21}^{(2)}$ , the right hand side in (78), with the integral over  $[0, t]$  is replaced by the integrals over  $[0, t/2]$  and  $[t/2, t]$  respectively.

In order to estimate  $I_{21}^{(1)}$ , we replace  $(t-s)$  by  $t$  since  $(t-s) \leq t$ , and we drop the term  $\frac{1}{(1+s)^{\frac{d}{\alpha}}}$  since it is less than 1. Then

$$I_{21}^{(1)} = \frac{Ct^{d/\alpha}}{e^{\nu t}} \int_0^{t/2} \frac{(t-s)}{(s+1)^{\frac{d}{\alpha}}} e^{\nu s} ds < \frac{Ct^{\frac{d}{\alpha}+1}}{e^{\nu t}} \int_0^{t/2} e^{\nu s} ds. \quad (79)$$

The right hand side of (79), decays exponentially as  $t \rightarrow \infty$ , so it is bounded.

In order to estimate  $I_{21}^{(2)}$ , we drop the 1 in  $(s+1)^{\frac{d}{\alpha}}$ , and make the change of variable,  $s = \tau t$ . Then

$$I_{21}^{(2)} = \frac{Ct^{d/\alpha}}{e^{\nu t}} \int_{t/2}^t \frac{(t-s)}{s^{\frac{d}{\alpha}}} e^{\nu s} ds \leq \frac{Ct^2}{e^{\nu t}} \int_{1/2}^1 \frac{(1-\tau)}{\tau^{\frac{d}{\alpha}}} e^{\nu \tau t} d\tau. \quad (80)$$

From Lemma 10, it follows that, the right hand side of (80) is bounded as  $t \rightarrow \infty$ .

The boundedness of both (79) and (80) prove that  $I_{21}$ , given in (78), is bounded in the first case when,  $2t^{\frac{1}{\alpha}} < |x| \leq Bt^\gamma$  and  $t \rightarrow \infty$ .

#### 4.7.2 Case 2: region $D_{21}$ when $|x| \leq 2t^{\frac{1}{\alpha}}$

For this case, we have that,  $m_1^2(t, x) \sim \frac{1}{t^{\frac{2d}{\alpha}}} e^{2t\nu}$ , as  $t \rightarrow \infty$ .

From (52),  $I_{21}$  satisfies that

$$I_{21} < \frac{Ct^{\frac{2d}{\alpha}}}{e^{2\nu t}} \int_0^t \sum_{z \in D_{21}^{(s)}} \frac{(t-s)}{|x-z|^{d+\alpha} (s+1)^{\frac{2d}{\alpha}}} e^{\nu(t+s)} ds. \quad (81)$$

For the estimation of the right side of (81), we use that  $\sum_{z \in D_{21}} \frac{1}{|x-z|^{d+\alpha}} < C$  for some  $x$ -independent constant  $C < \infty$  (since  $|x-z| \geq 1$ , see the details in the subsection on  $D_{22}$ ).

Thus for  $|x| \leq 2t^{\frac{1}{\alpha}}$  and  $t \rightarrow \infty$ , we have

$$I_{21} \leq \frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} \int_0^t \frac{(t-s)}{(s+1)^{\frac{2d}{\alpha}}} e^{\nu s} ds \leq C < \infty. \quad (82)$$

The boundedness of the right hand side of (82) can be proved exactly in the same way that was used to prove that (78) is bounded.

Results obtained in sections 4.7.1 and 4.7.2 imply that  $I_{21}$  is bounded as  $t \rightarrow \infty$ .

Now the boundedness of all  $I_{ij}$ , for  $i, j = 1, 2$  has been proved. This proves that  $\frac{m_2}{m_1}$ , given in (51) is bounded when  $|x| \leq Bt^\gamma, t \rightarrow \infty$ .

This completes the proof of the 2nd part of Theorem 9.

□

## REFERENCES

- [1] A. Agbor, S. Molchanov, B. Vainberg, Global limit theorems on the convergence of multidimensional random walks to stable processes, *Stochastics and Dynamics*, 15, 1550024 (2015)
- [2] Albeverio, S., Bogachev L., Yarovaya E., Asymptotics of branching symmetric random walk on the lattice with a single source, *C. R. Acad. Sci. Paris Sr. I Math.* 326 (1998), no. 8, 975980.
- [3] Albeverio, S., Bogachev, L., Yarovaya, E., Branching random walk with a single source, *Communications in difference equations (Poznan, 1998)*, 919, Gordon and Breach, Amsterdam, 2000.
- [4] Asmussen S., Hering H., Strong Limit Theorems for general supercritical branching processes with applications to branching diffusions, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 36 no 3, pp 195-212 (1976).
- [5] D. Bakry, R. Gill, S. Molchanov, *Lectures on Probability Theory, 1992 Summer School in Probability, Saint Flour, France, ed.1994. Springer Verlag lecture Notes, No 1581.*
- [6] Bogachev L., Yarovaya E., A limit theorem for a supercritical branching random walk on  $\mathbb{Z}^d$  with a single source, translation in *Russian Math. Surveys* 53 (1998), no. 5, 10861088.
- [7] Bramson M., Maximal displacement of branching brownian motion, *Communications on Pure and Applied Mathematics*, Vol 31, 531-581 (1978).
- [8] Cranston M., Korolov L., Molchanov S., Vainberg B., Continuous model for homopolymers, *Journal of Functional Analysis* 256 (2009), no. 8, 26562696.
- [9] Engländer J., Harris S. C., Kyprianou A.E., Strong Law of Large numbers for branching diffusions, *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, (2010), Vol.46, No. 1, 279-298.
- [10] Fedoryuk, M.V., *The saddle point method*, Moskow, Nauka, 1977, [Russian].
- [11] Fischer R.A., On the dominance ratio, *PRS Edinburgh*, 42 (1922), 321-341.
- [12] Fischer R.A., *The genetical theory of natural selection*, Oxford University Press, 1930.
- [13] Fischer R.A., The distribution of gene ratios for rare mutations *PRS Edinburgh*, 50 (1930) 204-219.
- [14] Haldane J.B.S., *A mathematical theory of natural and artificial selection, part V: Selection and mutation*, *PCPS*, 23 (1927), 838-844.

- [15] Hawkins D., Ulam S., Theory of Multiplicative Processes I, Los Alamos Scientific Laboratory, LADS-265 (1944).
- [16] A. Kolmogorov, I. Petrovski, N. Piskunov, A study of the diffusion operation with increase in the quantity of matter and its application to a biological problem, Bull.of Moscow University, Ser.A, No 2, (1937), 231-258.
- [17] L. Koralov, Branching diffusion in inhomogeneous media, submitted for publication.
- [18] S. Molchanov, L. Koralov, The structure of the population inside the propagating front. Journal of Mathematical Sciences (Problems In Mathematical Analysis), Vol 189, no 4 (2013), pp 637-658.
- [19] Marcel Ortgiese, Matthew I. Roberts Intermittency for branching random walk in heavy tailed environment, arXiv:1405.5449
- [20] Semenov N.N., Chain Reactions, (in Russian), Goshimizdat, 1934.
- [21] Watanabe S., A limit theorem of branching processes and continuous state branching processes, J. Math. Kyoto Univ. 8, 141-167 (1968).
- [22] Watson H.W., Galton F., On the probability of the extinction of families, J. Anthropol. Inst. Great Britain and Ireland, 4 (1874), 138-144.
- [23] Yarovaya E., Branching random walks with heavy tails, Comm. in Statistics - Theory and Methods, 42:16 (2013), pp 2301-2310.
- [24] Ya.Zeldovich, S. Molchanov, A.Ruzmaikin, D. Sokoloff. Intermittency, diffusion and generation of non-stationary random media. Sov. Sci. Reviews, Sec.C, 7, 1 - 110, Harvard Acad.Publish.