

BRAID INDEX BOUNDS ROPELENGTH FROM BELOW

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ABSTRACT. For an un-oriented link \mathcal{K} , let $L(\mathcal{K})$ be the ropelength of \mathcal{K} . It is known that in general $L(\mathcal{K})$ is at least of the order $O((Cr(\mathcal{K}))^{3/4})$, and at most of the order $O(Cr(\mathcal{K}) \ln^5(Cr(\mathcal{K})))$ where $Cr(\mathcal{K})$ is the minimum crossing number of \mathcal{K} . Furthermore, it is known that there exist families of (infinitely many) links with the property $L(\mathcal{K}) = O(Cr(\mathcal{K}))$. A long standing open conjecture states that if \mathcal{K} is alternating, then $L(\mathcal{K})$ is at least of the order $O(Cr(\mathcal{K}))$. In this paper, we show that the braid index of a link also gives a lower bound of its ropelength. More specifically, let $\mathbf{B}(\mathcal{K})$ be the largest braid index among all braid indices corresponding to all possible orientation assignments of the components of \mathcal{K} (called the *maximum braid index* of \mathcal{K}), we show that there exists a constant $a > 0$ such that $L(\mathcal{K}) \geq a\mathbf{B}(\mathcal{K})$ for any \mathcal{K} . Consequently, $L(\mathcal{K}) \geq O(Cr(\mathcal{K}))$ for any link \mathcal{K} whose absolute braid index is proportional to its crossing number. In the case of alternating links, the absolute braid indices for many of them are proportional to their crossing numbers hence the above conjecture holds for these alternating links.

1. INTRODUCTION

An important geometric property of a link with n components is its ropelength, defined as the minimum length of n unit thickness ropes needed to tie the link. Let \mathcal{K} be an un-oriented link, $Cr(\mathcal{K})$ be the minimum crossing number of \mathcal{K} and $L(\mathcal{K})$ be the ropelength of \mathcal{K} . In general the determination of the precise ropelength of a non-trivial link is a difficult problem with few exceptions (for example, the ropelength of a simple chain of $n \geq 2$ rings, namely the connected sum of $n - 1$ Hopf links, is $(4\pi + 4)n - 8 = 2(\pi + 1)Cr(\mathcal{K}) + 4(\pi - 1)$ [4]). In the case of a knot (namely a link with $n = 1$ component), it is known that $L(\mathcal{K}) \geq 31.32$ for any nontrivial knot \mathcal{K} [5], but the precise ropelength of any nontrivial knot is not known and may never be known. In the past two decades, much effort has been devoted to finding good estimates of the ropelength in the forms of lower or upper bounds of the ropelength. For example, it has been shown in [1, 2] that in general $L(\mathcal{K}) \geq 1.105(Cr(\mathcal{K}))^{3/4}$ and that this $3/4$ power can be attained by a family of infinitely many links [3, 7], and that $L(\mathcal{K})$ is at most of the order $O(Cr(\mathcal{K}) \ln^5(Cr(\mathcal{K})))$ [10]. Furthermore, not all links obey this $3/4$ power law since there exist families of infinitely many links such that the ropelength of a link from such a family grows linearly as the crossing number of the link [12]. On the other hand, no families of links are known to have a ropelength growth rate that is super linear in terms of the crossing numbers.

It is intuitive that links with smaller ropelengths tend to be highly non-alternating such as the $(n, n + 1)$ torus knots that obey the $3/4$ power law [3, 7], while the ones with larger ropelengths tend to be “more alternating”. The intuition here is that it is “cost effective” for creating multiple crossings if this is achieved by a (near) straight strand to cross over (under) many strands. One cannot help but wonder what happens to alternating links. In fact the following conjectures had been proposed more than a decade ago and are still open:

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Conjecture 1.1. *There exists a constant $a_0 > 0$ such that $L(\mathcal{K}) \geq a_0 Cr(\mathcal{K})$ for any alternating link \mathcal{K} .*

Conjecture 1.2. *There exists a constant $a_1 > 0$ such that $L(\mathcal{K}) \leq a_1 Cr(\mathcal{K})$ for any link \mathcal{K} .*

It is well known (and easy to prove) that the ropelength of a link is bounded below by the bridge number of the link (multiplied by some positive constant) and the links constructed in [12] have bridge numbers proportional to their crossing numbers. Of course, if a link has a small bridge number, then this approach would not provide us a good ropelength lower bound of the link. In this paper we show that for any oriented link, its braid index (corresponding to the given orientation) bounds the ropelength of the link from below (again up to the multiple of a positive constant). Note that different orientation assignments to components of a link may lead to different braid indices. Since this result applies to all of them, the largest braid index among all braid indices corresponding to all possible orientation assignments of the components of \mathcal{K} will thus give us the largest lower bound. We will call this largest braid index the *maximum braid index* of \mathcal{K} and denote it by $\mathbf{B}(\mathcal{K})$ in the rest of the paper. On the other hand, the smallest braid index among all braid indices corresponding to all possible orientation assignments of the components of \mathcal{K} is still greater than or equal to the bridge number of \mathcal{K} . This, together with the facts that the braid index is much easier to determine or estimate (due to its relationship with the HOMFLY-PT polynomial) than the bridge number in general, and that the braid indices for many classes of alternating links are now computable due to some recent results [9], makes our result much more applicable. As one will see in Section 4, Conjecture 1.1 holds for many classes of alternating links.

2. ELEMENTARY TANGLES AND THEIR SEIFERT DIAGRAMS

Definition 2.1. Let T_n be an n -string tangle with the end points of the strings $\alpha_1, \alpha_2, \dots, \alpha_n$ fixed on the boundary of the tangle. We say that T_n is *elementary* if (1) Geometrically T_n is of the shape of a cylinder; (2) if T_n is positioned in \mathbf{R}^3 so that its axis is parallel to the z -axis, then there exist constants z_j, z'_j for $j = 1, 2, \dots, n$ such that the “slabs” defined by $z'_j < z < z''_j$ do not overlap and that each α_j is bounded within the slab $z'_j < z < z''_j$ and (3) each α_j is a simple curve, that is, if its end points are joined by an arc on the boundary of T_n , then the result would be a trivial knot.

Figure 1 shows the top view of an elementary tangle. From this point on, we will assume that an elementary tangle T_n has been positioned so that the conditions in Definition 2.1 are satisfied.

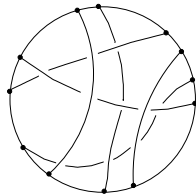


FIGURE 1. The top view of an elementary tangle.

Remark 2.2. The strings of an elementary tangle T_n can be ordered by a “height index” such that when the strings are projected to the xy -plane, the string with higher index is always the over strand when two strings cross each other in the projection. In fact, the definition of an elementary tangle given in Definition 2.1 is purely for the sake of convenience and simplicity as it suffices for the purpose of this paper. It certainly can be generalized to any tangle for which a projection direction can be

chosen such that the strings can be ordered by a height index so that when the strings are projected in this direction to a plane, the string with higher index is always the over strand when the projections of two strings intersect.

Remark 2.3. Let T'_n be an elementary tangle obtained from T_n by replacing each α_j with a simple curve α'_j that is bounded in $z'_j < z < z''_j$ and shares the same end points of α_j . Then by conditions (2) and (3) in Definition 2.1, there exists a boundary preserving ambient isotopy from T_n to T'_n that takes each α_j to α'_j . We say that T'_n is *equivalent* to T_n .

We say that T_n is oriented if each string in it is oriented. The *projection diagram* of an elementary tangle is the diagram obtained by projecting the tangle and its strings to the xy -plane. Two projection diagrams are said to be *equivalent* if their corresponding elementary tangles are equivalent. The *Seifert diagram* of an elementary tangle is the diagram obtained from its projection diagram by smoothing all crossings in the diagram as shown in Figure 2. Since there will be no crossings in the Seifert diagram of an elementary tangle, there are only two kinds of curves in the Seifert diagram: the ones that are topological circles (which we will call *Seifert circles*) and the ones that are open arcs whose end points are the projection points of the strings of the tangle (which we will call *partial Seifert circles*).



FIGURE 2. Smoothing a crossing in the projection of two oriented strands.

Remark 2.4. Consider a projection diagram P'_n with n strings in which each projected string share end points with one and only one projected string of the projection diagram P_n of T_n and the same height index. Then by Remark 2.3, there exists an elementary tangle T'_n that is equivalent to T_n with P'_n as its projection. That is, P'_n is equivalent to P_n .

Remark 2.5. Since the projection of each of the $2n$ end points of the strings of T_n belongs to one and only one partial Seifert circle in the Seifert diagram of T_n , we see that the number of partial Seifert circles in the Seifert diagram of T_n is exactly n . On the other hand, the number of Seifert circles in the Seifert diagram varies depending on the tangle itself. It is easy to see that some T_n may not have any Seifert circles in its Seifert diagram, but up to how many Seifert circles it may have? A generalization of the example shown in Figure 3 tells us that this number can be as large as the order of $n^2/8$.

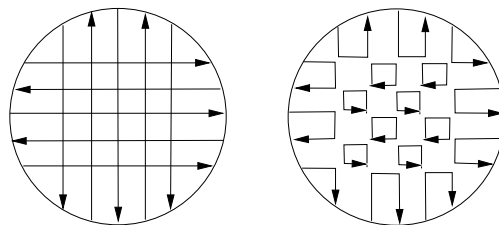


FIGURE 3. The projection diagram of an elementary tangle with 10 strings and its Seifert diagram.

Notice that as long as we are only interested in the Seifert diagrams of elementary tangles, the information about the over/under strands at the crossings in the diagrams is not important since that does not affect how we smooth the crossings. For this reason we shall omit the over/under strand information in the illustrative figures concerning the Seifert diagrams as we did in Figure 3. Furthermore, by applying a proper homeomorphism, we can also assume that the cross section of T_n

is a round disk so that C , the boundary of the projection diagram of T_n , is a geometric circle. This shall not affect our statement in Lemma 2.8.

Definition 2.6. Let P_n be the projection diagram of an elementary tangle T_n with β_1, \dots, β_n been the projections of the strings of T_n . Assume that C has been assigned an orientation. We call the arc β'_j of C that shares end points with β_j and is parallel to β_j (in terms of their orientations) the *companion* of β_j and the region bounded by β_j and β'_j the *domain* of β_j ($j = 1, 2, \dots, n$). See Figure 4. We say that P_n is *coherent* with respect to the given orientation of C if (1) the center O of C does not belong to the domain of β_j for any j and (2) for any point $x \in \beta_j$, the straight line going through and O and x intersects β_j only once (at x) and does so transversely.

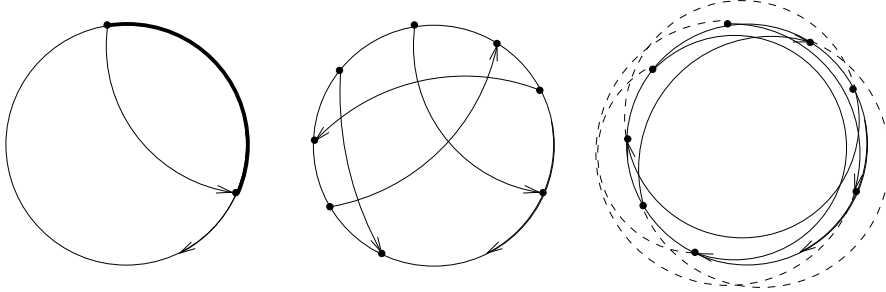


FIGURE 4. Left: An oriented arc in C with its companion on C (highlighted); Middle: A projection diagram of an elementary tangle that is not coherent; Right: A coherent projection diagram equivalent to the middle one.

Remark 2.7. The middle of Figure 4 shows a projection diagram that is not coherent since the union of the domains of the projected strings is the entire disk. By Remark 2.4, it is rather obvious that any projection diagram of an elementary tangle T_n admits an equivalent projection diagram that is coherent: one only needs to isotope each β_j (with its end points fixed) to a new curve so that is close enough to its companion, for example by an arc from a circle with radius much larger than that of C as shown in the right side of Figure 4.

A coherent projection diagram of T_n has an important property as stated in the following lemma, which provides a key fact in proving our main result in the next section.

Lemma 2.8. *Let P_n be a coherent projection an elementary tangle with n strings, then its Seifert diagram contains exactly n partial Seifert circles and at most $n - 1$ Seifert circles.*

Proof. Assume that C has the clockwise orientation. Connect the end points of each β_j by any simple arc outside of C results in an oriented simple closed curve (with its orientation induced by the orientation of β_j). Furthermore, if choose this arc to be close to the portion of C that is not the companion of β_j for each j , then we obtain a link diagram D_n with n components in which each component is a simple closed curve such that as one travels along β_j following its orientation, the center O of C is always on one's right hand side. It is well known that D_n is in a closed braid form with n strings hence its Seifert circle decomposition has exactly n Seifert circles. Since at least one of these Seifert circles will contain some end points of the strings β_j , at most $n - 1$ of them contain no end points of the β_j 's. Since a Seifert circle of P_n corresponds to a Seifert circle of D_n that contains no end points of the β_j 's, the statement of the lemma follows. \square

3. MAXIMUM BRAID INDEX BOUNDS THE ROPELENGTH FROM BELOW

Let us first consider links realized on the cubic lattice. Let \mathcal{K} be an un-oriented link and K_c a realization of \mathcal{K} on the cubic lattice. The length of K_c is denoted by $L(K_c)$ and the minimum of $L(K_c)$ over all lattice realization K_c of \mathcal{K} is called the *minimum step number* of \mathcal{K} and is denoted by $L_c(\mathcal{K})$. The discrete nature of the lattice polygons allows one to determine the precise value of $L_c(\mathcal{K})$ (at least for some small knots and links) through exhaustive search. For example, it has been shown that $L_c(\mathcal{K}) = 24$ for the trefoil [6], $L_c(\mathcal{K}) = 30$ for the figure 8 knot and $L_c(\mathcal{K}) = 34$ for the 5_1 knot [14]. For our purpose in this paper, its discrete structure also allows us to apply combinatorics methods for our analysis.

A line segment on K_c between two neighboring lattice points is called a *step*. A step that is parallel to the x -axis is called an x -step. y -steps and z -steps are similarly defined. Let $x(K_c)$, $y(K_c)$ and $z(K_c)$ be the total number of x -steps, y -steps and z -steps respectively, then $x(K_c) + y(K_c) + z(K_c) = L(K_c)$. Without loss of generality, let us assume that $z(K_c) \geq \max\{x(K_c), y(K_c)\}$ hence $z(K_c) \geq (1/3)L(K_c)$ and $x(K_c) + y(K_c) = L(K_c) - z(K_c) \leq (2/3)L(K_c)$. We now consider the projection of K_c to the xy -plane. Let p_1, p_2, \dots, p_m be all the square lattice points in the xy -plane occupied by the projection of K_c to the xy -plane. For each p_j , consider the unit square P_j centered at the lattice point p_j as shown in Figure 5, which is the projection of the square based infinite tube $Q_j = P_j \times (-\infty, \infty)$, together with the projections of the arcs of $K_c \cap Q_j$.

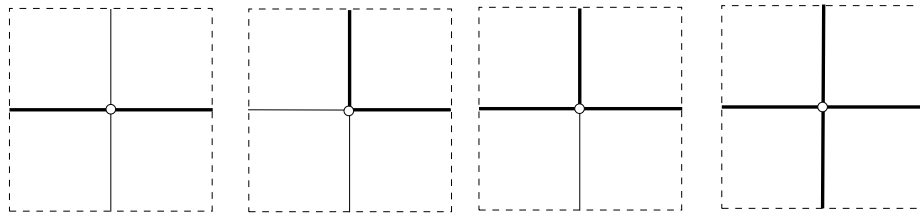


FIGURE 5. A unit square centered at a lattice point (indicated by the solid dot) occupied by the projection of K_c . The line segments that are part of the projection of K_c are marked by thick lines (they are on the square lattice), and the boundary of the unit square is marked by dashed line segments. Notice that the boundary of the square are not part of any lattice lines and are not part of the projection of K_c either, except the 4 mid points on its sides. This is not a exhaustive list: the ones that symmetric to one of the above are not listed.

Notice that each connected component of $K_c \cap Q_j$ consists of two half x, y steps and possibly several connected z steps as shown in Figure 6. It is thus apparent that these components are separated by planes parallel to the xy -plane.

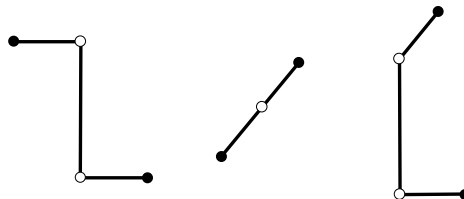


FIGURE 6. Several typical examples of connected components in $K_c \cap Q_j$ where a circle indicates a lattice point in the cubic lattice and a solid dot indicates points of K_c on the boundary of Q_j .

Furthermore, since the pre-image of an open x or y step (without its end points) in P_j consists of parallel (open) x or y steps of K_c (so they have different z coordinates), we can deform these steps without changing their z -coordinates as shown in Figure 7. This results in a link K' that is ambient isotopic to K_c . Furthermore, each Q_j bounds an elementary tangle (denoted by T^j) by Definition 2.1 whose strings are the connected components of $Q_j \cap K'$ as the end points of the strings project to different points on the boundary of P_j .

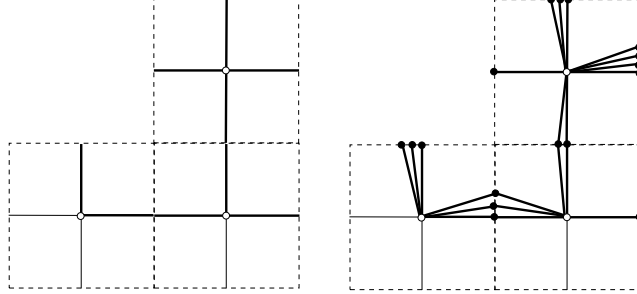


FIGURE 7. Left: A example of several P_j 's (with the projections of $K_c \cap Q_j$) sharing common boundaries; Right: The projection of corresponding $K' \cap Q_j$ (the deformed K_c) in these P_j 's. The circles are the square lattice points occupied by the projection of K_c and the solid dots indicate the projections of the strings of the corresponding elementary tangles defined by $Q_j \cap K'$.

Let n_j be the number of connected components in $Q_j \cap K_c$, namely the number of strings in the elementary tangle T^j . As noted before, each such component contains exactly two half x , y steps hence it makes a contribution of length 1 to the total length of the x and y steps in K_c . Thus the connected components in $Q_j \cap K_c$ makes a total contribution of n_j to the total length of the x and y steps in K_c . It follows that $\sum_{1 \leq j \leq m} n_j = x(K_c) + y(K_c)$. Now assign K_c an orientation so that it yields $\mathbf{B}(\mathcal{K})$. By Remark 2.7, we can modify each T^j by a boundary preserving ambient isotopy so that the result tangle has a projection diagram that is coherent. Thus we can modify the entire K' by an ambient isotopy such that the resulting link K'' has the property that $K'' \cap Q_j$ defines an elementary tangle whose projection diagram is coherent. By Lemma 2.8, each such projection diagram has n_j partial Seifert circles and at most $n_j - 1$ Seifert circles. Each partial Seifert circle must be connected to at least one other partial Seifert circle in order to form a complete Seifert circle, thus the total number of Seifert circles in the projection of K'' formed by the partial Seifert circles is at most $\frac{1}{2} \sum_{1 \leq j \leq m} n_j$. It follows that the total number of Seifert circles in the projection diagram of K'' (denoted by $s(K'')$) is bounded above by $\frac{1}{2} \sum_{1 \leq j \leq m} n_j + \sum_{1 \leq j \leq m} (n_j - 1) < \frac{3}{2} \sum_{1 \leq j \leq m} n_j = \frac{3}{2} (x(K_c) + y(K_c)) \leq \frac{3}{2} (2/3) L(K_c) = L(K_c)$. It is well known that for any oriented link diagram \mathcal{D} , we have $\mathbf{b}(\mathcal{D}) \leq s(\mathcal{D})$ where $s(\mathcal{D})$ is the number of Seifert circles in \mathcal{D} [15]. Since K_c has the orientation that yields $\mathbf{b}(K_c) = \mathbf{B}(\mathcal{K})$, we have $\mathbf{B}(\mathcal{K}) = \mathbf{b}(K_c) = \mathbf{b}(K'') \leq s(K'') < L(K_c)$. Since K_c is arbitrary, replacing it by a step length minimizer of \mathcal{K} yields $\mathbf{B}(\mathcal{K}) < L_c(\mathcal{K})$. Finally, it has been shown that $L_c(\mathcal{K}) < 14L(\mathcal{K})$ [11], thus we have proven the following theorem:

Theorem 3.1. *Let \mathcal{K} be an un-oriented link, then $\mathbf{B}(\mathcal{K}) < L_c(\mathcal{K}) < 14L(\mathcal{K})$, that is, $L(\mathcal{K}) > (1/14)\mathbf{B}(\mathcal{K})$.*

4. APPLICATIONS

In a recent paper, the author and his colleagues derived explicit formulas for braid indices of many alternating links including all alternating Montesinos links [9]. Using these formulas one can easily

identify many families of alternating links with small bridge numbers but with braid indices proportional to their crossings numbers, these provide us new examples of link families whose ropelengths grow at least linearly as their crossing numbers. The following are just a few such examples.

Example 4.1. Let \mathcal{K} be the $(2, 2n)$ torus link, a two component link with $2n$ crossings. There are two different choices for the orientations of the two components. One of them yields a braid index of 2 while the other yields a braid index of $n + 1$. Thus we have $\mathbf{B}(\mathcal{K}) = n + 1 = Cr(\mathcal{K})/2 + 1$, hence $L_c(\mathcal{K}) > n + 1$ and $L(\mathcal{K}) > (n + 1)/14 > Cr(\mathcal{K})/28$.

Example 4.2. Let \mathcal{K} be a twist knot with $n \geq 4$ crossings. We have $\mathbf{B}(\mathcal{K}) = \mathbf{b}(\mathcal{K}) = k + 1 = (Cr(\mathcal{K}) + 1)/2$ if $n = 2k + 1$ is odd, and $\mathbf{B}(\mathcal{K}) = \mathbf{b}(\mathcal{K}) = k + 2 = Cr(\mathcal{K})/2 + 1$ if $n = 2k + 2$ is even. It follows that $L(\mathcal{K}) > (Cr(\mathcal{K}) + 1)/28$ for any twist knot \mathcal{K} .

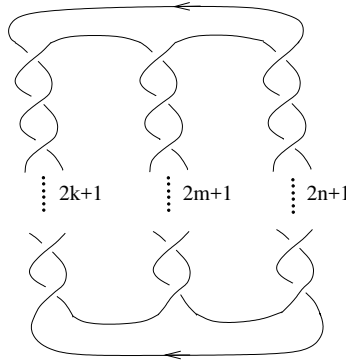


FIGURE 8. An alternating pretzel knot with three columns containing $2k + 1$, $2m + 1$ and $2n + 1$ crossings respectively (k , m and n are non-negative integers and the case of $k = m = n = 0$ gives the trefoil knot).

Example 4.3. Consider the pretzel knot \mathcal{K} a projection of which is given in Figure 8. $Cr(\mathcal{K}) = 2(k + m + n) + 3$ since it is alternating. It can be calculated from the formulas given in [9] that $\mathbf{B}(\mathcal{K}) = \mathbf{b}(\mathcal{K}) = 2 + k + m + n > (1/2)Cr(\mathcal{K})$. It follows that $L(\mathcal{K}) > Cr(\mathcal{K})/28$ as well.

Notice that in the above examples, the bridge numbers are either 2 or 3. Furthermore, since the link diagrams given in the above examples are all algebraic link diagrams, it is known that the ropelengths of these links grow at most linearly as their crossings numbers [8]. Thus the ropelengths of these links in fact grow linearly as their crossing numbers.

For an oriented link \mathcal{K} with a projection diagram \mathcal{D} , consider the HOMFLY-PT polynomial $H(\mathcal{D}, z, a)$ defined using the skein relation $aH(\mathcal{D}_+, z, a) - a^{-1}H(\mathcal{D}_-, z, a) = zH(\mathcal{D}_0, z, a)$ (and the initial condition $H(\mathcal{D}, z, a) = 1$ if \mathcal{D} is the trivial knot). Let $E(\mathcal{D})$ and $e(\mathcal{D})$ be the highest and lowest powers of a in $H(\mathcal{D}, z, a)$ and define $\mathbf{b}_0(\mathcal{K}) = (E(\mathcal{D}) - e(\mathcal{D}))/2 + 1$. It is a well known result that $\mathbf{b}_0(\mathcal{K}) \leq \mathbf{b}(\mathcal{K})$ where $\mathbf{b}(\mathcal{K})$ is the braid index of \mathcal{K} [13]. In the case that \mathcal{K} is un-oriented, similarly to the definition of $\mathbf{B}(\mathcal{K})$, we define $\mathbf{B}_0(\mathcal{K}) = \max\{\mathbf{b}_0(\mathcal{K}') : \mathcal{K}' \in O(\mathcal{K})\}$ where $O(\mathcal{K})$ is the set of oriented links obtained by assigning all possible orientations to the components of \mathcal{K} . Apparently we have $\mathbf{B}_0(\mathcal{K}) \leq \mathbf{B}(\mathcal{K})$ hence we have the following theorem, which is handy when we do not have a precise formula for the braid index of the link.

Theorem 4.4. *Let \mathcal{K} be an un-oriented link, then $\mathbf{B}_0(\mathcal{K}) < L_c(\mathcal{K}) < 14L(\mathcal{K})$ and $L(\mathcal{K}) > (1/14)\mathbf{B}_0(\mathcal{K})$.*

It has been conjectured that the ropelength of an alternating link \mathcal{K} is bounded below by a constant multiple of its crossing number. Our result shows that this conjecture holds for many alternating links.

A remaining challenge is about the alternating links whose absolute braid index is small, for example the $(2, 2n + 1)$ torus knot whose braid index is 2. While its minimum projection looks so much like the minimum projection of the $(2, 2n)$ torus link and it is quite plausible that its ropelength should behave linearly as its crossing number, we do not have a way to prove it! We end this paper with this problem as a challenge to our reader.

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