THE BRAID INDICES OF ALTERNATING LINKS

by

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ABSTRACT

PENGYU LIU. The Braid Indices of Alternating Links. (Under the direction of DR. YUANAN DIAO and DR. GÁBOR HETYEI)

It is well known in knot theory that any link can be represented by a closed braid and the braid index of a link is the invariant defined as the minimum number of strands in any closed braid representing the link. It is difficult in general to determine the braid index for a given link. However, in recent years, some progresses have been made due to the discovery of the HOMFLY polynomial. Yamada showed that the braid index of a link $\mathcal{L}$ equals the minimum number of Seifert circles of $\mathcal{L}$. With this connection, one might conjecture that the braid index of an alternating link equals the number of Seifert circles in any of its reduced alternating diagrams. However, this is not true in general. In this dissertation, we prove this conjecture is in fact true for a class of alternating links. Specifically, we prove that if D is a reduced alternating diagram of an alternating link $\mathcal{L}$, then the braid index $b(\mathcal{L})$ equals the number of Seifert circles in D if and only if the Seifert graph of D contains no edge of weight one where the Seifert graph $G(D)$ is a simple and edge weighted graph whose vertices correspond to Seifert circles in D and two vertices are connected by an edge of weight k if the corresponding Seifert circles share k crossings.

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CHAPTER 1: INTRODUCTION

In the late nineteenth century, Tait made a few famous conjectures in knot theory [20], one of which states that if a link $\mathcal{L}$ admits a reduced alternating diagram D, then $\mathcal{L}$ can not be represented by a diagram with fewer crossings than the number of crossings in D. In other words, the crossing number of an alternating link equals the number of crossings in any of its reduced alternating diagrams. After almost a hundred years, as one of the great applications of the Jones polynomial, Kauffman [9], Murasugi [13] and Thistlethwaite [21] proved this conjecture independently using the inequality that the span of the variable of the Jones polynomial is less than or equal to the number of crossings in the computed diagram. They proved that the span is actually equal to the number of crossings in the diagram for any alternating link.

In this dissertation, we discuss another invariant, the braid index of an oriented link. It is known that every link possesses a closed braid representation [1, 22]. The braid index of an oriented link is the minimum number of strands needed to represent the link in a closed braid form. The relationship between the braid index and the HOMFLY polynomial, a generalization of the Jones polynomial with two variables a and z [6, 17], is in some sense analogous to the relationship between the crossing number and the Jones polynomial as one can see from the following. It has been shown that a-span$/2+1$ is a lower bound of the number of Seifert circles [11].

Here, the a-span is the difference between the highest and the lowest a-degree in the HOMFLY polynomial of a given link. This inequality is called the Morton-Frank-Williams inequality. Since Yamada showed that there exists a transformation from any diagram to a closed braid without changing the number of Seifert circles [22] and the number of Seifert circles in a closed braid equals the number of its strands, a-span$/2+1$ is also a lower bound for the braid index of a given link. Murasugi thus conjectured that a-span$/2+1$ equals the braid index for any alternating link $\mathcal{L}$ [14]. This conjecture turned out to be false. A counterexample in [15] is displayed in Chapter 6. Since then, subsequent research focused on identifying specific link classes where the equality holds. Examples include the closed positive braids with a full twist, in particular the torus links [5], 2-bridge links, fibered alternating links [14] and a new class of links discussed in a more recent paper [10]. For more readings on this topic, see [2, 3, 4, 12, 16, 19].

The main result of this dissertation is that if D is a reduced alternating diagram of an alternating link $\mathcal{L}$, then the braid index $b(\mathcal{L})$ equals the number of Seifert circles in D if and only if its Seifert graph $G(D)$ contains no edge of weight one. This result completely characterizes the class of alternating links whose braid indices equal the numbers of Seifert circles in their corresponding diagrams, where the Seifert graph $G(D)$ is a simple and edge weighted graph whose vertices are Seifert circles in D and two vertices are connected by an edge of weight k if their corresponding Seifert circles in D share k crossings. It is worth noting that the main result dose not characterize the entire class the alternating links whose braid indices equal a-span$/2+1$ from their HOMFLY polynomials and to characterize this class is a much more difficult task.

This dissertation is organized in the following way. In Chapter 2, we introduce some basic definitions in knot theory and the prerequisites to understand the result. In Chapter 3, we introduce our core method and use it to compute the braid indices of alternating closed braids, which is a special case of the main result. In Chapter 4, we introduce several important concepts including Seifert graphs, ring diagrams and castles in order to extrapolate our method to link diagrams. Based on the castle structure, two generalized algorithms are developed to generate two resolving trees and the main theorem is proved in Chapter 5. Finally, in Chapter 6, we make a general conjecture about the braid indices of all alternating links by defining the reduction number of a diagram.

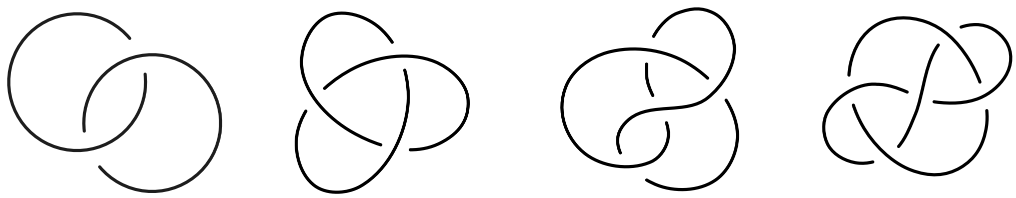
CHAPTER 2: BASIC DEFINITIONS

2.1 Links and Knots

Definition 1. A *link* is an embedding $L : S^1 \sqcup S^1 \sqcup ... \sqcup S^1 \to \mathbb{R}^3$, where $S^1 \sqcup S^1 \sqcup ... \sqcup S^1$ is the disjoint union of finitely many circles. In particular, a *knot* is an embedding $K : S^1 \to \mathbb{R}^3$.

A link is of n *components* if it is an embedding of disjoint union of n circles and it is trivial that any knot has only one component. Recall that an *embedding* is an injective continuous map such that the domain is homeomorphic to its image. If the circles in the domain are oriented, we define an *oriented link* to be an embedding such that the homeomorphism onto its image preserves the orientation.

Instead of studying the embeddings, we usually analyze the planar diagrams, that is, the projections of the images of the embeddings onto the two-dimensional plane. A point in a planar diagram is a *multiple point* if its pre-image under the projection contains more than one element. We also require that each pair of multiple points in a planar diagram is far enough from each other, that is, for each multiple point there exists a δ-neighborhood, where $\delta > 0$, such that no other multiple points are in this neighborhood. Such a neighborhood of a multiple point in a planar diagram is a *crossing*. A planar diagram is *regular* if the pre-image of any multiple point under the projection contains only two elements. It is necessary to define at each crossing in a regular planar diagram an *overpass* and an *underpass* to indicate how the embedding of circles are tangled in the space. From this point on, a *diagram* D will be refer to a regular planar diagram of a link unless otherwise specified.



Hopf link Trefoil knot Figure-eight knot Whitehead link

Figure 1: Examples of regular diagrams

In a diagram of an oriented link, there are two different species of crossings, the left one in Figure 2 is a *positive* crossing whereas the right one is a *negative* crossing.

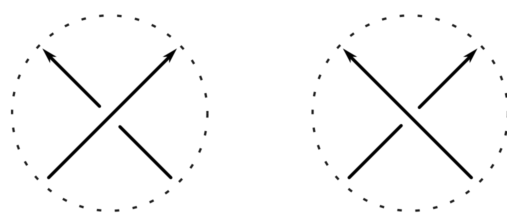


Figure 2: A positive crossing and a negative crossing

The *local writhe* for a positive crossing and a negative crossing to be $+1$ and -1 respectively and the *writhe* of a diagram is the sum of the local writhe of all the crossings. For example, the writhe of the diagram in Figure 1 is $+2$. We denote the writhe of a diagram D by $w(D)$ and the number of components in D by $\gamma(D)$.

Two links are equivalent if they can be transformed from one to another by an *ambient isotopy*.

Definition 2. Let $I = [0,1]$, $X = \bigsqcup_{i=1}^{n} S^1$ and L_1, L_2: $X \to \mathbb{R}^3$ be two links. L_1 and L_2 are *ambient isotopic* if there exists a continuous map $H : \mathbb{R}^3 \times I \to \mathbb{R}^3$ such that for any $i \in I$, $H(y, i) : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism, for any $y \in \mathbb{R}^3$, $H(y, 0) = id_{\mathbb{R}^3}$ and $L_2 = H(y, 1) \circ L_1$. The map H is an *ambient isotopy.*

If L_1 and L_2 are oriented links, we also need every $H(y, i)$ to be an orientation preserving homeomorphism. It's trivial to check that being ambient isotopic is an equivalence relation. Each equivalence class of links is a *link type*, in particular, each equivalence class of knots is a *knot type.* We denote a link type by $\mathcal{L}$ and a knot type by $\mathcal{K}$. In this dissertation, a link type or a knot type may also be referred to as a link or a knot. The reader should distinguish by the symbols and the context if necessary. If a link type contains the trivial embedding then the link type is the *unlink* with n components. Notably, If $n = 1$, the knot type is the *unknot.*

Since we can lift a diagram into the three-dimensional space, it is natural to define that two diagrams are equivalent if their corresponding embeddings are equivalent. Hence, we may also consider a link type as the set of all equivalent diagrams.

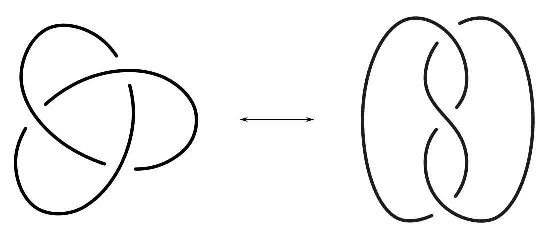


Figure 3: Equivalent diagrams of the trefoil knot

Theorem 1 (Reidemeister [18]). Two links L_1 and L_2 are equivalent if and only if a diagram of L_1 can be transformed in to a diagram of L_2 by a sequence of local moves of the following three types:

If L_1 and L_2 are oriented links, then each Reidemeister move should be oriented in all possible ways.

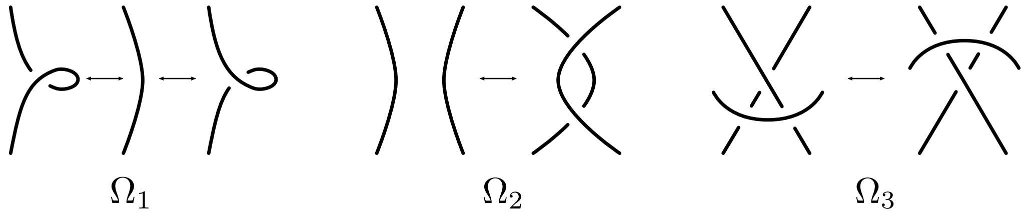


Figure 4: Reidemeister moves

Given an oriented knot K, let K' be its mirror image and K_* be the diagram constructed by inverting the orientation. It is not necessary that they represent the same knot type as K. For example, the trefoil knot is not equivalent to its mirror image. These two operations generate a group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, each subgroup represents a class of knots in terms of the symmetric properties. $\mathcal{K}$ is *totally asymmetric* if it is equivalent to none of K', K_* or K'_*, is *fully symmetric* If it is equivalent to all of K', K_* and K'_*, is *invertible* if K is equivalent to K_*, is *plus-amphicheiral* if K is equivalent to K' and is *minus-amphicheiral* if K_* is equivalent to K'. For instance, the trefoil knot is invertible and the figure-eight knot is fully symmetric.

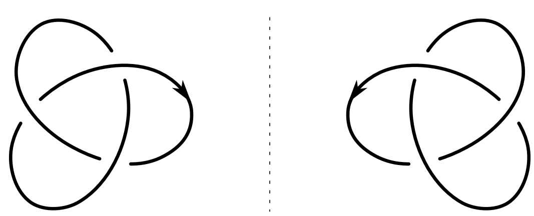


Figure 5: A left-handed trefoil knot and its mirror image

As for links with n components, since we can inverse the orientation of each component, the group generated by the symmetric operations is isomorphic to $\bigoplus_{i=1}^{n+1} \mathbb{Z}_2$. Similarly, each subgroup represents a symmetric class of links.

A very important class of links is the alternating links. A diagram is *alternating* if as we travel along the diagram, its overpass and underpass appear alternately.

Definition 3. A link $\mathcal{L}$ is *alternating* if it has an alternating diagram.

For example, the diagrams in Figure 1 are all alternating. Alternating links have several good properties in terms of diagrams. A crossing in a diagram is *nugatory* if it can be removed by a simple twist. A diagram is *reduced* if the diagram has no nugatory crossing.

Another class of links is the positive links. A diagram is *positive* if all the crossings in the diagram are positive. Analogously, A diagram is *negative* if all the crossings in the diagram are negative. Analogously, A link $\mathcal{L}$ is *positive* if it has a positive diagram and is *negative* if it has a negative diagram. There exists positive link that is also alternating, for example, the right-handed trefoil knot is both positive and alternating.

2.2 Invariants

An *invariant* of links is a function of links that is constant in the ambient isotopic equivalence classes. For example, the *crossing number* of a link $\mathcal{L}$ is the minimal number of crossings in the set of diagrams of $\mathcal{L}$. It's trivial to see the the crossing number is a link invariant from the space of links to the set of natural numbers. For instance, the crossing number of the unknot is 0 and of the trefoil knot is 3.

Another example of the invariants is the Jones polynomial. The *Jones polynomial* of an oriented link is a Laurent polynomial $J \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ such that $J(U) = 1$ and J satisfies the following *skein relation*:

$$tJ(L_+) - t^{-1}J(L_-) = (t^{-1/2} - t^{1/2})C(L_0) \tag{1}$$

where U is an oriented circle, that is, a trivial diagram of the unknot and L_+, L_- and L_0 are diagrams of a link that are identical everywhere except at a crossing where they are given by:

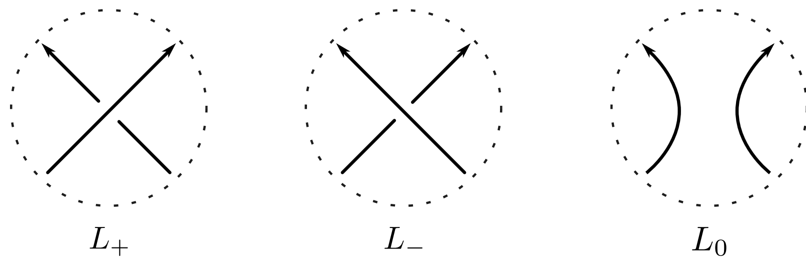


Figure 6: Notations in the skein relation

For instance, the Jones polynomial of the left-handed trefoil knot is $-t^4 + t^3 + t^1$ and of the Hopf link is $-t^{-1/2} - t^{-5/2}$.

2.3 Braid and Braid Index

Definition 4. A *braid* on n strands is a smooth embedding of n closed intervals, namely the *strands*, into the box $[-1, 1] \times [-a, a] \times [0, 1] \subset \mathbb{R}^3$ such that each interval connects a *starting point* in $\{0\} \times [-a, a] \times \{1\}$ and an *end point* in $\{0\} \times [-a, a] \times \{0\}$ and with the property that the z-coordinate of the tangent vector to each strand is never 0.

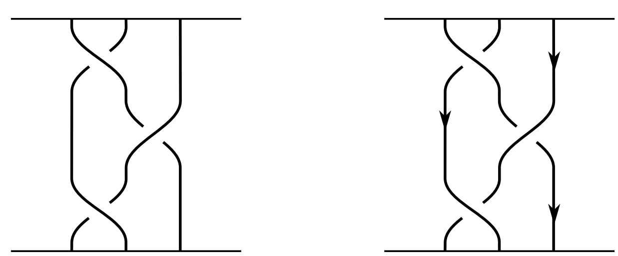


Figure 7: A braid and its orientation

If a braid is oriented, we always orient each strand downward, that is, from the starting point to the end point. Similarly, we're mainly interested in the regular planar diagrams of braids in this dissertation, instead of the embeddings. Hence, a braid always means a regular planar diagram of a braid unless otherwise specified.

Note that there are exactly n starting points and n end points for a braid on n strands as shown in Figure 7. Connect these points outside of the braid correspondingly, we have a diagram of a link. Such a diagram is a *closure* of a braid or a *closed braid*. Two braids are equivalent if their closures are equivalent link diagrams.

In 1920s, J. W. Alexander proved that any link can be represented by a braid [1]. Therefore, we can define a new invariant of links.

Definition 5. The *braid index* $b(\mathcal{L})$ of a link $\mathcal{L}$ is the minimal number strands among all the braid representations of $\mathcal{L}$.

2.4 HOMFLY Polynomial and Resolving Trees

Definition 6. The *HOMFLY polynomial* or *HOMFLY-PT polynomial* of an oriented link is a Laurent polynomial $P \in \mathbb{Z}[a, a^{-1}, z, z^{-1}]$ such that $P(U) = 1$ and P satisfies the following *skein relation*:

$$aP(L_+) - a^{-1}P(L_-) = zP(L_0) \tag{2}$$

where U, L_+, L_- and L_0 are the same as shown in the definition of the Conway polynomial.

Let U_n be an unlink with n components. It's easy to check, using the skein relation, that $P(U_n) = (a - a^{-1}/z)^{n-1}$. Moreover, let $L_1 \sqcup L_2 \sqcup ... \sqcup L_n$ be the split union of n links, that is, the union of these links such that each of these links is contained in a three-dimensional ball and each pair of balls have empty intersection. We have

$P(L_1 \sqcup L_2 \sqcup ... \sqcup L_n) = (a - a^{-1}/z)^{n-1} \prod_{i=1}^{n} P(L_i)$. Let L be a link and L' be its mirror image, then $P(L', a, z) = P(L, -a^{-1}, z)$.

The HOMFLY polynomial is a two-variable generalization of the Jones polynomial. Let $a = t$ and $z = t^{-1/2} - t^{1/2}$, then the skein relation is exactly the same as the one of the Jones polynomial.

The *Morton-Frank-Williams inequality* connects the HOMFLY polynomial to the braid index of a link.

$$b(L) \geq (E - e)/2 + 1 \tag{3}$$

where E and e are respectively the highest and lowest degree of variable a in the HOMFLY polynomial $P(L)$.

We introduce the resolving trees to compute the HOMFLY polynomial and in most cases, only E and e. First, we rewrite the skein relation in the following way.

$$P(L_+) = a^{-2}P(L_-) + a^{-1}zP(L_0) \tag{4}$$

$$P(L_-) = a^2P(L_+) - azP(L_0) \tag{5}$$

Definition 7. A *HOMFLY resolving tree* $\mathcal{T}$ of a link diagram D is a rooted binary tree constructed by recursively applying equation (4) or (5) to chosen crossings such that the root vertex is D and the HOMFLY polynomials of all leaf vertices are known.

According to the definition, there are infinitely many HOMFLY resolving trees. In this dissertation, we will use only those resolving trees whose leaf vertices are all unlinks. Hence we can compute the HOMFLY polynomial of a diagram by summing over all the branches of its resolving tree. Let D be a diagram and U be a leaf vertex

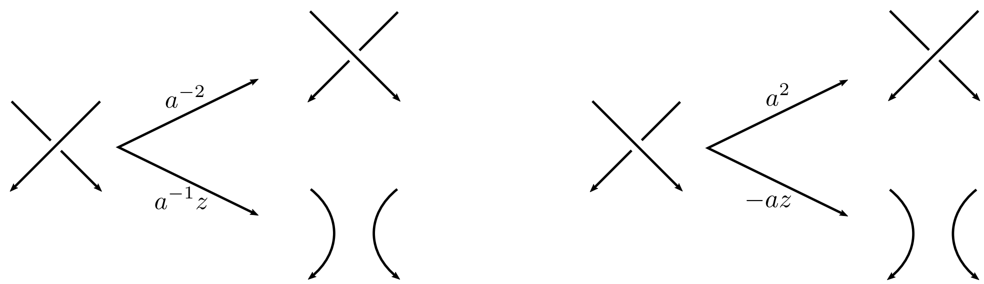


Figure 8: The HOMFLY skein relation

in a HOMFLY resolving tree of D, $t(U)$ be the number of smoothed crossings from D to U and $t^{-}(U)$ be the number of smoothed negative crossings. It follows that the total contribution of a leaf vertex U in $P(D,a,z)$ is

$$(-1)^{t^{-}(U)} z^{t(U)} a^{w(U)-w(D)} ((a-a^{-1})z^{-1})^{\gamma(U)-1}, \tag{6}$$

hence

$$P(D,a,z) = \sum_{U \in \mathcal{T}} (-1)^{t^{-}(U)} z^{t(U)} a^{w(U)-w(D)} ((a-a^{-1})z^{-1})^{\gamma(U)-1}. \tag{7}$$

It follows that the highest and lowest a-power terms that U contributes to $P(D,a,z)$ are

$$(-1)^{t^{-}(U)} z^{t(U)-\gamma(U)+1} a^{w(U)-w(D)+\gamma(U)-1} \tag{8}$$

and

$$(-1)^{t^{-}(U)+\gamma(U)-1} z^{t(U)-\gamma(U)+1} a^{w(U)-w(D)-\gamma(U)+1} \tag{9}$$

respectively.

It is well known that resolving trees exist for any given oriented link diagram D. We will describe several algorithms for constructing resolving trees with some special properties in Chapter 3 and Chapter 5.

CHAPTER 3: BRAID INDICES OF ALTERNATING CLOSED BRAIDS

In this chapter, we introduce our core method to the problems, that is, we develop two dual algorithms to generate two resolving trees for a reduced alternating braid D. The leaf vertices of the two resolving trees will provide information about the highest and the lowest a-degree respectively hence we can compute the a-span of the HOMFLY polynomial of its closure $\mathcal{L}$. If a-span$/2+1$ equals the number of strands in D, then we have the braid index of $\mathcal{L}$. We start by introducing the descending and ascending algorithms.

3.1 Descending and Ascending Resolving Trees

In this section, we will introduce the descending and the ascending algorithms to construct the descending resolving tree $\mathcal{T}^{\downarrow}(D)$ and the ascending resolving tree $\mathcal{T}^{\uparrow}(D)$ of a braid D. We shall start by stating a general approach to acquire a resolving tree.

We consider a HOMFLY resolving tree as s recorf of the branching process as shown locally in Figure 8. At each internal vertex of the resolving tree we take a crossing of the current diagram D and branch on smoothing or flipping the chosen crossing. We are growing our resolving tree by adding two children at a time to a vertex that was a leaf up until that point. A crossing is *descending* if during this process we travel along the overpassing strand first, otherwise it is *ascending*. A *descending operation* keeps a descending crossing unchanged (no branching happens when we encounter the crossing) and branches on flipping or smoothing an ascending crossing. An *ascending operation* does exactly the opposite. It keeps an ascending crossing currently visited and branches on flipping or smoothing a descending crossing.

Let D be a braid on n strands. We define the *standard way* of traveling the closure of D as follows. We start from the upper left starting point of D and start traveling. Once a component has been completely traveled, we start travel again from the next upper left starting point of D that has not been traveled before. We repeat this until all the starting points of D are traveled. A braid is *descending* if traveled in the standard way, every crossing is descending. A braid is *ascending* if traveled in the standard way, every crossing is descending. Now we define the descending and the ascending algorithms.

Algorithm D: Travel the closure of D in the standard way and apply the descending operation at the crossings encountered until the resolving tree branches. Repeat this process to the new vertices until every leaf vertices are descending.

Algorithm A: Travel the closure of D in the standard way and apply the ascending operation at the crossings encountered until the resolving tree branches. Repeat this process to the new vertices until every leaf vertices are ascending.

We denote the set of leaf vertices of $\mathcal{T}^{\downarrow}(D)$ by $\mathcal{F}^{\downarrow}(D)$ and the set of leaf vertice of $\mathcal{T}^{\uparrow}(D)$ by $\mathcal{F}^{\uparrow}(D)$. It is clear that the leaf vertices of the descending resolving tree are all descending and the leaf vertices of the ascending resolving tree are all ascending. The closures of descending braids are unlinks and the components are layered from top to bottom in the order that they are traveled. Similarly, the closures of ascending braids are all unlinks and the components are layered from bottom to top in the order that they are traveled. Moreover, they have special formulas for their HOMFLY polynomial which we will introduce in the next section.

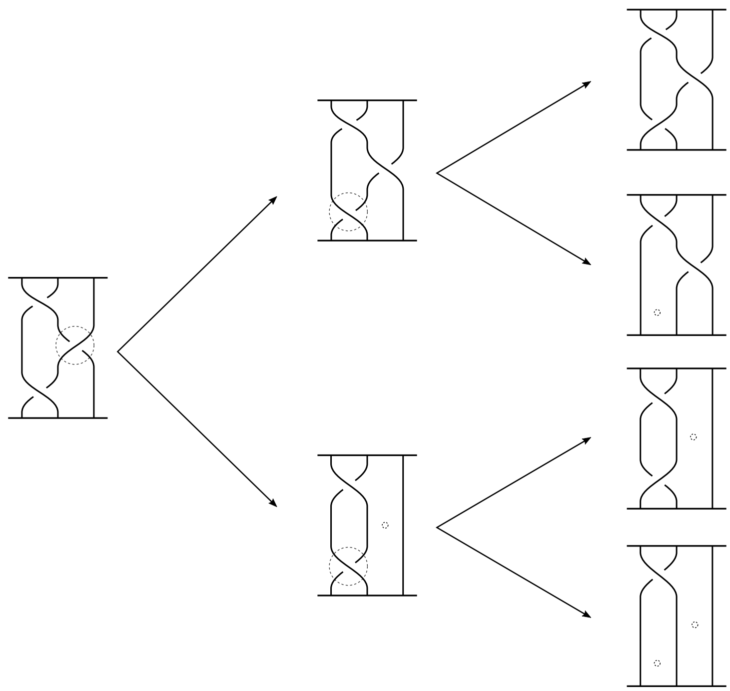


Figure 9: An example of descending resolving tree

Remark 1. [7] Let U be a braid constructed from D by flipping and smoothing some crossings of D, there is a simple way to check whether $U \in \mathcal{F}^{\downarrow}(D)$ by the way $\mathcal{T}^{\downarrow}(D)$ is generated. Note that if we travel the closure of U in the standard way, we will visit each crossing of D exactly twice. For each crossing of D that we encounter for the first time including the smoothed ones (marked by small circles in Figure 9), we perform the following test. If this crossing is smoothed in U, we check whether we would approach it from its underpass if the corresponding original crossing in D were not smoothed. On the other hand, for a crossing in U, which may or may not have been flipped, we check whether we approach it from its overpass. If all crossings pass this check, then $U \in \mathcal{F}^{\downarrow}(D)$, otherwise $U \notin \mathcal{F}^{\downarrow}(D)$.

3.2 The HOMFLY Polynomial of Closed Braids

In this section, we derive the formulas of the HOMFLY polynomial of a braid D based on $\mathcal{T}^{\downarrow}(D)$ and $\mathcal{T}^{\uparrow}(D)$ respectively.

Lemma 2. If U is a descending braid on n strands, then $\gamma(U) - w(U) = n$. On the other hand, if V is an ascending braid on n strands, then $\gamma(V) + w(V) = n$.

Proof. Since U is descending, its components are layered from top to bottom in the order that they are traveled, hence the writhe contribution of crossings whose strands belong to different components is zero. Let $\gamma(U) = k$, Γ_1, Γ_2, $\ldots$, Γ_k be the components of the closure of U and $l(\Gamma_i)$ be the number of times Γ_i travels through the braid U. We have $\sum_{1\leq i\leq k} l(\Gamma_i) = n$ and $\sum_{1\leq i\leq k} w(C_j) = w(U)$. We claim that for each component Γ_i, $l(\Gamma_i) = 1 - w(\Gamma_i)$. Since Γ_i is an unknot and is descending if traveled from the starting point, we can do the following transformation displayed in Figure 10 by sliding the strait lines in the braid within their layers using only Reidmeister move II and III, hence its writhe stays unchanged.

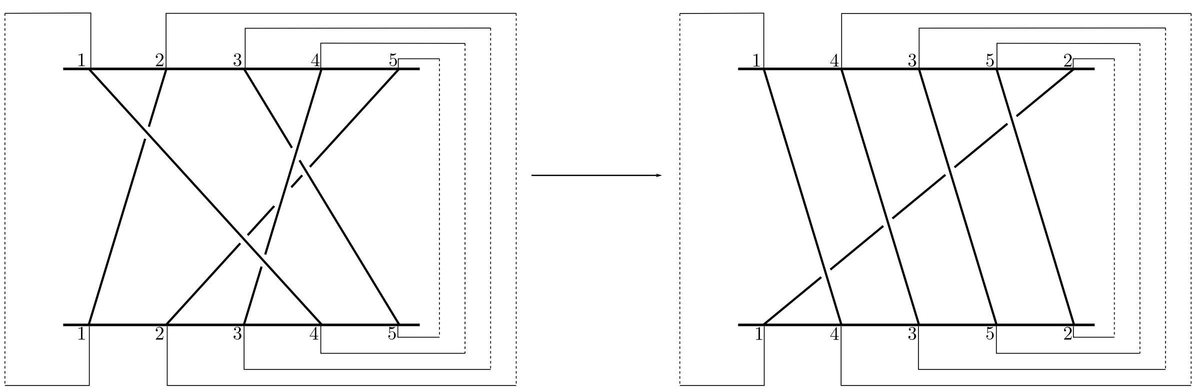


Figure 10: A transformation of a descending braid

We see that $w(\Gamma_i) = -(l(\Gamma_i) - 1)$ from the right part of Figure 10 since there are exactly $l(\Gamma_i) - 1$ negative crossings in the diagram. Thus $w(U) = \sum_{1\le i\le k} w(\Gamma_i) = -\sum_{1\le i\le k}(l(\Gamma_i)-1) = -n+k$, *i.e.*, $\gamma(U) - w(U) = n$. An ascending braid diagram V is the mirror image of a descending braid diagram U. It is known that $w(U) = -w(V)$ and it follows that $\gamma(V) + w(V) = n$. □

Apply Lemma 2 to the formula (7), we have the following theorem.

Theorem 3. Let D be a braid on n strands

$$P(D,a,z) = a^{1-n-w(D)} \sum_{U\in\mathcal{F}^{\downarrow}(D)} (-1)^{t'(U)} z^{t(U)} ((a^2-1)z^{-1})^{\gamma(U)-1} \tag{10}$$

$$P(D,a,z) = a^{n-1-w(D)} \sum_{V\in\mathcal{F}^{\uparrow}(D)} (-1)^{t'(V)} z^{t(V)} ((1-a^{-2})z^{-1})^{\gamma(V)-1} \tag{11}$$

Let $\mathcal{L}$ be a link and D be a braid on n strands which represents $\mathcal{L}$. Let E and e be the maximum and minimum a-degrees in $P(\mathcal{L},a,z)$. Since $\gamma(U) \le n$ for any $U \in \mathcal{F}^{\downarrow}(D)$, formulas (10) and (11) imply that $E \le 1-n-w(D)+2(n-1) = n-w(D)-1$ and $e \ge n-1-w(D)-2(n-1) = -n-w(D)+1$. It follows that a-span$/2+1 \le n$, that is, the Morton-Frank-Williams inequality is a direct consequence of Theorem 3.

3.3 The Braid Indices of Alternating Closed Braids

We say a braid is reduced if its closure is reduced and is alternating if its closure is alternating. In this section, we give an alternative proof of the following theorem.

Theorem 4. [14] Let D be the closure of a reduced alternating braid on n strands, then the braid index of D is n.

This theorem is a special case of a more general theorem on a class of oriented alternating fibered links proved by Murasugi. The alternating links in the class are the *-products of $(2,n)$ torus links [14].

Before we proceed to the proof of the theorem, note that if D is a split union of reduced alternating closed braids, that is, $D = D_1 \sqcup D_2 \sqcup \ldots \sqcup D_k$, then it suffice to prove the theorem for any of its non-splittable component given that the braid index is additive as well as the a-span of the HOMFLY polynomial. Hence, we assume a braid D is not a split union in the following proof. Note that there must be at least one crossing in each column of a braid D otherwise it is a split union. If D is reduced, then there must be two crossings at each column or the single crossing would be nugatory and the its closure is not reduced. Suppose D is alternating, the overpass (underpass) of a crossing x must be the underpass (overpass) at next crossing x'. So if x' is in the same column as x, x' must have the same sign as x and if x' is in an adjacent column, then x' must have the opposite sign. Therefore, if D is alternating, the crossings in every other column are of the same sign and the crossings in the rest of the columns are of the opposite sign. If the first column contains positive crossings, we say the reduced alternating braid D is *positive-leading* and if the first column contains negative crossings, then the reduced alternating braid D is *negative-leading.*

Lemma 5. Let D be a reduced alternating closed braid on n strands, E and e be the highest and the lowest a-degree in $P(D, a, z)$ respectively. Then $E = n - 1 - w(D)$ and $e = 1 - n - w(D)$. It follows that a-span$/2 + 1 = n$.

Since D is a closed braid on n strands, Theorem 4 follows the Morton-Frank-Williams inequality and Lemma 5. We now proceed to prove Lemma 5.

Proof. Assume that D is positive-leading, that is, all the odd columns contain only positive crossings and all the even colunms contain only negative crossings.

By (10), the highest possible degree of a is $n-1-w(D)$ and the leaf vertices $U \in \mathcal{F}^{\downarrow}(D)$ that can make contribution to the term of $P(D,a,z)$ with this a degree must satisfy the condition $\gamma(U)=n$. Let us consider the braid U^* obtained from D by smoothing all crossings except the first and the last one in each even column and flip the sign of the last crossings in each even column.

Claim 1. $U^* \in \mathcal{F}^{\downarrow}(D)$. This is obvious by the checking method in Remark 1.

Claim 2. U^* contribute to $P(D,a,z)$ with a term of form $\pm z^{t(U^*)-n+1}a^{n-1-w(D)}$.

By (10), the contribution of U^* to $P(D,a,z)$ is

$$a^{1-n-w(D)}(-1)^{t'(U^*)}z^{t(U^*)}((a^2-1)z^{-1})^{\gamma(U^*)-1}.$$

It is trivial to check that $\gamma(U^*)=n$ and Claim 2 follows.

Claim 3. For any $U \in \mathcal{F}^{\downarrow}(D)$, if $U \neq U^*$, then the contribution of U to $P(D,a,z)$ either has a maximum a-degree less than $n-1-w(D)$ or a z-degree less than $t(U^*)-n+1$.

If $\gamma(U) < n$ then the maximum a-degree is less than $n-1-w(D)$. Assume that $\gamma(U)=n$. The contribution of U to $P(D,a,z)$ is $a^{1-n-w(D)}(-1)^{t'(U)}z^{t(U)}((a^2-1)z^{-1})^{n-1}$ so the degree of z is $t(U)-n+1$. We need to show that $t(U)-n+1 < t(U^*)-n+1$, that is, $t(U) < t(U^*)$. Since $\gamma(U)=n$ and the first crossing in each even column is descending already for each component of the closure of U passes only one strand in U, which means the strand with starting point i must end at the corresponding end point i, by Algorithm D, U must have at least two crossings in each even column. If U also has some crossings in some odd column or has more than two crossings in some even column, then we have $t(U) < t(U^*)$ already. So the only case left is when U has no crossing in any odd column and exactly two crossings in each even column.

Claim 4. If $U \in \mathcal{F}^{\downarrow}(D)$ has no crossing in any odd columns and exactly two crossings in each even column, then $U = U^*$, that is, U^* is the only element in $\mathcal{F}^{\downarrow}(D)$ with this property.

By the proof of Claim 3, if $U \neq U^*$, then there exists an even column where the first crossing of D in U is with its original sign, and exactly one other crossing x of D in this column which is not the last one. By Remark 1 again, the sign of x in D has to be changed to make it descending in U. But as we travel the closure of U in the standard way through x and encounter the first crossing of D in the same column below x, keep in mind that this crossing exists because x is not the last crossing of D in this column and the other crossings of D in the adjacent columns have all been smoothed in U. This crossing has been smoothed in U but we are now approaching it from its overpass, so it fails the check in Remark 1 hence $U \notin \mathcal{F}^{\downarrow}(D)$.

The consequence of Claims 1 to 4 is that if we write $P(D, a, z)$ as a Laurent polynomial of a with coefficients in $\mathbb{Z}[z, z^{-1}]$, then the contribution of U^* contains a nontrivial term of the form $q(z)a^{n-1-w(D)}$ and all other terms have degrees less than $n - 1 - w(D)$, that is, $E = n - 1 - w(D)$. To obtain e, we will use $V^* \in \mathcal{T}^{\uparrow}(D)$ and (11), where V^* is constructed from D by keeping the first and flipping the last crossing in each odd column and smoothing all other crossings. The proof is analogous and the contribution of V^* contains a nontrivial term of the form $q(z)a^{1-n-w(D)}$ and all other terms have degrees higher than $1 - n - w(D)$, that is, $e = 1 - n - w(D)$.

Finally, if D is negative-leading, then its mirror image D' is positive-leading and we have $w(D) = -w(D')$. Let E' and e' be the highest and lowest a-degrees in $P(D', a, z)$. Then by the first part of the lemma, we have $E' = n - 1 - w(D')$ and $e' = 1 - n - w(D')$. That is, $E = -e' = -(1 - n - w(D')) = n - 1 - w(D)$ and $e = -E' = -(n - 1 - w(D')) = 1 - n - w(D)$. □

CHAPTER 4: SEIFERT CIRCLES, RING DIAGRAMS AND CASTLES

In this chapter, we make the preparations for the extrapolation of the core method introduced in Chapter 3 to link diagrams. The link diagrams, compared to closed braids, are more complicated in terms of the structure of the Seifert circles. The Seifert circles of a closed braid are concentric circles with the same orientation, however, those of a link diagram can be rather arbitrary. Therefore, the Seifert graphs and ring diagrams are introduced to analyze the link diagrams.

4.1 Seifert Circles and Seifert Graphs

Definition 8. Let D be an oriented diagram, we can smooth each crossing along its orientation and the circles left are *Seifert circles*. We denote the number of Seifert circles in D by $s(D)$.

The Seifert circles are related to the braid index. Yamada showed that any oriented diagram can be transformed into the closure of an oriented braid without changing the number of Seifert circles and the writhe of the diagram [22]. Note that in an oriented braid, the number of Seifert circles is the same as the number of strands. Hence, Yamada's result implies that the braid index of a link $\mathcal{L}$ equals the minimal number of Seifert circles of $\mathcal{L}$.

We can construct a graph based on the Seifert circles in a diagram D.

Definition 9. Let D be an oriented link diagram. Its *Seifert graph* $G(D)$ is an edge

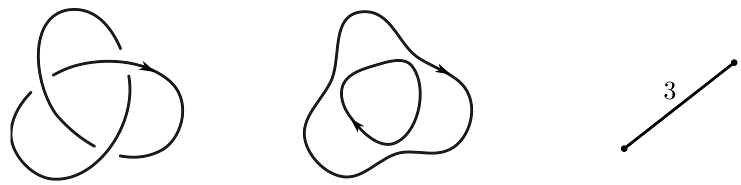


Figure 11: Seifert circles and the Seifert graph of the trefoil knot

weighted graph constructed in the following way. The vertices in the graph are the Seifert circles in D. Two vertices are connected by an edge if there are k crossings between the corresponding Seifert circles and the edge is of weight k.

4.2 Ring Diagrams

Definition 10. A *ring* in a diagram D is a component of D that has no self intersection. A *ring diagram* is a link diagram D such that each of its component is a ring. We denote the number of components in a ring diagram D by $c(D)$.

Let D be a link diagram and p be a point on D that is not in the neighborhood of a crossing. Consider the component of D that contains p and travel along this component starting from p following the orientation of the component. As we travel we ignore the crossings that we encounter the first time. Eventually we will arrive at the first crossing that we have already visited, which is in fact the first crossing involving strands of this component. We call this crossing a *loop crossing*. Since smoothing it results in two curves, the part that we had traveled between the two visits to this loop crossing, which is a ring since it does not contain crossings in itself and it does not contain p, the other part that still contains p is now a new component in the new link diagram obtained after smoothing the loop crossing. For this new link component that contains p, we will start from p and continue this process and obtain new rings. This process ends when the new link component containing p is itself an ring hence traveling along it will not create any new loop crossing. Thus, by smoothing the loop crossings encountered this way, we can decompose the component of D that contains p into a collection of rings. By choosing a point on each component of D and repeat the above process, we obtain an ring diagram of D.

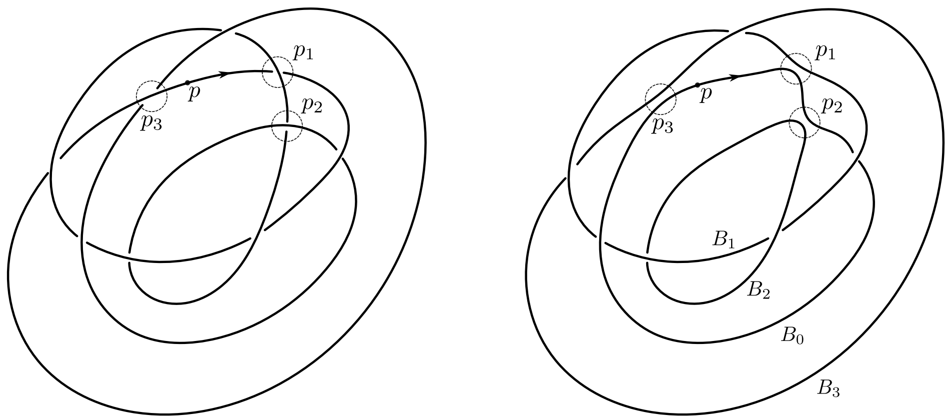


Figure 12: Loop crossings and a ring diagram

The loop crossings and the ring diagram obtained by smoothing them on a link component are uniquely determined by the starting point p. See Figure 12 for example. A different choice of p may result in different loop crossings and a different ring diagram of D. Certainly, there are other ways to obtain different ring diagrams of D. In this dissertation, we are interested in the diagrams in $\mathcal{R}(D)$, the set of all ring diagrams that can be constructed by smoothing crossings in a link digram D. It is clear that if we smooth all the crossings in D, the diagram consists of all Seifert circles is a ring diagram in $\mathcal{R}(D)$. We denote it by R_0.

Lemma 6. Let R_1 and R_2 be two ring diagrams in $\mathcal{R}(D)$. If R_2 can be obtained from R_1 by the operation described above, then $c(R_1) \leq c(R_2)$. In particular, $c(R_1) \leq s(D)$.

Proof. Let D be a diagram and R_1 be a ring diagram in $\mathcal{R}(D)$. If R_1 contains crossings, then we can obtain a new ring diagram R_2 in $\mathcal{R}(D)$ with less crossings than R_1 by the following operation. Let B_1 and B_2 be two rings in R_1 that intersect each other. They must intersect each other an even number of times. Consider two crossings between B_1 and B_2 that are consecutive as we travel along either B_1 or B_2. Smoothing these crossings results in two closed curves with two possible cases. First, These closed curves are rings. In this case, replacing B_1 and B_2 in R_1 by these two new rings results in the new ring diagram R_2 with $c(R_1) = c(R_2)$. If this is the case we say that R_1 is *reducible* and the new ring diagram is said to be obtained from R_1 by a *reduction operation*. (2) These closed curves contain self intersections, that is, loop crossings. If we smooth the loop crossings, then we get at least three rings. Thus replacing B_1 and B_2 in R_1 by these new rings results in the new ring diagram R_2 with $c(R_1) < c(R_2)$. Since each operation leads to a new ring diagram without decreasing the number of components (rings) and smoothes at least two crossings. □

Notice that in a ring diagram R in $\mathcal{R}(D)$, if a ring B has no crossings with any other rings, then it is a Seifert circle. If B has crossings with other rings, then these crossings divide B into arcs. We call these arcs the *dividing arcs of* B. As we travel along the ring B, we travel along these arcs in the order, say τ_1, τ_2, ..., τ_k and τ_k connects back to τ_1 where τ_j belongs to Seifert circle C_j $(1 \leq j \leq k)$ and obtain a directed and closed walk $C_1C_2 \cdots C_kC_1$ in $G(D)$. We call this closed walk the *Seifert circle walk* of B. Keep in mind that for any cycle $C_1C_2 \cdots C_mC_1$ in $G(D)$, $C_i \neq C_j$ if $i \neq j$ and C_j shares crossings with C_{j+1} in D, where $C_{m+1} = C_1$. It is necessary that m is even and $m \geq 4$. Furthermore, Seifert circles in a cycle cannot be concentric to each other with only one possible exception in which one Seifert circle in the cycle contains all other Seifert circles. Because of this, a cycle in $G(D)$ bounds a region in the plane if we consider $G(D)$ as a plane graph. A cycle $G(D)$ is said to be a *inner most cycle* if there are no other cycles in the region that it bounds. We say that a Seifert circle walk of B contains a cycle of $G(D)$ if there exists a cycle $C_1C_2\cdots C_mC_1$ in $G(D)$ such that B passes through the Seifert circles in the order of C_jC_{j+1} $(1 \leq j \leq m)$.

Lemma 7. If a ring diagram $R \in \mathcal{R}(D)$ contains a component B whose Seifert circle walk contains a cycle in $G(D)$, then $c(R) < s(D)$.

Proof. Let B be a component (ring) in R_1 whose Seifert circle walk contains a cycle in $G(D)$. Let D' be the diagram corresponding to R_1 with B removed. The rest of the rings form a ring diagram R' in $\mathcal{R}(D')$. Consider the ring diagram R_2 obtained from R_1 by smoothing all crossings not on B. Note that R_2 is a new ring diagram in $\mathcal{R}(D)$ and the diagram R_2 can also be obtained by smoothing all crossings in D' first and then adding B back to it. Thus every ring in R_2 is a Seifert circle of D' except B. By Lemma 6, $c(R_1)-1 = c(R') \leq s(D') = c(R_2)-1$, thus $c(R_1) \leq c(R_2) \leq s(D)$. Notice that smoothing crossings not on B does not change the Seifert circle walk of B and all crossings in R_2 are passed by B.

To prove the lemma, we make two additional observations. First, if a consecutive sub-walk $C_1C_2\cdots C_mC_1$ in the Seifert circle walk of B is a cycle in $G(D)$, keep in mind $m \geq 4$ and is even, then C_2 and C_3 can only share one crossing in R_2. This can be seen by traveling along the portion τ of B corresponding to this sub-walk, see Figure

13. The arcs $C_2 - \tau$ and $C_3 - \tau$ are in the two separate regions bounded by the simple closed curve defined by the union of τ and the arc $\widehat{qp}$ of C_1 as shown in Figure 13, hence there can be no crossings between them by the Jordan curve theorem. Notice that this observation is not affected if one of these Seifert circles contains the rest in its interior, in which case we may reroute a segment on one of the Seifert circles in an obvious way.

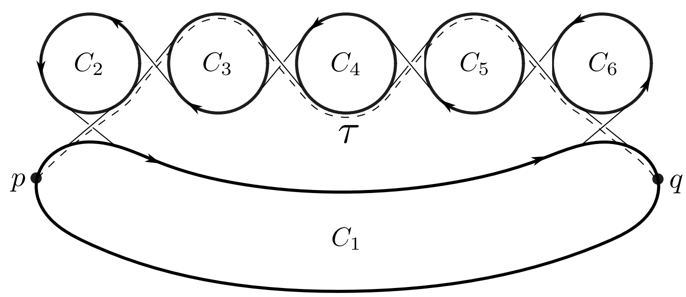


Figure 13: The union of the dividing arcs and the Seifert circle walk

Second, any operation on the Seifert circle walk of B as defined above is a reduction operation. Recall that such a reduction operation is performed at two consecutive crossings on B corresponding to a sub-walk of the form $C_1C_2C_1$ and it results in a new ring B' whose Seifert circle is obtained from that of B by reducing the sub-walk $C_1C_2C_1$ to C_1. Since a closed sub-walk that does not contain a cycle contains at least a sub-walk of the form $C_1C_2C_1$ and the corresponding sub-walk after the reduction operation results in a closed sub-walk that still does not contain a cycle, this reduction can be repeated until the sub-walk is reduced to a single Seifert circle.

Assume that the Seifert circle walk $C_1C_2 \cdots C_kC_1$ of B contains a cycle and consider a shortest closed sub-walk on it that contains a cycle of $G(D)$. By definition, any proper closed sub-walk of this sub-walk is cycle free and its underlying set of edges is a tree containing at least one leaf where we must have a reducible sub-walk of the form $C_1C_2C_1$ by the second observation above. These sub-walks do not contain cycles hence each can be reduced to a single Seifert circle. The Seifert circle walk of the resulting ring B' contains a sub-walk of the form $C_1C_2\cdots C_mC_1$ where $C_1C_2\cdots C_mC_1'$ is a cycle in $G(D)$. By the first observation, this sub-walk cannot be reduced to a single Seifert circle as the resulting Seifert circle walk does not contain a sub-walk of the form C_3C_2 since B' only pass between C_2 and C_3 once. Thus the reduction operations cannot reduce the Seifert circle walk $C_1C_2\cdots C_kC_1$ to a single Seifert circle. That is, at some point, the operation defined above can no longer be a reduction operation and will increase the number of components in the resulting ring diagram. This implies $c(R_1) < s(D)$. □

4.3 Castles

In this section, we describe a local braid structure called a *castle*, which exists in any link diagram.

Definition 11. Let D be a diagram. A Seifert circle C of D is *trapped* given C is bounded within a topological disk created by crossings and arcs of other Seifert circles and C is connected to one and only one Seifert circle in D.

As an example, consider three Seifert circles C_1, C_2 and C_3 in D as shown in Figure 17. Notice that C_3 is bounded within the topological disk created by arcs of C_1, C_2 and the two crossings and that C_3 is connected to C_2 but not to C_1 for the incoherent of the orientation. C_3 is trapped by C_1 and C_2. Similarly, C_4 is trapped by C_2 and C_3.

It is apparent from the definition and the Jordan curve theorem that if C_1 traps

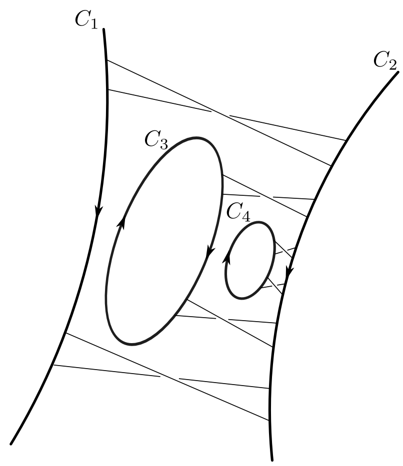


Figure 14: An example of trapped Seifert circles

C_3, then C_3 cannot trap C_1, in fact C_3 or any Seifert circle bounded within C_3 cannot trap any Seifert circle outside the disk bounded by C_3, C_2 and the two crossings as shown in Figure 17.

Definition 12. Seifert circles in a diagram D are said to form a *general concentric chain* if there is a topological embedding of a rectangle disk such that the intersection of the embedding and the arcs of these Seifert circles are with the same direction from one side of the rectangle to the opposite side. Seifert circles are said to form a *concentric chain* if they are concentric in D.

In Figure 15, C_1, C_2 and C_3 in the left part form a concentric chain of Seifert circles. C_1, C_2, C_3, C_4 in the right part form a general concentric chain of Seifert circles and C_1, C_2, C_3, C_6 form another general concentric chain of Seifert circles. Notice that between two concentric Seifert circles we may have other Seifert circles as shown in the right of Figure and the rectangle reveals the local braid structure in the diagram.

Let D be a link diagram. Consider an inner-most Seifert circle C of D, that is, C does not bound any other Seifert circles inside it. Choose a starting point on C away

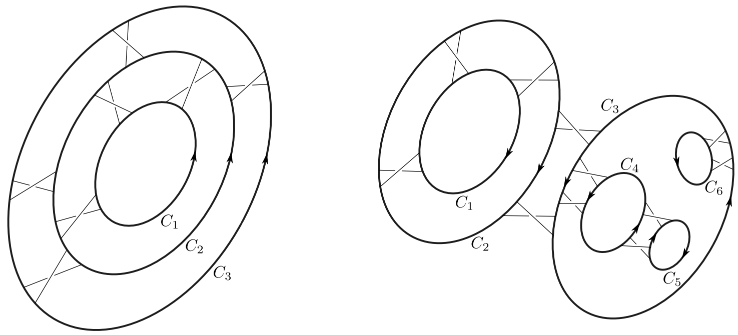


Figure 15: A concentric chain and some general concentric chains

from the places where crossings are placed. Following the orientation of C, we are able to order the crossings along C as shown in Figure 16, where a clockwise orientation is illustrated and for the purpose of illustration the part of C from the first to the last crossing is drawn in a horizontal manner bounded between the starting point p_0 and ending point q_0 marked on it. We call this segment of C the ground (0-th) floor of the *castle*, which will be defined next. We now describe the procedure to build a structure on top of the ground floor that we call a *castle*. If C is connected to another Seifert circle C_1 in $G(D)$, then there exists crossings between C and C_1 between p_0 and q_0 and they can be ordered by the orientation of C_1, which is coherent with the orientation of C. Let p_1 and q_1 be two points immediately before the first crossing and after the last crossing so that no other crossings are between p_1 and the first crossing or between q_1 and the last crossing. The segment of C_1 between p_1 and q_1 is defined to be a 1-st floor. If C_1 has no crossings with other Seifert circles on this floor, then this floor terminates. For example if there is only one crossing between C and C_1 then this floor should terminate. If C_1 shares crossings with another Seifert circle C_2 on this floor, keep in mind that the floor is the segment between p_1 and q_1, then we can define a second floor in a similar manner. We call the crossings between two floors *ladders* as we can only go up or down from a floor to the next through these crossings. This process is repeated until we reach a floor that terminates, meaning there are either no other floors on top or there are no ladders to reach the floor above. The castle is the structure that contains all possible floors and ladders between them constructed this way from the starting point. Notice that there may be more than one separate floor on top of any given floor. Let F_k be a top floor, F_{k-1} the floor below it, F_{k-2} the floor below F_{k-1} and so on, the collection of floors F_0, F_1, ..., F_k including all crossings between them is defined to be a *tower*. Notice that the Seifert circles corresponding to the floors in a tower form a general concentric chain hence the height of tower or the number of floors in it is bounded above by the number of Seifert circles in D. Finally, between two adjacent floors we may have trapped Seifert circles that may or may not be part of the castle as shown in Figure 16. However if there exist other floors between two adjacent floors, the Seifert circle corresponding to the top floor will not share any crossings with the Seifert circles with floors in between due to their opposite orientations.

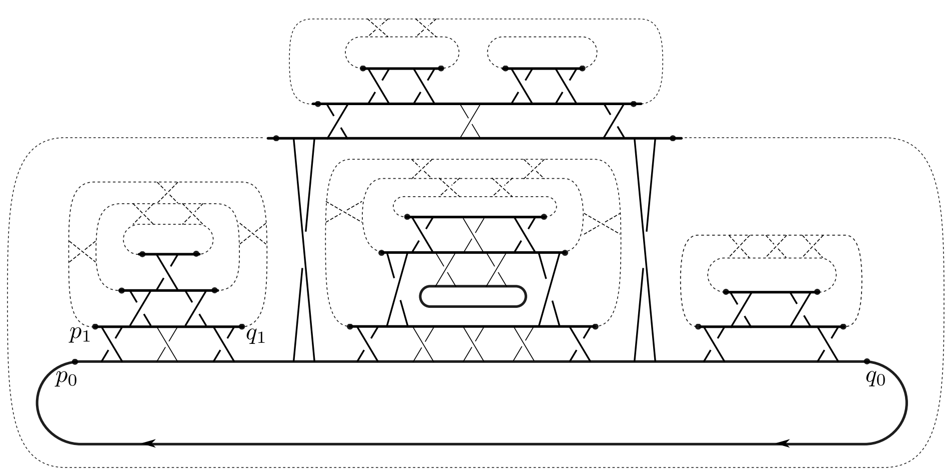


Figure 16: A castle built on top of an inner most Seifert circle

Lemma 8. For any link diagram D, there exists an inner most Seifert circle C such that a castle built on it contains no trapped Seifert circles.

Proof. Start with any inner most Seifert circle C_0 and build a castle on it. If it contains no trapped Seifert circles, we are done. If not, let C' be a Seifert circle trapped between floors F_i and F_{i+1} with C_i and C_{i+1} being their corresponding Seifert circles. Choose an inner most Seifert circle that is contained inside C' or use C' itself if it does not contain any and build a castle on it. This castle is bounded away from either F_i or F_{i+1} depending on the orientation of the new base Seifert circle. Since the base Seifert circle is contained in C', if the new castle is not contained within C' entirely, then C' will contribute a floor to the new castle. In fact, the curves in the new castle can only exit the region between F_i and F_{i+1} that traps C' through either F_i (if C' shares crossings with F_i) or F_{i+1} (if C' shares crossings with F_{i+1}), but not both. It follows that the new castle is completed contained within the towers that contain F_i and F_{i+1}. If the castle built on this new Seifert circle again contains trapped Seifert circles, we will repeat this process. Since this process starts with new trapped Seifert circles that are bounded within the previous castles, the process will end after finitely many steps and we reach a castle without trapped Seifert circles. □

CHAPTER 5: THE BRAID INDICES OF ALTERNATING LINKS

In this chapter we will extrapolate our core method to prove the following main theorem.

Theorem 9. Let D be a reduced alternating diagram of an alternating link $\mathcal{L}$. The braid index $b(\mathcal{L})$ equals the number of Seifert circles in D if and only if $G(D)$ contains no edge of weight one

The proof of the necessity of the theorem is relatively easier. We show that if the link diagram D of a link $\mathcal{L}$ contains an edge of weight one, then the braid index of $\mathcal{L}$ is less than the number of Seifert circles in D. In this part of the proof D does not have to be reduced nor alternating. As for the sufficiency, we show that if D is reduced, alternating and $G(D)$ is free of edges of weight one, then the equality in the Morton-Frank-Williams inequality holds hence the number of Seifert circles in D equals the braid index of $\mathcal{L}$. Similarly, We start by introducing the positive and negative algorithms and the positive and negative resolving trees.

5.1 Positive and Negative Resolving Trees

We now define two different algorithms used to derive the positive and negative resolving trees for link diagrams. These resolving trees will play a key role in proving our main theorem. Similar to the descending and ascending resolving trees introduced in Chapter 3, we think of a resolving tree as graph of a branching process. However, we will divide this process into several phases for positive and negative resolving trees.

Note that when we flip a crossing, the corresponding child has essentially the same components as the parent. Smoothing may merge two existing components or split one component into two. In order to maintain some control over this process, we select a starting point, start traversing the component from there and consider branching on the crossings as we encounter them in this traversing process. The first phase ends when all current leaf vertices have a component containing our starting point and all crossings (smoothed, kept, or flipped) of the original link diagram along the component have been visited at least once. Note that we visit a crossing twice exactly when it is an unchanged or flipped and the crossing is a loop crossing, that is, a crossing whose both strands end up in the same component at the end of phase.

In the first phase we will use either the ascending or the descending operation introduced in Chapter 3 at all crossings. Hence, in all leaf nodes at the end of phase one we have a component containing our starting point, along which all crossings of the original link diagram have been visited and the ones that were not smoothed are now either all descending or all ascending. Therefore the component containing our starting point is either below or above the other components and it is not linked to them. At this point we remove this component from consideration, we proceed as if it was not present any more. We select a new starting point and perform a next phase of adding new vertices to the resolving tree using only ascending or only descending operations. We continue adding new phases until there is no crossings to be considered left. The leaf vertices of the final tree will be unlinks.

Now, we will define two specific algorithms by choosing the starting point at the

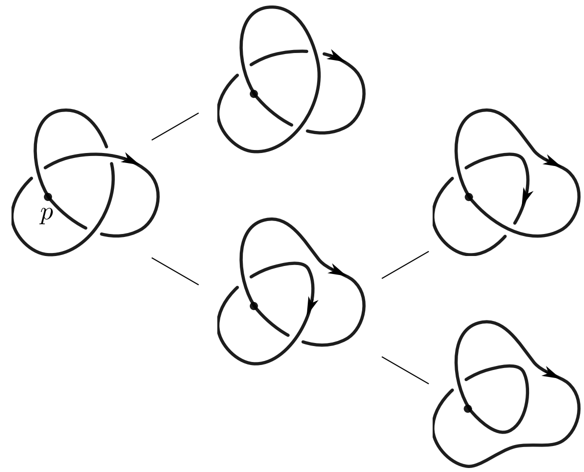


Figure 17: An example of a positive resolving tree

beginning of each phase and assigning the appropriate descending or ascending operation for that phase.

Algorithm P: The starting point at the beginning of each phase is chosen to be the starting point of the ground floor of a castle free of trapped Seifert circles. If the Seifert circle providing the ground floor is clockwise, the descending operation is applied, otherwise the ascending operation is applied throughout the entire phase.

Algorithm N: Exchange the descending and ascending operations in Algorithm P.

It is important to note that this is different from the approaches used in Chapter 3. Here, both the descending and ascending operations are used in the same algorithm depending on the starting point.

As before, we will use $\mathcal{T}^+(D)$ and $\mathcal{T}^-(D)$ to denote the resolving trees obtained by applying Algorithms P and N respectively, and use $\mathcal{F}^+(D)$ and $\mathcal{F}^-(D)$ to denote the set of leaf vertices of $\mathcal{T}^+(D)$ and $\mathcal{T}^-(D)$ respectively.

5.2 The Braid Indices of Alternating Links

In this section, we proceed to prove Theorem 9. We start by proving the following lemma.

Lemma 10. Let E and e be the maximum and minimum a-degree in $P(D, a, z)$, then $E \leq n - w(D) - 1$ and $e \geq -n - w(D) + 1$. If a component of $U \in \mathcal{F}^+(D)$ contains a negative loop crossing, then $\gamma(U) + w(U) < n$. Analogously, if a component of $V \in \mathcal{F}^-(D)$ contains a positive loop crossing, then $\gamma(V) - w(V) < n$.

Proof. Let U be a leaf vertex in either $\mathcal{T}^+(D)$ or $\mathcal{T}^-(D)$. Since each component is obtained with a fixed starting point, the loop crossings, if there is any, of the component are uniquely determined. Since the components are stacked over each other by the way they are obtained, the sum of the crossing signs between different components is zero. If we smooth the loop crossings, the resulting rings are also stacked over each other. So the sum of the crossing signs between these different rings is also zero. It follows that $w(U)$ equals the sum of the signs of the loop crossings. If U contains k components, that is, $\gamma(U) = k$, and the i-th component contains $m_i \geq 0$ loop crossings. Smoothing the loop crossings of the i-th component results in $m_i + 1$ rings. So smoothing all loop crossings of U results in $k + \sum_{1 \leq i \leq k} m_i$ rings. By Lemma 6, we have $k + \sum_{1 \leq i \leq k} m_j \leq n$ where n is the number of Seifert circles in D. Since $w(U)$ equals the sum of the signs at the loop crossings of its components, it follows that $E \leq n - w(D) - 1$ and $e \geq -n - w(D) + 1$ by (8) and (9), where E and e are maximum and minimum degrees of a in $P(D, a, z)$. Furthermore, if $U \in \mathcal{F}^+(D)$ contains a negative loop crossing, then $w(U)$ is strictly less than the total number of loop crossings in U and we will have $\gamma(U) + w(U) < n$. Similarly, if $V \in \mathcal{F}^-(D)$ contains a positive loop crossing, then $\gamma(V) - w(V) < n$. □

For any given component B of a leaf vertex in either $\mathcal{T}^+(D)$ or $\mathcal{T}^-(D)$, as we travel along it from its starting point, bear in mind that the starting point on the ground floor of the castle, we will eventually exit the castle for the first time through the end point of some floor. The arc of B from its starting point to this exiting point is called the *maximum path* of B. If the ending point B is the end point on the ground floor, it means that B is completely contained within the castle and the base Seifert circle. Notice that a maximum path consists of only three kind of line segments, ladders going up or down, which are parts of the crossings, and straight line segments parallel to the ground floor. In the case that the base Seifert circle has clockwise orientation, the ladders going up or down are both from left to right. Furthermore, on either side of a segment that is parallel to the ground floor, there are no crossings since if there were crossings previously there in the original diagram, they are smoothed in the process of obtaining B. This means that if we travel along a strand of the link diagram from a point outside the castle following its orientation, in order to enter a floor below this maximum path, it is necessary for the strand to pass the path through a ladder from the left in the case of clockwise orientation for the base Seifert circle or from the right in the case of counter clockwise orientation for the base Seifert circle.

Lemma 11. If $U \in \mathcal{F}^+(D)$ contains a maximum path that does not end on the ground floor, then the maximum a-degree in the contribution of U to $P(D, a, z)$ is smaller than $n - w(D) - 1$. Analogously, if any component of $V \in \mathcal{T}^-(D)$ contains a maximum path that does not end on the ground floor, then the minimum a-degree in the contribution of V to $P(D, a, z)$ is larger than $-n - w(D) + 1$.

Proof. Consider the case of $U \in \mathcal{T}^+(D)$ first. Assume B is the first component of U that contains such a maximum path. Notice that a component that is bounded within the castle and its base Seifert circle contains no loop crossings. So the components before B are all rings in D. Suppose there are k components before B, then by Lemma 7, there are at most $n - k$ Seifert circles in D' where n is the number of Seifert circles in D and D' is the link diagram obtained from D after the first k components are removed from it, since Seifert circles in D' are rings in D. Let C_0 be the base Seifert circle on which the castle used to derive B is built.

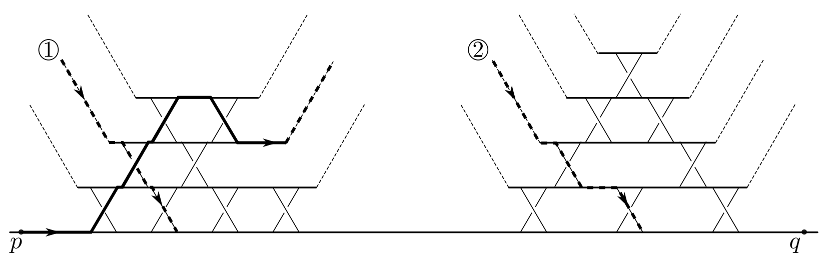


Figure 18: Two possible cases of a maximum path

Without loss of generality, we assume that C_0 has clockwise orientation. Let F_i, $i \geq 1$, be the floor where the maximum path of B exits from its end point and let T_1 be the tower that houses the maximum path. If B is to get back to C_0 within T_1, it will have to cross the maximum path from the left side as shown in the left part of Figure 18 where the maximum path is drawn in double lines, creating a negative loop crossing since in this case we are applying a descending algorithm. Hence $\gamma(U) + w(U) < n$ by Lemma 10 and it follows that the a-degree in the contribution of U to $P(D, a, z)$

is less than $n - w(D) - 1$ by (8). If B does not go back to the base floor within T_1, then it has to do so through another tower T_2. Let us first smooth the loop crossings of B that are defined by the starting point on C_0. Let B^* be the resulting ring that contains the starting point, which contains the part of B that is within T_1, and let R_2 be as defined in Lemma 7. Let F_j be the highest common floor T_1 and T_2 share, it is necessary that $j < i$ and that B returns to F_j from F'_{j+1}, the floor above it in T_2 within T_2. Consider the sub-walk of the Seifert circle walk of B^*, defined by traversing B^*, starting from its last dividing arc on C_j within T_1 and ending at the first dividing arc on C_j within T_2. This sub-walk has the form of $C_jC_{j+1}\cdots C'_{j+1}C_j$. If this sub-walk contains a sub-walk starting and ending at C_j, by the proof of Lemma 7, it cannot contain a cycle, hence can be reduced to a single Seifert circle C_j. However the last step of the reduction would be performed on a sub-walk of the form $C_jC_{j+1}C_j$, which is impossible since the other crossing is either within T_1 which is again impossible since we started on the last dividing arc on C_j in T_1, or the new dividing arc will contain the entire portion of C_j that is outside of T_1 which is also impossible since B^* would then not go through C'_{j+1} at all. It follows that $C_jC_{j+1}\cdots C'_{j+1}C_j$ contains no other closed sub-walk starting and ending at C_j, hence it must contain a cycle of $G(D)$. By Lemma 7, it follows that the total number of components in U plus the total number of loop crossings in them is less than n, so $\gamma(U) + w(U) < n$ again, and the maximum a-degree in the contribution of U to $P(D, a, z)$ is also less than $n - w(D) - 1$. The case of $V \in \mathcal{T}^-(D)$ is similar. Since we are using Algorithm N, in the first situation it will result in a positive loop crossing and the second situation is the same. Hence, $\gamma(U) - w(U) > n$ for both types of maximum paths. $\square$

Lemma 11 leads to the following two corollaries.

Corollary 12. If the contribution of $U \in \mathcal{F}^+(D)$ to $P(D, a, z)$ contains an $a^{n-w(D)-1}$ term, then each component of U is a ring in D. Similarly, if the contribution of $V \in \mathcal{F}^-(D)$ to $P(D, a, z)$ contains an $a^{-n-w(D)+1}$ term, then each component of V is also a ring in D.

Proof. This is direct from the proof of Lemma 11. Since every such component has a maximum path ending at the end point of the ground floor, the component contains no loop crossings. □

The following result provides the proof for the only if part of Theorem 9. We feel that it is significant enough on its own so we state it as a theorem. Notice that it implies the inequality in the Morton-Frank-Williams inequality is strict, that is, a-span$/2 + 1 < n$.

Theorem 13. Let D be a diagram of a link $\mathcal{L}$ such that $G(D)$ contains an edge of weight one ($\mathcal{L}$ and D need not be alternating) and $s(D) = n$, then we have $b(\mathcal{L}) < n$.

Proof. Let C' and C'' be two Seifert circles sharing only one crossing between them. If C'' shares no crossings with any other Seifert circles, then the crossing between C' and C'' is nugatory and the statement of the theorem holds. So assume that this is not the case and let C_1, C_2, ..., C_j be the other Seifert circles sharing crossings with C''. The orientations of C_1, C_2, ..., C_j are the same as that of C' and there are no crossings between any two of them. A case of $j = 2$ is shown in Figure 20. We will reroute the overpass at the crossing between C' and C'' along C_1, C_2, ..., C_j keeping the strand over the crossings we encounter as shown in Figure 20.

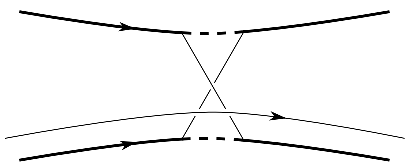


Figure 19: The local effect of rerouting the overpass

Rerouting the overpass this way will only create new crossings over some crossings between the C_i's and its neighbors other than C''. The effect of this rerouted strand to the Seifert circle structure locally is shown in Figure 19, which does not change the Seifert circles C_1, C_2, ..., C_j, but the weights of the edges connecting to the vertices corresponding to them in $G(D')$ where D' is the new link diagram after the rerouting may have changed from those in $G(D)$. Figure 21 displays the change of Seifert graph after the rerouting shown in Figure 20.

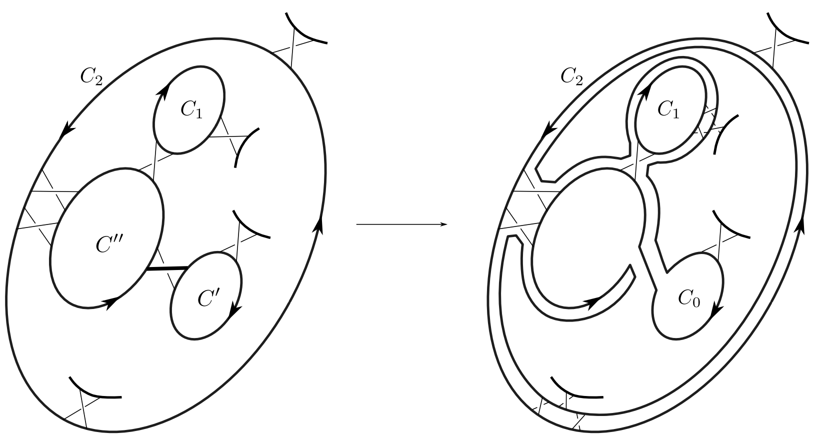


Figure 20: Rerouting the overpass of the single crossing

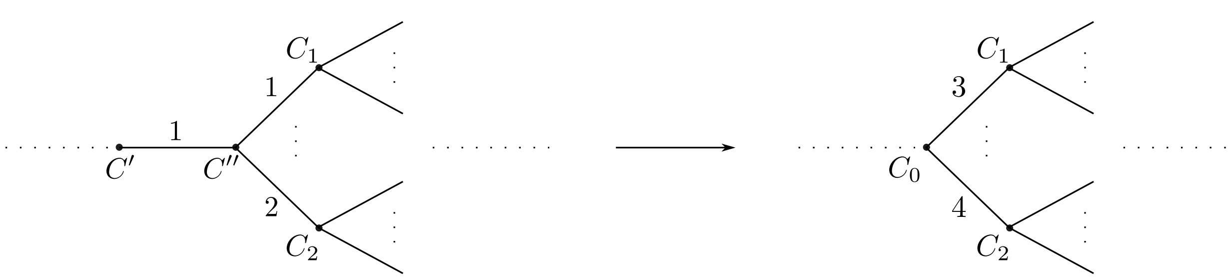


Figure 21: The change of Seifert graph after the rerouting

At the end, we arrive at a new link diagram D' that is equivalent to D, but with one less Seifert circle. The result then follows since the number of Seifert circles in D' is an upper bound of the braid index of D', hence D. □

If D is a reduced alternating link diagram and $G(D)$ contains no edges of weight one, then any pair of Seifert circles in D that are adjacent in $G(D)$ share at least two crossings and these crossings are of the same signs. In fact, all crossings on one side of any Seifert circle of D are of the same sign. Let α^+ be the total number of positive crossings in D, σ^+ be the number of pairs of Seifert circles in D that share positive crossings, and let α^- be the total number of negative crossings in D, σ^- be the number of pairs of Seifert circles in D that share negative crossings. We are now ready to prove our main theorem.

Proof. The necessity is direct from Theorem 13. We will prove the sufficiency in two steps. In the first step, we construct a specific leaf vertex in $U \in \mathcal{F}^+(D)$ whose contribution to $P(D, a, z)$ contains a term of the form $(-1)^{t^-(U)} z^{t(U)-n+1} a^{n-w(D)-1}$, where $t^-(U) = \alpha^- - 2\sigma^-$ and $t(U) = \alpha^+ + \alpha^- - 2\sigma^-$. Similarly we construct a specific leaf vertex in $V \in \mathcal{F}^-(D)$ whose contribution to $P(D, a, z)$ contains a term of the form $(-1)^{t^-(V)+n-1} z^{t(V)-n+1} a^{-n-w(D)+1}$, where $t^-(V) = \alpha^+ - 2\sigma^+$ and $t(V) = \alpha^- + \alpha^+ - 2\sigma^+$. In the second step, we show that if a leaf vertex $U' \in \mathcal{F}^+(D)$ makes a contribution to the $a^{n-w(D)-1}$ term in $P(D, a, z)$, then $t^-(U') \leq \alpha^- - 2\sigma^-$. Similarly, if a leaf vertex $V' \in \mathcal{F}^-(D)$ makes a contribution to the $a^{-n-w(D)+1}$ term in $P(D, a, z)$, then $t^-(V') \leq \alpha^+ - 2\sigma^+$.

Combining the results of the these two steps will then lead to the conclusion of the theorem since the result from the second step implies that $t(U') \leq \alpha^+ + \alpha^- - 2\sigma^-$ and $t(V') \leq \alpha^- + \alpha^+ - 2\sigma^+$. So $\alpha^+ + \alpha^- - 2\sigma^-$ is the maximum degree of z in the coefficient of the $a^{n-w(D)-1}$ term in $P(D,a,z)$ and $\alpha^- + \alpha^+ - 2\sigma^+$ is the maximum degree of z in the coefficient of the $a^{-n-w(D)+1}$ term in $P(D,a,z)$. Furthermore, these maximum degrees can only be contributed from $U' \in \mathcal{F}^+(D)$ and $V' \in \mathcal{F}^-(D)$ that are obtained by smoothing all positive crossings of D and all negative crossings of D except two between each pair of Seifert circles sharing negative crossings in the case of U' and by smoothing all negative crossings of D and all positive crossings of D except two between each pair of Seifert circles sharing positive crossings in the case of V'. Apparently any such U', V' will make exactly the same contributions to $P(D,a,z)$ as that of U and V. Thus $E = n - w(D) - 1$ and $e = -n - w(D) + 1$. So $E - e = 2(n-1)$ and $(E-e)/2 + 1 = a\text{-span}/2 + 1 = n$ and the theorem follows.

Step 1. Choose a castle that is free of trapped Seifert circles. Let C_0 be the base Seifert circle of the castle with starting point p and ending point q on its floor and start travel along D from p.

Case 1. The crossings between C_0 and its adjacent Seifert circles are all positive. If C_0 is clockwise, then we need to apply the descending rule. We will encounter the first crossing from its under strand. We will stay with the component obtained by smoothing this crossing. So we are still traveling on C_0 after this crossing is smoothed. We then encounter the next crossing from its under strand and we can again smooth this crossing. Repeating this process, we arrive the first component of U by smoothing all the crossings between C_0 and its adjacent neighbors. If C_0 is counter-clockwise then we will be applying the ascending rule and we can also obtain a component of

U by smoothing all crossings along C_0. It is apparent that after we remove this new component from the diagram, the resulting new diagram is still alternating and has $n-1$ Seifert circles.

Case 2. The crossings between C_0 and its adjacent Seifert circles are all negative. If C_0 is clockwise, then we need to apply the descending rule. Let C_1 be the first Seifert circle with which C_0 shares a crossing as we travel along C_0 from p. In this case we encounter the first crossing from its over strand. Therefore we have no choice but to keep this crossing. This moves us to C_1. Keep in mind that by the given condition, C_0 and C_1 share at least two crossings and all crossings between C_1 and C_0 are on the floor of C_1 above F_0. It is easy to check that as we travel on F_1 toward the last crossing between C_1 and C_0, we encounter each crossing from an under strand. So we can smooth all crossings we encounter, either between C_1 and C_0 or between F_1 and F_2 before we reach the last crossing between C_1 and C_0. We then flip the last crossing to return to C_0. If C_0 is adjacent to more Seifert circles, we repeat the same procedure. Finally we return to the ending point of F_0 and back to the starting point. See Figure 22 for an illustration, where a case of two floors on top of F_0 is shown.

Since smoothing crossings does not change the alternating nature of a diagram, removing this newly created component will keep the resulting diagram alternating as one can easily see from Figure 22. In fact, the new diagram is equivalent to the one obtained by smoothing all crossings encountered by the maximum path of the new component when the new component is removed. So it contains $n-1$ Seifert circles. If C_0 is counter clockwise, the above argument is the same after we replace the descending algorithm by the ascending algorithm.

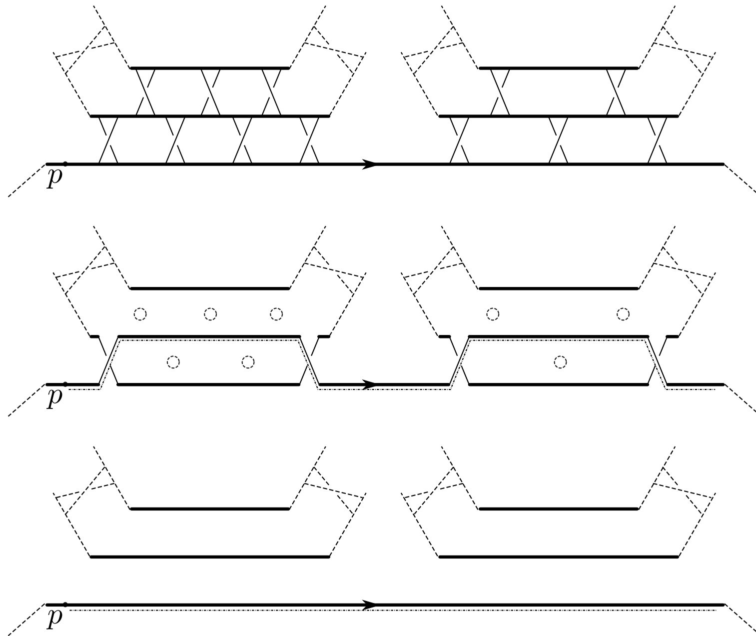


Figure 22: An example of constructing the target leaf vertices

So in both cases we created a component and the new diagram is still alternating and contains one less Seifert circle than before. Thus this process can be repeated, at the end we obtain U, which contains n components each of which is a ring and by the way U is obtained, all positive crossings have been smoothed and between any pair of Seifert circles that share negative crossings, all but two crossings are smoothed. V is obtained in a similar manner in which all negative crossings are smoothed and between any pair of Seifert circles that share positive crossings, all but two crossings are smoothed. This finishes the first step.

Step 2. Consider a leaf vertex $U' \in \mathcal{F}^+(D)$ that makes a contribution to the $a^{n-w(D)-1}$ term in $P(D,a,z)$. By Lemma 11, the maximum path of each component of U' is bounded within its defining castle. Let B_1 be the first component of U'. Consider a horizontal segment of the maximum path that represents a local maximum. It is easy to verify that we will never encounter a negative crossing to the left side of B_1, and all crossings to the right of B_1 are positive and are smoothed. Thus, for a given pair of Seifert circles in D that share negative crossings, if B_1 crosses from one to the other, then at least two crossings between them are not smoothed. If B_1 does not cross from one to the other, then removing B_1 may change parts of these two Seifert circles but will not affect the crossings between these two Seifert circles. The same argument can then be applied to the next component B_2 and so on. It follows that for each pair of Seifert circles sharing negative crossings, at least two crossings cannot be smoothed in U'. That is, $t^-(U') \leq \alpha^- - 2\sigma^-$. Similarly, we have $t^-(V') \leq \alpha^+ - 2\sigma^+$. □

CHAPTER 6: A GENERAL CONJECTURE

Having that the braid indices of alternating links equal the number of Seifert circles in their reduced alternating diagrams if and only if their Seifert graphs contain no edge of weight one, an immediate question is what if there exists an edge of weight one in the Seifert graph of a reduced alternating diagram. Note that an edge of weight one in a Seifert graph must be on a cycle otherwise the single crossing is nugatory and the diagram is not reduced. For such a single crossing, we can do a rerouting operation to its over strand as shown in Figure 23. There will be one less Seifert circle after a rerouting operation. We define the *reduction number* $r(D)$ of a diagram D to be the maximal number of rerouting operations we can apply to the diagram. The diagram in Figure 23 has reduction number equal to one.

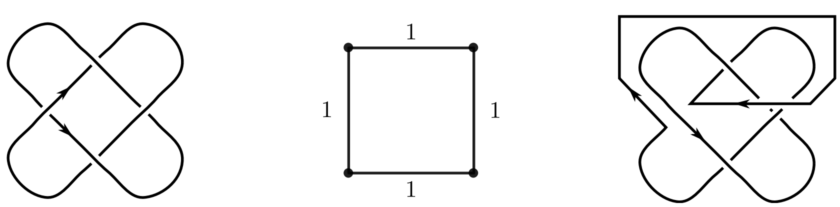


Figure 23: An example of rerouting a single crossing.

Conjecture 14. If a link $\mathcal{L}$ possesses a reduced alternating link diagram D with only one single crossing connecting a pair of Seifert circles, then $b(\mathcal{L}) = s(D) - 1$.

Let D be a diagram of a link $\mathcal{L}$. It is not hard to prove that if there is only one single crossing in the diagram, then we can always reroute the crossing and the reduction number of such a diagram is always one. This implies that we can represent $\mathcal{L}$ by a diagram D' with one less Seifert circle than the number of Seifert circles in D. While the hard part of proving this conjecture is that to show a-span$/2 + 1 = s(D) - 1$.

Since we see that the diagrams with only one single crossing have reduction number equal to one, it is natural to consider the following general conjecture.

Conjecture 15. If a link $\mathcal{L}$ possesses a reduced alternating link diagram D, then $b(\mathcal{L}) = s(D) - r(D)$.

This conjecture is of the same importance as the crossing number conjecture for alternating links. To prove or disprove it, we need other tools since the equality in the Morton-Frank-Williams inequality does not always hold to compute the braid index. For instance, in [15], Murasugi and Przytycki showed that the alternating link $\mathcal{L}$ with a diagram D, as displayed in Figure 24, has the braid index equal to six. However, the Morton-Frank-Williams inequality only provides the lower bound equal to five. Nevertheless, we know that the reduction number of the diagram is one and the number of Seifert circles in the diagram is seven. Therefore, Conjecture 15 is true for this example. As another example, the knot 8_1 in Rolfsen's table has braid index equal to five and its reduced alternating diagram D has $s(D) = 7$ and $r(D) = 2$.

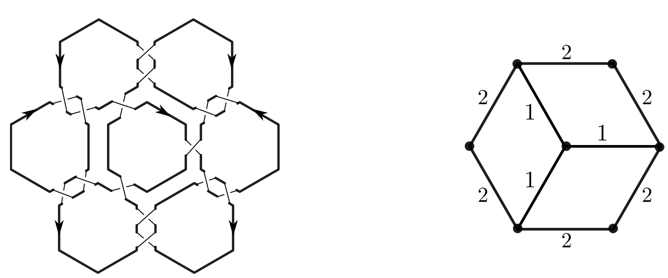


Figure 24: An example for which new methods are needed.

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