On the discrete spectrum of exterior elliptic problems

by

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ABSTRACT

RAJAN PURI. On the discrete spectrum of exterior elliptic problems.
(Under the direction of Dr. BORIS VAINBERG)

In this dissertation, we present three new results in the exterior elliptic problems with the variable coefficients that describe the process in inhomogeneous media in the presence of obstacles. These results concern perturbations of the operator $H_0 = -\text{div}((a(x)\nabla)$ in an exterior domain with a Dirichlet, Neumann, or FKW boundary condition. We study the critical value β_{cr} of the coupling constant (the coefficient at the potential) that separates operators with a discrete spectrum and those without it. Our main technical tool of the study is the resolvent operator $(H_0 - \lambda)^{-1} : L^2 \to \mathring{H}^2$ near point $\lambda = 0$. The dependence of β_{cr} on the boundary condition and on the distance between the boundary and the support of the potential is described. The discrete spectrum of a non-symmetric operator with the FKW boundary condition (that appears in diffusion processes with traps) is also investigated.

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CHAPTER 1: INTRODUCTION OF THE PROBLEM

1.1 Motivation

Many equations of Mathematical Physics has been characterized as the the major bridge between central issues of applied mathematics and physical sciences. An increasing number of publications written on elliptic problems and their spectral analysis is an evidence of the continuing interests of mathematicians and physicists. The main reason for studying elliptic problems is that they arise in a wide variety of physical systems and describe a large class of natural phenomena. The rigorous mathematical analysis of physical models and applications has played a major role in many branches of mathematics. Also, the stability and dynamics of the solutions of many physical problems have been carefully investigated in various mathematical applications including quantum mechanics, optics, acoustics, geophysics, and population dynamics.

Many quantum mechanical systems are descried by a Hamiltonian of the form $H = -\Delta + V$. Different systems are distinguished by the potential V. We have a significant amount of information concerning the eigenvalues of H. However, for more general elliptic operators , i.e. $H_\beta = H_0 - \beta V$, where $H_0 = -\nabla(a(x)\nabla)$, spectral stability and dynamics have not been exhaustively studied. Therefore, we decided to study elliptic problems in the exterior domain with the variable coefficient that describes the process in inhomogeneous media with the presence of obstacles. This framework covers conceptually any physical system governed by an elliptic equation.

1.2 Statement of the problem

Let $\Omega = R^d \backslash \bar{B}$, where B is a bounded domain in R^d with smooth boundary. i.e $\partial\Omega$ is smooth.

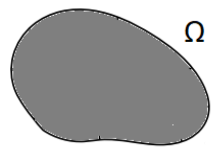

Figure 1.1: Exterior domain

Consider the following elliptic problems in Ω.

$$H_0 u - \beta V(x) u - \lambda u = f, \quad x \in \Omega, \tag{1.1}$$

where $H_0 = -\text{div}(a(x)\nabla)$, the potential $V(x) \geq 0$ is compactly supported and continuous, $\beta \geq 0$, $a(x) > 0$, $a(x) \in C^1(\Omega)$, and $a = 1$ when $|x| >> 1$. We assume that $f \in L^2(R^d)$ and the solutions belong to the space $H^2(\Omega)$ and satisfies a Dirichlet, Neumann, or FKW-boundary condition. The latter boundary condition will be introduced later.

One can reformulate the problem in the operator setting. Let $H_\beta : \overset{\circ}{H}{}^2 \to L^2(\Omega)$, be the operator which maps each $u \in \overset{\circ}{H}{}^2$ into

$$f = H_\beta u - \lambda u,$$

where $H_\beta = (H_0 - \beta V(x))$ and $\overset{\circ}{H}{}^2$ is the set of functions from the Sobolev space $H^2(\Omega)$ that satisfy the boundary condition, and it can be a Dirichlet, Neumann, or FKW boundary condition.

The main question under the investigation is whether the discrete spectrum appears for arbitrary small perturbations (arbitrary small $\beta > 0$), or β must be large enough to create negative eigenvalues. Thus we would like to know when $\beta_{cr} = 0$ and when $\beta_{cr} > 0$. The answer is known [2] for the Schrödinger operator $-\Delta - \beta V(x)$ in R^d and depends only on dimension: $\beta_{cr} = 0$ if $d = 1, 2$ and $\beta_{cr} > 0$ if $d \geq 3$. The purpose of our study is to present some new results on the critical value of the coupling constant in exterior elliptic problems. The β_{cr} remains positive for all boundary conditions if $d \geq 3$. This fact follows immediately from the Cwikel-Lieb-Rozenblum inequality:

$$\#\{\lambda_j < 0\} \leq C_d \int_{\partial\Omega} (\beta V)^{d/2} dx, \quad d \geq 3.$$

The inequality above implies that the negative eigenvalues do not exist if β is so small that the right-hand side above is less than one. We will show that the answer to the main question for the problem (1.1) in dimension $d = 1$ and 2 is different from the answer in the case of the Schrödinger operator and depends on the boundary condition. The dependence of β_{cr} on the boundary condition and on the distance between the boundary and the support of the potential will be described. The discrete spectrum of a non-symmetric operator with the FKW boundary condition (that appears in diffusion processes with traps) will be also investigated.

1.3 Layout of Dissertation

Chapter 2 starts with some necessary prerequisite information needed for our results. We will provide the background needed for our results, but will not go into details. For more information, we refer the reader to books on the subject, such as [23, 24, 22, 25]. In Chapter 3, we will recall the results of the Schrödinger operator and discuss some properties of the operators associated with the Schrödinger equation to put our results in perspective. The statements of our results are given in the the Chapter 4, 5, and 6. Chapter 4 contains a discussion of the structure of the discrete spectrum of the operator H_β and related results considering Drichlet and Neumann boundary conditions. In Chapter 5, we will provide the detailed introduction of the FKW boundary condition and provide related results. Chapter 6 contains a study of the dependence of β_{cr} on the distance between the support of the potential and the boundary of the domain.

CHAPTER 2: BACKGROUND INFORMATION

In this chapter, we will discuss the known general facts of the Laplace operator in a bounded domain Ω. We will not go into details, but rather remind the reader some understanding of the basic definitions and facts.

2.1 Some definitions

Definition 2.1. Laplace Operator

Laplace operator is a differential operator denoted by Δ and is given by

$$\Delta u = (\nabla \cdot \nabla)u = \left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_d}\right)^2,$$

where u is a sufficiently smooth real valued functions, $u : \Omega \to \mathbb{R}$ and $x_1, x_2, \cdots, x_d$ are the coordinates for $\Omega \subset \mathbb{R}^d$.

Boundary Conditions(BCs)

- Dirichlet BC means $u = 0$ in $\partial\Omega$,
- Neumann BC means $\frac{\partial u}{\partial n} = 0$ in $\partial\Omega$ where n denotes the normal vector to the boundary $\partial\Omega$.
- FKW BC means $u = \alpha$ in $\partial\Omega$, $\int_{\partial\Omega} \frac{\partial u}{\partial n} d\mu(x) = 0$ where α is some constant in $\mathbb{R}$ and $d\mu$ is a positive measure on $\partial\Omega$.

The idea of this boundary condition, FKW boundary condition was first introduced by M. Freidlin, L. Koralov, and A. Wentzell in their paper "On the behaviour of diffusion processes with traps", 2015. When large drifts of a diffusion process direct towards to a point in the interior of the domain (see fig 2.2) then the FKW boundary problem occurs as the limiting problem describing the behaviour of the process as the magnitude of the drift tends to infinity.

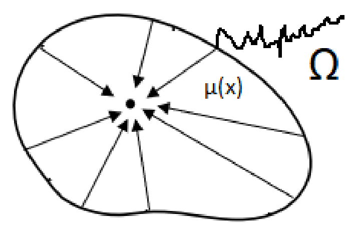

Figure 2.2: FKW boundary problem.

It is a common physical phenomenon that occurs when the vector field is large and the domain becomes trapping. We will provide more information of this boundary condition later in the Chapter 5.

Definition 2.2. Resolvent set and Spectrum

Let A be a densely defined linear operator on a Hilbert space $\mathcal{H}$. The operator $A-\lambda$ has domain $D(A)$, for each $\lambda \in \mathbb{C}$. Define the resolvent set

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is invertible (has a bounded inverse defined on } \mathcal{H})\}.$$

For λ in the resolvent set, we call the inverse $(A-\lambda)^{-1}$ the resolvent operator. The spectrum is defined as the complement of the resolvent set:

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

The discrete spectrum consists of λ where $A - \lambda$ fails to be an injective. Continuous spectrum consists of λ with $A-\lambda$ injective and with dense image, but not surjective.

The classification of the spectrum into discrete and continuous parts usually corresponds to a classification of the dynamics into localized (bound) states and locally decaying states when time increases (scattering), respectively.

If λ is an eigenvalue of A then $\lambda \in \sigma(A)$, because if $Af = \lambda f$ for some $f \neq 0$, then $(A - \lambda)f = 0$ and so $A - \lambda$ is not injective, and hence is not invertible. The resolvent set is open, and hence the spectrum is closed.

Definition 2.3. Compact Operator

Let X and Y be Hilbert spaces. A linear operator $T : X \to Y$ is said to be compact if for each bounded sequence $\{x_i\}_{i\in\mathbb{N}} \subset X$, there is a subsequence of $\{Tx_i\}_{i\in\mathbb{N}}$ that is convergent.

Definition 2.4. Fredholm operator

Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Then T is said to be Fredholm if dimension of Ker(T) $< \infty$, and co-dimension Im(T)$< \infty$.

Definition 2.5. We define the Sobolev spaces as

$$H^m(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega), 0 \leq |\alpha| \leq m\} \tag{2.2}$$

and its norm is defined by

$$\|u\|^2_{H^m(\Omega)} = \Sigma_{0\leq|\alpha|\leq m} \|D^\alpha u\|^2_{L^2(\Omega)}$$

where m is a non-negative integer.

Analytic Fredholm Theory Let Ω be a connected open subset of $\mathbb{C}$ and suppose $T(\lambda)$ is an analytic family of Fredholm operators on a Hilbert Space H. Then either:

a. $T(\lambda)$ is not invertible for any $\lambda \in \mathbb{C}$ OR

b. There exists a discrete set $S \subset \Omega$ such that $T(\lambda)$ is invertible $\forall\lambda \notin S$ and furthermore, every operator appearing as a coefficients of a term of negative order is finite rank.

2.2 Some remarks

Remark 2.1. From the Green formulas it follows that the Laplace operator is symmetric and positive on the space S= $C_c^\infty(\Omega)$ of all smooth and compactly supported functions on Ω.

Remark 2.2. The Dirichlet problem $-\Delta = f, x \in \Omega, u|_{\partial\Omega} = 0$, in a bounded domain with the a smooth boundary has a solution $u \in H^2(\Omega)$ for each $f \in L^2$, and the solution is unique.

Remark 2.3. If Ω is bounded, then the inverse Laplace operator with the Dirichlet boundary condition, Δ_d^{-1} is a compact operator in $L^2(\Omega)$, and its eigenvalues $\mu_j \to 0$.

Remark 2.4. The $-\Delta$ on a bounded domain with a smooth boundary and with Dirichlet or Neumann boundary condition has discrete spectrum.

Remark 2.5. Operator $-\Delta : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ does not have eigenvalues.

Proof. We have,

$$-\Delta u = \lambda u, \ u \in L^2(\mathbb{R}^d)$$

Taking a Fourier Transform on both sides, we get,

$$|\xi^2|\hat{u}(\xi) = \lambda \hat{u}(\xi), \ \hat{u} \in L^2(\mathbb{R}^d)),$$

$$(|\xi^2| - \lambda)\hat{u}(\xi) = 0,$$

$$\hat{u}(\xi) = 0 \ \ a.e.$$

This shows that $\hat{u} = 0$ as element of $L^2(\mathbb{R}^d)$. Thus, operator $-\Delta : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ does not have eigenvalues. $\square$

Definition 2.6. (Weyl Sequence): A Weyl sequence exists for $-\Delta$ and λ, if there exist functions w_n such that the following three conditions hold:

WS1. $\|(-\Delta-\lambda)w_n\|_{L^2} \to 0$ as $n \to \infty$.

WS2. $\|w_n\|_{L^2} = 1$.

WS3. $w_n \stackrel{w}{\to} 0$ in L^2 as $n \to \infty$. The continuous spectrum coincides with those λ values for which Weyl sequence exists.

Lemma 2.6. *The operator* $(-\Delta-\lambda) : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *has a bounded inverse*

$$(-\Delta-\lambda)^{-1} : L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)$$

for each $\lambda \notin [0, \infty)$.

Proof. Let $(-\Delta-\lambda)u = f$ where $u \in H^2(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$. Then after the Fourier transform on both sides, we have,

$$(|\xi^2| - \lambda)\hat{u}(\xi) = \hat{f}(\xi),$$

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{(|\xi^2| - \lambda)}.$$

Hence,

$$\begin{aligned}
\|u(x)\|_{H^2}^2 &= \int_{\mathbb{R}^d} (1+|\xi|^2)^2 |\hat{u}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^d} \left| \frac{(1+|\xi|^2)}{(|\xi^2|-\lambda)} \right|^2 \hat{f}(\xi)|^2 d\xi \\
&\leq \sup \left| \frac{(1+|\xi|^2)}{(|\xi^2|-\lambda)} \right|^2 \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi \\
&= \sup \left| \frac{(1+|\xi|^2)}{(|\xi^2|-\lambda)} \right|^2 \left\| \hat{f}(\xi) \right\|_{L^2}^2 .
\end{aligned}$$

Since $\lambda \notin [0, \infty)$, the function

$$\psi(\xi) = \left| \frac{(1+|\xi|^2)}{(|\xi^2|-\lambda)} \right|^2$$

is continuous.The limit at infinity is equal to one:

$$\lim_{\xi \to \infty} \psi(\xi) = 1.$$

Thus, $\psi(\xi)$ is bounded, i.e $|\psi(\xi)| \leq M(\lambda)$, and therefore

$$\|u(x)\|_{H^2} \leq \sqrt{M(\lambda)} \left\| \hat{f}(\xi) \right\|_{L^2} = \sqrt{M(\lambda)} \|f(x)\|_{L^2}.$$

□

Lemma 2.7. *The following estimates are valid :*

$$\left\|(\Delta - \lambda)^{-1} f\right\|_{H^1} \leq \frac{C}{|\lambda|} \|f\|_{L^2}, \ \left\|(\Delta - \lambda)^{-1} f\right\|_{L^2} \leq \frac{C}{|\lambda|^2} \ \|f\|_{L^2}, \ \lambda \to -\infty$$

Proof. Arguments that were used to prove Lemma 2.6 lead to the estimates

$$\left\|(\Delta - \lambda)^{-1} f\right\|_{H^1} \leq \max \, \psi_1(\xi, \lambda) \, \|f\|_{L^2},$$

$$\left\|(\Delta - \lambda)^{-1} f\right\|_{L^2} \leq \max \, \psi_0(\xi, \lambda) \, \|f\|_{L^2},$$

where $\psi_1(\xi, \lambda) = \frac{|\xi|^2+1}{(|\xi|^2-\lambda)^2}$, $\psi_0(\xi, \lambda) = \frac{1}{(|\xi|^2-\lambda)^2}$. This immediately implies the statement of the lemma.

□

Theorem 2.8. *The spectrum of operator $-\Delta$ in $\mathbb{R}^d$ is continuous and coincides with the positive semi-axis.*

Proof. From Lemma 2.6, it follows that $\mathbb{R}^d \backslash [0, \infty)$ does not belong to the spectrum of $-\Delta$. Hence, it remains to show that the semi-axis $[0, \infty)$ belongs to the continuous spectrum. It will be done by showing the existence of a Weyl Sequence.

Fix $\lambda \in [0, \infty)$ and choose $\omega \in \mathbb{R}^d$ with $|\omega|^2 = \lambda$. Take a cut-off function $\alpha \in C_0^\infty(\mathbb{R}^d)$

such that $\alpha \equiv 1$ on the unit ball $B(1)$ and $\alpha \equiv 0$ on $\mathbb{R}^d \setminus B(2)$. Define a cut-off version of the generalized eigenfunction, by

$$w_n = c_n \alpha(\frac{x}{n}) v_\omega(x), \quad \text{where } v_\omega(x) = e^{i\omega\cdot x},$$

and the normalizing constant is

$$c_n = \frac{1}{n^{d/2} \left\|\alpha\right\|_{L^2}}.$$

First we prove (WS1). We have

$$\begin{aligned}
&(\Delta + \lambda) w_n \\
&= c_n(\lambda v_\omega + \Delta v_\omega)\alpha(\frac{x}{n}) + 2\frac{c_n}{n}\nabla v_\omega(x)\cdot(\nabla\alpha)(\frac{x}{n}) + \frac{c_n}{n^2} v_\omega(x)(\Delta\alpha)(\frac{x}{n}).
\end{aligned}$$

The first term vanishes because $\Delta v_\omega = -|\omega|^2 v_\omega$ pointwise. In the third term, note that v_ω is a bounded function, and that a change of variable shows

$$\frac{c_n}{n^2}\left\|(\Delta\alpha)(\frac{x}{n})\right\|_{L^2} = \frac{1}{n^2}\frac{\left\|\Delta\alpha\right\|_{L^2}}{\left\|\alpha\right\|_{L^2}} \to 0 \text{ as } n \to \infty.$$

The second term similarly vanishes in the limit, as $n \to \infty$. Hence $(\lambda + \Delta)w_n \to 0$ in L^2, which is (WS1).

For (WS2) we simply observe that $|v_\omega(x)| = 1$ pointwise, so that $\left\|w_n\right\|_{L^2} = 1$ is a consequence of the change of variable, $\frac{x}{n} \to x$ and the definition of c_n.

To prove (WS3), we take $f \in L^2$ and fix $\varepsilon > 0$. We decompose f into "near" and "far" components, as $f = g + h$ where $g = f 1_{B(R)}$ and $h = f 1_{\mathbb{R}^d \setminus B(R)}$.

$$(f, w_n)_{L^2} = (g, w_n)_{L^2} + (h, w_n)_{L^2}.$$

We choose $R > 0$ such that $\|h\| < \frac{\varepsilon}{2}$. Then by the Cauchy-Schwartz inequality

$$\lim_{n\to\infty} \left|(h, w_n)_{L^2}\right| \leq \|h\|_{L^2} \cdot 1 < \frac{\varepsilon}{2}.$$

We have,

$$\left|(g, w_n)_{L^2}\right| \leq c_n \|\alpha\|_{L^\infty} \|g\|_{L^1} \to 0$$

as $n \to \infty$, since $c_n \to 0$ and R is fixed. So,

$$\lim_{n\to\infty} (f, w_n)_{L^2} = 0 \ ,$$

i.e., $w_n \xrightarrow{w} 0$ weakly (WS3). Hence, the spectrum of operator $-\Delta$ is continuous and coincides with the positive semi-axis.

$\square$

CHAPTER 3: SCHRÖDINGER OPERATOR

In this chapter, we will study some important results in the spectral theory of Schrödinger operators. More specifically, we would need these results to put our results in perspective. We will consider the operator, $\hat{H}_\beta = -\Delta - \beta V : L^2(\Omega) \to L^2(\Omega)$, where the potential $V(x) \geq 0$, is compactly supported and continuous, and $\beta \geq 0$. It is known that the spectrum of $\hat{H}_\beta$ consists of absolutely continuous part $[0, \infty)$ and at most a finite number of negative eigenvalues:

$$\sigma(\hat{H}_\beta) = \{\lambda_j\} \cup [0, \infty), \ 0 \leq j \leq N, \quad \lambda_j \leq 0.$$

One can define β_{cr} as the value of β such that the eigenvalues λ of the operator $\hat{H}_\beta$ exist when $\beta > \beta_{cr}$ and does not exist when $\beta < \beta_{cr}$ i.e

$$\beta_{cr} = \inf\{\beta : \ \inf \sigma(\hat{H}_\beta) < 0\}.$$

It is known [2] that $\beta_{cr} = 0$ for d $=$ 1, 2 and $\beta_{cr} > 0$ for d ≥ 3 . The later fact follows immediately from the Cwikel-Lieb-Rozenblum inequality:

$$\#\{\lambda_j < 0\} \leq C_d \int_{\partial\Omega} (\beta V)^{d/2} dx, \quad d \geq 3.$$

The inequality above implies that the negative eigenvalues do not exist if β is so small that the right-hand side above is less than one. If $d = 1$ or 2, then the answer to the main question for our problem is different from the answer in the case of the

Schrödinger operator and depends on the boundary condition.

3.1 Operators associated with $\hat{H}_\beta$.

Let u be an eigenfunction of the operator $\hat{H}_\beta$ with an eigenvalue $\lambda < 0$. We have,

$$\hat{H}_\beta = -\Delta - \beta V,$$
$$\hat{H}_\beta u = \lambda u,$$
$$(-\Delta - \lambda)u = \beta V u.$$

We know that the operator $(-\Delta - \lambda)^{-1} : L^2 \to H^2$ exists and bounded for $\lambda < 0$. Hence,

$$(-\Delta - \lambda)^{-1}(-\Delta - \lambda)u = (-\Delta - \lambda)^{-1}\beta V u,$$
$$u = (-\Delta - \lambda)^{-1}\beta V u.$$

One can multiply both sides by $\beta\sqrt{V}$, this leads to

$$\beta\sqrt{V}u = \beta\sqrt{V}(-\Delta - \lambda)^{-1}\beta V u,$$
$$\beta\sqrt{V}u = \beta\sqrt{V}(-\Delta - \lambda)^{-1}\sqrt{V}\beta\sqrt{V}u.$$

For simplicity, let $w = \beta\sqrt{V}u$, then we have:

$$w = \beta\sqrt{V}(-\Delta - \lambda)^{-1}\sqrt{V}w. \tag{3.3}$$

Now, we will define operator T_λ and A_λ as follows:

$$T_\lambda = \beta\sqrt{V}(-\Delta - \lambda)^{-1}\sqrt{V} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \tag{3.4}$$

$$A_\lambda = \sqrt{V}(-\Delta - \lambda)^{-1}\sqrt{V} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d). \tag{3.5}$$

Lemma 3.9. *$\mu = 1$ is an eigenvalue of T_λ if and only if λ is an eigenvalue of $\hat{H}_\beta$. Moreover, there is a one to one correspondence between eigenspaces of T_λ and $\hat{H}_\beta$.*

Namely, if $u \in L^2(\mathbb{R}^d)$ *is an eigenfunction of* $\hat{H}_\beta$ *then* $w = \beta\sqrt{V}u$ *is the eigenfunction of* T_λ*, and vice versa, if* w *is an eigenfunction of* T_λ *with the eigenvalue* $\mu = 1$*, then* $u = (-\Delta - \lambda)^{-1}\sqrt{V}$ *is the eigenfunction of* $\hat{H}_\beta$*.*

Proof. Let $u \in L^2(\mathbb{R}^d)$ be an eigenfunction of $\hat{H}_\beta$ corresponding to an eigenvalue λ. Then equations (3.3) and (3.4) imply that $w = \beta\sqrt{V}u$ satisfies

$$(I - T_\lambda)\ w = 0.$$

Let us assume now that $w \in L^2(\mathbb{R}^d)$ is an eigenfunction of T_λ with the eigenvalue $\mu = 1$, i.e,

$$(I - \beta\sqrt{V}(-\Delta - \lambda)^{-1})\sqrt{V}w = 0. \tag{3.6}$$

Then define

$$u = (-\Delta - \lambda)^{-1}\sqrt{V}w \in L^2(\mathbb{R}^d)\ ,$$

and it follows that

$$(-\Delta - \lambda - \beta V)u = 0\ ,$$

i.e, u is the eigenfunction of $\hat{H}_\beta$ with the eigenvalue λ. Hence, there is one to one correspondence between the kernel of the operator $I - T_\lambda$ and the eigenspace of the operator $\hat{H}_\beta$ corresponding to the eigenvalue λ. □

Remark: Since operators T_λ and A_λ differ by a factor β, one can use above lemma to conclude that, $\mu = \frac{1}{\beta}$ is an eigenvalue of A_λ if and only if λ is an eigenvalue of $\hat{H}_\beta$, and there is a one to one correspondence between eigenspaces of A_λ and $\hat{H}_\beta$. Namely, if $u \in L^2(\mathbb{R}^d)$ is an eigenfunction of the $\hat{H}_\beta$ then $w = \beta\sqrt{V}u$ is the eigenfunction of A_λ, and vice versa, if w is an eigenfunction of A_λ with the eigenvalue $\mu = \frac{1}{\beta}$, then $u = (-\Delta - \lambda)^{-1}\sqrt{V}$ is the eigenfunction of $\hat{H}_\beta$.

Lemma 3.10. T_λ *is a compact operator in* $L^2(\Omega)$ *that depends analytically on* λ *when* $\lambda \notin [0, \infty]$*, i.e* $\frac{d}{d\lambda}(T_\lambda)$ *exists.*

Proof. We have

$$T_\lambda f = \beta\sqrt{V}(-\Delta-\lambda)^{-1}\sqrt{V}f. \tag{3.7}$$

Operator $(-\Delta-\lambda)^{-1} : L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)$ is bounded (see Lemma 2.6). Since V has a compact support, the product $T_\lambda = \beta\sqrt{V}(-\Delta-\lambda)^{-1}\sqrt{V}$ is compact due to the Sobolev imbedding theorem.

To prove the analyticity, it is enough to prove for the operator $(-\Delta-\lambda)^{-1}$. The analyticity of $(-\Delta-\lambda)^{-1}$ is obvious if we use the Fourier transform $\mathcal{F}$.

$$\mathcal{F}(-\Delta-\lambda)^{-1}f = \frac{\mathcal{F}f}{|\xi|^2-\lambda},\ \lambda \notin [0,\infty). \tag{3.8}$$

One can apply the differentiation in λ in the operator norm to the element of $L^2(\mathbb{R}^d)$ defined by (3.8) above. Hence, T_λ is compact operator in $L^2(\Omega)$ that depends analytically on λ. □

Theorem 3.11. *For each $\lambda < 0$, eigenvalues $\mu = \mu_j(\lambda)$ of the operator T_λ are discrete and positive, if $\beta > 0$.*

Proof. From Lemma 2.7, it follows that:

$$\left\|(-\Delta-\lambda)^{-1}\right\|_{L^2(\mathbb{R}^d)} \to 0 \text{ as } \lambda \to -\infty.$$

Thus, $\|T_\lambda\|_{L^2(\mathbb{R}^d)} \to 0$ as $\lambda \to -\infty$. Lemma 3.10 and the Analytic Fredholm theorem imply that eigenvalues of T_λ are discrete. The eigenvalues are real since T_λ is symmetric. Moreover, if $w = \sqrt{V}u$ then

$$(T_\lambda u, u) = (\beta\sqrt{V}(-\Delta-\lambda)^{-1})\sqrt{V}u, u) = \beta((-\Delta-\lambda)^{-1}w, w) = \beta\int \frac{|\hat{w}|^2}{|\xi|^2-\lambda}d\xi > 0.$$

Thus, operator T_λ is positive and its eigenvalues are positive. □

Let us define, $\mu_0(\lambda) = \max_{\|u\|=1}(T_\lambda u, u)$.

Lemma 3.12. *$\mu_0(\lambda)$ is a continuous and increasing function of λ on the semi-axis $(-\infty, 0)$.*

Proof. As we defined,

$$\mu_0(\lambda) = \max_{\|u\|=1}(T_\lambda u, u).$$

It is easy to check that $(T_{\lambda_1}u, u) \leq (T_{\lambda_2}u, u),\ \forall u \in L^2$ by using the Fourier transform if $\lambda_1 \leq \lambda_2 < 0$. Indeed, if $w = \sqrt{V}u$ then

$$(T_\lambda u, u) = (\beta\sqrt{V}(-\Delta - \lambda)^{-1})\sqrt{V}u, u) = \beta((-\Delta - \lambda)^{-1}w, w) = \beta \int \frac{|\hat{w}|^2}{|\xi|^2 - \lambda} d\xi.$$

This implies that $\mu_0(\lambda)$ is an increasing function of λ on the semi-axis $(-\infty, 0)$. Also, the operation of taking the maximum value preserves the continuity which implies the continuity of $\mu_0(\lambda)$. □

Lemma 3.13. *If $d = 1, 2$, then $\mu_0(\lambda) \to \infty$ when $\lambda \to 0^-$.*

Proof. We know the asymptotic behavior of the operator T_λ as $\lambda \to 0^-$ from [2]. Now,

$$(T_\lambda u, u) \asymp \begin{cases} \frac{\beta}{\sqrt{|\lambda|}} \int V|u|^2 + o(1) & \text{for d=1} \\ \beta \ln|\lambda| \int V|u|^2 + o(1) & \text{for d=2} \end{cases}$$

Which concludes that $\mu_0(\lambda) \to \infty$ as $\lambda \to 0^-$. □

Lemma 3.14. *$\mu_0(\lambda) \to 0$ as $\lambda \to -\infty$.*

Proof. We will use Parseveals's identity to proof the lemma. Here,

$$(T_\lambda u, u) = (\widehat{T_\lambda u}, \widehat{u}) = (\beta\frac{\widehat{\sqrt{V}u}}{|\xi|^2 + |\lambda|}, \widehat{\sqrt{V}u}).$$

Hence, by using the definition of $\mu_0(\lambda)$ one can conclude that $\mu_0(\lambda) \to 0$ as $\lambda \to -\infty$. □

CHAPTER 4 : DEPENDENCE OF β_{cr} ON THE BOUNDARY CONDITION

4.1 Spectrum of the Operator H_β

Consider the following elliptic problems in Ω.

$$H_0 u - \beta V(x) u - \lambda u = f, \quad x \in \Omega, \tag{4.9}$$

where $H_0 = -\text{div}(a(x)\nabla)$, the potential $V(x) \geq 0$ is compactly supported and continuous, $\beta \geq 0$, $a(x) > 0$, $a(x) \in C^1(\Omega)$, and $a = 1$ when $|x| >> 1$. We assume that $f \in L^2(R^d)$ and the solutions belong to the space $H^2(\Omega)$ and satisfies a Dirichlet, Neumann, or FKW-boundary condition. We consider the truncated resolvent A_λ of operator H_0 with the cut-off function $\chi = \sqrt{V(x)}$:

$$A_\lambda = \sqrt{V}(H_0 - \lambda)^{-1}\sqrt{V} : L^2(\Omega) \to L^2(\Omega), \ \lambda < 0. \tag{4.10}$$

The main question of our study is whether the discrete spectrum appears for arbitrary small perturbations (arbitrary small $\beta > 0$), or β must be large enough to create negative eigenvalues. As we defined β_{cr} in the chapter three, we defined β_{cr} as the value of β such that operator $H_\beta = H_0 - \beta V(x)$ does not have negative eigenvalues for $\beta < \beta_{cr}$ and has them if $\beta > \beta_{cr}$. Thus, we would like to know when $\beta_{cr} = 0$ and when $\beta_{cr} > 0$ considering the Dirichlet, Neumann, and FKW boundary condition.

Lemma 4.15. *There is a one-to-one correspondence between the eigenspaces of operators H_β, $\beta > 0$, and A_λ, $\lambda < 0$. Namely, if $u \in H^2(\Omega)$ is an eigenfunction of H_β*

with an eigenvalue $\lambda < 0$, *then* $w = \sqrt{V}u$ *is an eigenfunction of operator* A_λ *with the eigenvalue* $\frac{1}{\beta}$. *Vice versa, if* $w \in L^2(\Omega)$ *is an eigenfunction of* A_λ, $\lambda < 0$, *with an eigenvalue* μ, *then* $\mu > 0$ *and* $u = (H_0 - \lambda)^{-1}(\sqrt{V}w)$ *is an eigenfunction of* $H_{1/\mu}$ *with the eigenvalue* λ.

Proof. Let $H_\beta u = \lambda u$. Then $(H_0 - \lambda)u = \beta V u$ and $u = \beta(H_0 - \lambda)^{-1}(Vu)$. After multiplying both sides by $\sqrt{V}$, we obtain $w = \beta A_\lambda w$, i.e., w is an eigenfunction of A_λ with the eigenvalue $1/\beta$.

Conversely, let $A_\lambda w = \mu w$, $\lambda < 0$, i.e.,

$$\sqrt{V}(H_0 - \lambda)^{-1}(\sqrt{V}w) = \mu w. \tag{4.11}$$

Since operator A_λ, $\lambda < 0$, is positive, we have $\mu > 0$. Define $u = (H_0 - \lambda)^{-1}(\sqrt{V}w)$. Then $u \in H^2, Bu|_{\partial\Omega} = 0$, and $(H_0 - \lambda)u = \sqrt{V}w$. We multiply both sides of (4.11) by $\sqrt{V}$ and express $\sqrt{V}w$ through u. This leads to $Vu = \mu(H_0 - \lambda)u$, i.e., u is an eigenfunction of $H_{1/\mu}$ with the eigenvalue λ.

□

Theorem 4.16. *The spectrum of* H_β *consists of the absolutely continuous part* $[0, \infty)$ *and at most a finite number of negative eigenvalues.*

$$\sigma(H_\beta) = \{\lambda_j\} \cup [0, \infty), \; 0 \le j \le N, \quad \lambda_j \le 0.$$

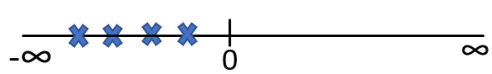

Figure 4.3: Spectrum of the operator H_β.

Proof. Let $\Omega_{R+1} := \Omega \bigcap \{|x| < R+1\}$, and denote by $A - \lambda_0 : H^2(\Omega_{R+1}) \to L^2(\Omega_{R+1})$

the operator that corresponds to the problem

$$-\mathrm{div} a(x)\nabla u - \beta V(x)u - \lambda_0 u = g, \quad x \in \Omega_{R+1}; \tag{4.12}$$

$$Bu|_{\partial\Omega} = 0, \ u|_{|x|=R+1} = 0. \tag{4.13}$$

We construct the resolvent $R_\lambda : L^2(\Omega) \to H^2(\Omega)$ using the following parametrix P (almost inverse operator) for $H_\beta - \lambda$ which consists of two terms that "invert" operator $H_\beta - \lambda$ in a bounded part Ω_{R+1} of Ω and in a neighborhood of infinity, respectively. Namely,

$$P : L^2(\Omega) \to H^2(\Omega), \quad Ph = \psi_1(A-\lambda_0)^{-1}(\varphi_1 h) + \psi_2(-\Delta-\lambda)^{-1}(\varphi_2 h) := \psi_1 P_1 h + \psi_2 P_2 h, \tag{4.14}$$

where $\{\varphi_1, \varphi_2\}$ is a partition of unity (i.e., $\varphi_1(x) + \varphi_2(x) = 1$ in Ω) such that $\varphi_1 \in C^\infty$, $\varphi_1 = 1$ when $|x| < R + 1/3$, $\varphi_1 = 0$ when $|x| > R + 2/3$; $\psi_1, \psi_2 \in C^\infty$, $\psi_1 = 1$ for $|x| < R + 2/3$, $\psi_1 = 0$ for $|x| > R + 1$; $\psi_1 = 1$ for $|x| > R + 1/3$, $\psi_1 = 0$ for $|x| < R$ (i.e., $\psi_1\varphi_1 = \varphi_1$, $\psi_2\varphi_2 = \varphi_2$); $\lambda_0 < -\beta \max V$ will be chosen later.

The Green formula implies that operator $A - \lambda_0$ is positive, and therefore it is invertible. Hence P is bounded for $\lambda \notin [0, \infty)$. We will look for the solution of the equation $(H_\beta - \lambda)u = f$, $\lambda \notin [0, \infty)$, in the form $u = Ph$ with some $h \in L^2(\Omega)$. Since $\psi_1 = 1$ and $\psi_2 = 0$ in a neighborhood of $\partial\Omega$, function Ph satisfies the boundary condition $Bu = 0$ on $\partial\Omega$. Thus $R_\lambda f = Ph$ if

$$(-\mathrm{div} a(x)\nabla - \beta V(x) - \lambda)Ph = f \in L^2(\Omega). \tag{4.15}$$

If we apply the differential operator above only to the second factors $P_i h$ in the terms of the expression (4.14) for P, then we will get the following contribution to the left-hand side of (4.15):

$$\psi_1[(\lambda_0-\lambda)(A-\lambda_0)^{-1}(\varphi_1 h)+\varphi_1 h]+\psi_2\varphi_2 h = h+\psi_1(\lambda_0-\lambda)P_1 h.$$

Hence (4.15) has the form

$$h+F_\lambda h = f, \quad f,h\in L^2(\Omega), \tag{4.16}$$

where

$$F_\lambda h=\psi_1(\lambda_0-\lambda)P_1h-(\mathrm{div}a\nabla\psi_1)P_1h-2a\nabla\psi_1\cdot\nabla P_1h-\Delta\psi_2P_2h-2\nabla\psi_2\cdot\nabla P_2h. \tag{4.17}$$

Since F_λ does not contain derivatives of the second order of P_ih, it follows that F_λ is a bounded operator from $L^2(\Omega)$ into $H^1(\Omega)$. Additionally, one can easily see that the support of $F_\lambda h$ belongs to $\overline{\Omega_{R+1}}$. Thus, the Sobolev imbedding theorem implies that operator $F_\lambda : L^2(\Omega)\to L^2(\Omega)$ is compact.

Obviously, operator F_λ is analytic in $\lambda\in\mathbb{C}\backslash[0,\infty)$. We will show that λ_0 can be chosen in such a way that $\|F_{\lambda_0}\|_{L^2(\Omega)}<1$. Then analytic Fredholm theorem can be applied to the operator $I+F_\lambda$ from which it follows that operator $(I+F_\lambda)^{-1}$ is meromorphic in $\lambda\in\mathbb{C}\backslash[0,\infty)$, and therefore

$$R_\lambda = P(I+F_\lambda)^{-1}, \quad \lambda\in\mathbb{C}\backslash[0,\infty). \tag{4.18}$$

Let us justify the possibility to choose an appropriate λ_0. The following parameter-elliptic estimate is valid for $v=(A-\lambda_0)^{-1}g$:

$$\|v\|_{H^2(\Omega_{R+1})}+\sqrt{|\lambda_0|}\|v\|_{H^1(\Omega_{R+1})}+|\lambda_0|\|v\|_{L^2(\Omega_{R+1})}\le C\|g\|_{L^2(\Omega_{R+1})}, \quad \lambda_0\to-\infty. \tag{4.19}$$

Since the supports of $\varphi_1,\ \psi_1$ belong to Ω_{R+1}, from (4.19) it follows that the norms of the second and third operators in the right-hand side of (4.17) go to zero as $\lambda_0 \to -\infty$. Since

$$\|(-\Delta - \lambda)^{-1} g\|_{H^1(R^d)} \leq C\|g\|_{L^2(R^d)}/\sqrt{|\lambda|}, \quad \lambda < 0,$$

the norms of the last two operators in the right-hand side of (4.17) go to zero as $\lambda = \lambda_0 \to -\infty$. The first term in the right-hand side of (4.17) vanishes as $\lambda = \lambda_0$. Thus if $-\lambda_0$ is large enough, then $\|F_{\lambda_0}\| < 1$ and (4.18) is valid.

Formula (4.18) proves that the negative poles of R_λ are discrete. Since operator H_β is bounded from below, the first statement of the theorem will be proved if one shows that $I + F_\lambda$ is invertible when $0 < |\lambda| < \varepsilon$ for some $\varepsilon > 0$. The latter follows from specific properties of F_λ at $\lambda = 0$. In fact, using Lemma 1 of [18], the operator F_0 is of finite rank, i.e bounded linear operator in the $\varepsilon-$ neighbourhood of the point $\lambda = 0$. Hence, there is not an eigenvalue in the neighbourhood of $\lambda = 0$, which states the invertibility of $I + F_\lambda$.

□

The truncated resolvent $\widehat{R}_\lambda$ is defined by $\widehat{R}_\lambda = \chi R_\lambda \chi$ where $\chi = \sqrt{V(x)}$ and $R_\lambda = P(I + F_\lambda)^{-1}$, $\lambda \in \mathbb{C}\backslash[0, \infty)$.

Theorem 4.17. *The truncated resolvent $\widehat{R}_\lambda$ admits a meromorphic continuation through the continuous spectrum. If d is odd, then the continuation is a meromorphic function in the complex plane $k = \sqrt{\lambda}$; if d is even, then it is meromorphic in $k \in \mathbb{C}$ with a logarithmic branching point at $k = 0$.*

Proof. From (4.18) it follows that

$$\widehat{R}_\lambda = \chi P(I + F_\lambda)^{-1}\chi, \quad \lambda \in \mathbb{C}\backslash[0, \infty). \tag{4.20}$$

Let $\chi_1(x) \in C_{com}(\overline{\Omega})$ be a cut-off function which is equal to one on Ω_{R+1} and on the support of χ. Since $F_\lambda h = 0$ outside of Ω_{R+1}, we have $\chi_1 F_\lambda = F_\lambda$. If we also take into account that $h = \chi_1 h$ for the solutions h of (4.16) with χf in the right-hand side, then we get that $(I + F_\lambda)^{-1}\chi f = \chi_1(I + \chi_1 F_\lambda \chi_1)^{-1}\chi f$. Thus, (4.20) can be rewritten in the form

$$\widehat{R}_\lambda = \chi P \chi_1 (I + \chi_1 F_\lambda \chi_1)^{-1}\chi, \quad \lambda \in \mathbb{C}\backslash[0,\infty). \tag{4.21}$$

Formula (4.21) allows one to extend $\widehat{R}_\lambda$ through the continuous spectrum. Indeed, the integral kernel of the operator $(-\Delta - \lambda)^{-1} : L^2(R^d) \to H^2(R^d)$, $\mathrm{Im}\sqrt{\lambda} > 0$, is a Bessel function which is decaying at infinity and which grows exponentially at infinity if $k = \sqrt{\lambda}$ is extended into the lower half plane. After the truncation, the kernel becomes bounded uniformly with respect to space variables and analytic in k, so that the following statement is valid: for any cut-off functions $\chi_{(1)}, \chi_{(2)} \in C_{com}(R^d)$, the truncated resolvent

$$\widehat{R}^0_{k^2} = \chi_{(1)}(-\Delta - \lambda)^{-1}\chi_{(2)} : L^2(R^d) \to H^2(R^d)$$

admits the analytic continuation into the lower half plane $\mathrm{Im}k < 0$ with the properties: $\widehat{R}^0_{k^2} = \frac{N(k)}{k}$ if d is odd, $\widehat{R}^0_{k^2} = N(k)\ln k + M(k)$ if d is even. Here $N(k), M(k)$ are entire operator functions, $N(0)$ is a one-dimensional operator when $d = 1, 2$, and $N(0) = 0$ when $d \geq 3$ (thus $\|\widehat{R}^0_{k^2}\|$ is bounded when $|k|$ and $|\arg k|$ are bounded and $d \geq 3$.)

From (4.14) and (4.17) it follows that operators $\chi P \chi_1$ and $\chi_1 F_\lambda \chi_1$ have the same analytic properties with respect to variable k as operator $\widehat{R}^0_{k^2}$ has. Further, we note that the proof of the compactness of F_λ and the proof of the estimate $\|F_{\lambda_0}\| < 1$, $-\lambda_0 \gg 1$, remain valid for analytic continuation of $\chi_1 F_{k^2}\chi_1$, $k \in \mathbb{C}\backslash\{0\}$ (one need first to check the latter statements when $\chi_1 \in C^\infty_{com}(\overline{\Omega})$, $\chi_1 = 1$ in Ω_ρ, $\rho \gg 1$; then these statements are obviously valid for continuous χ_1 with the support in Ω_ρ.) Therefore, the analytic Fredholm theorem can be applied to operator $I + \chi_1 F_{k^2}\chi_1, k \in$

$\mathbb{C}\backslash\{0\}$. Hence (4.21) implies the second statement of the theorem □

4.2 Dirichlet Problem

Consider the Dirichlet problem in an exterior domain $\Omega \subset \mathbb{R}^d$.

$$H_\beta - \lambda u = f, \ x \in \Omega, \ u|_{\partial\Omega} = 0, \ \lambda < 0. \tag{4.22}$$

We know from equation (4.10) the truncated resolvent operator A_λ is defined by

$$A_\lambda = \sqrt{V}(H_0 - \lambda)^{-1}\sqrt{V} : L^2(\Omega) \to L^2(\Omega), \ \lambda < 0.$$

It will be shown that the choice $\beta_{cr} > 0$ or $\beta_{cr} = 0$ depends on whether the truncated resolvent A_{0^-} is bounded or goes to infinity when $\lambda \to 0^-$. In fact, β_{cr} will be expressed through $\|A_\lambda\|$. Note that operator $H_\beta - \lambda$ decays when λ grows, and therefore $\|A_\lambda\|$ is monotone in λ and the limit $\lim_{\lambda\to 0^-}[1/\|A_\lambda\|]$ exists.

Theorem 4.18. $\beta_{cr} = 1/\|A_{0^-}\|$.

Proof. The operator $(H_0 - \lambda)^{-1} : L^2(\Omega) \to H^2(\Omega), \ \lambda < 0$, (where $H^2(\Omega)$ is the Sobolev space) is bounded and the potential V has a compact support. Hence, Sobolev's imbedding theorem implies that operator (4.10) is compact. Since operator $A_\lambda, \ \lambda < 0$, is positive, depends continuously on λ, and increases when λ increases, its principal (largest) eigenvalue $\mu_0(\lambda), \ \lambda < 0$, is a positive, continuous, and monotonically increasing function of λ. Let $\mu^* = \lim_{\lambda\to 0^-}\mu(\lambda) = \|A_{0^-}\|$. Obviously, $\|A_\lambda\| \to 0$ as $\lambda \to -\infty$. Thus, the range of the function $\mu_0(\lambda), \ -\infty < \lambda < 0$, is $(0, \mu^*)$. Hence, for each $\mu \in (0, \mu^*)$, there is a $\lambda = \lambda_0 < 0$ such that $\mu_0(\lambda_0) = \mu$, and therefore $H_{1/\mu}$ has the eigenvalue $\lambda = \lambda_0$ due to Lemma 4.15. Since $1/\mu \in (1/\mu^*, \infty)$, operator H_β has at least one negative eigenvalue when $\beta > 1/\mu^*$.

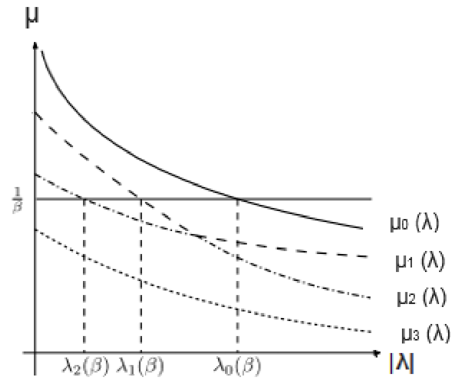

Figure 4.4: Graph of $\mu_0(\lambda)$.

Since $A_\lambda,\ \lambda < 0$, can not have eigenvalues larger than μ^*, Lemma 4.15 implies that H_β does not have eigenvalues $\lambda < 0$ if $\beta < 1/\mu^*$. It concludes that $\beta_{cr} = 1/\mu^* = 1/\|A_{0-}\|$. □

Theorem 4.19. *Let $d = 1$ or 2. Then $\|A_{0-}\| < \infty$ and $\beta_{cr} > 0$ in the case of the Dirichlet boundary condition.*

In order to prove Theorem 4.19, we will need the following lemma.

Lemma 4.20. *Let $\omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary. Let H be a non-zero closed subspace of Sobolev space $H^1(\omega)$ that does not contain non-zero constant functions. Then there exists a constant $C > 0$ that depends on ω such that*

$$\|u\|_{L^2} \leq C\|\nabla u\|_{L^2}, \quad u \in H. \tag{4.23}$$

Proof. Assume that (4.23) is not true. Then there is a sequence $v_n \in H,\ n \in \mathbb{N}$, such that $\|v_n\|_{L^2} \geq n\|\nabla v_n\|_{L^2}$. Define $u_n = \frac{v_n}{\|v_n\|_{L^2}}$. Then $\|u_n\| = 1$ and $\|\nabla u_n\|_{L^2} \leq \frac{1}{n}$, i.e., $\{u_n\}$ is a bounded sequence in $H^1(\omega)$. Since the imbedding $H^1(\omega) \subset L^2(\omega)$ is compact, there exists a subsequence of $\{u_n\}$ that converges in $L^2(\omega)$. Without loss of generality we can assume that $\{u_n\}$ converges in $L^2(\omega)$ as $n \to \infty$. Since $\|\nabla u_n\|_{L^2} \to 0$

as $n \to \infty$, the sequence $\{u_n\}$ converges in $H^1(\omega)$. The limiting function u belongs to H since $u_n \in H$ and H is a closed subspace of $H^1(\omega)$. Relation $\|\nabla u_n\|_{L^2} \to 0$ as $n \to \infty$ implies that u is a constant. This constant must be zero since H does not contain non-zero constant functions. The latter contradicts the fact that $\|u_n\|_{L^2} = 1$. Hence, our assumption is wrong. □

Proof of Theorem 4.19. Consider first the case of the Dirichlet boundary condition. We would like to show that $\|A_\lambda\| < C < \infty,\ \lambda \to 0^-$. From (4.10) it follows that $A_\lambda f = \sqrt{V}u$, where $u = (H_0 - \lambda)^{-1}\sqrt{V}f$, i.e., $(H_0 - \lambda)u = \sqrt{V}f$. From the Green formula it follows that

$$\int_\Omega (a(x)|\nabla u|^2 - \lambda |u|^2)dx = \int_{\Omega_R} \sqrt{V} f u dx.$$

Hence,

$$\int_\Omega |\nabla u|^2 dx \le \int_{\Omega_R} |\sqrt{V} f u| dx, \quad \lambda < 0.$$

Lemma 4.20 implies that

$$\int_{\Omega_R} |u|^2 dx \le C \int_\Omega |\nabla u|^2 dx \le C \int_{\Omega_R} |\sqrt{V} f u| dx \le \frac{1}{2} \int_{\Omega_R} |u|^2 dx + \frac{C^2}{2} \int_{\Omega_R} |\sqrt{V} f|^2 dx, \quad \lambda < 0.$$

Thus

$$\|u\|_{L^2(\Omega_R)} \le C\|\sqrt{V} f\|_{L^2(\Omega_R)} \le C_1 \|f\|_{L^2(\Omega_R)}, \quad \lambda < 0,$$

and

$$\|A_\lambda f\|_{L^2(\Omega)} = \|\sqrt{V} u\|_{L^2(\Omega_R)} \le C_2 \|f\|_{L^2(\Omega)}, \quad \lambda < 0,$$

Hence, $\|A_\lambda\| \le C,\ \lambda \to 0^-$, and Theorem 4.18 implies that $\beta_{cr} > 0$.

□

4.3 Neumann Problem

Consider the Neumann problem in an exterior domain $\Omega \subset \mathbb{R}^d$.

$$H_\beta - \lambda u = f,\ x \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0,\ \lambda < 0. \tag{4.24}$$

From Theorem 4.18 it follows that the proof of Theorem 4.21 will be complete if we show that $\|A_\lambda\| \to \infty$ when $\lambda \to 0^-$. We will focus only on the case $d = 2$ since one-dimensional case can be studied similarly or independent simpler proof can be given.

Theorem 4.21. *Let $d = 1$ or 2. Then $\beta_{cr} = 0$ in the cases of the Neumann boundary condition.*

Proof. Denote $R_{\lambda,0} = (H_0 - \lambda)^{-1}$, where the sub-index zero indicates that $\beta = 0$. Consider

$$\widehat{R}_{\lambda,0} = \chi R_{\lambda,0}\chi \text{ with } \chi \in C_0^\infty(\overline{\Omega}).$$

An asymptotic expansion of this operator as $\lambda \to 0$ was obtained in [17], [19] in a more general setting than here (for operators H_0 of higher order). In particular, the following result is valid:

$$\widehat{R}_{\lambda,0} = \nu_\lambda(P + Q(\lambda)), \quad \|Q(\lambda)f\|_{H^2(\Omega_R)} \leq C|\ln\lambda|^{-1}\|f\|_{L^2(\Omega_R)},\ \lambda \to 0^-, \tag{4.25}$$

where $\widehat{R}_{\lambda,0} : L^2(\Omega_R) \to H^2(\Omega_R)$ is a non-zero bounded operator, $\nu_\lambda = |\lambda|^{\alpha/2}|\ln\lambda|^\beta$, α and β are integers.

It will be convenient for us to consider the truncated resolvent $\widehat{R}_{\lambda,0}$ with a non-smooth function χ which is the indicator of the domain Ω_{R+1}, i.e., $\chi = 1$ in Ω_{R+1}, $\chi = 0$ for $|x| \geq R + 1$. Expansion (4.25) was proved in [17], [19] for this χ, and below we assume that χ is the indicator of the domain Ω_{R+1}.

Let us show that $\nu_\lambda \to \infty$ as $\lambda \to 0$. Let

$$u = \varphi(x)K_0(\sqrt{|\lambda|}|x|) - (1-\varphi(x))\ln\sqrt{|\lambda|}, \tag{4.26}$$

where K_0 is the modified Bessel function (it can be expressed through the Hankel function: $K_0(r) = \frac{\pi i}{2}H_0^{(1)}(ir)$), and $\varphi \in C^\infty$, $\varphi(x) = 1$ when $|x| > R + \frac{1}{2}$, $\varphi(x) = 0$ when $|x| < R$. Since operator H_0 coincides with $-\Delta$ when $\varphi \neq 1$, it follows that $H_0 u = f$, $\lambda < 0$, where

$$f = -\Delta\varphi(x)K_0(\sqrt{|\lambda|}|x|) - 2\nabla\varphi(x)\cdot\nabla K_0(\sqrt{|\lambda|}|x|) - \ln\sqrt{|\lambda|}H_0(1-\varphi(x)). \tag{4.27}$$

Function u decays exponentially at infinity. Thus $u = R_{\lambda,0}f$, $\lambda < 0$. Support of f belongs to the layer $R \leq |x| \leq R + \frac{1}{2}$, and $\chi = 1$ there. Thus $u = R_{\lambda,0}(\chi f)$, $\lambda < 0$. Since $K_0(\sqrt{|\lambda|}|x|) = -\ln\sqrt{\lambda} + v$, where $v \in C^2$ uniformly in λ when $|x| > R$, $\lambda \to 0^-$, the terms with $\ln\sqrt{|\lambda|}$ are canceled in (4.27), and this implies that $\|f\| \leq C$, $\lambda \to 0^-$. On the other hand, from (4.26) it follows that $u = -\ln\sqrt{|\lambda|}$ when $|x| < R$. Hence $\|\widehat{R}_{\lambda,0}f\| = \|\chi u\| \sim -\ln\sqrt{|\lambda|}$, $\lambda \to 0^-$, i.e., operator $\|\widehat{R}_{\lambda,0}\|$ is unbounded as $\lambda \to 0^-$. This fact does not prove yet that A_λ is unbounded, but it proves that $\nu_\lambda \to \infty$ as $\lambda \to 0$. Since $\chi = 1$ when $|x| < R$, we have $A_\lambda h = \sqrt{V}R_{\lambda,0}(\sqrt{V}h)$. Thus (4.25) will follow that $\|A_\lambda\| \to \infty$ as $\lambda \to 0^-$ if we show that $\sqrt{V}P\sqrt{V} \neq 0$.

The Green function $R_{\lambda,0}(x,\xi)$, $\lambda < 0$, of operator H_0 (with the Neumann boundary condition) is defined uniquely (due to the decay of the Green function at infinity). It coincides with the integral kernel of operator $R_{\lambda,0}$. Since $R_{\lambda,0}(x,\xi) \geq 0$, $\lambda < 0$, and symmetric, from (4.25) it follows that the same properties hold for the integral kernel $P(x,\xi)$ of operator P: $P(x,\xi) \geq 0$, $P(x,\xi) = P(\xi,x)$.

Since $\chi = 1$ when $|x| < R+1$, from (4.25) it follows that

$$H_0(Ph + Q(\lambda)h) = \nu_\lambda^{-1}H_0R_{\lambda,0}(\chi h) = \chi h, \quad \text{for } |x| < R+1 \quad \text{and any } h \in L^2(\Omega).$$

We pass to the limit as $\lambda \to 0^-$ and obtain that $H_0Ph = 0,\ |x| < R+1$. Let $h \in L^2(\Omega)$ be a function with the support in Ω_{R+1} and such that $h \geq 0$, and $u := Ph \not\equiv 0$ (recall that $P(x,\xi) \geq 0$ and P is a non-zero operator). The uniqueness of the solutions of the Cauchy problem for elliptic equations of the second order implies that u can not vanish identically on an open set in Ω_{R+1}. Since $u \geq 0$ it follows that

$$\int_{\Omega_{R+1}} u(x)\sqrt{V}(x)dx \neq 0.$$

Consider now function $w(\xi) = \int_\Omega P(x,\xi)\sqrt{V}(x)dx$. The relation above implies that $w \not\equiv 0$. Due to the symmetry of the $P(x,\xi)$, function w satisfies the equation $H_0 w = 0,\ |\xi| < R$, and therefore, it cant vanish on an open set. Since $w \geq 0$, it follows that $\int_\Omega w(\xi)\sqrt{V}(\xi)d\xi > 0$, i.e.,

$$\int_\Omega \int_\Omega \sqrt{V}(x)P(x,\xi)\sqrt{V}(\xi)d\xi > 0.$$

Hence $\sqrt{V}P\sqrt{V}$ is a non-zero operator, and $\|A_\lambda\| \to \infty$ as $\lambda \to 0^-$. This complete the proof of the theorem.

For the convenience of readers, we will provide another proof of this statement. Consider now operator (4.24) with the Neumann boundary condition in dimensions $d = 1$ and 2. It was shown in [2] that the Schrödinger operator $H = c\Delta - \beta V$ in $R^d,\ d = 1,2$, with arbitrary constants $c,\beta > 0$ has negative eigenvalues. Let ψ be its eigenfunction with an eigenvalue $\lambda < 0$. Then

$$< H\psi, \psi >= \int_{R^d} (c|\nabla\psi|^2 - \beta V|\psi|^2)dx = \lambda\|\psi\|^2 < 0.$$

We choose $c = \max a(x)$. Since the support of V belongs to $\overline{\Omega}$, we have

$$< H_\beta\psi, \psi >= \int_\Omega (a(x)|\nabla\psi|^2 - \beta V|\psi|^2)dx \leq \int_{R^d} (c|\nabla\psi|^2 - \beta V|\psi|^2)dx < 0.$$

Thus H_β, with an arbitrary $\beta > 0$, has negative eigenvalues. Hence $\beta_{cr} = 0$ (and, therefore, $\|A_{0^-}\| = \infty$.) $\square$

CHAPTER 5: FKW EXTERIOR BOUNDARY PROBLEM

In this chapter, we will study an exterior boundary problems with the FKW condition of the form,

$$-\mathrm{div}(a(x)\nabla u) - \beta V(x)u - \lambda u = f, \quad x \in \Omega \subset R^d; \quad u|_{\partial\Omega} = \alpha, \quad \int_{\partial\Omega} \frac{\partial u}{\partial n} d\mu = 0, \tag{5.28}$$

with $f \in H^s(\Omega)$ and $u \in H^{s+2}(\Omega)$, where $s > [\frac{d}{2}]$. This FKW condition appears in the description of the diffusion process in Ω that is a limit, as $A \to \infty$, of the process in R^d with a large drift $A\overrightarrow{F}(x) \cdot \nabla u$ in $R^d \backslash \Omega$, where the vector field $\overrightarrow{F}$ is directed to an interior point of $R^d \backslash \Omega$, and the time spent by the process outside of Ω is not taken into account. For simplicity, we will assume in the last section that $\partial\Omega$ and $a(x)$ are infinitely smooth. The results below can be easily extended to a more general situation (which is considered in [8], [9]) when $R^d \backslash \Omega$ is a union of several non-intersecting domains, and FKW conditions (with different α, μ) are imposed on the boundaries of these domains.

We will prove that the spectrum of problem (5.28) consists of the continuous component $[0, \infty)$ and a discrete set of eigenvalues with the only possible limiting point at infinity. It will be shown that $\|A_{0^-}\| < \infty$ for problem (5.28) if $d \geq 3$ and $\|A_{0^-}\| = \infty$ if $d = 2$. Problem (5.28) is not symmetric and may have complex eigenvalues. Moreover, eigenvalues can be imbedded into the continuous spectrum (compare with [10]). If μ is the Lebesgue measure on the boundary, then the problem is symmetric and may have only real eigenvalues $\lambda \leq 0$. In the latter case, Theorem

4.18 and its proof remain valid, and therefore $\beta_{cr} > 0$ for problem (5.28) when $d > 2$, $\beta_{cr} = 0$ when $d = 1$ or 2.The last restriction and the Sobolev imbedding theorem imply the inclusion $u \in C^1(\overline{\Omega})$, which makes the last condition in (5.28) meaningful.

We will use the same notation H_0 for the operator related to problem (5.28):

$$H_0 : H^{s+2}(\Omega) \to H^s(\Omega),$$

where the domain of H_0 consists of functions $u \in H^{s+2}(\Omega)$ satisfying the last two conditions in (5.28). Obviously, $[0, \infty)$ belongs to the continuous spectrum of $H_\beta = H_0 - \beta V(x)$ since one can use the same Weyl sequence for operator $H_\beta - \lambda, \lambda > 0$, as the one in the case of the Dirichlet or Neumann boundary conditions. To study the spectrum outside of $[0, \infty)$ (and eigenvalues on $[0, \infty)$), consider the resolvent $R_\lambda = (H_\beta - \lambda)^{-1}$ and the truncated resolvent $\widehat{R}_\lambda = \chi(x)R_\lambda\chi(x)$, where $\chi \in C_0^\infty$.

Theorem 5.22. *1) The resolvent R_λ is meromorphic in $\lambda \in \mathbb{C}\backslash[0, \infty)$. Its poles have finite orders and do not have limiting points except, possibly, at infinity.*

2) If $k = \sqrt{\lambda}, \mathrm{Im}k > 0$, then the truncated resolvent $\widehat{R}_{k^2}$, $\mathrm{Im}k > 0$, has a meromorphic continuation to the whole complex k-plane when d is odd or to the Riemann surface of $\ln k$ when d is even. The poles in the regions $|\arg k| < C$ may have a limiting point only at infinity.

3) The truncated resolvent $\widehat{R}_{k^2}$ has a pole at a real $k \neq 0$ (with $\arg k = 0$ or π) if and only if the homogeneous problem (5.28) has a non-trivial solution satisfying the radiation condition:

$$|u| < Cr^{-(d-1)/2}, \quad |\frac{\partial u}{\partial r} - iku| < Cr^{-(d+1)/2}e^{ikr}, \quad r = |x| \to \infty.$$

Remarks. 1) The first statement implies that the spectrum of H_β outside of $[0, \infty)$ consists of a discrete set of eigenvalues of finite multiplicity with the only possible limiting point at infinity. While the last statement of the theorem indicates the possibility of the existence of spectral singularises on the continuous spectrum, see [10], operator H_β does not have eigenvalues imbedded into the continuous spectrum. The latter follows from the arguments used in [16, Theorem 3.3].

2) The FKW problem has a non-local boundary condition, and therefore it is not elliptic. It is also non-symmetric, unless μ in (5.28) is the Lebesgue measure. Theorem 5.22 is known [19] for general (non-symmetric) exterior elliptic problems with fast stabilizing at infinity coefficients (see also [20]). There is a wide literature concerning estimates on eigenvalues of non-symmetric elliptic problems, see for example [4, 7, 1, 11, 21, 5, 6] and references therein. In particular, [21] contains the proof of the finiteness of the number of eigenvalues for the Schrödinger operators with complex potentials in R^d under certain assumptions on the potential with a minimal requirement on the decay rate at infinity. Note that a similar result is not valid for exterior problems with fast decaying potentials, where the number of eigenvalues can be infinite even in the one-dimensional case [14].

Proof of Theorem 5.22. As we mentioned above, the statement of the theorem is well known [19] for the resolvent $R_{\lambda,D}$ (and the truncated resolvent $\widehat{R}_{\lambda,D}$) of the problem with the Dirichlet boundary condition (as well as for other elliptic boundary conditions). In particular, from [19] it follows that the problem

$$-\text{div}(a(x)\nabla v)-\beta V(x)v-\lambda v = f, \quad x \in \Omega \subset R^d; \quad v|_{\partial\Omega} = 1, \quad \lambda \in \mathbb{C}\backslash[0,\infty), \tag{5.29}$$

with $f \in H^s(\Omega_{com})$ has a meromprphic in λ solution $v \in H^{s+2}(\Omega)$, and $\chi(x)v$ has a meromorphic continuation in $k = \sqrt{\lambda}$ with the properties described in the second statement of the theorem above. These properties of v follow immediately from the properties of $\widehat{R}_{\lambda,D}$ after the substitution $v = \varphi + w$, where $\varphi \in C_0^\infty$, $\varphi = 1$ in a neighborhood of $\partial\Omega$, and w is the solution of the corresponding Dirichlet problem.

Let us look for the solution $u \in H^{s+2}(\Omega)$ of (5.28) in the form

$$u = \alpha v + R_{\lambda,D} f, \;\; \lambda \in \mathbb{C}\backslash[0,\infty). \tag{5.30}$$

Obviously, u satisfies (5.28) if and only if

$$\alpha = -\gamma(\lambda, f)/\gamma_1(\lambda), \quad \text{where} \quad \gamma(\lambda, f) = \int_{\partial\Omega} \frac{\partial}{\partial n} R_{\lambda,D} f d\mu, \;\; \gamma_1(\lambda) = \int_{\partial\Omega} \frac{\partial v}{\partial n} d\mu. \tag{5.31}$$

From the analytic properties of v it follows that γ_1 is meromorphic in $\lambda \in \mathbb{C}\backslash[0,\infty)$ and admits a meromorphic continuation to the whole complex k-plane if d is odd or to the Riemann surface of $\ln k$ if d is even. When $\lambda \in \mathbb{C}\backslash[0,\infty)$, function v decays at infinity. If $\lambda < -\beta \max V(x)$, then the maximum principle is valid for solutions of (5.29), v achieves its maximum value at all the points of the boundary, and therefore, $\frac{\partial v}{\partial n} > 0$ on $\partial\Omega$. Thus $\gamma_1(\lambda) > 0$ when $\lambda < -\beta \max V(x)$. Hence $\gamma_1(\lambda) \not\equiv 0$, and therefore $\gamma_1^{-1}(\lambda)$ is meromorphic in the complex k-plane if d is odd or on the Riemann surface of $\ln k$ if d is even. Moreover, from the asymptotics of $\widehat{R}_{\lambda,D}$ as $\lambda \to 0$ [19, Theorem 10] it follows that the origin is not a limiting point for zeroes of $\gamma_1^{-1}(\lambda)$ located in a region $|\arg k| < C$. Thus, the poles of $\gamma_1^{-1}(\lambda)$ in this region may converge only to infinity, and therefore the first two statements of Theorem 5.22 follow from (5.30), (5.31), and the validity of these statements for $R_{\lambda,D}$. The last statement of the theorem can be proved in the same way as a similar statement for $\widehat{R}_{\lambda,D}$ was proved in [17].

□

Denote by A_λ the operator $\widehat{R}_\lambda$ for problem (5.28) with $\beta = 0$ and $\chi = \sqrt{V}$, i.e.,

$$A_\lambda = \sqrt{V}(x)(H_0 - \lambda)^{-1}\sqrt{V}(x) : H^s(\Omega) \to H^{s+2}(\Omega), \tag{5.32}$$

where H_0 is defined by (5.28).

Lemma 5.23. *The following relations are valid for operator (5.32):*

$$\|A_\lambda\| \le C < \infty \quad \text{as } \lambda \to 0^- \text{ if } d \ge 3; \quad \lim_{\lambda \to 0^-} \|A_\lambda\| = \infty \quad \text{if } d = 2.$$

Proof. Below we assume that $\beta = 0$.

Let $d \ge 3$. For each $\rho < \infty$, $\Omega_\rho = \Omega \bigcap |x| < \rho$, and $\lambda \to 0^-$, the solution $v \in H^{s+2}(\Omega)$ of (5.29) converges in $H^{s+2}(\Omega_\rho)$ to a decaying at infinity solution of the same equation with $\beta = \lambda = 0$. Hence, the arguments used in the proof of Theorem 5.22 to show that $\gamma_1(\lambda) > 0$ for $\lambda < -\beta \max V(x)$ remain valid when $\beta = 0,\ \lambda = 0^-$, i.e., $\gamma_1(0^-) > 0$. Hence, the first statement of the lemma follows immediately from (5.30), (5.31), and the boundedness of $\widehat{R}_{\lambda,D}$ as $\lambda \to 0^-$.

If $d = 1$ or 2, then v converges in each $H^{s+2}(\Omega_\rho)$ to a constant (equal to one) as $\lambda \to 0^-$, and therefore $\gamma_1(0^-) = 0$. Thus the second statement of the lemma will follow from (5.30), (5.31) if we show the existence of $f \in H^s$ such that

$$\gamma(0^-, f) > c > 0 \quad \text{and} \quad \|\widehat{R}_{0^-,D} f\| < \infty. \tag{5.33}$$

To construct such an f, we consider an arbitrary $u \in H^{s+2}(\Omega_{com})$ with a compact support and such that $u|_{\partial\Omega} = 0,\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 1$. We have

$$H_0 u - \lambda u = f - \lambda u, \quad \text{where } f = -\text{div}(a(x)\nabla u) \in H^s(\Omega_{com}).$$

One can assume that the cut-off function χ in the definition of $\widehat{R}_{\lambda,D}$ is chosen in such a way that $\chi = 1$ on the support of u. Then $\widehat{R}_{\lambda,D} f = u - \lambda \widehat{R}_{\lambda,D} u$. If $d = 1, 2$, then $\|\widehat{R}_{\lambda,D} u\|_{L^2} < C < \infty,\ \lambda \to 0^-$, due to Theorem 4.21. From a priori estimates for elliptic equations, it follows that the same estimate holds in the space H^{s+2}. Hence $\|\widehat{R}_{\lambda,D} f - u\|_{H^{s+2}} \to 0$ as $\lambda \to 0^-$, and this implies (5.33).

□

As me mentioned earlier, Theorem 4.18 remains valid for symmetric FKW problems. Thus Lemma 5.23 implies the following statement.

Theorem 5.24. *If μ is the Lebesgue measure on the boundary in FKW problem, then $\beta_{cr} = 0$ in dimensions one and two and $\beta_{cr} > 0$ if $d \geq 3$.*

CHAPTER 6: POTENTIALS WITH THE SUPPORTS NEAR THE BOUNDARY

This chapter is devoted to the dependence of β_{cr} on the distance between the support of the potential and the boundary of the domain. In fact, it is obvious that moving the support of the potential to the boundary does not affect β_{cr} essentially, but we will shrink the size of the support, increase the height of the potential appropriately, and move the potential toward to the boundary.

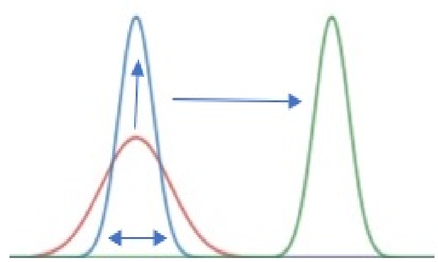

Figure 6.5: Potentials with the supports near the boundary.

It will be shown that this process, in the case of the Dirichlet boundary condition, will imply the blowing up of β_{cr} in dimension $d = 1$. In dimension two (and the Dirichlet boundary condition), the behavior of β_{cr} depends on the relation between the rates of the shrinking of the support of the potential and the rate of its motion to the boundary. We do not consider the Neumann boundary condition when $d = 1$ or 2 since β_{cr} is always zero in this case. We will show that β_{cr} is not very sensitive to the location of the support of the potential for both Dirichlet and Neumann problems if

$d \geq 3$.

For the sake of the transparency, we will assume that $a(x) \equiv 1$, i.e., problem

$$H_0 u - \beta V(x)u = \lambda u, \quad x \in \Omega, \tag{6.34}$$

will be in the form

$$-\Delta u - \beta V_n(x)u = \lambda u, \quad x \in \Omega \subset R^d; \quad u|_{\partial\Omega} = 0; \ \beta \geq 0, \quad n \to \infty. \tag{6.35}$$

We will consider the potential V_n of the form $V_n(x) = h_d(n)W((x - x(n))n)$, where $W \in C_0(R^d)$, $W \geq 0$, the support of W belongs to the unit ball, $x(n) \to x_0 \in \partial\Omega$ as $n \to \infty$, the support of $V_n(x)$ belongs to Ω, and $h_d(n)$ will be chosen in the next paragraph.

Let $d \geq 3$, so that $\beta_{cr} > 0$. In order to study the dependence of β_{cr} on the location of the potential, we consider the problem in the whole space R^d, assume that $x(n) = 0$, and choose $h_d(n)$ in such a way that β_{cr} does not depend on n. This value of $h_d(n)$ will be used in (6.35) to study the dependence of β_{cr} on the location of the potential. We will proceed similarly when $d = 1, 2$. By Theorem 4.18, $h_d(n)$, with $d \geq 3$, must be chosen in such a way that the norm of the operator $A_{0^-} = A_{0^-}(n)$ with the integral kernel

$$A_{0^-}(x, y, n) = \sqrt{h_d W(xn)} \frac{c_d}{|x - y|^{d-2}} \sqrt{h_d W(yn)}$$

(where c_d is a constant) does not depend on n. The substitution $xn = x'$, $yn = y'$ implies that $h_d(n) = n^2$, $d \geq 3$. Indeed, this substitution immediately implies that if $u(x)$ is an eigenfunction of the operator $A_{0^-}(n)$ with an eigenvalue $\lambda(n)$, then $u(x'/n)$ is an eigenfunction of the operator $A_{0^-}(1)$ with the eigenvalue $\lambda(1) = \frac{h_d(1)n^2}{h_d(n)}\lambda(n)$. The converse relation is also valid. Hence, the choice $h_d(n) = n^2$, $d \geq 3$, implies that

$\|A_{0^-}(n)\|$ does not depend on n.

A small change is needed in dimensions one and two. We can't consider the operator $A_{0^-}(n)$ in the whole space for small dimensions (the operator is not defined), but we can consider a similar operator for the Dirichlet problem in Ω. Its integral kernel is bounded when $d = 1$ and has the singularity $\frac{1}{2\pi} \ln \frac{1}{|x-y|}$ if $d = 2$. The same substitution implies that $h_1(n) = n,\ h_2(n) = \frac{n^2}{\ln n}$. The norm of $A_{0^-}(n)$ depends on n in this case, but approaches a constant as $n \to \infty$.

For transparency, we will not study β_{cr} in the general setting, but focus our attention on the case when $\partial\Omega$ contains a flat part Γ, x_0 is an interior point of Γ, and $x(n)$ moves toward x_0 in the direction perpendicular to Γ.

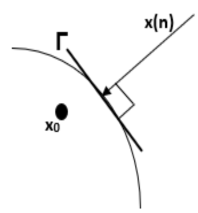

Figure 6.6: Flat part of the boundary.

Theorem 6.25. *If $d = 1$, then β_{cr} (for operator (6.35)) goes to infinity as $n \to \infty$. The same is true if $d = 2$ and $|x(n) - x_0| < C/n,\ n \to \infty$. If $d = 2$ and $|x(n) - x_0| \to 0,\ |x(n) - x_0| > C/n^\delta,\ n \to \infty$, with some $\delta \in (0, 1)$, then β_{cr} remains bounded as $n \to \infty$. If $d \geq 3$, then β_{cr} remains bounded as $n \to \infty$ for both the Dirichlet and Neumann boundary conditions.*

Remarks. The arguments in the proof allow one to estimate the rate with which β_{cr} tends to infinity. This rate depends on the rate of the convergence of $x(n)$ to x_0.

Proof. Let $d = 1$. Since the exterior of an interval is a union of two half-lines, it is enough to prove the statement for the Dirichlet problem on $(0, \infty)$. The Green function G_λ for the operator

$$H_0 u = -u'' - \lambda u, \quad x > 0, \quad u(0) = 0, \quad \lambda < 0,$$

has the form $G_\lambda = \frac{e^{-k|x-\xi|} - e^{-k|x+\xi|}}{-2k}$, $x, \xi > 0$, $k = \sqrt{|\lambda|}$, and its limiting value as $\lambda \to 0^-$ is $|x+\xi| - |x-\xi|$. Hence, the operator $A_{0^-}(n)$ defined by (4.10) has the integral kernel

$$A_{0^-}(x, \xi, n) = n\sqrt{W((x - x(n))n)}(|x + \xi| - |x - \xi|)\sqrt{W((\xi - x(n))n)}, \quad x, \xi > 0.$$

Since $|x - x(n)| + |\xi - x(n)| < \frac{c}{n}$ on the support of $A_{0^-}(x, \xi, n)$, and $x(n) \to 0,\ n \to \infty$, there exists $\alpha(n)$ such that $\alpha(n) \to 0,\ n \to \infty$, and

$$|x + \xi| - |x - \xi| \le \alpha(n)$$

on the support of $A_{0^-}(x, \xi, n)$. Then one can easily see that

$$\|A_{0^-}(n)\| \le \left[\int_0^\infty \int_0^\infty A_{0^-}^2(x, \xi, n) dx d\xi\right]^{1/2} \le C\ \alpha(n) \to 0,\ n \to \infty,$$

and the statement of Theorem 6.25 for $d = 1$ follows from Theorem 4.18.

Let us consider the case $d \ge 3$. Without loss of generality, we can assume that Γ is a part of the hyperplane $x_1 = 0,\ x_0 = 0$, and there exists a ball B_ε of radius ε centered at the origin such that its right half B_ε^+, where $x_1 > 0$, belongs to Ω, and the other half does not contain points of Ω. Hence $x(n)$ moves to the origin along the positive x_1-semi-axis as $n \to \infty$. Let $E(x) = \frac{c_d}{|x|^{d-2}}$ be a fundamental solution of $-\Delta$. For $\xi \in B_\varepsilon$, denote by ξ^* the point symmetrical to ξ with respect to the plane $x_1 = 0$.

Lemma 6.26. *The Green function $G = G_\mp(x, \xi)$ of the Dirichlet (Neumann) problem* *in* Ω *for the operator* $-\Delta$ *has the form* $G_{\mp} = E(x-\xi) \mp E(x-\xi^*) + F(x,\xi)$*, where* F *is uniformly bounded when* $x \in \Omega,\ \xi \in B_{\varepsilon/2}$.

Remark. Additional smoothness of $\partial\Omega$ is needed to prove this statement in the case of the Neumann boundary condition. For example, one can assume that $\partial\Omega \in C^{2,\alpha}$.

Proof. In the case of the Dirichlet problem, F is the solution of the homogeneous equation $\Delta F = 0$ with the boundary condition

$$F = E(x-\xi^*) - E(x-\xi), \quad x \in \partial\Omega,\ \xi \in B_{\varepsilon/2}.$$

Since $F|_{\partial\Omega}$ is bounded uniformly in $\xi \in B_{\varepsilon/2}$, the maximum principle implies that $|F| < C$. In the case of the Neumann boundary condition, the normal derivative of F on the boundary belongs to $C^{1,\alpha}$ (if $\partial\Omega \in C^{2,\alpha}$). From local a priori estimates for elliptic equations, it follows that $F|_{\partial\Omega} \in C^{\alpha}$, and the maximum principle can be applied again.

□

In order to prove the theorem in the case $d \geq 3$, it is enough to show that $\|A_{0^-}(n)\| \geq c > 0$ when $n \to \infty$ (see Theorem 4.18). Let $\widehat{F}$ be the operator in $L^2(\Omega)$ with the integral kernel

$$\widehat{F}(x,\xi) = \sqrt{n^2 W((x-x(n))n)}F(x,\xi)\sqrt{n^2 W((\xi-x(n))n)},$$

where F is defined in lemma above. The support of $\sqrt{n^2 W((\xi-x(n))n)}$ belongs to $B_{\varepsilon/2}$ when n is large enough, and therefore Lemma 6.26 and the substitution

$$x - x(n) = y/n,\ \xi - x(n) = \sigma/n \tag{6.36}$$

imply that

$$\int_\Omega \int_\Omega \widehat{F}^2(x,\xi)dxd\xi \le \frac{C}{n} \to 0, \quad n \to \infty.$$

Hence, $\|\widehat{F}\| \to 0,\ n \to \infty$, and it remains to show that the norm of the operators $\widehat{E}_\mp$ in $L^2(\Omega)$ with the integral kernel

$$c_d n^2 \sqrt{W((x-x(n))n)}[|x-\xi|^{2-d} \mp |x-\xi^*|^{2-d}]\sqrt{W((\xi-x(n))n)}$$

is bounded from below when $n \to \infty$. One can consider these operators in $L^2(R^d_+),\ R^d_+ = \{x : x_1 > 0\}$ instead of $L^2(\Omega)$ since the integral kernel vanishes if x or ξ are not in $B^+_{\varepsilon/2}$ and n is large enough. The norm remains the same after substitution (6.36) (since the principal eigenvalues are the same). Hence, it is enough to show that the norm of the integral operator $G_\mp$ in $L^2(R^d_+)$ with the integral kernel

$$c_d \sqrt{W(y)}[|y-\sigma|^{2-d} \mp |y-\sigma^* + 2nx(n)|^{2-d}]\sqrt{W(\sigma)}$$

is bounded from below when $n \to \infty$.

Consider an arbitrary ball $B \in R^d_+$ such that its distance from the origin is positive and $W(x) \ge \alpha > 0$ when $x \in B$. There exists $\rho > 0$ such that $|y-\sigma^* + 2nx(n)|^{2-d} \le (1-\rho)|y-\sigma|^{2-d},\ y, \sigma \in B$, i.e.,

$$|y-\sigma|^{2-d} \mp |y-\sigma^* + 2nx(n)|^{2-d} \ge \rho|y-\sigma|^{2-d}.$$

Hence $\|G_\mp\|$ is not smaller than the norm of the operator in $L^2(B)$ with the integral kernel $c_d \alpha \rho |y-\sigma|^{2-d}$, which does not depend on n. This completes the proof of the theorem in the case of $d \ge 3$.

The Dirichlet problem when $d = 2$ is treated absolutely similarly to the case $d \ge 3$.

The only difference is that operator G_- now has the following integral kernel:

$$\frac{1}{2\pi \ln n}\sqrt{W(y)}\ln\frac{|y-\sigma^*+2nx(n)|}{|y-\sigma|}\sqrt{W(\sigma)}. \tag{6.37}$$

If $n|x(n)| < C$, then operator G_- converges strongly to zero as $n \to \infty$. Hence, $A_{0^-}(n)$ has the same property, and $\beta_{cr} \to 0$ due to Theorem 4.18. If $|x(n)| > Cn^{-\delta}$, $0 < \delta < 1$, then we write the logarithm of the quotient in (6.37) as the difference of the logarithms and represent the operator G_-as $G_- = G_1 - G_2$. Obviously, G_2 converges strongly to zero as $n \to \infty$, and G_- is bounded from below for large n by the operator with the kernel

$$\frac{\ln|2nx(n)|}{2\pi \ln n}\sqrt{W(y)}\sqrt{W(\sigma)} \geq \frac{1-\delta}{4\pi}\sqrt{W(y)}\sqrt{W(\sigma)}.$$

Hence, $\|A_{0^-}(n)\|$ is bounded from below as $n \to \infty$, and β_{cr} is bounded.

□

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