Isomorphism in Wavelets II

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Abstract. A scaling function φ_A associated with a $d\times d$ expansive dyadic integral matrix A can be isomorphically embedded into the family of scaling functions associated with a $s\times s, d \le s$, expansive dyadic integral matrix B. On the other hand, a scaling function φ_A associated with a $d\times d$ expansive dyadic integral matrix A and a finite two scaling relation can be isomorphically embedded into the family of scaling functions associated with expansive dyadic integral $s \times s$ matrix B, for any s. In particular, for $s = 1$ and $B = [2]$. We provide examples for such isomorphisms.

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1. Introduction

For a vector $\vec{\ell} \in \mathbb{R}^d$, the *translation operator* $T_{\vec{\ell}}$ is defined as

$$(T_{\vec{\ell}} f)(\vec{t}) \quad \equiv \quad f(\vec{t} - \vec{\ell}), \ \forall f \in L^2(\mathbb{R}^d), \ \forall \vec{t} \in \mathbb{R}^d \ .$$

Let A be a $d \times d$ integral matrix with eighenvalues $\beta_1, \cdots, \beta_d$. A is called *expansive* if $\min\{|\beta_1|, \cdots, |\beta_d|\} > 1$. A is called *dyadic* if $|\det(A)| = 2$. We define the operator D_A as

$$(D_A f)(\vec{t}) \quad \equiv \quad (\sqrt{2}) f(A\vec{t}), \ \forall f \in L^2(\mathbb{R}^d), \ \forall \vec{t} \in \mathbb{R}^d \ .$$

The operators $T_{\vec{\ell}}$ and D_A are unitary operators on $L^2(\mathbb{R}^d)$.

Let $\{s_{\vec{n}} \mid \vec{n} \in \mathbb{Z}^d\}$ be a solution to the following system of equations (1.1) associated with a $d \times d$ expansive dyadic integral matrix A:

$$\begin{cases} \sum_{\vec{n}\in\mathbb{Z}^d} h_{\vec{n}} \overline{h_{\vec{n}+\vec{k}}} = \delta_{\vec{0}\vec{k}}, \ \vec{k} \in A\mathbb{Z}^d, \\ \sum_{\vec{n}\in\mathbb{Z}^d} h_{\vec{n}} = \sqrt{2}. \end{cases} \tag{1.1}$$

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The set $\Lambda = \{\vec{n} \in \mathbb{Z}^d \mid s_{\vec{n}} \neq 0\}$ is the support of $\{s_{\vec{n}} \mid \vec{n} \in \mathbb{Z}^d\}$. If Λ is a finite set, then $\{s_{\vec{n}}\}$ is called a *finite* solution. Define the operator Ψ on $L^2(\mathbb{R}^d)$ as

$$\Psi \equiv \sum_{\vec{n} \in \Lambda} s_{\vec{n}} D_A T_{\vec{n}}.$$

When Λ is finite the operator Ψ has a non-zero fixed point φ_A (Lawton [18] and Bownik [3]),

$$\varphi_A = \Psi \varphi_A. \tag{1.2}$$

This φ_A is the scaling function associated with matrix A and it induces a Parseval frame wavelet ψ_A associated with matrix A. It satisfies the two-scale relation:

$$\varphi_A = \sum_{\vec{n} \in \Lambda} s_{\vec{n}} D_A T_{\vec{n}} \varphi_A. \tag{1.3}$$

We will say that φ_A is *derived* from the solution $\mathcal{S}$. This scaling function φ_A associated with matrix A is generated by a solution $\mathcal{S} = \{s_{\vec{n}}\}$ to the system of equations (1.1). The scaling function φ_A induces a Parseval frame wavelet ψ_A associated with matrix A as defined in Definition 1.1.

Definition 1.1. Let A be an expansive dyadic integral matrix. A function $\psi_A \in L^2(\mathbb{R}^d)$ is called a Parseval frame wavelet associated with A, if the set

$$\{D_A^n T_{\vec{\ell}}\, \psi_A \mid n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^d\}$$

forms a normalized tight frame for $L^2(\mathbb{R}^d)$. That is

$$\|f\|^2 = \sum_{n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^d} |\langle f, D_A^n T_{\vec{\ell}}\, \psi_A \rangle|^2, \ \forall f \in L^2(\mathbb{R}^d).$$

If the set is also orthogonal, then ψ_A is an orthonormal wavelet for $L^2(\mathbb{R}^d)$ associated with A.

2. Definition of Isomorphisms

Let A be an expansive dyadic integral matrix. Let $\mathcal{W}(A,d)$ be the collection of all scaling functions in $L^2(\mathbb{R}^d)$ associated with A and solutions to the system of equation (1.1). Define $\mathcal{W}(d) \equiv \bigcup_A \mathcal{W}(A,d)$. The union is for all $d \times d$ expansive dyadic integral matrices. Define $\mathcal{W} \equiv \bigcup_{d \geq 1} \mathcal{W}(d)$. This is the set of scaling functions in all dimensions.

In particular, let $\mathcal{W}_0(A,d)$ be the collection of all scaling functions in $L^2(\mathbb{R}^d)$ associated with A and *finite* solutions to the system of equation (1.1). Define $\mathcal{W}_0(d) \equiv \bigcup_A \mathcal{W}_0(A,d)$. The union is for all $d \times d$ expansive dyadic integral matrices. Define $\mathcal{W}_0 \equiv \bigcup_{d \geq 1} \mathcal{W}_0(d)$.

Let A be a $d \times d$ expansive dyadic integral matrix and $\varphi_A \in \mathcal{W}(A,d)$ which is derived from the solution $\{a_{\vec{n}} \mid \vec{n} \in \mathbb{Z}^d\}$ to (1.1). Denote the support of this solution as Λ_A, and $\mathcal{S}_A = \{a_{\vec{n}} \mid \vec{n} \in \Lambda_A\}$.

A *reduced system of equations* $\mathcal{E}_{(\Lambda_A, A, d)}$ from system of equations (1.1) can be obtained by the following steps:

Step 1. For $\vec{n} \in \mathbb{Z}^d \backslash \Lambda_A$, replace all variables $h_{\vec{n}}$ in (1.1) by 0.

Step 2. Then remove all trivial equations "$0 = 0$".

Step 3. If there are redundant equations, choose and keep one and remove the other identical equations.

Note that, the discussion of the reduced system of equations $\mathcal{E}_{(\Lambda_A, A, d)}$ from the support Λ_A does not depend on the existence of a solution $\mathcal{S}_A$. This gives flexibility in the discussion.

Denote the family of all such reduced systems of equations by $\mathfrak{E}$. A reduced system of equations $\mathcal{E}_{(\Lambda_A, A, d)}$ has the following form:

$$\begin{cases} \sum_{\vec{n} \in \Lambda_A} h_{\vec{n}} \overline{h_{\vec{n}+\vec{k}}} = \delta_{\vec{0}\vec{k}}, \ \vec{k} \in \Lambda_A^E, \\ \sum_{\vec{n} \in \Lambda_A} h_{\vec{n}} = \sqrt{2}. \end{cases} \tag{2.1}$$

The *index set* Λ_A^E in (2.1) is a subset of $A\mathbb{Z}^d$. The equation in $\mathcal{E}_{(\Lambda_A, A, d)}$ that corresponding to $\vec{k} \in \Lambda_A^E$ is

$$\sum_{\vec{n} \in \Lambda_A} h_{\vec{n}} \overline{h_{\vec{n}+\vec{k}}} = \delta_{\vec{0}\vec{k}}.$$

This set Λ_A^E might not be unique due to the Step 3 above. However, it is fixed in discussion. It is clear that $\mathcal{S}_A$ is a solution to (2.1).

Similarly, for an $s \times s$ expansive dyadic integral matrix B and $\Lambda_B \subset \mathbb{Z}^s$, the reduced system of equation is

$$\mathcal{E}_{(\Lambda_B, B, s)} : \begin{cases} \sum_{\vec{m} \in \Lambda_B} h'_{\vec{m}} \overline{h'_{\vec{m}+\vec{\ell}}} = \delta_{\vec{0}\vec{\ell}}, \ \vec{\ell} \in \Lambda_B^E, \\ \sum_{\vec{m} \in \Lambda_B} h'_{\vec{m}} = \sqrt{2}. \end{cases} \tag{2.2}$$

Definition 2.1. $\mathcal{E}_{(\Lambda_A, A, d)}, \mathcal{E}_{(\Lambda_B, B, s)} \in \mathfrak{E}$ are isomorphic, or $\mathcal{E}_{(\Lambda_A, A, d)} \sim \mathcal{E}_{(\Lambda_B, B, s)}$ if there exist

(A). a bijection $\theta : \Lambda_A \to \Lambda_B$ and

(B). a bijection η from an index set Λ_A^E of $\mathcal{E}_{(\Lambda_A, A, d)}$ onto an index set Λ_B^E of $\mathcal{E}_{(\Lambda_B, B, s)}$

with the following properties: for each $\vec{k} \in \Lambda_A^E$, the equation in $\mathcal{E}_{(\Lambda_B, B, s)}$ generated by $\vec{\ell} \equiv \eta(\vec{k})$ is obtained by replacing $h_{\vec{n}}$ by $h'_{\theta(\vec{n})}$ and $\delta_{\vec{0}\vec{k}}$ by $\delta_{\vec{0}\vec{\ell}}$ in the equation in $\mathcal{E}_{(\Lambda_A, A, d)}$ generated by $\vec{k}$.

In each of the examples in Sections 4 and 5 we will list the corresponding matrices A and B, sets Λ_A, Λ_A, mappings θ, η. The related systems of equations $(\mathcal{S}_A, \mathcal{E}_{(\Lambda_A, A, d)})$ and $(\mathcal{S}_B, \mathcal{E}_{(\Lambda_B, B, s)})$ are reduced. We check each of the cases with computer programs. For simplicity we omit the details.

Let $\mathcal{S}_A = \{a_{\vec{n}} \mid \vec{n} \in \Lambda_A\}$ be a solution to (2.1) and $\mathcal{S}_B = \{b_{\vec{m}} \mid \vec{m} \in \Lambda_B\}$ be a solution to (2.2). Let $\varphi_A, \varphi_B \in \mathcal{W}$ be the scaling functions derived from $(\mathcal{S}_A, \mathcal{E}_{(\Lambda_A, A, d)})$ and $(\mathcal{S}_B, \mathcal{E}_{(\Lambda_B, B, s)})$ respectively. Notice that d and s can be different.

Definition 2.2. The scaling functions φ_A, φ_B are *algebraically isomorphic*, or $\varphi_A \simeq \varphi_B$, if $\mathcal{E}_{(\Lambda_A, A, d)} \sim \mathcal{E}_{(\Lambda_B, B, s)}$ with bijection θ and η. And

$$b_{\theta(\vec{n})} = a_{\vec{n}}, \forall \vec{n} \in \Lambda_A.$$

It is clear that the isomorphism of the reduced system of equations guarantees the isomorphism of the scaling functions derived from the solutions of the reduced system of equations. We have

Lemma 2.3. *For isomorphic systems $\mathcal{E}_{(\Lambda_A,A,d)}$ and $\mathcal{E}_{(\Lambda_B,B,s)}$ with bijection θ from Λ_A to Λ_B, if $\mathcal{S}_A = \{a_{\vec{n}} \mid \vec{n} \in \Lambda_A\}$ is a solution to $\mathcal{E}_{(\Lambda_A,A,d)}$, then the set $\mathcal{S}_B \equiv \{b_{\vec{m}} = a_{\theta^{-1}(\vec{m})} \mid \vec{m} \in \Lambda_B\}$ is a solution to $\mathcal{E}_{(\Lambda_B,B,s)}$. Moreover, the scaling functions derived from $(\mathcal{S}_A, \mathcal{E}_{(\Lambda_A,A,d)})$ and $(\mathcal{S}_B, \mathcal{E}_{(\Lambda_B,B,s)})$ are algebraically isomorphic.*

Definition 2.4. Let $\mathcal{U}$ and $\mathcal{V}$ be subsets of $\mathcal{W}$.

1. The set $\mathcal{U}$ is *isomorphically embedded into* $\mathcal{V}$,
$$\mathcal{U} \sqsubseteq \mathcal{V},$$
if for each φ_U in $\mathcal{U}$ there is an element $\varphi_V \in \mathcal{V}$ such that $\varphi_U \simeq \varphi_V$.
2. If $\mathcal{U} \sqsubseteq \mathcal{V}$ and $\mathcal{V} \sqsubseteq \mathcal{U}$, $\mathcal{U}$ and $\mathcal{V}$ are isomorphically identical, or
$$\mathcal{U} \cong \mathcal{V}.$$

We have

Theorem 2.5.
$$\mathcal{W}(1) \sqsubseteq \mathcal{W}(2) \sqsubseteq \mathcal{W}(3) \sqsubseteq \cdots,$$
that is, the sequence $\{\mathcal{W}(d) \mid d \in \mathbb{N}\}$ is an ascending sequence.

In [11], we proved that

Theorem 2.6.
$$\mathcal{W}_0(1) \cong \mathcal{W}_0(2) \cong \mathcal{W}_0(3) \cong \cdots,$$
that is, each $\varphi \in \mathcal{W}_0$ is isomorphic to a one dimensional scaling function in $\mathcal{W}_0(1)$.

The purpose of this paper is to prove Theorem 2.5 and present examples for both Theorem 2.5 and Theorem 2.6.

3. Proof of Theorem 2.5

Let s be a natural number and $d \leq s$ and B be a $s \times s$ expansive dyadic integral matrix. To prove Theorem 2.5, we need to find a function $\varphi_B \in \mathcal{W}_0(s)$ for any given $\varphi_A \in \mathcal{W}_0(d)$ such that $\varphi_A \simeq \varphi_B$.

By the Smith Normal Form for integral matrices [2] $A = UDV$, where U, V are integral matrices of determinant ± 1, and D a diagonal matrix with the last diagonal entry 2 and all other diagonal entries 1. Let $\vec{e}_1, ..., \vec{e}_d$ be the standard basis for $\mathbb{Z}^d$. Note that $V\mathbb{Z}^d = \mathbb{Z}^d$ and $U\mathbb{Z}^d = \mathbb{Z}^d$. We have

$$\begin{aligned}
\mathbb{Z}^d &= span\{\vec{e}_1, ..., \vec{e}_{d-1}, 2\vec{e}_d\} \mathbin{\dot\cup} \big(span\{\vec{e}_1, ..., \vec{e}_{d-1}, 2\vec{e}_d\} + \vec{e}_d\big) \\
&= D\mathbb{Z}^d \mathbin{\dot\cup} (D\mathbb{Z}^d + \vec{e}_d) = DV\mathbb{Z}^d \mathbin{\dot\cup} (DV\mathbb{Z}^d + \vec{e}_d) \\
&= UDV\mathbb{Z}^d \mathbin{\dot\cup} U(DV\mathbb{Z}^d + \vec{e}_d) = A\mathbb{Z}^d \mathbin{\dot\cup} \big(A\mathbb{Z}^d + U\vec{e}_d\big).
\end{aligned}$$

Let $\vec{\ell}_A \equiv U\vec{e}_d$. It follows that, for any $d \times d$ expansive dyadic integral matrix A, there exists a vector $\vec{\ell}_A \in \mathbb{Z}^d \backslash A\mathbb{Z}^d$ such that

$$\mathbb{Z}^d = A\mathbb{Z}^d \dot{\cup} (\vec{\ell}_A + A\mathbb{Z}^d).$$

The same proof shows that there exists a vector $\vec{\ell}_B \in \mathbb{Z}^s \backslash B\mathbb{Z}^s$ such that

$$\mathbb{Z}^s = B\mathbb{Z}^s \dot{\cup} (\vec{\ell}_B + B\mathbb{Z}^s).$$

Since $d \leq s$, we can consider $\mathbb{R}^d$ as subspace of $\mathbb{R}^s$. Let $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_s\}$ be the standard basis for $\mathbb{R}^s$. We will further assume that the first d vectors of the basis, $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_d\}$ be the standard basis for $\mathbb{R}^d$.

Define the mapping Θ from $\mathbb{Z}^d$ to $\mathbb{Z}^s$.

$$\Theta(\vec{n}) = \begin{cases} BA^{-1}(\vec{n}), & \text{if } \vec{n} \in A\mathbb{Z}^d, \\ \vec{\ell}_B + BA^{-1}(\vec{n} - \vec{\ell}_A), & \text{if } \vec{n} \in \vec{\ell}_A + A\mathbb{Z}^d. \end{cases} \tag{3.1}$$

This is a well-defined mapping on $\mathbb{Z}^d$ since $\mathbb{Z}^d = A\mathbb{Z}^d \dot{\cup} (\vec{\ell}_A + A\mathbb{Z}^d)$ with range $\Theta(\mathbb{Z}^d) \subset \mathbb{Z}^s$. Since $\det B \neq 0$ and A has an inverse on $A\mathbb{Z}^d$ with range contained in $\Theta(\mathbb{Z}^d) \subseteq \mathbb{Z}^s$, the mapping Θ is an injection. We have $\Theta(\vec{0}) = \vec{0}$. Also, if $\Theta(\vec{x}) = \vec{0}$ for some $\vec{x} \in \mathbb{Z}^d$ then $\vec{x} = \vec{0}$.

Lemma 3.1. *For $\vec{n} \in \mathbb{Z}^d$ and $\vec{k} \in A\mathbb{Z}^d$, we have*

$$\Theta(\vec{n} + \vec{k}) = \Theta(\vec{n}) + \Theta(\vec{k}).$$

Proof. Since $\vec{k} \in A\mathbb{Z}^d$, $\Theta(\vec{k}) = BA^{-1}(\vec{k})$. We have

Since $\vec{k} \in A\mathbb{Z}^d$, $\vec{n} + \vec{k} \in A\mathbb{Z}^d$ iff $\vec{n} \in A\mathbb{Z}^d$. Also $\vec{n} + \vec{k} \in \vec{\ell}_A + A\mathbb{Z}^d$ iff $\vec{n} \in \vec{\ell}_A + A\mathbb{Z}^d$. So we have

$$\begin{aligned} \Theta(\vec{n}) + \Theta(\vec{k}) = BA^{-1}(\vec{k}) + &\begin{cases} BA^{-1}(\vec{n}), \ \vec{n} \in A\mathbb{Z}^d \\ \vec{\ell}_B + BA^{-1}(\vec{n} - \vec{\ell}_A), \ \vec{n} \in \vec{\ell}_A + A\mathbb{Z}^d \end{cases} \\ &= \begin{cases} BA^{-1}(\vec{n} + \vec{k}), \ \vec{n} \in A\mathbb{Z}^d \\ \vec{\ell}_B + BA^{-1}((\vec{n} + \vec{k}) - \vec{\ell}_A), \ \vec{n} \in \vec{\ell}_A + A\mathbb{Z}^d \end{cases} \\ &= \Theta(\vec{n} + \vec{k}). \end{aligned}$$

□

Lemma 3.2. *If $\vec{n}_1, \vec{n}_2 \in \mathbb{Z}^d$ and $\vec{\ell} \equiv \Theta(\vec{n}_2) - \Theta(\vec{n}_1) \in B\mathbb{Z}^s$, then there exists a vector $\vec{k} \in A\mathbb{Z}^d$ such that $\vec{\ell} = \Theta(\vec{k})$, and $\vec{n}_2 = \vec{n}_1 + \vec{k}$.*

Proof. Since $\Theta(\vec{n}_2) - \Theta(\vec{n}_1) = \vec{\ell} \in B\mathbb{Z}^d$, by Equation (3.1) we have only two cases.

Case (1), both $\Theta(\vec{n}_1), \Theta(\vec{n}_2)$ are in $B\mathbb{Z}^s$. By Equation (3.1), $n_1 = A\lambda_1, n_2 = A\lambda_2$ for some vectors $\lambda_1, \lambda_2 \in \mathbb{Z}^d$. Denote $\vec{k} = A\lambda_2 - A\lambda_1 \in A\mathbb{Z}^d$. So $\vec{\ell} = \Theta(\vec{n}_2) - \Theta(\vec{n}_1) = BA^{-1}(A\lambda_2) - BA^{-1}(A\lambda_1) = BA^{-1}(A\lambda_2 - A\lambda_1) = \Theta(\vec{k})$. We have $\vec{\ell} = \Theta(\vec{k})$ and $\vec{n}_2 - \vec{n}_1 = \vec{k}$.

Case (2). Both $\Theta(\vec{n}_1), \Theta(\vec{n}_2)$ are in $\vec{\ell}_B + B\mathbb{Z}^s$. By Equation (3.1), $n_1 = \vec{\ell}_A + A\lambda_1, n_2 = \vec{\ell}_A + A\lambda_2$ for some vectors $\lambda_1, \lambda_2 \in \mathbb{Z}^d$. Denote $\vec{k} = A\lambda_2 - A\lambda_1 \in$

$A\mathbb{Z}^d$. So $\vec{\ell} = \Theta(\vec{n}_2) - \Theta(\vec{n}_1) = (\vec{\ell}_A + BA^{-1}(A\lambda_2)) - (\vec{\ell}_A + BA^{-1}(A\lambda_1)) = BA^{-1}(A\lambda_2 - A\lambda_1) = \Theta(\vec{k})$. We have $\vec{\ell} = \Theta(\vec{k})$ and $\vec{n}_2 - \vec{n}_1 = \vec{k}$. $\square$

For matrix B, the system of equation in (1.1) becomes

$$\begin{cases} \sum_{\vec{m}\in\mathbb{Z}^s} h'_{\vec{m}}\overline{h'_{\vec{m}+\vec{\ell}}} = \delta_{\vec{0}\vec{\ell}}, \ \vec{\ell} \in B\mathbb{Z}^s, \\ \sum_{\vec{m}\in\mathbb{Z}^s} h'_{\vec{m}} = \sqrt{2}. \end{cases} \tag{3.2}$$

Consider the reduced system $\mathcal{E}_{(\Lambda_A,A,d)}$ with index set Λ_A^E. Define $\theta \equiv \Theta|_{\Lambda_A}$ and $\eta \equiv \Theta|_{\Lambda_A^E}$. Denote $\Lambda_B \equiv \theta(\Lambda_A)$. Since Θ is an injection from $\mathbb{Z}^d$ to $\mathbb{Z}^s$, θ is a bijection from Λ_A to Λ_B.

Let $\mathcal{S}_A = \{a_{\vec{n}} \mid \vec{n} \in \Lambda_A\}$ be a solution to $\mathcal{E}_{(\Lambda_A,A,d)}$.

Define

$$\begin{aligned} b_{\vec{m}} &\equiv \begin{cases} a_{\theta^{-1}(\vec{m})}, & \text{if } \vec{m} \in \Lambda_B, \\ 0, & \text{if } \vec{m} \in \mathbb{Z}^s \setminus \Lambda_B. \end{cases} \\ \mathcal{S}_B &\equiv \{b_{\vec{m}} \mid \vec{m} \in \Lambda_B\}. \end{aligned}$$

To prove that $\varphi_B \simeq \varphi_A$, by Definition 2.2, we only need to show that

1. The system of equation

$$\begin{cases} \sum_{\vec{m}\in\Lambda_B} h'_{\vec{m}}\overline{h'_{\vec{m}+\vec{\ell}}} = \delta_{\vec{0}\vec{\ell}}, \ \vec{\ell} \in \eta(\Lambda_A^E), \\ \sum_{\vec{m}\in\Lambda_B} h'_{\vec{m}} = \sqrt{2}. \end{cases} \tag{3.3}$$

   is the reduced system of equations $\mathcal{E}_{(\Lambda_B,B,s)}$. Or equivalently, the set $\eta(\Lambda_A^E)$ is an index set for $\mathcal{E}_{(\Lambda_B,B,s)}$, denoted as Λ_B^E. This is Lemma 3.3 below.
2. The set $\mathcal{S}_B \equiv \{b_{\vec{m}} \mid \vec{m} \in \Lambda_B\}$ is a solution to (3.3) by Lemma 2.3.

Lemma 3.3. *The set $\eta(\Lambda_A^E)$ is an index set for $\mathcal{E}_{(\Lambda_B,B,s)}$.*

Proof. Let $\vec{k} \in \Lambda_A^E$. A reduced equation in $\mathcal{E}_{(\Lambda_A,A,d)}$ generated by $\vec{k}$ has the following form:

$$\sum_{\vec{n}\in\Lambda_A} h_{\vec{n}}\overline{h_{\vec{n}+\vec{k}}} = \delta_{\vec{0}\vec{k}}. \tag{3.4}$$

We will show that $\vec{\ell} \equiv \eta(\vec{k}) \in B\mathbb{Z}^s$ generates an reduced equation in $\mathcal{E}_{(\Lambda_B,B,s)}$. We write

$$\sum_{\vec{m}\in\mathbb{Z}^s} h'_{\vec{m}}\overline{h'_{\vec{m}+\vec{\ell}}} = \delta_{\vec{0}\vec{\ell}}$$

Note that $h'_{\vec{m}} = 0$ for $\vec{m} \notin \Lambda_B$, so the above equation is the same as

$$\sum_{\vec{m}\in\Lambda_B} h'_{\vec{m}}\overline{h'_{\vec{m}+\vec{\ell}}} = \delta_{\vec{0}\vec{\ell}}. \tag{3.5}$$

By definition of θ, η and the fact that $\Lambda_B \equiv \theta(\Lambda_A)$, we have

$$\sum_{\vec{n}\in\Lambda_A} h'_{\theta(\vec{n})}\overline{h'_{\theta(\vec{n})+\eta(\vec{k})}} = \delta_{\vec{0}\eta(\vec{k})}$$

By Lemma 3.1, $\theta(\vec{n}+\vec{k}) = \theta(\vec{n}) + \Theta(\vec{k}) = \theta(\vec{n}) + \eta(\vec{k})$, thus

$$\sum_{\vec{n}\in\Lambda_A} h'_{\theta(\vec{n})}\overline{h'_{\theta(\vec{n}+\vec{k})}} = \delta_{\vec{0}\eta(\vec{k})}$$

Replace $h'_{\theta(\vec{n})}$ with $h_{\vec{n}}$, $h'_{\theta(\vec{n}+\vec{k})}$ with $h_{\vec{n}+\vec{k}}$ and $\delta_{\vec{0}\eta(\vec{k})}$ with $\delta_{\vec{0}\vec{k}}$, We obtained the the same equation as (3.4). Since (3.4) is non-trivial, (3.5) is non-trivial as well. Furthermore, (3.5) is a reduced equation in $\mathcal{E}_{(\Lambda_B,B,s)}$. It is clear that different elements in $\eta(\Lambda_A^E)$ generate different equations in $\mathcal{E}_{(\Lambda_B,B,s)}$.

Next, we will show that every (non-trivial) equation in $\mathcal{E}_{(\Lambda_B,B,s)}$ can be generated by an element in $\eta(\Lambda_A^E)$. Let the following be a non-trivial equation in $\mathcal{E}_{(\Lambda_B,B,s)}$ generated by $\vec{\ell}_0 \in B\mathbb{Z}^s$:

$$\sum_{\vec{m}\in\Lambda_B} h'_{\vec{m}}\overline{h'_{\vec{m}+\vec{\ell}_0}} = \delta_{\vec{0}\vec{\ell}_0}. \tag{3.6}$$

Denote $\vec{m} = \theta(\vec{n})$, where $\vec{n} \in \Lambda_A \subset \mathbb{Z}^d$:

$$\sum_{\theta(\vec{n})\in\Lambda_B} h'_{\theta(\vec{n})}\overline{h'_{\theta(\vec{n})+\vec{\ell}_0}} = \delta_{\vec{0}\vec{\ell}_0}.$$

By Lemma 3.2, there exists $\vec{k}_0 \in A\mathbb{Z}^d$ such that $\vec{\ell}_0 = \Theta(\vec{k}_0)$:

$$\sum_{\Theta(\vec{n})\in\Lambda_B} h'_{\Theta(\vec{n})}\overline{h'_{\Theta(\vec{n})+\Theta(\vec{k}_0)}} = \delta_{\vec{0}\Theta(\vec{k}_0)}.$$

By Lemma 3.1,

$$\sum_{\Theta(\vec{n})\in\Lambda_B} h'_{\Theta(\vec{n})}\overline{h'_{\Theta(\vec{n}+\vec{k}_0)}} = \delta_{\vec{0}\Theta(\vec{k}_0)}.$$

Replace $h'_{\Theta(\vec{n})}$ with $h_{\vec{n}}$, $h'_{\Theta(\vec{n}+\vec{k}_0)}$ with $h_{\vec{n}+\vec{k}_0}$ and $\delta_{\vec{0}\Theta(\vec{k}_0)}$ with $\delta_{\vec{0}\vec{k}_0}$, we have

$$\sum_{\vec{n}\in\Lambda_A} h_{\vec{n}}\overline{h_{\vec{n}+\vec{k}_0}} = \delta_{\vec{0}\vec{k}_0}.$$

It is clear that this is a reduced non-trivial equation in $\mathcal{E}_{(\Lambda_A,A,d)}$ generated by $\vec{k}_0$. On the other hand, since this is a reduced non-trivial equation in $\mathcal{E}_{(\Lambda_A,A,d)}$, it is generated by an element $\vec{k}$ in its index set Λ_A^E. It follows that $\vec{\ell} \equiv \eta(\vec{k}) \in \Lambda_B^E$ generates the same equation as (3.6). Hence $\Lambda_B^E = \eta(\Lambda_A^E)$ is an index set for $\mathcal{E}_{(\Lambda_B,B,s)}$.

$\square$

The proof of Theorem 2.5 is completed.

4. Examples from higher dimensions to one dimension

Examples for Theorem 2.6 are presented in this section.

The sublattice $A\mathbb{Z}^d$ generated by the $d \times d$ expansive dyadic integral matrix A can be further simplified by changing of basis:

Proposition 4.1. [11] *Let $d \geq 1$ be a natural number and A a $d \times d$ expansive dyadic integral matrix. Then $\mathbb{R}^d$ has a basis $\{\vec{f_j} \mid j = 1, ..., d\}$ with properties that, under this new basis, a vector $\vec{k}$ is in $A\mathbb{Z}^d$ if and only if the last coordinate of $\vec{k}$ is an even number. That is, under this new basis, we have*

$$A\mathbb{Z}^d = \{(\vec{x}, 2n) \mid \vec{x} \in \mathbb{Z}^{d-1}, n \in \mathbb{Z}\}. \tag{4.1}$$

Hence, for simplicity, all matrices discussed in the examples in this section will have this property (4.1). Let A be a $d \times d$ expansive dyadic integral matrix with properites (4.1).

For a natural number $N \geq 1$, define

$$\Lambda_{d,N} \equiv [0, 2^N)^d \cap \mathbb{Z}^d = \big\{(n_1, \cdots, n_d) \mid 0 \leq n_1, \cdots, n_d \leq 2^N - 1\big\}. \tag{4.2}$$

The set $\Lambda_{d,N}$ contains 2^{dN} elements in $\mathbb{Z}^d$.

For vector $\vec{n} = (n_1, n_2, \cdots, n_{d-1}, n_d) \in \mathbb{Z}^d$, define the function $\sigma_{_{d,N}} : \mathbb{Z}^d \to \mathbb{Z}$ as :

$$\sigma_{_{d,N}}(\vec{n}) = \sum_{j=1}^{d} n_j \cdot 4^{(j-1)N}. \tag{4.3}$$

Define $f_{d,N} : \mathbb{Z}^d \to \mathbb{Z}$:

$$f_{d,N}(\vec{x}, y) \equiv \lfloor \frac{y}{2} \rfloor 2^{(2d-3)N+2} + \begin{cases} 2\sigma_{_{d-1,N}}(\vec{x}) & y \text{ even} \\ 2\sigma_{_{d-1,N}}(\vec{x}) + 1 & y \text{ odd} \end{cases} \quad \forall \vec{x} \in \mathbb{Z}^{d-1}, y \in \mathbb{Z} \tag{4.4}$$

where $\lfloor \frac{y}{2} \rfloor$ gives the greatest integer that is less than or equal to $\frac{y}{2}$.

Define mappings $\theta_{d,N}$ and $\eta_{d,N}$ as follows:

$$\theta_{d,N}\big((\vec{x}, y)\big) \equiv f_{d,N}(\vec{x}, y), \ (\vec{x}, y) \in \Lambda_{d,N} \tag{4.5}$$

$$\eta_{d,N}\big((\vec{x}, y)\big) \equiv f_{d,N}(\vec{x}, y), \ (\vec{x}, y) \in \Lambda^E_{d,N}. \tag{4.6}$$

$\theta_{d,N}, \eta_{d,N}$ are injections on $\Lambda_{d,N}$ and $\Lambda^E_{d,N}$ respectively.

Denote

$$\begin{aligned} \Lambda_A &= \Lambda_{d,N}. \\ \Lambda^E_A &= \{\vec{n} = (\vec{x}, 2j) \in \mathbb{Z}^d \mid \sigma_{_{d,N}}(\vec{n}) \geq 0; \vec{n} \in (-2^N, 2^N)^d \cap \mathbb{Z}^d\}. \\ \theta &= \theta_{d,N}. \\ \eta &= \eta_{d,N}. \\ \Lambda_1 &= \theta(\Lambda_A). \\ \Lambda^E_1 &= \eta(\Lambda^E_A). \end{aligned}$$

With the above settings, the following Theorem collects some results from Section 4 of [11]. This is a special version of Theorem 2.6.

Theorem 4.2. 1. *The systems of equations $\mathcal{E}_{(\Lambda_A, A, d)}$ is a reducing system and Λ^E_A is an index set.*

2. *The systems of equations $\mathcal{E}_{(\Lambda_1, [2], 1)}$ is a reducing system and Λ^E_1 is an index set.*

3. *The systems of equations* $\mathcal{E}_{(\Lambda_A,A,d)}$ *and* $\mathcal{E}_{(\Lambda_1,[2],1)}$ *are isomorphic with bijections* θ *and* η*:*

$$\mathcal{E}_{(\Lambda_A,A,d)} \sim \mathcal{E}_{(\Lambda_1,[2],1)}.$$

Example. Let $A = \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix}$ and $B = [2]$.
Choose $\Lambda_A = \Lambda_{2,1} = \{(0,0),(0,1),(1,0),(1,1)\}$. It is clear that $\mathcal{E}_{(\Lambda_A,A,2)}$ below is a reduced system of equation:

$$\mathcal{E}_{(\Lambda_A,A,2)} : \begin{cases} h_{00}+h_{10}+h_{01}+h_{11} &= \sqrt{2} \\ h_{00}^2+h_{10}^2+h_{01}^2+h_{11}^2 &= 1 \\ h_{00}\cdot h_{10}+h_{01}\cdot h_{11} &= 0. \end{cases}$$

The bijections defined in (4.5) and (4.6) become

$$\theta(x,y) = \lfloor \frac{y}{2} \rfloor 4 + \begin{cases} 2x & y \text{ even} \\ 2x+1 & y \text{ odd} \end{cases} \quad (x,y)\in\Lambda_A;$$
$$\eta(x,y) = \lfloor \frac{y}{2} \rfloor 4 + \begin{cases} 2x & y \text{ even} \\ 2x+1 & y \text{ odd} \end{cases} \quad (x,y)\in\Lambda_A^E = \{(0,0),(1,0)\}.$$

The mappings are:

Λ_A	$\Lambda_B = \theta(\Lambda_A)$	Λ_A^E	$\Lambda_B^E = \eta(\Lambda_A^E)$
$(0,0)$	0	$(0,0)$	0
$(0,1)$	1		
$(1,0)$	2	$(1,0)$	2
$(1,1)$	3		

Under the above mapping ,the corresponding isomorphic systems of equations are

$$\mathcal{E}_{(\Lambda_A,A,2)} : \begin{cases} h_{00}+h_{10}+h_{01}+h_{11} &= \sqrt{2} \\ h_{00}^2+h_{10}^2+h_{01}^2+h_{11}^2 &= 1 \\ h_{00}\cdot h_{10}+h_{01}\cdot h_{11} &= 0. \end{cases}$$

$$\mathcal{E}_{(\Lambda_B,B,1)} : \begin{cases} h_0+h_1+h_2+h_3 &= \sqrt{2} \\ h_0^2+h_1^2+h_2^2+h_3^2 &= 1 \\ h_0\cdot h_2+h_1\cdot h_3 &= 0. \end{cases}$$

Example. Let $A = \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix}$ and $B = [2]$. Choose $\Lambda_A = \Lambda_{2,3} = \{(x,y) \mid 0 \le x,y \le 2^3-1\}$. The index set Λ_A^E for $\mathcal{E}_{(\Lambda_A,A,2)}$ is $\{(x,y) \mid -7 \le x \le 7, y \in \{2,4,6\} \text{ or } 0 \le x \le 7, y=0\}\}$.

The bijections defined in (4.5) and (4.6) become

$$\theta(x,y) = \lfloor \frac{y}{2} \rfloor 2^{3+2} + \begin{cases} 2x & y \text{ even} \\ 2x+1 & y \text{ odd} \end{cases} \quad (x,y)\in\Lambda_A;$$
$$\eta(x,y) = \lfloor \frac{y}{2} \rfloor 2^{3+2} + \begin{cases} 2x & y \text{ even} \\ 2x+1 & y \text{ odd} \end{cases} \quad (x,y)\in\Lambda_A^E.$$

The mappings are:

$\theta(x,y)$	0	1	2	3	4	5	6	7
0	0	1	32	33	64	65	96	97
1	2	3	34	35	66	67	98	99
2	4	5	36	37	68	69	100	101
3	6	7	38	39	70	71	102	103
4	8	9	40	41	72	73	104	105
5	10	11	42	43	74	75	106	107
6	12	13	44	45	76	77	108	109
7	14	15	46	47	78	79	110	111

$\eta(x,y)$	0	2	4	6
-7		18	50	82
-6		20	52	84
-5		22	54	86
-4		24	56	88
-3		26	58	90
-2		28	60	92
-1		30	62	94
0	0	32	64	96
1	2	34	66	98
2	4	36	68	100
3	6	38	70	102
4	8	40	72	104
5	10	42	74	106
6	12	44	76	108
7	14	46	78	110

For example, $\theta(4,3) = 41$ according to the above mapping table. $\Lambda_B = \theta(\Lambda_A)$ is the content listed in the table for θ and $\Lambda_B^E = \eta(\Lambda_A^E)$ is the content listed in the table for η. The corresponding isomorphic systems of equations can be obtained:

$$\mathcal{E}_{(\Lambda_A,A,2)}: \quad \begin{cases} \sum\limits_{\vec{n}\in\Lambda_A} h_{\vec{n}} & = \sqrt{2} \\ \sum\limits_{\vec{n}\in\Lambda_A} h_{\vec{n}}^2 & = 1 \\ \sum\limits_{\vec{n}\in\Lambda_A} h_{\vec{n}} \cdot h_{\vec{n}+\vec{k}} & = 0,\ \vec{k} \in \Lambda_A^E \end{cases} \qquad \mathcal{E}_{(\Lambda_B,B,1)}: \quad \begin{cases} \sum\limits_{m\in\Lambda_B} h_m & = \sqrt{2} \\ \sum\limits_{m\in\Lambda_B} h_m^2 & = 1 \\ \sum\limits_{m\in\Lambda_B} h_m \cdot h_{m+\ell} & = 0, \ell \in \Lambda_B^E. \end{cases}$$

Example. Let $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ and $B = [2]$. Choose $\Lambda_A = \Lambda_{3,1} = \{(\vec{x}, y) \mid \vec{x} = (n_1, n_2), y = n_3, 0 \le n_1, n_2, n_3 \le 2^1 - 1\}$. The index set Λ_A^E for $\mathcal{E}_{(\Lambda_A,A,3)}$ is $\{(0,0,0), (1,0,0), (-1,1,0), (0,1,0), (1,1,0)\}$.

The bijections defined in (4.5) and (4.6) become

$$\theta(\vec{x},y) = \lfloor\frac{y}{2}\rfloor 2^{3+2} + \begin{cases} 2\sigma_{2,1}(\vec{x}) & y \text{ even} \\ 2\sigma_{2,1}(\vec{x})+1 & y \text{ odd} \end{cases} \quad (\vec{x},y)\in\Lambda_A;$$
$$\eta(\vec{x},y) = \lfloor\frac{y}{2}\rfloor 2^{3+2} + \begin{cases} 2\sigma_{2,1}(\vec{x}) & y \text{ even} \\ 2\sigma_{2,1}(\vec{x})+1 & y \text{ odd} \end{cases} \quad (\vec{x},y)\in\Lambda_A^E.$$

Where $\sigma_{2,1}(n_1,n_2) = \sum_{j=1}^{2} n_j \cdot 4^{(j-1)}$ by Equation (4.3).

The mappings are:

Λ_A	$\Lambda_B = \theta(\Lambda_A)$
$(0,0,0)$	0
$(0,0,1)$	1
$(1,0,0)$	2
$(1,0,1)$	3
$(0,1,0)$	8
$(0,1,1)$	9
$(1,1,0)$	10
$(1,1,1)$	11

Λ_A^E	$\Lambda_B^E = \eta(\Lambda_A^E)$
$(0,0,0)$	0
$(1,0,0)$	2
$(-1,1,0)$	6
$(0,1,0)$	8
$(1,1,0)$	10

Example. Let $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ and $B=[2]$. Choose $\Lambda_A = \Lambda_{3,2} = \{(\vec{x},y) \mid \vec{x}=(n_1,n_2), y=n_3, 0\le n_1,n_2,n_3 \le 2^2-1\}$. The index set Λ_A^E for $\mathcal{E}_{(\Lambda_A,A,3)}$ contains 74 elements as shown later.

The bijections defined in (4.5) and (4.6) become

$$\theta(\vec{x},y) = \lfloor\frac{y}{2}\rfloor 2^{6+2} + \begin{cases} 2\sigma_{2,2}(\vec{x}) & y \text{ even} \\ 2\sigma_{2,2}(\vec{x})+1 & y \text{ odd} \end{cases} \quad (\vec{x},y)\in\Lambda_A;$$
$$\eta(\vec{x},y) = \lfloor\frac{y}{2}\rfloor 2^{6+2} + \begin{cases} 2\sigma_{2,2}(\vec{x}) & y \text{ even} \\ 2\sigma_{2,2}(\vec{x})+1 & y \text{ odd} \end{cases} \quad (\vec{x},y)\in\Lambda_A^E.$$

Where $\sigma_{2,2}(n_1,n_2) = \sum_{j=1}^{2} n_j \cdot 4^{(j-1)}$ by Equation (4.3).

The mappings are:

$\theta(\vec{x},y)$	0	1	2	3
(0,0)	0	1	256	257
(1,0)	2	3	258	259
(2,0)	4	5	260	261
(3,0)	6	7	262	263
(0,1)	32	33	288	289
(1,1)	34	35	290	291
(2,1)	36	37	292	293
(3,1)	38	39	294	295
(0,2)	64	65	320	321
(1,2)	66	67	322	323
(2,2)	68	69	324	325
(3,2)	70	71	326	327
(0,3)	96	97	352	353
(1,3)	98	99	354	355
(2,3)	100	101	356	357
(3,3)	102	103	358	359

$\eta(\vec{x},y), y=0$ $\vec{x}=(x_1,x_2)$	0	1	2	3
-3		26	58	90
-2		28	60	92
-1		30	62	94
0	0	32	64	96
1	2	34	66	98
2	4	36	68	100
3	6	38	70	102

$\eta(\vec{x},y), y=2$ $\vec{x}=(x_1,x_2)$	-3	-2	-1	0	1	2	3
-3	154	186	218	250	282	314	346
-2	156	188	220	252	284	316	348
-1	158	190	222	254	286	318	350
0	160	192	224	256	288	320	352
1	162	194	226	258	290	322	354
2	164	196	228	260	292	324	356
3	166	198	230	262	294	326	358

For example, $\theta(3,2,1)=71, \eta(2,1,0)=36, \eta(2,1,2)=292$ according to the above mapping tables. $\Lambda_B=\theta(\Lambda_A)$ is the content listed in the table for θ and $\Lambda_B^E=\eta(\Lambda_A^E)$ is the content listed in the 2 tables for η. We omit the corresponding isomorphic systems of equations as it can be easily populated from the table content of η.

So far, all examples are with Λ_A of the form $\Lambda_{d,N}$. Next we will show an example with Λ_A a proper subset of $\Lambda_{d,N}$.

Example. Let $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ and $B = [2]$.
Choose $\Lambda_A = \{(0,0,0),(0,0,1),(1,0,0),(1,0,1),(2,3,2),(2,3,3),(3,3,2),(3,3,3)\}$. Notice that this support set Λ_A is properly contained in $\Lambda_{3,2}$, which is the support of the previous example. The index set Λ_A^E for $\mathcal{E}_{(\Lambda_A,A,3)}$ contains 5 elements as shown later.

The mappings are:

Λ_A	$\Lambda_B = \theta(\Lambda_A)$	Λ_A^E	$\Lambda_B^E = \eta(\Lambda_A^E)$
$(0,0,0)$	0	$(0,0,0)$	0
$(0,0,1)$	1		
$(1,0,0)$	2	$(1,0,0)$	2
$(1,0,1)$	3		
		$(1,3,2)$	354
$(2,3,2)$	356	$(2,3,2)$	356
$(2,3,3)$	357		
$(3,3,2)$	358	$(3,3,2)$	358
$(3,3,3)$	359		

The corresponding isomorphic systems of equations are:

$\mathcal{E}_{(\Lambda_A,A,3)}$:

$$\begin{cases} h_{0,0,0}+h_{0,0,1}+h_{1,0,0}+h_{1,0,1}+h_{2,3,2}+h_{2,3,3}+h_{3,3,2}+h_{3,3,3} & =\sqrt{2} \\ h_{0,0,0}^2+h_{0,0,1}^2+h_{1,0,0}^2+h_{1,0,1}^2+h_{2,3,2}^2+h_{2,3,3}^2+h_{3,3,2}^2+h_{3,3,3}^2 & =1 \\ h_{0,0,0}h_{1,0,0}+h_{0,0,1}h_{1,0,1}+h_{2,3,2}h_{3,3,2}+h_{2,3,3}h_{3,3,3} & =0 \\ h_{1,0,0}h_{2,3,2}+h_{1,0,1}h_{2,3,3} & =0 \\ h_{0,0,0}h_{2,3,2}+h_{0,0,1}h_{2,3,3}+h_{1,0,0}h_{3,3,2}+h_{1,0,1}h_{3,3,3} & =0 \\ h_{0,0,0}h_{3,3,2}+h_{0,0,1}h_{3,3,3} & =0; \end{cases}$$

$\mathcal{E}_{(\Lambda_B,B,1)}$:

$$\begin{cases} h_0+h_1+h_2+h_3+h_{356}+h_{357}+h_{358}+h_{359} & =\sqrt{2} \\ h_0^2+h_1^2+h_2^2+h_3^2+h_{356}^2+h_{357}^2+h_{358}^2+h_{359}^2 & =1 \\ h_0h_2+h_1h_3+h_{356}h_{358}+h_{357}h_{359} & =0 \\ h_2h_{356}+h_3h_{357} & =0 \\ h_0h_{356}+h_1h_{357}+h_2h_{358}+h_3h_{359} & =0 \\ h_0h_{358}+h_1h_{359} & =0. \end{cases}$$

5. From lower dimensions to higher dimensions

In this section we provide an example for Theorem 2.5.

Example. Let $A = \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix}$, $\ell_A = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ 0 & -1 & 0 \end{bmatrix}$,

$\ell_B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Choose $\Lambda_A = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 3\}$.

The mappings are:

Λ_A	$\Lambda_B=\theta(\Lambda_A)$	Λ_A^E	$\Lambda_B^E=\eta(\Lambda_A^E)$
$(0,0)$	$(0,0,0)$	$(0,0)$	$(0,0,0)$
$(0,1)$	$(1,\text{-}2,0)$		
$(0,2)$	$(0,\text{-}2,\text{-}1)$	$(0,2)$	$(0,\text{-}2,\text{-}1)$
$(0,3)$	$(1,\text{-}4,\text{-}1)$		
$(1,0)$	$(0,1,1)$	$(1,0)$	$(0,1,1)$
$(1,1)$	$(1,\text{-}1,1)$		
$(1,2)$	$(0,\text{-}1,0)$	$(1,2)$	$(0,\text{-}1,0)$
$(1,3)$	$(1,\text{-}3,0)$		

The corresponding isomorphic systems of equations are:

$\mathcal{E}_{(\Lambda_A,A,2)}$:

$$\begin{cases} h_{0,0}+h_{0,1}+h_{0,2}+h_{0,3}+h_{1,0}+h_{1,1}+h_{1,2}+h_{1,3} & =\sqrt{2} \\ h_{0,0}^2+h_{0,1}^2+h_{0,2}^2+h_{0,3}^2+h_{1,0}^2+h_{1,1}^2+h_{1,2}^2+h_{1,3}^2 & =1 \\ h_{0,0}h_{0,2}+h_{0,1}h_{0,3}+h_{1,0}h_{1,2}+h_{1,1}h_{1,3} & =0 \\ h_{0,0}h_{1,0}+h_{0,1}h_{1,1}+h_{0,2}h_{1,2}+h_{0,3}h_{1,3} & =0 \\ h_{0,0}h_{1,2}+h_{0,1}h_{1,3} & =0; \end{cases}$$

$\mathcal{E}_{(\Lambda_B,B,3)}$:

$$\begin{cases} h_{0,0,0}+h_{1,\text{-}2,0}+h_{0,\text{-}2,\text{-}1}+h_{1,\text{-}4,\text{-}1}+h_{0,1,1}+h_{1,\text{-}1,1}+h_{0,\text{-}1,0}+h_{1,\text{-}3,0} & =\sqrt{2} \\ h_{0,0,0}^2+h_{1,\text{-}2,0}^2+h_{0,\text{-}2,\text{-}1}^2+h_{1,\text{-}4,\text{-}1}^2+h_{0,1,1}^2+h_{1,\text{-}1,1}^2+h_{0,\text{-}1,0}^2+h_{1,\text{-}3,0}^2 & =1 \\ h_{0,0,0}h_{0,\text{-}2,\text{-}1}+h_{1,\text{-}2,0}h_{1,\text{-}4,\text{-}1}+h_{0,1,1}h_{0,\text{-}1,0}+h_{1,\text{-}1,1}h_{1,\text{-}3,0} & =0 \\ h_{0,0,0}h_{0,1,1}+h_{1,\text{-}2,0}h_{1,\text{-}1,1}+h_{0,\text{-}2,\text{-}1}h_{0,\text{-}1,0}+h_{1,\text{-}4,\text{-}1}h_{1,\text{-}3,0} & =0 \\ h_{0,0,0}h_{0,\text{-}1,0}+h_{1,\text{-}2,0}h_{1,\text{-}3,0} & =0. \end{cases}$$

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