ABSOLUTELY CONTINUOUS SPECTRUM OF A TYPICAL SCHRÖDINGER OPERATOR WITH AN OPERATOR VALUED POTENTIAL

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1. Main results

Let $\mathfrak{H}$ be a separable Hilbert space and let V be a measurable function from $\mathbb{R}_+$ to the set of bounded self-adjoint operators on $\mathfrak{H}$. We study the absolutely continuous spectrum of the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + \alpha V, \tag{1}$$

acting in the space $L^2(\mathbb{R}_+; \mathfrak{H})$. Here, α is a real parameter. We impose the condition

$$\int_{\mathbb{R}_+} \|V(x)\|^2 dx < \infty. \tag{2}$$

The domain of H consists of $W_0^2(\mathbb{R}_+, \mathfrak{H})$-functions. The generalized second derivatives of these functions are square integrable and the functions themselves vanish at $x = 0$.

Definition. We say that the absolutely continuous spectrum of the operator H is essentially supported on a set containing $[0, \infty)$, if the spectral projection $E_\alpha(\Omega)$ of H corresponding to any set $\Omega \subset [0, \infty)$ is different from zero $E_\alpha(\Omega) \neq 0$ as soon as the Lebesgue measure of Ω is positive.

Operators with square integrable potentials were studied by P. Deift and R. Killip [1] in the case where $\mathfrak{H} = \mathbb{R}$. The main result of [1] states that absolutely continuous spectrum of the operator $-d^2/dx^2 + V$ covers the positive half-line $[0, \infty)$, if $V \in L^2(\mathbb{R}_+)$.

We consider the case where the space $\mathfrak{H}$ is infinitely dimensional and give a different proof of the following theorem by Denisov [2].

Theorem 1.1. *Let V satisfy the condition* (2). *Then the absolutely continuous spectrum of the operator* (1) *is essentially supported on a set containing* $[0, \infty)$ *for almost every* $\alpha \in \mathbb{R}$.

Besides the article [2], one can also find a close discussion of similar operator families in the papers [6] and [7]. I all mentioned publications, the properties of the absolutely continuous spectrum are established for almost every value of a real parameter α. However, if $\|V(x)\| \leq C(1 + |x|)^{-3/2-\delta}$ with $\delta > 0$, then the absolutely continuous spectrum fills the positive half-line $\mathbb{R}_+$ for all α (see [3]).

2. Auxiliary lemma

Notations. Throughout the text, $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary parts of a complex number z. For a self-adjoint operator $B = B^*$ and a vector g of a Hilbert space the expression $((B - k - i0)^{-1} g, g)$ is always understood as the limit

$$\Big((B - k - i0)^{-1} g, g\Big) = \lim_{\varepsilon \to 0} \Big((B - k - i\varepsilon)^{-1} g, g\Big), \qquad \varepsilon > 0,\ k \in \mathbb{R}.$$

The following simple and very well known statement plays very important role in our proof.

Lemma 2.1. *Let B be a self-adjoint operator in a separable Hilbert space $\mathfrak{H}$ and let $g \in \mathfrak{H}$. Then the function*

$$\eta(k) := \mathrm{Im}\Big((B-k-i0)^{-1}g, g\Big) \geq 0$$

is integrable over $\mathbb{R}$. Moreover,

$$\int_{-\infty}^{\infty} \eta(k)dk \leq \pi||g||^2.$$

and

$$\int_{-\infty}^{\infty} \frac{\eta(k)}{k^2+1} dk \leq \pi||(B^2+I)^{-1/2}g||^2.$$

3. Entropy

Let μ be a positive finite measure on the real line $\mathbb{R}$. As any other measure it is decomposed uniquely into a sum of three terms

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sc},$$

where the first term is pure point, the second term is absolutely continuous and the last term is a continuous but singular measure on $\mathbb{R}$. Obviously, $\mu(-\infty, \lambda)$ is a monotone function of λ, therefore, it is differentiable almost everywhere. In particular, the limit

$$\mu'(\lambda) = \lim_{\epsilon \to 0} \frac{\mu(\lambda - \epsilon, \lambda + \epsilon)}{2\epsilon}$$

exists for almost every $\lambda \in \mathbb{R}$. It is also clear that

$$\mu_{ac}(\Omega) = \int_\Omega \mu'(\lambda)\, d\lambda, \qquad \forall \Omega \subset \mathbb{R},$$

which means $\mu' = \mu'_{ac}$.

Let $\Omega_0 = \{\lambda : \ \mu'(\lambda) > 0\}$ A measurable set $\Omega \subset \mathbb{R}$ is called an essetial support of μ_{ac}, if the Lebesgue measure of the symmetric difference

$$\Omega_0 \triangle \Omega := \Big(\Omega_0 \setminus \Omega\Big) \cup \Big(\Omega \setminus \Omega_0\Big)$$

is zero. So, an essential support of μ_{ac} coincides with the set where $\mu' > 0$ up to a set of measure zero. As we see, the study of the essential support of the a.c. part of the measure μ is reduced to the study of the set $\Omega_0 = \{\lambda : \ \mu'(\lambda) > 0\}$. Let Ω be a measurable set. One of the ways to show that $\mu(\lambda) > 0$ for almost every $\lambda \in \Omega$ relies on the study of the quantity

$$S_\Omega(\mu) := \int_\Omega \log \mu'(\lambda)\, d\lambda.$$

Due to Jenssen's inequality, $S_\Omega < \infty$, if $|\Omega| < \infty$. So, the entropy can diverge only to the negative infinity.

But if $|\Omega| < \infty$ and

$$S_\Omega(\mu) > -\infty,$$

then

$$\mu'(\lambda) > 0, \qquad \text{a.e. on } \Omega.$$

Very often one can obtain an estimate for μ' by an analytic function from below. In this case we will use the following statement

Proposition 3.1. *Let a function $F(\lambda) \neq 0$ be analytic in the neighborhood of an interval $[a,b] \subset \mathbb{R}$. Suppose that*

$$\mu'(\lambda) > c_0|F(\lambda)|^2, \qquad \text{for all } \lambda \in \Omega \subset [a,b]. \tag{3}$$

Then

$$S_\Omega(\mu) := \int_\Omega \log \mu'(\lambda)\, d\lambda \geq C > -\infty,$$

where the constant $C = C([a,b], c_0, F)$ depends on the interval $[a,b]$, c_0 and F.

The proof is left to the reader as an exercise. We only mention that zeros of an analytic functions are always isolated zeros of a finite order.

In applications to Schrödinger operators, one often has an estimate of the form (3) for a sequence of measures μ_n that converges to μ weakly

$$\mu_n \to \mu \qquad \text{weakly}.$$

In this situation, one can still derive a certain information about the limit measure μ from the information about μ_n.

Definition. Let ρ, ν be finite Borel measures on a compact Hausdorff space, X. We define the entropy of ρ relative to ν by

$$S(\rho|\nu) = \begin{cases} -\infty, & \text{if } \rho \text{ is not } \nu-\text{ac} \\ -\int_X \log(\frac{d\rho}{d\nu})d\rho, & \text{if } \rho \text{ is } \nu-\text{ac}. \end{cases} \tag{4}$$

Theorem 3.1. (cf.[4]) *The entropy $S(\rho|\nu)$ is jointly upper semi-continuous in ρ and ν with respect to the weak topology. That is, if $\rho_n \to \rho$ and $\nu_n \to \nu$ as $n \to \infty$, then*

$$S(\rho|\nu) \geq \limsup_{n\to\infty} S(\rho_n|\nu_n).$$

Now, we will use this theorem in order to prove the following statement.

Proposition 3.2. *Let $a < b$. Let $F(\lambda) \neq 0$ be a function analytic in the neighborhood of $[a,b]$. Let μ_n be a sequence of positive finite measures on the real line $\mathbb{R}$ converging to μ weakly. Suppose that*

$$\mu_n'(\lambda) > c_0|F(\lambda)|^2, \qquad \text{for all } \lambda \in \Omega_n \subset [a,b],$$

where the measurable sets Ω_n satisfy

$$\Big|[a,b] \setminus \Omega_n\Big| < b - a - \varepsilon.$$

Then $\mu'(\lambda) > 0$ on a subset of $[a,b]$ whose measure is not smaller than $b - a - \varepsilon$

Proof. Let us denote the characteristic function of the set Ω_n by χ_n. Since L^2-norms of χ_n are uniformly bounded, this sequence of functions has a weakly convergent subsequence. Therefore without loss of generality, one can assume that

$$\chi_n \to \chi, \qquad \text{weakly in} \quad L^2(\mathbb{R}).$$

This, of cause, implies that the corresponding measures $\chi_n d\lambda$ also converge weakly to $\chi d\lambda$. Even though, $\mathbb{R}$ is not compact, we can still use Theorem 3.1 and show (see [6]) that

$$\int_\mathbb{R} \log\Big(\frac{\mu'(\lambda)}{\chi(\lambda)}\Big)\, \chi(\lambda)\, d\lambda \geq \limsup_{n\to\infty} \int_\mathbb{R} \log\Big(\frac{\mu_n'(\lambda)}{\chi_n}\Big)\, \chi_n(\lambda)\, d\lambda > -\infty$$

Thus, we see that $\mu' > 0$ on the support of the function χ. However, we still need to know how big this set is. On the one hand,

$$\int_a^b \chi(\lambda)\, d\lambda = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_n(\lambda)\, d\lambda \geq b - a - \varepsilon.$$

On the other hand, it is easy to show that $0 \leq \chi \leq 1$. Therefore, the Lebesgue measure of the support of the function χ is not smaller than $b - a - \varepsilon$. $\square$

Since we deal with a family of operators depending on a parameter α, we also need a modification of the previous statement, suitable in the case when measures depend on the parameter α as well.

Proposition 3.3. *Let $a < b$. Let $F(\lambda) \neq 0$ be a function analytic in the neighborhood of $[a, b]$. Let $\mu_n(\cdot, \alpha)$ be a sequence of α-dependent families of positive finite measures on $\mathbb{R}$ converging to $\mu(\cdot, \alpha)$ weakly for every $\alpha \in \mathbb{R}$. Suppose that the derivatives of μ_n with respect to $d\lambda$ satisfy*

$$\mu_n'(\lambda, \alpha) > c_0 |F(\lambda)|^2, \qquad \text{for all } (\lambda, \alpha) \in \Omega_n \subset [a, b] \times [\alpha_1, \alpha_2],$$

where the measurable sets Ω_n obey

$$\Big|[a, b] \times [\alpha_1, \alpha_2] \setminus \Omega_n\Big| < (b - a)(\alpha_2 - \alpha_1) - \varepsilon.$$

Then $\mu'(\lambda, \alpha) > 0$ on a subset of $[a, b] \times [\alpha_1, \alpha_2]$ whose measure is not smaller than $(b-a)(\alpha_2-\alpha_1)-\varepsilon$.

The proof of this statement is a counterpart of the proof above and it is left to the reader as an exercise. A similar statement is proven in [6].

We conclude this section by a discussion of the following simple claim.

Proposition 3.4. *Let $a < b$. Let $F(\lambda) \neq 0$ be a function analytic on a neighborhood of the interval $[a, b]$. Let $\mu(\cdot, \alpha)$ be an α-dependent family of positive finite measures on $\mathbb{R}$. Suppose that the derivatives of μ with respect to the Lebesgue measure $d\lambda$ satisfy the estimate*

$$\mu'(\lambda, \alpha) \geq |F(\lambda)|^2 (1 - \Psi(\lambda, \alpha)), \qquad \textit{where} \quad \int_{\alpha_1}^{\alpha_2} \int_a^b |\Psi(\lambda, \alpha)| d\lambda d\alpha < \varepsilon/2.$$

Then

$$\mu'(\lambda, \alpha) \geq \frac{1}{2} |F(\lambda)|^2, \qquad \textit{for all} \quad (\lambda, \alpha) \in \Omega,$$

where the measureable set Ω obeys

$$\Big|[a, b] \times [\alpha_1, \alpha_2] \setminus \Omega\Big| \leq \varepsilon. \tag{5}$$

Proof. According to Chebyshev's inequality,

$$\Psi(\lambda, \alpha) \leq 1/2$$

on a set satisfying the condition (5). $\square$

4. The case of a compactly supported V

In this section, we assume that V belongs to the class $\mathfrak{V}$ described below.

Definition. We say that V belongs to the class $\mathfrak{V}$ if

1) there is a bounded the interval $[0, R]$ containing the support of V and such that $V(x + R/2)$ is an odd function of x:

$$V(x + R/2) = -V(-x + R/2), \qquad \forall x \in [0, R/2]. \tag{6}$$

2) the range of the operator $V(x)$ is a finite dimensional sub-space $\mathfrak{H}_0 \subset \mathfrak{H}$ which stays the same when one changes x.

Our proof of Theorem 1.1 is based on the relation between the derivative of the spectral measure and the so called scattering amplitude. Both objects should be introduced properly. While the spectral measure can be defined for any self-adjoint operator, the scattering coefficient will be introduced only for a Schrödinger operator. Let f be a square integrable function from $\mathbb{R}_+$ to $\mathfrak{H}$. It is very well known that the quadratic form of the resolvent of H can be written as a Cauchy integral

$$((H-z)^{-1}f,f)=\int_{-\infty}^{\infty}\frac{d\mu(t)}{t-z},\qquad \operatorname{Im} z\neq 0.$$

The measure μ in this representation is called the spectral measure of H corresponding to the element f.

Let us introduce the scattering amplitude. Since the support of the potential V is compact, there exists an R.), such that $V(x)=0$ for $x>R$. Take any compactly supported function f that also vanishes for $x>R$. Then

$$\big[(H-z)^{-1}f\big](x)=e^{ik|x|}A_f(k),\qquad \text{for } x>R,\ k^2=z,\ \operatorname{Im} k\geq 0,\ A_f(k)\in\mathfrak{H}.$$

Clearly, the relation

$$\mu'(\lambda)=\pi^{-1}\lim_{z\to\lambda+i0}\operatorname{Im}((H-z)^{-1}f,f)=\pi^{-1}\lim_{z\to\lambda+i0}\operatorname{Im} z||(H-z)^{-1}f||^2$$

implies that

$$\pi\mu'(\lambda)=\sqrt{\lambda}\|A_f(k)\|^2,\qquad k^2=\lambda>0. \tag{7}$$

Formula (7) is a very important equality that relates the absolutely continuous spectrum to so-called extended states. The rest of the proof will be devoted to a lower estimate of $\|A_f(k)\|$.

For our purposes, it is sufficient to assume that f is the product of the characteristic function of the unit interval $[0,1]$ times a unit vector $\tau\in\mathfrak{H}$. Traditionally, H is viewed as an operator obtained by a perturbation of

$$H_0=-\frac{d^2}{dx^2}.$$

In its turn, $(H-z)^{-1}$ can be viewed as an operator obtained by a perturbation of $(H_0-z)^{-1}$. The theory of such perturbations is often based on the second resolvent identity

$$(H-z)^{-1}=(H_0-z)^{-1}-(H-z)^{-1}\,\alpha V(H_0-z)^{-1}, \tag{8}$$

which turns out to be useful for our reasoning. As a consequence of (8), we obtain that

$$A_f(k)=F_0(k)\tau-A_g(k),\qquad z=k^2+i0,\ k>0, \tag{9}$$

where $g(x)=\alpha V(H_0-z)^{-1}f$ and the number $F_0(k)\in\mathbb{C}$ is defined by

$$(H_0-z)^{-1}f=e^{ik|x|}F_0(k)\tau,\qquad \text{for } x>1. \tag{10}$$

We will shortly show that, without loss of generality, one can assume that $V(x)\tau=0$ inside the unit interval $[0,1]$. In this case,

$$g=F_0(k)h_k,\quad \text{where}\quad h_k(x)=\alpha e^{ik|x|}V\tau. \tag{11}$$

According to (9),

$$2\|A_f(k)\|^2\geq|F_0(k)|^2-2\|A_g(k)\|^2,$$

which can be written in the form

$$2\pi\mu'(\lambda)\geq|F_0(k)|^2\Big(\sqrt{\lambda}-2\operatorname{Im}\Big(\big((H-z)^{-1}h_k,\,h_k\big)\Big)\Big),\qquad z=\lambda+i0, \tag{12}$$

due to (7) and (11). Therefore, in order to establish the presence of the absolutely continuous spectrum, we need to show that the quantity $\operatorname{Im}\Big((H-z)^{-1}h_k,\,h_k\Big)$ is small.

Let us define η setting

$$\alpha^2 k^{-2}\eta(k,\alpha) := \frac{1}{k}\mathrm{Im}\Big((H-z)^{-1}h_k,\, h_k\Big) \geq 0, \qquad z = k^2 + i0.$$

Obviously, η is positive for all real $k \neq 0$, because we agreed that $z = k^2 \pm i0$ if $\pm k > 0$. This is very convenient. Since $\eta \geq 0$, we can conclude that η is small on a rather large set if the integral of this function is small. That is why we will estimate

$$J(V) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta(k,\alpha)}{(\alpha^2+k^2)}\frac{|k|\, dk d\alpha}{(k^2+1)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta(k,tk)}{(k^2+1)(t^2+1)}\, dkdt. \tag{13}$$

We will employ a couple of tricks, one of which is related to the involvment of an additional parameter ε. Instead of dealing with the operator H, we will deal with $H + \varepsilon I$ where $\varepsilon > 0$ is small. We will first obtain an integral estimate for the quantity

$$\eta_\varepsilon(k,\alpha) = \frac{k}{\alpha^2}\mathrm{Im}\Big((H+\varepsilon-z)^{-1}h_k,\, h_k\Big).$$

The latter estimate will not be uniform in ε, but we can still pass to the limit $\varepsilon \to 0$ according to Fatou's lemma, because

$$\eta(k,\alpha) = \lim_{\varepsilon\to 0}\eta_\varepsilon(k,\alpha) \qquad \text{a.e. on } \mathbb{R}\times\mathbb{R}.$$

The second trick is to set $\alpha = kt$ and represent η_ε in the form

$$\eta_\varepsilon(k,kt) = \mathrm{Im}\Big((B+1/k)^{-1}H_\varepsilon^{-1/2}v,\, H_\varepsilon^{-1/2}v\Big) \tag{14}$$

where $v = V\tau$, $H_\varepsilon = -d^2/dx^2 + \varepsilon I$ and B is the bounded selfadjoint operator defined by

$$B = H_\varepsilon^{-1/2}\Big(-2i\frac{d}{dx} + tV\Big)H_\varepsilon^{-1/2}.$$

The reader can easily establish that B is not only self-adjoint but bounded as well. Note that it is the parameter ε that makes B bounded.

In order to justify (14) at least formally, one has to introduce the operator U of multiplication by the function $\exp(ikx)$. Using this notation, we can represent η_ε in the following form

$$\eta_\varepsilon(k,tk) = k\mathrm{Im}\Big(U^{-1}(H+\varepsilon-z)^{-1}Uv,\, v\Big).$$

Since we deal with a unitary equivalence of operators, we can employ the formula

$$U^{-1}(H+\varepsilon-z)^{-1}U = (U^{-1}HU+\varepsilon-z)^{-1}.$$

On the other hand, since H is a differential operator and U is an operator of multiplication, the commutator $[H,U] := HU - UH$ can be easily found

$$\Big[H,U\Big] = kU\Big(-2i\frac{d}{dx} + k\Big).$$

The latter equality implies that

$$U^{-1}HU + \varepsilon - z = H_\varepsilon + k\Big(-2i\frac{d}{dx} + tV\Big) = H_\varepsilon^{1/2}(I+kB)H_\varepsilon^{1/2}.$$

Consequently,

$$kU^{-1}(H+\varepsilon-z)^{-1}U = H_\varepsilon^{-1/2}(B+1/k)^{-1}H_\varepsilon^{-1/2}. \tag{15}$$

Let us have a look at the formula (14). If k belongs to the upper half plane then so does $-1/k$. Since B is a self-adjoint operator, $\pi^{-1}\eta_\varepsilon(k,kt)$ coincides with the derivative of the spectral measure of the operator B corresponding to the element $H_\varepsilon^{-1/2}v$. According to Lemma 2.1, the latter observation implies that

$$\int_{-\infty}^{\infty} \frac{\eta_\varepsilon(k,kt)}{(1+k^2)}dk \leq \pi\Big((B^2+I)^{-1}H_\varepsilon^{-1/2}v, H_\varepsilon^{-1/2}v\Big),$$

which leads to

$$\int_{-\infty}^{\infty} \frac{\eta_\varepsilon(k,kt)}{(1+k^2)}dk \leq \pi\Big(B^{-1}H_\varepsilon^{-1/2}v, B^{-1}H_\varepsilon^{-1/2}v\Big) = \pi||B^{-1}H_\varepsilon^{-1/2}v||^2. \tag{16}$$

Our further arguments will be related to the estimate of the quantity in the right hand side of (16). We will show now that

$$\lim_{\varepsilon\to 0} ||B^{-1}H_\varepsilon^{-1/2}v||^2 \leq \int_{\mathbb{R}_+} \|V(x)\|^2 dx. \tag{17}$$

In order to do that we use the representation

$$B^{-1}H_\varepsilon^{-1/2} = H_\varepsilon^{1/2}T^{-1}, \tag{18}$$

where $T \subset T^*$ is the first order differential operator, defined by

$$T = -2i\frac{d}{dx} + tV, \qquad D(T) = D(H_\varepsilon^{1/2}).$$

The representation (18) is a simple consequence of the fact that $B = H_\varepsilon^{-1/2}TH_\varepsilon^{-1/2}$.

Let us discuss the basic properties of the operator T. Since it is a first order differential operator, one can derive an explicit formula for the resolvent of T. For that purpose, one needs to recall the theory of ordinary differential equations, which says that the equation

$$y' + p(t)y = f(t), \qquad y = y(t),\ t \in \mathbb{R},$$

is equivalent to the relation

$$\Big(e^{\int p\,dt}y\Big)' = e^{\int p\,dt}f.$$

Put differently,

$$y' + p(t)y = e^{-\int p\,dt}\Big(e^{\int p\,dt}y\Big)'.$$

This gives us an idea of how to handle the operator T. Let U_0 be the unitary operator of multiplication by the solution of the equation

$$U_0'(x) = \frac{it}{2}U_0(x)V(x), \qquad U_0(0) = I.$$

Then

$$T = -2iU_0^{-1}\Big[\frac{d}{dx}\Big]U_0, \qquad \text{and} \qquad T^{-1} = \frac{i}{2}U_0^{-1}\Big[\frac{d}{dx}\Big]^{-1}U_0.$$

Since $[\frac{d}{dx}]^{-1}$ is just the simple integration with respect to x and $U_0'\tau = \frac{i}{2}tU_0V\tau$,

$$\begin{aligned}\big[T^{-1}v\big](x) &= \frac{i}{2}U_0^{-1}(x)\int_0^x U_0(y)V(y)\tau dy = \\ &\frac{1}{t}U_0^{-1}(x)(U_0(x)-I)\tau = \frac{1}{t}(I - U_0^{-1}(x))\tau.\end{aligned} \tag{19}$$

Note that due to the condition (6), the function $T^{-1}v$ is compactly supported, which leaves no doubt about the relation $v \in D(T^{-1})$. Combining (18) with (19), we conclude that

$$\lim_{\varepsilon\to 0} ||B^{-1}H_\varepsilon^{-1/2}v||^2 = \lim_{\varepsilon\to 0} ||H_\varepsilon^{1/2}T^{-1}v||^2 = \int_{\mathbb{R}_+} \|V(x)U_0^{-1}(x)\tau\|^2 dx.$$

Thus, (17) is established. The relations (16), (17) lead to the inequality

$$J(V) \leq \pi^2 \int_{\mathbb{R}_+} \|V(x)\|^2 dx.$$

where the quantity $J(V)$ from (13). However, we can say more:

Lemma 4.1. *Let $T > 0$. Let V be a potential of the class $\mathfrak{V}$ such that*

$$V(x)\tau = 0, \qquad \textit{for all} \quad x < T. \tag{20}$$

Then

$$J(V) \leq \pi^2 \int_T^\infty \|V(x)\|^2 dx. \tag{21}$$

5. Approximations of potentials and spectral measures

Proposition 5.1. *Let $T > 0$. Let $\tilde{V}$ be the potential*

$$\tilde{V}(x) = V(x) - (\cdot,\tau)V(x)\tau - (\cdot, V(x)\tau)\tau + (V(x)\tau,\tau)(\cdot,\tau)\tau, \qquad \textit{for all} \quad x < T. \tag{22}$$

and

$$\tilde{V}(x) = V(x), \qquad \textit{for all} \quad x > T. \tag{23}$$

Then

$$\Big(H - z\Big)^{-1} - \Big(-\frac{d^2}{dx^2} + \alpha\tilde{V} - z\Big)^{-1} \in \mathfrak{S}_1 \tag{24}$$

is a trace class operator for any z with $\operatorname{Im} z > 0$.

Proof. Using the Hilbert identity, we obtain

$$\Big(H - z\Big)^{-1} - \Big(-\frac{d^2}{dx^2} + \alpha\tilde{V} - z\Big)^{-1} = \alpha\Big(H - z\Big)^{-1}(\tilde{V} - V)\Big(-\frac{d^2}{dx^2} + \alpha\tilde{V} - z\Big)^{-1}.$$

Consequently, it is sufficient to prove that

$$\Gamma := \Big(-\frac{d^2}{dx^2} - z\Big)^{-1}(\tilde{V} - V)\Big(-\frac{d^2}{dx^2} - z\Big)^{-1} \in \mathfrak{S}_1.$$

Observe now that $\tilde{V}(x) - V(x)$ is a finite rank operator of the form

$$\tilde{V}(x) - V(x) = w_1(x)(\cdot, e_1(x))e_1(x) + w_2(x)(\cdot, e_2(x))e_2(x),,$$

where $w_j \in L^1(\mathbb{R}_+)$ are real valued compactly supported functions and $e_j(x)$ are unit vectors in $\mathfrak{H}$. Since $(-\frac{d^2}{dx^2} - z)^{-1}$ is an integral operator whose integral kernel $r(x,y)$ satisfies

$$\sup_x \int_0^\infty |r(x,y)|^2 dy + \sup_y \int_0^\infty |r(x,y)|^2 dx < \infty,$$

the operators $G_j(z)$ defined by

$$\big[G_j(z)u\big](x) = \int_0^\infty |w_j(x)|^{1/2}\big(r(x,y)u(y), e_j(x)\big)e_j dy$$

are Hilbert-Schmidt operators. It remains to note that

$$\Gamma = G_1^*(\bar{z})\Omega_1 G_1(z) + G_2^*(\bar{z})\Omega_2 G_2(z)$$

where Ω_j are bounded. $\square$

According to one of the fundamental theorems of Scattering Theory, we can now state the following result.

Proposition 5.2. *Let $\tilde V$ be defined as in (22). Then the absolutely continuous parts of the operators H and $-\frac{d^2}{dx^2}+\alpha\tilde V$ are unitary equivalent.*

The latter proposition allows one to assume that V has the following properties:
1) $V(x)\tau = 0$ for all $x < T$.
2) the number T is so large that $\int_T^\infty \|V(x)\|^2dx < \delta$ is small.

Let us use the inequality (12) and employ Proposition 3.4 with

$$F(\lambda) = (2\pi)^{-1/2}F_0(\sqrt{\lambda})\lambda^{1/4} \qquad \text{and} \qquad \Psi(\lambda) = \frac{2\mathrm{Im}((H-z)^{-1}h_k, h_k)}{\sqrt{\lambda}}.$$

According to Lemma 4.1, we obtain the following result.

Theorem 5.1. *Let $0 < a < b < \infty$, let $0 < \alpha_1 < \alpha_2 < \infty$ and let $T > 1$. For any $\varepsilon > 0$ there is a number $\delta > 0$ such that for any potential V of the class $\mathfrak{V}$ having the properties*

$$1) \quad V(x)\tau = 0 \qquad \text{for all} \quad x < T, \qquad \text{and} \quad 2) \quad \int_T^\infty \|V(x)\|^2dx < \delta,$$

the derivative $\mu'(\lambda) = \mu'(\lambda,\alpha)$ of the spectral measure satisfies the inequality

$$\mu'(\lambda,\alpha) \geq (4\pi)^{-1}|F_0(\sqrt{\lambda})|^2\lambda^{1/2}, \qquad \text{for all} \quad (\lambda,\alpha)\in\Omega,$$

where the measurable set Ω obeys

$$\Big|[a,b]\times[\alpha_1,\alpha_2]\setminus\Omega\Big| \leq \varepsilon.$$

The proof of the next statement is left to the reader as an exercise.

Proposition 5.3. *Let V be a measurable operator-valued function obeying*

$$\int_{\mathbb{R}_+} \|V(x)\|^2dx < \infty.$$

Assume that

$$V(x)\tau = 0, \qquad \text{for all} \quad x < T, \tag{25}$$

where $T > 0$ is a fixed number. Then there is a sequence of compactly supported operator-valued functions $V_n \in \mathfrak{V}$ having the following three properties:
1)

$$V_n(x)\tau = 0, \qquad \text{for all} \quad x < T,$$

2)

$$\int_T^\infty \|V_n(x)\|^2dx \leq 2\int_T^\infty \|V(x)\|^2dx,$$

and
3)

$$\int_0^K \|\big(V_n(x)-V(x)\big)u(x)\|^2dx \to 0, \qquad \text{as} \quad n\to\infty, \qquad \text{for any } u\in L^\infty(\mathbb{R}_+,\mathfrak{H}) \quad \text{and any } K>0.$$

Another statement, that we are going to use, deals with the spectral measures of operators whose potentials V_n approximate the function V.

Proposition 5.4. *Let $V \in L^2(\mathbb{R}_+, \mathfrak{H})$ and $V_n \in L^2(\mathbb{R}_+, \mathfrak{H})$. Let μ_n and μ be the spectral measures of the operators with potentials αV_n and αV, correspondingly. Assume that*

$$\int_0^K \|\big(V_n(x) - V(x)\big)u(x)\|^2 dx \to 0, \qquad \text{as} \quad n \to \infty, \qquad \text{for any } u \in L^\infty(\mathbb{R}_+, \mathfrak{H}) \quad \text{and any } K > 0.$$

Then

$$\mu_n \to \mu \qquad \text{weakly}, \quad \text{as} \quad n \to \infty, \quad \text{for all} \quad \alpha \in \mathbb{R}.$$

According to Proposition 3.2, the assertion below follows from Theorem 5.1 and Propositions 5.3 and 5.4.

Theorem 5.2. *Let $0 < a < b < \infty$, let $0 < \alpha_1 < \alpha_2 < \infty$ and let $T > 1$. For any $\varepsilon > 0$ there is a number $\delta > 0$ such that for any potential $V \in L^2(\mathbb{R}_+, \mathfrak{H})$ having the properties*

$$1) \quad V(x)\tau = 0 \qquad \text{for all} \quad x < T, \qquad \text{and} \quad 2) \quad \int_T^\infty \|V(x)\|^2 dx < \delta,$$

the derivative $\mu'(\lambda) = \mu'(\lambda, \alpha)$ of the spectral measure is positive

$$\mu'(\lambda, \alpha) > 0, \qquad \text{for all} \quad (\lambda, \alpha) \in \Omega, \tag{26}$$

where the measurable set Ω obeys

$$\big|[a, b] \times [\alpha_1, \alpha_2] \setminus \Omega\big| \leq \varepsilon.$$

Let $E_\alpha(\cdot)$ be the operator-valued spectral measure of H. Let also

$$\Omega_\alpha = \{\lambda \in [a, b] : \quad (\lambda, \alpha) \in \Omega\}$$

be the cross-section of Ω. One can conclude from the inequality (26) that, for any measurable subset $X \subset [a, b]$, the condition $E_\alpha(X) = 0$ implies the relation

$$\big|\Omega_\alpha \cap X\big| = 0.$$

Using the unitary equivalence claimed by Proposition 5.2, we obtain

Theorem 5.3. *Let $0 < a < b < \infty$, let $0 < \alpha_1 < \alpha_2 < \infty$. Assume that $V \in L^2(\mathbb{R}_+, \mathfrak{H})$. Then for any $\varepsilon > 0$, there is a measurable set $\Omega(\varepsilon) \subset [a, b] \times [\alpha_1, \alpha_2]$ obeying*

$$\big|[a, b] \times [\alpha_1, \alpha_2] \setminus \Omega(\varepsilon)\big| \leq \varepsilon,$$

such that, for any Borel set $X \subset [a, b]$ and the cross-section $\Omega_\alpha(\varepsilon)$ defined by

$$\Omega_\alpha(\varepsilon) = \{\lambda \in [a, b] : \quad (\lambda, \alpha) \in \Omega(\varepsilon)\},$$

the condition $E_\alpha(X) = 0$ implies the equality

$$\big|\Omega_\alpha(\varepsilon) \cap X\big| = 0.$$

Take now a monotonically decreasing sequence ε_n converging to 0, as $n \to \infty$, and set

$$\tilde{\Omega} = \bigcup_{n=1}^\infty \Omega(\varepsilon_n).$$

Obviously, $\tilde{\Omega}$ is a subset of full measure in $[a, b] \times [\alpha_1, \alpha_2]$. Consequently,

$$\tilde{\Omega}_\alpha = \{\lambda \in [a, b] : \quad (\lambda, \alpha) \in \tilde{\Omega}\}$$

is a subset of full measure in $[a, b]$ for almost every $\alpha \in [\alpha_1, \alpha_2]$.

Take now an arbitrary Borel subset $X \subset [a, b]$. If $|X \cap \tilde{\Omega}_\alpha| > 0$ then there is an integer number n for which

$$\big|\Omega_\alpha(\varepsilon_n) \cap X\big| > 0.$$

The latter condition implies that $E_\alpha(X) \neq 0$. Thus, the essential support of the absolutely continuous spectrum of H contains the interval $[a, b]$ for all α such that

$$|\tilde{\Omega}_\alpha| = b - a. \tag{27}$$

It remains to note that (27) holds for almost every $\alpha \in [\alpha_1, \alpha_2]$.

This completes the proof of Theorem 1.1

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