MULTIVARIATE DICKMAN DISTRIBUTION AND ITS APPLICATIONS

by

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ABSTRACT

XINGNAN ZHANG. Multivariate Dickman distribution and its applications.
(Under the direction of DR. MICHAEL GRABCHAK)

In this dissertation, we develop multivariate Dickman distribution and explore its properties. In addition, we utilize the Dickman distribution to model the small jumps within a broad class of Lévy processes, building upon the work presented in [1] for the univariate case. Our central theorem establishes that the limit distribution of an appropriately transformed truncated Lévy process with finite variation exhibits a Dickman-type Lévy measure. We also provide equivalent conditions to further characterize this result. Drawing inspiration from this, we partition the Lévy process into small and large jumps. Small jumps are effectively modeled by the Dickman distribution, while the remaining large jumps follow a compound Poisson distribution. This approach enables us to simulate Lévy processes within the p-temper α-stable class. Further, we extend our findings to Ornstein-Uhlenbeck (OU) processes, which have extensive applications in finance and economics, including interest rate modeling, option pricing, commodity pricing, and risk management. Our investigation encompasses two scenarios: the truncated OU process and the OU process driven by a truncated Lévy process. In general, employing the same transformation outlined in our main theorem, we observe that the limit distribution of the truncated OU process aligns with a Dickman-type Lévy measure. Notably, for the OU process with a truncated driving process, the limit distribution remains consistent with that of the OU process with a truncated driving process having a Dickman-type Lévy measure.

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CHAPTER 1: INTRODUCTION

The Dickman function, first introduced by Karl Dickman, satisfies the delay differential equation $x\mathrm{D}'(x) + \mathrm{D}(x-1) = 0$ with the initial condition $\mathrm{D}(x) = 1$ for any $x \in [0,1]$. Then, the Dickman distribution can be defined as $F(x) = 1 - \mathrm{D}(x)$. Differently, Penrose and Wade [2] studied the Dickman distribution using random variables as the following:

Definition 1. *A random variable X has the **Dickman distribution** if*

$$X = U_1 + U_1U_2 + U_1U_2U_3 + \cdots \tag{1.1}$$

where $U_i \overset{\text{iid}}{\sim} U(0,1)$ and $U(0,1)$ stands for uniform distribution on $(0,1)$.

Remark 1. *There is also an equivalent definition:*

$$X \overset{d}{=} U(1+X) \tag{1.2}$$

where $U \sim U(0,1)$ and U and X are independent on the right-hand side. The Dickman distribution is the only one with this property. If we factor out U_1 in equation (1.1), we can get the form of equation (1.2); if we recursively substitute X in Equation (1.2), then we can have the form of Equation (1.1).

They also introduced the generalized Dickman distribution, studied its probabilistic properties, and provided the Laplace transform. Let us introduce the definition in their work.

Definition 2. *A random variable X has a **generalized Dickman distribution** with* *parameter $\theta > 0$ if*

$$X = U_1^{\frac{1}{\theta}} + (U_1U_2)^{\frac{1}{\theta}} + (U_1U_2U_3)^{\frac{1}{\theta}} + \cdots \overset{d}{=} U^{\frac{1}{\theta}}(1+X) \tag{1.3}$$

The Dickman distribution has been extensively studied in the literature. It can be applied in various domains such as number of species, random graphs, small jumps observed in Lévy processes, and Hoare's quickselect algorithm, see [3], [4], [1], [2], [5], and [6].

One probabilistic property of the Dickman distribution is that it is infinitely divisible. See Proposition 3 in [2].

Definition 3. *A probability measure μ on $\mathbb{R}^d$ is* ***infinitely divisible****, if $\forall n \in \mathbb{N}$, there exists μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^n$, where μ_n^n stands for the n-fold convolution.*

Remark 2. *The equivalent way to say the infinitely divisible distribution is that a distribution F is infinitely divisible if, $\forall n \in \mathbb{N}$, there exists n i.i.d. random variables $X_1, X_2, \cdots, X_n$ such that $X_1 + X_2 + \cdots + X_n \sim F$.*

The characteristic function can uniquely describe the infinitely divisible distribution, which makes it the most important tool for proofs in our work.

Definition 4. *The* ***characteristic function*** *of an infinitely divisible distribution μ on $\mathbb{R}^d$ has the form*

$$\begin{aligned}\hat{\mu}(z) = \mathrm{E}(e^{i\langle z,x\rangle}) = \exp\Big\{&-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle \\ &+ \int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle \mathbb{I}_{|x|\le 1}(x))\nu(\mathrm{d}x)\Big\}, z \in \mathbb{R}^d\end{aligned} \tag{1.4}$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$, ν is a Lévy measure satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty \tag{1.5}$$

We write it as $X \sim \mathrm{ID}(A, \nu, \gamma)$ where (A, ν, γ) is called the generating triplet of the distribution of X. The A is the Gaussian covariance matrix, and the ν is the Lévy measure of the distribution.

Remark 3. *When ν satisfies $\int_{|x|\leq 1} |x|\nu(\mathrm{d}x) < \infty$, we have the following form of the characteristic function:*

$$\hat{\mu}(z) = \exp\left\{-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_0, z\rangle \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(\mathrm{d}x)\right\}$$

where $\gamma_0 = \gamma - \int_{\mathbb{R}^d} x\mathbb{I}_{|x|\leq 1}(x)\nu(\mathrm{d}x)$ is called the drift. In this case, we write it as $X \sim \mathrm{ID}_0(A, \nu, \gamma_0)$.

Remark 4. *When we write $X \sim \mathrm{ID}_0(0, \nu, 0)$ in this paper, we mean $A = 0$ and $\gamma_0 = 0$.*

Remark 5. *The characteristic function of the generalized Dickman distribution on $\mathbb{R}$ has the following form*

$$\mathrm{E}(e^{izx}) = \exp\left\{\theta \int_{\mathbb{R}} (e^{izx} - 1)\mathbb{I}_{(0,1]}(x)x^{-1}\mathrm{d}x\right\} \tag{1.6}$$

Lévy process is closely related to the infinitely divisible distribution. For a Lévy process $\{X_t : t \geq 0\}$, when we fix t, X_t is a random variable and its distribution is infinitely divisible. Since the Lévy process has been studied in the literature ever since the 1940's, we will not strengthen its importance anymore, but we will give the definition.

Definition 5. *A cadlag stochastic process $\{X_t : t \geq 0\}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ such that $X_0 = 0$ is called a* ***Lévy process*** *if*

1. ***Independent increments:*** *for every increasing sequence of times $t_0, t_1, \cdots,$ t_n, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.*

2. ***Stationary increments:*** *the distribution of $X_{t+h} - X_t$ does not depend on t, i.e. $X_{t+h} - X_t \stackrel{d}{=} X_h$.*

3. ***Stochastic continuity:*** $\forall \epsilon > 0$, $\lim_{h \to 0} \mathrm{P}(|X_{t+h} - X_t| \geq \epsilon) = 0$.

Remark 6. *The term "cadlag" in the Definition 5 means right continuous with left limits, i.e. $\lim_{s \downarrow t} X(s) = X(t)$ and $\lim_{s \uparrow t} X(s)$ exists.*

For the distribution of the Lévy process, there is a Lévy measure associated with the distribution.

Definition 6. *Let $\{X_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ and $A \in \mathscr{B}(\mathbb{R}^d)$. The **Lévy measure** ν of the distribution of X is defined by:*

$$\nu(A) = \mathrm{E}(\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\})$$

i.e. $\nu(A)$ is the expected number, per unit time, of jumps whose size belongs to A.

For the Lévy process having the Dickman distribution, there must be a Lévy measure associated with this distribution. To see what this Lévy measure looks like, let us start with the next lemma and gradually process the definition of the Dickman-type Lévy measure.

Lemma 1. *Let D be a σ-finite measure on $(0, \infty)$. If, for $B \in \mathscr{B}(\mathbb{R} \setminus \{0\})$, we have, $\forall a \in \mathbb{R}$, $\mathrm{D}(aB) = \mathrm{D}(B)$, then for all $c \in \mathbb{R}$, $\mathrm{D}(\{c\}) = 0$.*

Proof of Lemma 1. For the sake of contradiction, we assume that there exists $c \in \mathbb{R}$ *s.t.* $\mathrm{D}(\{c\}) > 0$. Then, for any $a \in \mathbb{R}$, $\mathrm{D}(\{c\}) = \mathrm{D}(a\{c\}) = \mathrm{D}(a^2\{c\}) = ... > 0$, i.e. $\mathrm{D}(\{c\}) = \mathrm{D}(\{ac\}) = \mathrm{D}(\{a^2c\}) = ... > 0$. Note that the set $\{a^n c : n \in \mathbb{N}, a \in \mathbb{R}\}$ is uncountable, by Theorem 10.2 (iv) of [7], this is a contradiction. ■

Proposition 1. *Let D be a σ-finite measure on $(0, \infty)$. If, for $B \in \mathscr{B}(\mathbb{R} \setminus \{0\})$, we have, $\forall a \in \mathbb{R}$, $\mathrm{D}(aB) = \mathrm{D}(B)$, then $\mathrm{D}(B) = \int_0^\infty \theta \mathbb{I}_B(x) x^{-1} \mathrm{d}x$, for some $\theta > 0$.*

Proof of Proposition 1. Define $\theta = \mathrm{D}((1,e])$, since $\mathrm{D}(aB) = \mathrm{D}(B)\ \forall a \in \mathbb{R}$, then, for any $n \in \mathbb{N}$,

$$\mathrm{D}((1,e]) = \mathrm{D}((e,e^2]) = ... = \mathrm{D}((e^{n-1},e^n])$$

Thus,

$$\begin{aligned} n\theta &= n\mathrm{D}((1,e]) \\ &= \mathrm{D}((1,e]) + \mathrm{D}((e,e^2]) + ... + \mathrm{D}((e^{n-1},e^n]) \\ &= \mathrm{D}((1,e] \cup (e,e^2] \cup ... \cup (e^{n-1},e^n]) \\ &= \mathrm{D}((1,e^n]) \end{aligned}$$

Note that, $n = \ln e^n - \ln 1$, then

$$\begin{aligned} \mathrm{D}((1,e^n]) &= (\ln e^n - \ln 1)\theta \\ &= \int_0^\infty \theta \mathbb{I}_{(1,e^n]}(x) x^{-1} \mathrm{d}x \end{aligned}$$

Similarly, for any $m \in \mathbb{N}$,

$$\mathrm{D}(\left(1, e^{\frac{1}{m}}\right]) = \mathrm{D}(\left(e^{\frac{1}{m}}, e^{\frac{2}{m}}\right]) = ... = \mathrm{D}(\left(e^{\frac{m-1}{m}}, e\right])$$

then

$$m(\left(1, e^{\frac{1}{m}}\right]) = \mathrm{D}((1,e]) = \theta$$

Thus,

$$\begin{aligned} \mathrm{D}(\Big(1, e^{\frac{1}{m}}\Big]) &= \frac{\theta}{m} \\ &= \theta(\ln e^{\frac{1}{m}} - \ln 1) \\ &= \int_0^\infty \theta \mathbb{I}_{(1,e^{\frac{1}{m}}]}(x) x^{-1} \mathrm{d}x \end{aligned}$$

Hence,

$$\theta \frac{n}{m} = n\mathrm{D}((1, e^{\frac{1}{m}}]) = \mathrm{D}((1, e^{\frac{n}{m}}])$$

So, for any rational number $q \in \mathbb{Q}$, $\theta q = \mathrm{D}((1, e^q])$

1. For any $a \in \mathbb{R}$ *s.t.* $a > 1$, $\ln a \in \mathbb{R}$ and $\ln a > 0$, there exists an increasing sequence $\{q_k\}$ of rational numbers *s.t.* $\lim_{k\to\infty} q_k = \ln a$, then

$$\begin{aligned} \theta \ln a &= \theta \lim_{k\to\infty} q_k \\ &= \lim_{k\to\infty} \theta q_k \\ &= \lim_{k\to\infty} \mathrm{D}((1, e^{q_k}]) \end{aligned}$$

Since $\{q_k\}$ is increasing, then $\{(1, e^{q_k}]\}$ is increasing, by continuity of measure

$$\begin{aligned} \lim_{k\to\infty} \mathrm{D}((1, e^{q_k}]) &= \mathrm{D}\left(\bigcup_{k=1}^{\infty} (1, e^{q_k}]\right) \\ &= \mathrm{D}((1, e^{\lim_{k\to\infty} q_k}]) \\ &= \mathrm{D}((1, e^{\ln a})) \\ &= \mathrm{D}((1, e^{\ln a}]) \\ &= \mathrm{D}((1, a]) \end{aligned}$$

So,

$$\begin{aligned}\mathrm{D}((1,a]) &= \theta \ln a \\ &= \int_0^\infty \theta \mathbb{I}_{(1,a]}(x) x^{-1} \mathrm{d}x\end{aligned}$$

2. For any $a \in \mathbb{R}$ *s.t.* $0 < a < 1$, then $\ln a \in \mathbb{R}$ and $\ln a < 0$

$$\begin{aligned}\mathrm{D}((a,1]) &= \mathrm{D}((e^{\ln a}, 1]) \\ &= \mathrm{D}(e^{\ln a}(1, e^{-\ln a}]) \\ &= \mathrm{D}((1, e^{-\ln a}]) \\ &= \theta(-\ln a) \\ &= \int_0^\infty \theta \mathbb{I}_{(a,1]}(x) x^{-1} \mathrm{d}x\end{aligned}$$

3. For any $0 < a < 1 < b$,

$$\begin{aligned}\mathrm{D}((a,b]) &= \mathrm{D}((a,1] \cup (1,b]) \\ &= \mathrm{D}((a,1]) + \mathrm{D}((1,b]) \\ &= \int_0^\infty \theta \mathbb{I}_{(a,1]}(x) x^{-1} \mathrm{d}x + \int_0^\infty \theta \mathbb{I}_{(1,b]}(x) x^{-1} \mathrm{d}x \\ &= \int_0^\infty \theta \mathbb{I}_{(a,b]}(x) x^{-1} \mathrm{d}x\end{aligned}$$

From above we can conclude that, for any $a < b$, $\mathrm{D}((a,b]) = \int_0^\infty \theta \mathbb{I}_{(a,b]}(x) x^{-1} \mathrm{d}x$.

Let $\mathscr{A} = \{(a,b] : a \in \mathbb{R}, b \in \mathbb{R}\}$, by Proposition 1.15 of [8], we have $\sigma(\mathscr{A}) \subset \mathscr{B}(\mathbb{R})$, since $\sigma(\mathscr{A})$ is the collection of all Borel sets, it follows that $\mathrm{D}(B) = \int_0^\infty \theta \mathbb{I}_B(x) x^{-1} \mathrm{d}x$, for all $B \in \mathscr{B}(\mathbb{R} \setminus \{0\})$. ■

Definition 7. *We call* D_θ^a *with parameter* θ *and* a *the* ***generalized Dickman-type Lévy measure on*** $\mathbb{R}$, *if* $\forall B \in \mathscr{B}(\mathbb{R})$,

$$\mathrm{D}_\theta^a(B) = \int_0^a \theta \mathbb{I}_B(x) x^{-1} \mathrm{d}x, \ \theta > 0 \tag{1.7}$$

Remark 7. *From now on, when we use the notation* D, *we refer to* $\theta = 1$ *and* $a = 1$ *in the Definition 7; if we use* D^ϵ, *we mean* $\theta = 1$ *and* $a = \epsilon$.

As mentioned at the beginning, Dickman distribution can be applied to approximate the small jumps of the Lévy process. Here, small jumps mean those jumps in the process whose magnitudes are capped by a constant number ϵ, i.e. we truncate the whole Lévy process by this ϵ. Associated with this truncated Lévy process, there is a Lévy measure. Let ν be a Lévy measure on $(0, \infty)$. For all $\epsilon \in (0, 1]$, define

$$\nu^\epsilon(A) = \int_{\mathbb{R}} \mathbb{I}_A(x) \mathbb{I}_{(0,\epsilon]}(x) \nu(\mathrm{d}x) \tag{1.8}$$

where $A \in \mathscr{B}((0, \infty))$, and

$$\mathrm{D}^\epsilon(B) = \int_0^1 \mathbb{I}_B(x) \mathbb{I}_{(0,\epsilon]}(x) \mathrm{D}(\mathrm{d}x) \tag{1.9}$$

where $B \in \mathscr{B}((0, 1])$.

From Proposition 3.(i) in [2], we know the Lévy process having the Dickman distribution is a pure jump process, i.e. $X \sim \mathrm{ID}_0(0, \nu, 0)$.

Proposition 2. *let* $\{X_t^\epsilon\}$ *be the pure jump Lévy process consisting of jumps of* $\{X_t\}$ *not exceeding* ϵ *with Lévy measure* ν^ϵ. *Assume* $\int_{x \le 1} x \nu^\epsilon(\mathrm{d}x) < \infty$. *If, for all* $\epsilon \in (0, 1]$, $\epsilon^{-1} X_t^\epsilon \overset{d}{=} X_t^1$, *then,* $\nu^\epsilon(B) = \mathrm{D}_\theta^\epsilon(B)$, *for all* $B \in \mathscr{B}(\mathbb{R} \setminus \{0\})$

Proof of Proposition 2. If $\int_{x \le 1} x \nu^\epsilon(\mathrm{d}x) < \infty$, then by equation (8.7) in [8] we have the following characteristic function:

$$\hat{\mu}_{X_t^\epsilon}(z) = \exp\left\{t\int_{\mathbb{R}}(e^{izx}-1)\nu^\epsilon(\mathrm{d}x)\right\}$$

then,

$$\hat{\mu}_{X_1^\epsilon}(z) = \exp\left\{\int_{\mathbb{R}}(e^{izx}-1)\nu^\epsilon(\mathrm{d}x)\right\}$$

$$\begin{aligned}\hat{\mu}_{\epsilon^{-1}X_1^\epsilon}(z) &= \hat{\mu}_{X_1^\epsilon}(\frac{z}{\epsilon}) \\ &= \exp\left\{\int_{\mathbb{R}}(e^{i\frac{z}{\epsilon}x}-1)\nu^\epsilon(\mathrm{d}x)\right\}\end{aligned}$$

Let M be the Lévy measure of $\epsilon^{-1}X_1^\epsilon$, then $\mathrm{M}(B) = \int_{\mathbb{R}} \mathbb{I}_B(\frac{x}{\epsilon})\nu^\epsilon(\mathrm{d}x) = \int_{\mathbb{R}} \mathbb{I}_{\epsilon B}(x)\nu^\epsilon(\mathrm{d}x) = \nu^\epsilon(\epsilon B)$ where $\epsilon B = \{\epsilon y : y \in B\}$. Since $\epsilon^{-1}X_t^\epsilon \stackrel{d}{=} X_t^1$, then, by Theorem 7.10(iii) of [8], $P_{\epsilon^{-1}X_t^\epsilon} = P_{X_t^1}$, so $\hat{\mu}_{\epsilon^{-1}X_t^\epsilon}(z) = \hat{\mu}_{X_t^1}(z)$. By Theorem 8.1 of [8], the characteristic function is unique, hence $\mathrm{M}(B) = \nu^1(B)$, i.e. $\nu^\epsilon(\epsilon B) = \nu^1(B)$. Note that,

$$\begin{aligned}\nu^\epsilon(\epsilon B) &= \int_{\mathbb{R}} \mathbb{I}_{\epsilon B}(x)\mathbb{I}_{(0,\epsilon]}(x)\nu(\mathrm{d}x) \\ &= \int_{\mathbb{R}} \mathbb{I}_{\epsilon B}(x)\mathbb{I}_{\epsilon(0,1]}(x)\nu(\mathrm{d}x) \\ &= \int_{\mathbb{R}} \mathbb{I}_{\epsilon(B\cap(0,1])}(x)\nu(\mathrm{d}x) \\ &= \nu(\epsilon(B\cap(0,1]))\end{aligned}$$

Similarly, $\nu^1(B) = \nu((B\cap(0,1]))$. So, we have $\nu(\epsilon(B\cap(0,1])) = \nu((B\cap(0,1]))$. By Proposition 1, we have $\nu = \mathrm{D}_\theta$. Then, we conclude that $\nu^\epsilon = \mathrm{D}_\theta^\epsilon$. ■

CHAPTER 2: Multivariate Dickman Distribution

In this section, we extend the univariate Dickman distribution to the multivariate case. We also study its properties and develop several limit theorems. First, we introduce polar decomposition, which we will use soon.

Suppose $B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$, then the **polar decomposition** of a Lévy measure ν has the following form:

$$\nu(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_B(r\xi)\nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi) \tag{2.1}$$

where $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the $(d-1)$-sphere; σ is a finite measure on $\mathbb{S}^{d-1}$; ν_ξ is a measure on $(0, \infty)$ which depends on ξ. See Lemma 2.1 in [9] for details.

Now, we introduce some notation. Define:

$$\mathbb{R}^+ = \{a \in \mathbb{R} : a > 0\} \tag{2.2}$$

Let $C \in \mathscr{B}(\mathbb{S}^{d-1})$ and $a, b \in \mathbb{R}^+$, we define the following cross product set:

$$(a, b]C = \{x \in \mathbb{R}^d : |x| \in (a, b], \frac{x}{|x|} \in C\} \tag{2.3}$$

In Bhattacharjee [10], they introduced a generalized multivariate Dickman distribution.

Definition 8. *A random variable X follows a **multivariate Dickman distribution** if*

$$X \stackrel{d}{=} U^{\frac{1}{\theta}}(V + X) \tag{2.4}$$

where $\theta > 0$, $U \sim U([0,1])$, $V \sim \dfrac{\sigma}{\sigma(\mathbb{S}^{d-1})}$, *and* U, V, X *are independent on the right-hand side.*

From Theorem 5 in [11], we know the Lévy process having the multivariate Dickman distribution is a pure jump process with finite variation. Let us recall the definition of finite variation.

Definition 9. *We say the Lévy process* $\{X_t\}$ *are of* ***finite variation****, if*

$$V_t(X) = \sup_{\Delta} \sum_{j=1}^{n} |X_{s_j} - X_{s_{j-1}}| < \infty \tag{2.5}$$

where Δ *is any partition of* $(0,t]$*, i.e.* $0 = s_0 \leq s_1 \leq s_2 \leq \cdots \leq s_n = t$

Remark 8. *If* $\{X_t\}$ *is a Lévy process with finite variation, then* $A = 0$ *and* $\int_{\mathbb{R}^d} (1 \wedge |x|)\nu(\mathrm{d}x) < \infty$*, so the characteristic function of* X_1 *can be written as*

$$\hat{\mu}(z) = \mathrm{E}(e^{i\langle z,x\rangle}) = \exp\left\{ i\langle \gamma_0, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(\mathrm{d}x) \right\}, z \in \mathbb{R}^d \tag{2.6}$$

where $\gamma_0 = \gamma - \int_{\mathbb{R}^d} x\mathbb{I}_{|x|\leq 1}(x)\nu(\mathrm{d}x)$*. And we denote it as* $X \sim \mathrm{ID}_0(0, \nu, \gamma_0)$*.*

The same as the univariate case, the following will lead us to the definition of the multivariate Dickman-type Lévy measure.

Proposition 3. *Let* D *be a* σ*-finite measure on* $\mathbb{R}^d \setminus \{0\}$*. If,* $\forall a \in \mathbb{R}^+$ *and* $\forall B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$*,* $\mathrm{D}(aB) = \mathrm{D}(B)$*, then there exists a finite measure* σ *defined on* $\mathbb{S}^{d-1}$*, s.t.*

$$\mathrm{D}(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_B(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi) \tag{2.7}$$

Proof of Proposition 3. For any $C \in \mathscr{B}(\mathbb{S}^{d-1})$, define $\sigma(C) = \mathrm{D}((1,e]C)$. For any $n \in \mathbb{N}$, we have

$$\mathrm{D}((1,e]C) = \mathrm{D}((e,e^2]C) = \ldots = \mathrm{D}((e^{n-1},e^n]C)$$

then

$$n\sigma(C) = \mathrm{D}((1,e]C) + \mathrm{D}((e,e^2]C) + ... + \mathrm{D}((e^{n-1},e^n]C) = \mathrm{D}((1,e^n]C)$$

Note that $n = \ln e^n - \ln 1$, thus

$$\begin{aligned}
\mathrm{D}((1,e^n]C) &= \sigma(C)(\ln e^n - \ln 1) \\
&= \sigma(C)\int_0^\infty \mathbb{I}_{(1,e^n]}(r)r^{-1}\mathrm{d}r \\
&= \int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbb{I}_{(1,e^n]C}(r\xi)r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)
\end{aligned}$$

Similarly, for any $m \in \mathbb{N}$,

$$\mathrm{D}((1,e^{\frac{1}{m}}]C) = \mathrm{D}((e^{\frac{1}{m}},e^{\frac{2}{m}}]C) = ... = \mathrm{D}((e^{\frac{m-1}{m}},e]C)$$

then

$$\begin{aligned}
m\mathrm{D}((1,e^{\frac{1}{m}}]C) &= \mathrm{D}((1,e^{\frac{1}{m}}]C) + \mathrm{D}((e^{\frac{1}{m}},e^{\frac{2}{m}}]C) + ... + \mathrm{D}((e^{\frac{m-1}{m}},e]C) \\
&= \mathrm{D}((1,e]C) \\
&= \sigma(C)
\end{aligned}$$

thus

$$\begin{aligned}
\mathrm{D}((1,e^{\frac{1}{m}}]C) &= \frac{1}{m}\sigma(C) \\
&= (\ln e^{\frac{1}{m}} - \ln 1)\sigma(C) \\
&= \int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbb{I}_{C(1,e^{\frac{1}{m}}]}(r\xi)r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)
\end{aligned}$$

Hence, we have

$$\begin{aligned} \mathrm{D}((1, e^{\frac{n}{m}}]C) &= \frac{n}{m}\sigma(C) \\ &= (\ln e^{\frac{n}{m}} - \ln 1)\sigma(C) \\ &= \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_{C(1,e^{\frac{n}{m}}]}(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi) \end{aligned}$$

Then, we conclude that, for any $q \in \mathbb{Q}$,

$$\mathrm{D}((1, e^q]C) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_{C(1,e^q]}(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi)$$

1. For any $a \in \mathbb{R}$ s.t. $a > 1$, we have $\ln a \in \mathbb{R}$, so there exists an increasing sequence $\{q_k\}$ of rational numbers s.t.

$$\lim_{k\to\infty} q_k = \ln a$$

Then

$$\begin{aligned} \sigma(C) \ln a &= \sigma(C) \lim_{k\to\infty} q_k \\ &= \lim_{k\to\infty} \sigma(C) q_k \\ &= \lim_{k\to\infty} \mathrm{D}((1, e^{q_k}]C) \end{aligned}$$

Since $\{q_k\}$ is increasing, then $\{(1, e^{q_k}])\}$ is increasing. Thus,

$$\begin{aligned} \lim_{k\to\infty} \mathrm{D}((1, e^{q_k}]C) &= \mathrm{D}\left(\bigcup_{k=1}^{\infty}((1, e^{q_k}]C)\right) \\ &= \mathrm{D}((1, e^{\lim_{k\to\infty} q_k}]C) \\ &= \mathrm{D}((1, e^{\ln a})C) \\ &= \mathrm{D}((1, a)C) \end{aligned}$$

The same as the proof of Lemma 1, since D is a σ-finite measure, then $\mathrm{D}(\{a\}) = 0$. Hence, we get

$$\lim_{k\to\infty} \mathrm{D}((1, e^{q_k}]C) = \mathrm{D}((1, a]C)$$

So

$$\begin{aligned}
\mathrm{D}((1, a]C) &= \sigma(C)\ln a \\
&= \sigma(C)(\ln a - \ln 1) \\
&= \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_{(1,a]C}(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi)
\end{aligned}$$

2. For any $a \in \mathbb{R}$ s.t. $0 < a < 1$,

$$\begin{aligned}
\mathrm{D}((a, 1]C) &= \mathrm{D}\left(a\left(1, \frac{1}{a}\right] C\right) \\
&= \mathrm{D}\left(\left(1, \frac{1}{a}\right] C\right) \\
&= \sigma(C)\ln \frac{1}{a} \\
&= \sigma(C)(\ln 1 - \ln a) \\
&= \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_{(a,1]C}(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi)
\end{aligned}$$

3. For any $0 < a < 1 < b$,

$$\begin{aligned}
\mathrm{D}((a, b]C) &= \mathrm{D}((a, 1]C) + \mathrm{D}((1, b]C) \\
&= \sigma(C)\ln \frac{1}{a} - \sigma(C)\ln \frac{1}{b} \\
&= \sigma(C)(\ln b - \ln a) \\
&= \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_{(a,b]C}(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi)
\end{aligned}$$

Then we conclude that, for any $0 < a < b$,

$$\mathrm{D}((a,b]C) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_{(a,b]C}(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi)$$

Let $\mathscr{A} = \{(a,b]C : C \in \mathscr{B}(\mathbb{R}^{d-1}), 0 \le a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$, then $\sigma(\mathscr{A})$ is the collection of all Borel sets in $\{x \in \mathbb{R}^d : |x| > 0\}$, by Proposition 1.15 of [8], it follows that

$$\mathrm{D}(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_B(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi), \text{ for all } B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$$

■

Definition 10. *We call the measure* D_σ *is the* ***multivariate Dickman Lévy measure with parameter*** σ*, if,* $\forall B \in \mathscr{B}(\mathbb{R}^d)$,

$$\mathrm{D}(B) = \int_{\mathbb{S}^{d-1}} \int_0^1 \mathbb{I}_B(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi) \tag{2.8}$$

Definition 11. *Let* $\epsilon > 0$ *be any constant. We call* $\mathrm{D}_\sigma^\epsilon$ *a* ***multivariate Dickman-type Lévy measure with parameters*** σ ***and*** ϵ*, if ,* $\forall B \in \mathscr{B}(\mathbb{R}^d)$,

$$\mathrm{D}^\epsilon(B) = \int_{\mathbb{S}^{d-1}} \int_0^\epsilon \mathbb{I}_B(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi) \tag{2.9}$$

Remark 9. *Let* $\{X_t\}$ *be the Lévy process with the multivariate Dickman-type Lévy measure* D^ϵ*, then the cummulant function of* X *is:*

$$C_\mu(z) = t\int_{\mathbb{R}} (e^{i\langle z,x\rangle} - 1)\mathrm{D}^\epsilon(\mathrm{d}x) = t\int_{\mathbb{S}^{d-1}} \int_0^\epsilon (e^{i\langle z,r\xi\rangle} - 1) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi) \tag{2.10}$$

Since

$$\begin{aligned} E(X_{jt}) &= (-i)\frac{\partial}{\partial z_j}C_\mu(z)\Big|_{z=0} \\ &= (-i)t\int_{\mathbb{S}^{d-1}}\int_0^\epsilon e^{i\langle z,r\xi\rangle}ir\xi_j r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\Big|_{z=0} \\ &= t\epsilon\int_{\mathbb{S}^{d-1}}\xi_j\sigma(\mathrm{d}\xi) \end{aligned}$$

then the mean vector is $E(X_t) = t\epsilon\int_{\mathbb{S}^{d-1}}\xi\sigma(\mathrm{d}\xi)$.

$$\begin{aligned} E(X_{jt}X_{kt}) &= (-i)^2\frac{\partial^2}{\partial z_j\partial z_k}C_\mu(z)\Big|_{z=0} \\ &= (-i)^2 t\int_{\mathbb{S}^{d-1}}\int_0^\epsilon e^{i\langle z,r\xi\rangle}(ir)^2\xi_j\xi_k r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\Big|_{z=0} \\ &= \frac{1}{2}t\epsilon^2\int_{\mathbb{S}^{d-1}}\xi_j\xi_k\sigma(\mathrm{d}\xi) \end{aligned}$$

So the covariance matrix is $Cov(X_t) = \frac{1}{2}t\epsilon^2\int_{\mathbb{S}^{d-1}}\xi\xi^T\sigma(\mathrm{d}\xi)$

Let $\{X_t\}$ be a pure jump Lévy process with Lévy measure ν, and consider $\{X_t^\epsilon\}$ as the truncated process with Lévy measure ν^ϵ as defined in Equation (1.8), but in $\mathbb{R}^d$ instead of in $\mathbb{R}$. So $\{X_t^\epsilon\}$ consists of jumps of $\{X_t\}$ bounded by ϵ. Assume $\int_{|x|\le 1}|x|\nu^\epsilon(\mathrm{d}x) < \infty$, then by equation (8.7) in [8] we have the following characteristic function:

$$\hat{\mu}_{X_t^\epsilon}(z) = \exp\left\{t\int_{\mathbb{R}^d}(e^{i\langle z,x\rangle}-1)\nu^\epsilon(\mathrm{d}x)\right\}$$

where ν is a Lévy measure on $\mathbb{R}^d$, and $\forall\epsilon > 0$, $\nu^\epsilon(B) = \int_{|x|\le\epsilon}\mathbb{I}_B(x)\nu(\mathrm{d}x)$, $B \in \mathscr{B}(\mathbb{R}^d)$. Next, consider the transformation $\epsilon^{-1}X^\epsilon$, then all the jumps will be bounded by 1. Let M^ϵ be the Lévy measure of $\epsilon^{-1}X_1^\epsilon$, then $\mathrm{M}^\epsilon(B) = \int_{\mathbb{R}^d}\mathbb{I}_B(\frac{x}{\epsilon})\nu^\epsilon(\mathrm{d}x) = \int_{\mathbb{R}^d}\mathbb{I}_{\epsilon B}(x)\nu^\epsilon(\mathrm{d}x) = \nu^\epsilon(\epsilon B)$ where $\epsilon B = \{\epsilon y : y \in B\}$. Furthermore, we can have $\int_B f(x)\mathrm{M}^\epsilon(\mathrm{d}x) = \int_{\epsilon B} f(\frac{x}{\epsilon})\nu^1(\mathrm{d}x)$ for all bounded and continuous function f. In this paper, we will always use the notation M^ϵ as the Lévy measure of this transformation.

Proposition 4. *Let $\{X_t^\epsilon\}$ be the pure jump truncated Lévy process consisting of jumps of $\{X_t\}$ bounded by ϵ with Lévy measure ν^ϵ. Assume $\int_{|x|\le 1} |x|\nu^\epsilon(\mathrm{d}x) < \infty$. If $\epsilon^{-1}X^\epsilon \overset{d}{=} X^1$, then $\nu^\epsilon(B) = \mathrm{D}^\epsilon(B)$ for all $B \in \mathscr{B}(\mathbb{R}^d)$.*

Proof of Proposition 4.

$$\hat{\mu}_{X_1^\epsilon}(z) = \exp\left\{\int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu^\epsilon(\mathrm{d}x)\right\}$$

$$\begin{aligned}\hat{\mu}_{\epsilon^{-1}X_1^\epsilon}(z) &= \hat{\mu}_{X_1^\epsilon}\Big(\frac{z}{\epsilon}\Big) \\ &= \exp\left\{\int_{\mathbb{R}^d} (e^{i\langle \frac{z}{\epsilon},x\rangle} - 1)\nu^\epsilon(\mathrm{d}x)\right\}\end{aligned}$$

Since $\epsilon^{-1}X_t^\epsilon \overset{d}{=} X_t^1$, then, by Theorem 7.10(iii) of [8], $P_{\epsilon^{-1}X_t^\epsilon} = P_{X_t^1}$, so $\hat{\mu}_{\epsilon^{-1}X_t^\epsilon}(z) = \hat{\mu}_{X_t^1}(z)$. By Theorem 8.1 of [8], the characteristic function is unique, hence $\mathrm{M}(B) = \nu^1(B)$, i.e. $\nu^\epsilon(\epsilon B) = \nu^1(B)$. Note that,

$$\begin{aligned}\nu^\epsilon(\epsilon B) &= \int_{\mathbb{R}^d} \mathbb{I}_{\epsilon B}(x)\mathbb{I}_{(0,\epsilon]}(|x|)\nu(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \mathbb{I}_{\epsilon B}(x)\mathbb{I}_{\epsilon(0,1]}(|x|)\nu(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \mathbb{I}_{\epsilon(B\cap(0,1]\mathbb{S}^{d-1})}(x)\nu(\mathrm{d}x) \\ &= \nu(\epsilon(B\cap(0,1]\mathbb{S}^{d-1}))\end{aligned}$$

Similarly, $\nu^1(B) = \nu((B\cap(0,1]\mathbb{S}^{d-1}))$. So, we have $\nu(\epsilon(B\cap(0,1]\mathbb{S}^{d-1})) = \nu((B\cap(0,1]\mathbb{S}^{d-1}))$. By Proposition 3, we have $\nu = \mathrm{D}$. Then, we conclude that $\nu^\epsilon = \mathrm{D}^\epsilon$. ∎

After studying the distributional property of the multivariate distribution, we proceed to its limiting property. Let $\{X_t\}$ be a pure jump Lévy process with a finite variation. Define $\mu(\epsilon, C) = \int_{\mathbb{R}^d} |x|\nu^\epsilon(\mathrm{d}x) = \int_{\mathbb{R}^d} |x|\mathbb{I}_{(0,\epsilon]C}(x)\nu(\mathrm{d}x)$, for every $C \in \mathscr{B}(\mathbb{S}^{d-1})$. Consider another pure jump Lévy process $\{Y_t^1 : t \geq 0\}$ with the Dickman Lévy measure D^1 as defined in Equation (1.9), and we denote this as $Y^1 \sim \mathrm{ID}(0, \mathrm{D}^1, 0)$ which means it is a pure jump Lévy process with finite variation. Before we arrive at the limiting property, we start with two lemmas that will be used later.

Lemma 2. *Let* M^ϵ *and* D^1 *be defined as above. If* $\forall C \in \mathscr{B}(\mathbb{S}^{d-1})$ *with* $\sigma(\partial C) = 0$, $\frac{\mu(\epsilon, C)}{\epsilon} \to \sigma(C)$ *as* $\epsilon \downarrow 0$, *then,* $\mathrm{M}^\epsilon \xrightarrow{v} \mathrm{D}^1$.

Proof of Lemma 2. Since all jumps of $\epsilon^{-1}X^\epsilon$ are bounded by 1, then by Lemma 4.9 in [12], $\mathrm{M}^\epsilon \xrightarrow{v} \mathrm{D}^1$ if and only if, for every $0 < h \leq 1$ and every $C \in \mathscr{B}(\mathbb{S}^{d-1})$, $\mathrm{M}^\epsilon(|x| > h, \frac{x}{|x|} \in C) \to \mathrm{D}^1(|x| > h, \frac{x}{|x|} \in C)$ with $\mathrm{D}^1(\{x \in \mathbb{R}^d : |x| = h\}) = 0$. Define, for $B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$,

$$\eta_\epsilon(B) = \int_{\mathbb{R}^d} |x|\mathbb{I}_B(x)\mathrm{M}^\epsilon(\mathrm{d}x)$$

$$\eta(B) = \int_{\mathbb{R}^d} |x|\mathbb{I}_B(x)\mathrm{D}^1(\mathrm{d}x)$$

then, for all $C \in \mathscr{B}(\mathbb{S}^{d-1})$ and $h \in (0, 1]$,

$$\begin{aligned}
\eta_\epsilon\left((0, h]C\right) &= \int_{(0,h]C} |x|\mathrm{M}^\epsilon(\mathrm{d}x) \\
&= \int_{(0,\epsilon h]C} \frac{|x|}{\epsilon}\nu^1(\mathrm{d}x) \\
&= \frac{1}{\epsilon}\int_{\mathbb{R}^d} |x|\mathbb{I}_{(0,\epsilon h]C}(x)\nu(\mathrm{d}x) \\
&= \frac{\mu(\epsilon h, C)}{\epsilon}
\end{aligned}$$

$$\begin{aligned}
\eta\left((0,h]C\right) &= \int_{(0,h]C} |x|\mathrm{D}^1(\mathrm{d}x) \\
&= \int_C \int_0^\infty \mathbb{I}_{(0,h]C}(r\xi)|r\xi|r^{-1}\mathrm{d}r\mathrm{d}\xi \\
&= \sigma(C)\int_0^h \mathrm{d}r \\
&= h\sigma(C)
\end{aligned}$$

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}\eta_\epsilon\left((0,h]C\right) &= \lim_{\epsilon\downarrow 0}\frac{\mu(\epsilon h, C)}{\epsilon} \\
&= \lim_{\epsilon\downarrow 0} h\frac{\mu(\epsilon h, C)}{\epsilon h} \\
&= h\sigma(C) \\
&= \eta\left((0,h]C\right)
\end{aligned}$$

Specifically, when $h = 1$, $\lim_{\epsilon\downarrow 0}\eta_\epsilon\left((0,1]C\right) = \eta\left((0,1]C\right)$. Then $\forall 0 < h < 1$

$$\begin{aligned}
\eta_\epsilon\left((h,1]C\right) &= \eta_\epsilon\left((0,1]C\right) - \eta_\epsilon\left((0,h]C\right) \\
&\to \eta\left((0,1]C\right) - \eta\left((0,h]C\right) \\
&= \eta\left((h,1]C\right)
\end{aligned}$$

Note,

$$\begin{aligned}
\int_{(h,1]C}\frac{1}{|x|}\eta_\epsilon(\mathrm{d}x) &= \int_{(h,1]C}\frac{1}{|x|}|x|\mathrm{M}^\epsilon(\mathrm{d}x) = \mathrm{M}^\epsilon\left((h,1]C\right) \\
\int_{(h,1]C}\frac{1}{|x|}\eta(\mathrm{d}x) &= \int_{(h,1]C}\frac{1}{|x|}|x|\mathrm{D}^1(\mathrm{d}x) = \mathrm{D}^1\left((h,1]C\right)
\end{aligned}$$

By theorem 1 of [13], since $\frac{1}{|x|}$ is bounded continuous on $(h,1]C$ and $\eta(\partial((0,h]C)) = 0$, then

$$\int_{(h,1]C} \frac{1}{|x|}\eta_\epsilon(\mathrm{d}x) \;\to\; \int_{(h,1]C} \frac{1}{|x|}\eta(\mathrm{d}x) \text{ as } \epsilon \to 0$$

So, $\mathrm{M}^\epsilon\left((h,1]C\right) \to \mathrm{D}^1\left((h,1]C\right)$, $\forall 0 < h < 1$, $\forall C \in \mathscr{B}(\mathbb{S}^{d-1})$. ■

Remark 10. *In Lemma 2,* $\mathrm{M}^\epsilon \xrightarrow{v} \mathrm{D}^1$ *means vague convergence. For the definition of vague convergence, we can refer to Definition* 4.2 *in [12]. Lévy processes need this kind of convergence for finite variation. Because, near* 0*, small jumps can accumulate and, at last, generate a Guassin part even if the original process does not have the Guassin part.*

Lemma 3. *Let* M^ϵ *and* D^1 *be defined as above. If* $\forall C \in \mathscr{B}(\mathbb{S}^{d-1})$ *with* $\sigma(\partial C) = 0$*,* $\dfrac{\mu(\epsilon, C)}{\epsilon} \to \sigma(C)$ *as* $\epsilon \downarrow 0$*, then for any* $h \in (0,1]$*,*

$$\int_{|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} xx^T\mathrm{D}^1(\mathrm{d}x) \iff \lim_{h\to 0}\lim_{\epsilon\to 0}\int_{|x|\le h}\langle z,x\rangle^2\mathrm{M}^\epsilon(\mathrm{d}x) = 0$$

Proof of Lemma 3. ($\Rightarrow$) Suppose, for any $h \in (0,1]$, $\lim_{\epsilon\to 0}\int_{|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) = \int_{|x|\le h} xx^T\mathrm{D}^1(\mathrm{d}x)$, then, when $h = 1$, $\lim_{\epsilon\to 0}\int_{|x|\le 1} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) = \int_{|x|\le 1} xx^T\mathrm{D}^1(\mathrm{d}x)$

$$\lim_{\epsilon\to 0}\int_{0<|x|\le 1} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) \;=\; \lim_{h\to 0}\lim_{\epsilon\to 0}\left(\int_{0<|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) + \int_{h<|x|\le 1} xx^T\mathrm{M}^\epsilon(\mathrm{d}x)\right)$$

By Lemma 2 and Theorem 1 in [13], $\int_{h<|x|\le 1} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{h<|x|\le 1} xx^T\mathrm{D}^1(\mathrm{d}x)$, so

$$\lim_{\epsilon\to 0}\int_{0<|x|\le 1} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) \;=\; \lim_{h\to 0}\lim_{\epsilon\to 0}\int_{0<|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) + \lim_{h\to 0}\int_{h<|x|\le 1} xx^T\mathrm{D}^1(\mathrm{d}x))$$

note,

$$\begin{aligned}
&\int_{|x|\le 1} xx^T \mathrm{D}^1(\mathrm{d}x)\\
=\;& \lim_{h\to 0}\int_{0<|x|\le h} xx^T\mathrm{D}^1(\mathrm{d}x) + \lim_{h\to 0}\int_{h<|x|\le 1} xx^T\mathrm{D}^1(\mathrm{d}x)\\
=\;& \lim_{h\to 0}\int_{\mathbb{S}^{d-1}}\int_0^h r\xi\xi^T\mathrm{d}r\sigma(\mathrm{d}\xi) + \lim_{h\to 0}\int_{h<|x|\le 1} xx^T\mathrm{D}^1(\mathrm{d}x)\\
=\;& \lim_{h\to 0}\frac{h^2}{2}\int_{\mathbb{S}^{d-1}}\xi\xi^T\sigma(\mathrm{d}\xi) + \lim_{h\to 0}\int_{h<|x|\le 1} xx^T\mathrm{D}^1(\mathrm{d}x)\\
=\;& \lim_{h\to 0}\int_{h<|x|\le 1} xx^T\mathrm{D}^1(\mathrm{d}x)
\end{aligned}$$

So $\lim_{h\to 0}\lim_{\epsilon\to 0}\int_{0<|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) = 0$.

($\Leftarrow$) Suppose, for any $h\in(0,1]$, $\lim_{h\to 0}\lim_{\epsilon\to 0}\int_{0<|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) = 0$. Then, for $0 < \delta < h$,

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) &= \lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{|x|\le\delta} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) + \lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{\delta<|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x)\\
&= \lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{\delta<|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x)\\
&= \lim_{\delta\to 0}\int_{\delta<|x|\le h} xx^T\mathrm{D}^1(\mathrm{d}x)\\
&= \int_{|x|\le h} xx^T\mathrm{D}^1(\mathrm{d}x)
\end{aligned}$$

Therefore, $\int_{|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} xx^T\mathrm{D}^1(\mathrm{d}x) \iff \lim_{h\to 0}\lim_{\epsilon\to 0}\int_{0<|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) = 0$.

Next, we prove $\int_{|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} xx^T\mathrm{D}^1(\mathrm{d}x) \iff \int_{|x|\le h}\langle z,x\rangle^2\mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h}\langle z,x\rangle^2\mathrm{D}^1(\mathrm{d}x)$.

($\Rightarrow$) Suppose $\int_{|x|\le h} xx^T \mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} xx^T \mathrm{D}^1(\mathrm{d}x)$, then, for any $z \in \mathbb{R}$,

$$\begin{aligned}
\langle z, \int_{|x|\le h} xx^T \mathrm{M}^\epsilon(\mathrm{d}x) z\rangle &\to \langle z, \int_{|x|\le h} xx^T \mathrm{D}^1(\mathrm{d}x) z\rangle \\
\langle z, \int_{|x|\le h} xx^T \mathrm{M}^\epsilon(\mathrm{d}x) z\rangle &= \int_{|x|\le h} \langle z, xx^T z\rangle \mathrm{M}^\epsilon(\mathrm{d}x) \\
&= \int_{|x|\le h} \langle z, x\langle z, x\rangle\rangle \mathrm{M}^\epsilon(\mathrm{d}x) \\
&= \int_{|x|\le h} \langle z, x\rangle^2 \mathrm{M}^\epsilon(\mathrm{d}x)
\end{aligned}$$

Similarly, $\langle z, \int_{|x|\le h} xx^T \mathrm{D}^1(\mathrm{d}x) z\rangle = \int_{|x|\le h} \langle z, x\rangle^2 \mathrm{D}^1(\mathrm{d}x)$. Thus, $\int_{|x|\le h} \langle z, x\rangle^2 \mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} \langle z, x\rangle^2 \mathrm{D}^1(\mathrm{d}x)$.

($\Leftarrow$) Suppose $\int_{|x|\le h} \langle z, x\rangle^2 \mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} \langle z, x\rangle^2 \mathrm{D}^1(\mathrm{d}x)$, then by Corollary 2.1.9 in [14], $\int_{|x|\le h} xx^T \mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} xx^T \mathrm{D}^1(\mathrm{d}x)$. Thus, $\int_{|x|\le h} xx^T \mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} xx^T \mathrm{D}^1(\mathrm{d}x) \iff \int_{|x|\le h} \langle z, x\rangle^2 \mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} \langle z, x\rangle^2 \mathrm{D}^1(\mathrm{d}x)$. Therefore $\lim_{h\to 0}\lim_{\epsilon\to 0} \int_{0<|x|\le h} xx^T \mathrm{M}^\epsilon(\mathrm{d}x) = 0 \iff \lim_{h\to 0}\lim_{\epsilon\to 0} \int_{|x|\le h} \langle z, x\rangle^2 \mathrm{M}^\epsilon(\mathrm{d}x) = 0$.

Combine these two necessary and sufficient conditions, and we conclude that

$$\int_{|x|\le h} xx^T \mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} xx^T \mathrm{D}^1(\mathrm{d}x) \iff \lim_{h\to 0}\limsup_{\epsilon\to 0} \int_{|x|\le h} \langle z, x\rangle^2 \mathrm{M}^\epsilon(\mathrm{d}x) = 0$$

■

Theorem 1. *Let σ be a finite measure defined on $\mathbb{S}^{d-1}$. If, $\forall C \in \mathscr{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial C) = 0$ and $\forall \epsilon \in (0,1]$, $\frac{\mu(\epsilon, C)}{\epsilon} \to \sigma(C)$ as $\epsilon \downarrow 0$, then $\epsilon^{-1} X^\epsilon \xrightarrow{d} Y^1$.*

Proof of Theorem 1. By Theorem 13.14 of [15] , to prove $\epsilon^{-1} X^\epsilon \xrightarrow{d} Y^1$ is equivalent to prove:

1. $\mathrm{M}^\epsilon \xrightarrow{v} \mathrm{D}^1$.

2. $\displaystyle\int_{h<|x|\le 1} x\mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{h<|x|\le 1} x\mathrm{D}^1(\mathrm{d}x)$ and $\displaystyle\int_{|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} xx^T\mathrm{D}^1(\mathrm{d}x)$, for every $h > 0$.

By Lemma 2, condition 1 holds. Since x is continous and bounded on $(h, 1]$, by Theorem 1 in [13], $\displaystyle\int_{h<|x|\le 1} x\mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{h<|x|\le 1} x\mathrm{D}^1(\mathrm{d}x)$. And, by Lemma 3, it suffices to prove

$$
\begin{aligned}
0 &\le \lim_{h\to 0}\lim_{\epsilon\to 0}\int_{|x|\le h}\langle z, x\rangle^2\mathrm{M}^\epsilon(\mathrm{d}x)\\
&\le \lim_{h\to 0}\limsup_{\epsilon\to 0}\int_{|x|\le h}|z|^2|x|^2\mathrm{M}^\epsilon(\mathrm{d}x)\\
&\le \lim_{h\to 0}\limsup_{\epsilon\to 0}|z|^2 h\int_{|x|\le h}|x|\mathrm{M}^\epsilon(\mathrm{d}x)\\
&\le \lim_{h\to 0}\limsup_{\epsilon\to 0}|z|^2 h\int_{|x|\le 1}|x|\mathrm{M}^\epsilon(\mathrm{d}x)\\
&= |z|^2\lim_{h\to 0}\limsup_{\epsilon\to 0} h\int_{|x|\le\epsilon}\frac{|x|}{\epsilon}\nu^1(\mathrm{d}x)\\
&= |z|^2\lim_{h\to 0}\limsup_{\epsilon\to 0}\frac{1}{\epsilon}h\int_{\mathbb{R}^d}|x|\mathbb{I}_{|x|\le\epsilon}(x)\nu(\mathrm{d}x)\\
&= |z|^2\lim_{h\to 0}\limsup_{\epsilon\to 0}\frac{1}{\epsilon}h\int_{\mathbb{R}^d}|x|\mathbb{I}_{(0,\epsilon]\mathbb{S}^{d-1}}(x)\nu(\mathrm{d}x)\\
&= |z|^2\lim_{h\to 0}\limsup_{\epsilon\to 0} h\frac{\mu(\epsilon,\mathbb{S}^{d-1})}{\epsilon}\\
&= |z|^2\sigma(\mathbb{S}^{d-1})\lim_{h\to 0} h\\
&= 0
\end{aligned}
$$

So, condition 2 holds. ■

Corollary 1. *Let σ be a finite measure defined on $\mathbb{S}^{d-1}$. If, $\forall C \in \mathscr{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial C) = 0$ and $\forall\epsilon \in (0, 1]$, $\dfrac{\mu(\epsilon, C)}{\epsilon} \to \sigma(C)$ as $\epsilon \downarrow 0$ and $\sigma(C) \neq 0$, then $\dfrac{X^\epsilon}{\mu(\epsilon, C)} \xrightarrow{d} \dfrac{Y^1}{\sigma(C)}$.*

Proof of Corollary 1. $\dfrac{X^\epsilon}{\mu(\epsilon, C)} = \dfrac{X^\epsilon}{\epsilon}\dfrac{\epsilon}{\mu(\epsilon, C)}$, then by Theorem 1 and the Slutsky's theorem, we can get

$$\frac{X^\epsilon}{\mu(\epsilon, C)} \xrightarrow{d} \frac{Y^1}{\sigma(C)}$$

■

In the following proposition, we provide broad cases that we can use to get the convergence condition in the Theorem 1.

Proposition 5. *Let σ be a finite measure defined on $\mathbb{S}^{d-1}$. For every $C \in \mathscr{B}(\mathbb{S}^{d-1})$, the following statements are equivalent:*

1. *For all $p > 0$, $\displaystyle\frac{1}{\epsilon^p}\int_{(0,\epsilon]C} |x|^p \nu^1(\mathrm{d}x) \to \frac{\sigma(C)}{p}$ as $\epsilon \downarrow 0$.*
2. *For some $p > 0$, $\displaystyle\frac{1}{\epsilon^p}\int_{(0,\epsilon]C} |x|^p \nu^1(\mathrm{d}x) \to \frac{\sigma(C)}{p}$ as $\epsilon \downarrow 0$.*
3. *For all $0 < h < 1$, $\nu^1((\epsilon h, \epsilon]C) \to \sigma(C) \ln\dfrac{1}{h}$ as $\epsilon \downarrow 0$.*

Proof of Proposition 5. It suffices to show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

1. $(1 \Rightarrow 2)$ This is obvious.
2. $(2 \Rightarrow 3)$ Assume $\exists\, p > 0$, s.t. $\displaystyle\frac{1}{\epsilon^p}\int_{(0,\epsilon]C} |x|^p \nu^1(\mathrm{d}x) \to \frac{\sigma(C)}{p}$. Define

$$\begin{aligned} \eta_\epsilon(\mathrm{d}x) &= |x|^p \mathrm{M}^\epsilon(\mathrm{d}x) \\ \eta(\mathrm{d}x) &= |x|^p \mathrm{D}^1(\mathrm{d}x) \end{aligned}$$

$\forall 0 < h \leq 1$ and $C \in \mathscr{B}(\mathbb{R}^{d-1})$

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}\int_{(0,h]C}\eta_\epsilon(\mathrm{d}x) &= \lim_{\epsilon\downarrow 0}\int_{(0,h]C}|x|^p\mathrm{M}^\epsilon(\mathrm{d}x)\\
&= \lim_{\epsilon\downarrow 0}\int_{(0,\epsilon h]C}\frac{|x|^p}{\epsilon^p}\nu^1(\mathrm{d}x)\\
&= \lim_{\epsilon\downarrow 0}\frac{h^p}{h^p\epsilon^p}\int_{(0,\epsilon h]C}|x|^p\nu^1(\mathrm{d}x)\\
&= h^p\frac{\sigma(C)}{p}\\
\int_{(0,h]C}\eta(\mathrm{d}x) &= \int_{(0,h]C}|x|^p\mathrm{D}^1(\mathrm{d}x)\\
&= \int_C\int_0^\infty \mathbb{I}_{(0,h]C}(r\xi)|r\xi|^p r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\\
&= \sigma(C)\int_0^h r^{p-1}\mathrm{d}r\\
&= h^p\frac{\sigma(C)}{p}
\end{aligned}$$

So, $\lim_{\epsilon\downarrow 0}\eta_\epsilon\left((0,h]C)\right) = \eta\left((0,h]C\right)$. Then similar to the proof of theorem 1, we have $\eta_\epsilon((h,1]C) \to \eta((h,1]C)$. Therefore,

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}\nu^1((\epsilon h,\epsilon]C) &= \lim_{\epsilon\downarrow 0}\mathrm{M}^\epsilon((h,1]C)\\
&= \lim_{\epsilon\downarrow 0}\int_{(h,1]C}\frac{1}{|x|^p}\eta_\epsilon(\mathrm{d}x)\\
&= \int_{(h,1]C}\frac{1}{|x|^p}\eta(\mathrm{d}x)\\
&= \int_{(h,1]C}\frac{1}{|x|^p}|x|^p\mathrm{D}^1(\mathrm{d}x)\\
&= \mathrm{D}^1((h,1]C)\\
&= \sigma(C)\ln\frac{1}{h}
\end{aligned}$$

So, condition 3 holds.

3. (3 $\Rightarrow$ 1) Assume $\lim_{\epsilon\downarrow 0}\nu^1((\epsilon h,\epsilon]C) = \sigma(C)\ln\frac{1}{h}$, $\forall 0 < h < 1$. Fix $p > 0$ and $N \in \mathbb{N}$

$$\begin{aligned}
\frac{1}{\epsilon^p}\int_{(\frac{\epsilon}{2^N},\epsilon]C}|x|^p\nu^1(\mathrm{d}x) &= \int_{(\frac{1}{2^N},1]C}|x|^p\mathrm{M}^\epsilon(\mathrm{d}x)\\
&= \int_{\bigcup_{k=1}^{N}\left(\frac{1}{2^k},\frac{1}{2^{k-1}}\right]C}|x|^p\mathrm{M}^\epsilon(\mathrm{d}x)\\
&= \sum_{k=1}^{N}\int_{\left(\frac{1}{2^k},\frac{1}{2^{k-1}}\right]C}|x|^p\mathrm{M}^\epsilon(\mathrm{d}x)\\
&= \sum_{k=1}^{N}\int_{(\frac{1}{2},1]C}\left|\frac{x}{2^{k-1}}\right|^p\mathrm{M}^\epsilon(\mathrm{d}x)\\
&= \sum_{k=1}^{N}\frac{1}{2^{p(k-1)}}\int_{(\frac{1}{2},1]C}|x|^p\mathrm{M}^\epsilon(\mathrm{d}x)
\end{aligned}$$

From the process of proving theorem 1 we know that condition 3 implies $\mathrm{M}^\epsilon \xrightarrow{v} \mathrm{D}^1$. Therefore, $\forall\theta > 0$, $\exists\delta > 0$, if $0 < \epsilon < \delta$, then

$$\left|\int_{(\frac{1}{2},1]C}|x|^p\mathrm{M}^\epsilon(\mathrm{d}x) - \int_{(\frac{1}{2},1]C}|x|^p\mathrm{D}^1(\mathrm{d}x)\right| < \frac{\theta}{\sum_{k=1}^{\infty}\frac{1}{2^{p(k-1)}}}$$

$$\begin{aligned}
&\Rightarrow \sum_{k=1}^{N}\frac{1}{2^{p(k-1)}}\left|\int_{(\frac{1}{2},1]C}|x|^p\mathrm{M}^\epsilon(\mathrm{d}x) - \int_{(\frac{1}{2},1]C}|x|^p\mathrm{D}^1(\mathrm{d}x)\right|\\
&\quad\leq \sum_{k=1}^{\infty}\frac{1}{2^{p(k-1)}}\left|\int_{(\frac{1}{2},1]C}|x|^p\mathrm{M}^\epsilon(\mathrm{d}x) - \int_{(\frac{1}{2},1]C}|x|^p\mathrm{D}^1(\mathrm{d}x)\right|\\
&\quad< \theta
\end{aligned}$$

So, $\left|\frac{1}{\epsilon^p}\int_{\left(\frac{\epsilon}{2^N},\epsilon\right]C}|x|^p\nu^1(\mathrm{d}x)-\sum_{k=1}^{N}\frac{1}{2^{p(k-1)}}\int_{(\frac{1}{2},1]C}|x|^p\mathrm{D}^1(\mathrm{d}x)\right|<\theta$. Since

$$\begin{aligned}\int_{(\frac{1}{2},1]C}|x|^p\mathrm{D}^1(\mathrm{d}x) &= \int_C\int_0^\infty \mathbb{I}_{(\frac{1}{2},1]C}(r\xi)|r\xi|^p r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\\ &= \sigma(C)\int_{\frac{1}{2}}^{1} r^{p-1}\mathrm{d}r\\ &= \frac{\sigma(C)}{p}(1-\frac{1}{2^p})\end{aligned}$$

take the limit as $N\to\infty$

$$\begin{aligned}&\lim_{N\to\infty}\left|\frac{1}{\epsilon^p}\int_{\left(\frac{\epsilon}{2^N},\epsilon\right]C}|x|^p\nu^1(\mathrm{d}x)-\sum_{k=1}^{N}\frac{1}{2^{p(k-1)}}\int_{(\frac{1}{2},1]C}|x|^p\mathrm{D}^1(\mathrm{d}x)\right|\\ =&\left|\lim_{N\to\infty}\frac{1}{\epsilon^p}\int_{\left(\frac{\epsilon}{2^N},\epsilon\right]C}|x|^p\nu^1(\mathrm{d}x)-\lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{2^{p(k-1)}}\int_{(\frac{1}{2},1]C}|x|^p\mathrm{D}^1(\mathrm{d}x)\right|\\ =&\left|\lim_{N\to\infty}\frac{1}{\epsilon^p}\int_{\left(\frac{\epsilon}{2^N},\epsilon\right]C}|x|^p\nu^1(\mathrm{d}x)-\lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{2^{p(k-1)}}\frac{\sigma(C)}{p}(1-\frac{1}{2^p})\right|\\ =&\left|\frac{1}{\epsilon^p}\int_{\bigcup_{N=1}^{\infty}\left(\frac{\epsilon}{2^N},\epsilon\right]C}|x|^p\nu^1(\mathrm{d}x)-\frac{1}{1-\frac{1}{2^p}}\frac{\sigma(C)}{p}(1-\frac{1}{2^p})\right|\\ =&\left|\frac{1}{\epsilon^p}\int_{(0,\epsilon]C}|x|^p\nu^1(\mathrm{d}x)-\frac{\sigma(C)}{p}\right|\\ <&\ \theta\end{aligned}$$

i.e. $\forall\theta>0,\ \exists\delta>0,\ \text{if } 0<\epsilon<\delta,\ \text{then}$

$$\left|\frac{1}{\epsilon^p}\int_{(0,\epsilon]C}|x|^p\nu^1(\mathrm{d}x)-\frac{\sigma(C)}{p}\right|<\theta$$

So, condition 1 holds.

■

Proposition 6. *If for every $C \in \mathscr{B}(\mathbb{S}^{d-1})$ such that $\sigma(\partial C) = 0$, then any of the statements in Proposition 5 are equivalent to:*

4. $\frac{X^\epsilon}{\epsilon} \xrightarrow{d} Y^1$ *as* $\epsilon \downarrow 0$

Proof of Proposition 6.

1. $(1 \Rightarrow 4)$ Assume condition 1 is true, when $p = 1$

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{(0,\epsilon]C} |x| \nu^1(\mathrm{d}x) = \lim_{\epsilon \downarrow 0} \frac{\mu(\epsilon, C)}{\epsilon} = \sigma(C)$$

   By theorem1, $\frac{X^\epsilon}{\epsilon} \xrightarrow{d} Y^1$.

2. $(4 \Rightarrow 3)$ Assume $\frac{X^\epsilon}{\epsilon} \xrightarrow{d} Y^1$. According to the proof of the theorem 1 $\mathrm{M}^\epsilon((h,1]C) \to \mathrm{D}^1((h,1]C)$

$$\begin{aligned}
\lim_{\epsilon \downarrow 0} \nu^1((\epsilon h, \epsilon]C) &= \lim_{\epsilon \downarrow 0} \mathrm{M}^\epsilon((h,1]C) \\
&= \mathrm{D}^1((h,1]C) \\
&= \int_C \int_0^\infty \mathbb{I}_{(h,1]C}(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi) \\
&= \sigma(C) \int_h^1 r^{-1} \mathrm{d}r \\
&= \sigma(C) \ln \frac{1}{h}
\end{aligned}$$

■

We can generalize Theorem 1 in many ways. First, it is not necessary that we truncate the Lévy process by a constant ϵ. In the next proposition, we generalize the truncating constant to a non-negative function.

Proposition 7. *Assume* f *is nonnegative and* $\lim_{x\to 0}\frac{f(x)}{x} = a, a \in (0,\infty)$. *Let* $C \in \mathscr{B}(\mathbb{R}^d)$ *and* $\{X_t^{f(\epsilon)}\}$ *be the Lévy process truncated by* $f(\epsilon)$. *If* $\frac{\mu(\epsilon, C)}{f(\epsilon)} \to \frac{\sigma(C)}{a}$, *as* $\epsilon \to 0$, *then* $\frac{X^{f(\epsilon)}}{f(\epsilon)} \xrightarrow{d} Y^1$, *as* $\epsilon \to 0$.

Proof of Proposition 7.

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\mu(\epsilon, C)}{\epsilon} &= \lim_{\epsilon\to 0}\frac{\mu(\epsilon, C)}{f(\epsilon)}\frac{f(\epsilon)}{\epsilon}\\ &= \frac{\sigma(C)}{a}a\\ &= \sigma(C)\end{aligned}$$

Then, by Theorem 1, $\frac{X^{f(\epsilon)}}{f(\epsilon)} \xrightarrow{d} Y^1$ ■

Next, we generalize the Lévy measure. Originally, for every $B \in \mathscr{B}(\mathbb{S}^{d-1})$, $\nu(B) = \int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbb{I}_B(r\xi)\mathrm{d}r\sigma(\mathrm{d}\xi)$. Now, we generalize it by a Borel function ρ, $\nu(B) = \int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbb{I}_B(r\xi)\rho(r,\xi)\mathrm{d}r\sigma(\mathrm{d}\xi)$.

Proposition 8. *Let* σ *be a finite measure defind on* $\mathbb{S}^{d-1}$ *and* X^ϵ *be the pure jump Lévy process such that* $X^\epsilon \sim \mathrm{ID}_0(0, \nu^\epsilon, 0)$, *where, for every* $B \in \mathscr{B}(\mathbb{S}^{d-1})$,

$$\nu(B) = \int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbb{I}_B(r\xi)\rho(r,\xi)\mathrm{d}r\sigma(\mathrm{d}\xi)$$

Define, for any measurable function h, $\sigma_h(B) = \int_B h(\xi)\sigma(\mathrm{d}\xi) < \infty$. *Define* $\mathrm{D}_h^1(\mathrm{d}x) = \mathbb{I}_{|x|\le 1}(x)\mathrm{D}_h(\mathrm{d}x)$, *where* $\mathrm{D}_h(B) = \int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbb{I}_B(r\xi)r^{-1}\mathrm{d}r\sigma_h(\mathrm{d}\xi)$. *Let* $\{Y_t^1 : t \ge 0\}$ *be a Lévy process with generating triplet* $(0, \mathrm{D}^1, 0)$. *If* $r\rho(r,\xi) \to h(\xi)$ *in* $L^1(\sigma)$ *as* $r \to 0$, *i.e.* $\int_{\mathbb{S}^{d-1}} |r\rho(r\xi) - h(\xi)|\sigma(\mathrm{d}\xi) \to 0$ *as* $r \to 0$, *then* $\frac{X^\epsilon}{\epsilon} \xrightarrow{d} Y^1$ *as* $\epsilon \downarrow 0$

Proof of Proposition 8. For any $C \in \mathbb{S}^{d-1}$ such that $\sigma(\partial C) = 0$, obviously $\sigma_h(\partial C) =$

$\int_{\partial C} h(\xi)\sigma(\mathrm{d}\xi) = 0.$

$$
\begin{aligned}
\lim_{\epsilon\to 0}\frac{\mu(\epsilon, C)}{\epsilon} &= \lim_{\epsilon\to 0}\frac{\int_{\mathbb{R}^d}|x|\mathbb{I}_{(0,\epsilon]C}(x)\nu(\mathrm{d}x)}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{\int_C\int_0^\epsilon r\rho(r,\xi)\mathrm{d}r\sigma(\mathrm{d}\xi)}{\epsilon}\\
&= \lim_{\epsilon\to 0}\int_C\frac{\int_0^\epsilon r\rho(r,\xi)\mathrm{d}r}{\epsilon}\sigma(\mathrm{d}\xi)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^\epsilon\int_C r\rho(r,\xi)\sigma(\mathrm{d}\xi)\mathrm{d}r
\end{aligned}
$$

Note $\int_C |r\rho(r,\xi)-h(\xi)|\sigma(\mathrm{d}\xi) \le \int_{\mathbb{S}^{d-1}} |r\rho(r,\xi)-h(\xi)|\sigma(\mathrm{d}\xi)$, since $\lim_{r\to 0}\int_{\mathbb{S}^{d-1}} |r\rho(r,\xi)-h(\xi)|\sigma(\mathrm{d}\xi) = 0$, then $\lim_{r\to 0}\int_C |r\rho(r,\xi)-h(\xi)|\sigma(\mathrm{d}\xi) = 0$. $\forall\eta > 0\ \exists\delta > 0$, if $r < \epsilon < \delta$, then $\int_C |r\rho(r,\xi)-h(\xi)|\sigma(\mathrm{d}\xi) < \eta$, thus $\left|\frac{1}{\epsilon}\int_0^\epsilon\int_C (r\rho(r,\xi)-h(\xi))\sigma(\mathrm{d}\xi)\mathrm{d}r\right| \le \frac{1}{\epsilon}\int_0^\epsilon\int_C |r\rho(r,\xi)-h(\xi)|\sigma(\mathrm{d}\xi)\mathrm{d}r \le \frac{1}{\epsilon}\int_0^\epsilon \eta\mathrm{d}r = \eta$. So $\lim_{\epsilon\to 0}\frac{\mu(\epsilon,C)}{\epsilon} = \int_C h(\xi)\sigma(\mathrm{d}\xi) = \sigma_h(C)$, then by Theorem 1, $\frac{X^\epsilon}{\epsilon}\xrightarrow{d} Y^1$ as $\epsilon\downarrow 0$. ■

Remark 11. *If σ is a finite measure, then $r\rho(r,\xi)\to h(\xi)$ uniformly on $\mathbb{S}^{d-1}$ always implies $r\rho(r,\xi)\to h(\xi)$ in $L^1(\sigma)$ as $r\to 0$. Since $r\rho(r,\xi)\to h(\xi)$ uniformly on $\mathbb{S}^{d-1}$, then $\forall\epsilon > 0$, $\exists\delta > 0$, $\forall\xi\in\mathbb{S}^{d-1}, r > 0$, if $r < \delta$, then $|r\rho(r,\xi)-h(\xi)| < \frac{\epsilon}{\sigma(\mathbb{S}^{d-1})}$. Thus $\int_{\mathbb{S}^{d-1}} |r\rho(r,\xi)-h(\xi)|\sigma(\mathrm{d}\xi) \le \int_{\mathbb{S}^{d-1}}\frac{\epsilon}{\sigma(\mathbb{S}^{d-1})}\sigma(\mathrm{d}\xi) = \epsilon$.*

For further generalization, we consider the p-tempered α-stable distribution. The class of TS^p_α relates to many important subclasses that have been well-studied such as tempered stable distributions [16], the $J_{\alpha,p}$ class [17], the Thorin class [9], the Goldie-Steutel-Bondesson class [9], and the class of type-G distributions [18]. For details of these infinitely divisible distribution classes, reference to [18]. So, it deserves to pay attention to this class.

Definition 12. *For $\alpha < 2$ and $p > 0$, an infinitely divisible distribution with no Gaussian part and Lévy measure $\nu(B) = \int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbb{I}_B(r\xi)q(r^p,\xi)r^{-\alpha-1}\mathrm{d}r\sigma(\mathrm{d}\xi)$, $B\in$*

$\mathscr{B}(\mathbb{R}^d)$ *is called a* p***-tempered*** α***-stable*** (TS^p_α) *distribution, where* σ *is a finite measure on* $\mathbb{S}^{d-1}$ *and* $q:(0,\infty)\times\mathbb{S}^{d-1}\mapsto(0,\infty)$ *is a Borel function such that* $\forall\xi\in\mathbb{S}^{d-1}$*,* $q(\cdot,\xi)$ *is completely monotone and* $\lim_{r\to\infty} q(r,\xi)=0$*.*

In Grabchak [19], the author provided the condition under which ν is a Lévy measure; see Equation ($8b$) in the paper. In the following, we provide the condition in a direct way without defining a new measure. Recall ν is a Lévy measure if and only if $\int_{\mathbb{R}^d}(|x|^2\wedge 1)\nu(\mathrm{d}x)<\infty$. Define $Q(r,\xi)=q(r^p,\xi)r^{-\alpha}=\int_{(0,\infty)} r^{-\alpha}e^{-r^p s}Q_\xi(\mathrm{d}s)$. Then $\nu(B)=\int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbb{I}_B(r\xi)Q(r,\xi)r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)$.

$$
\begin{aligned}
\int_{\mathbb{R}^d}(|x|^2\wedge 1)\nu(\mathrm{d}x) &= \int_{\mathbb{S}^{d-1}}\int_0^\infty (r^2\wedge 1)\int_0^\infty e^{-r^p x}Q_\xi(\mathrm{d}x)r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\\
&= \int_{\mathbb{S}^{d-1}}\int_0^\infty\int_0^\infty (r^2\wedge 1)r^{-1}e^{-r^p x}\mathrm{d}rQ_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi)\\
&= \int_{\mathbb{S}^{d-1}}\int_0^\infty\left(\int_0^1 re^{-r^p x}\mathrm{d}r+\int_1^\infty r^{-1}e^{-r^p x}\mathrm{d}r\right)Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi)
\end{aligned}
$$

Let $t=r^p x$, then $r=\left(\frac{t}{x}\right)^{\frac{1}{p}}$, $\mathrm{d}r=\frac{1}{p}\left(\frac{t}{x}\right)^{\frac{1}{p}-1}\frac{1}{x}\mathrm{d}t$

$$
\begin{aligned}
& \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^1 r e^{-r^p x} \mathrm{d}r Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\
= & \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^x x^{-\frac{2}{p}} \frac{1}{p} t^{\frac{2}{p}-1} e^{-t} \mathrm{d}t Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\
\leq & \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^x x^{-\frac{2}{p}} \frac{1}{p} t^{\frac{2}{p}-1} \mathrm{d}t Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\
= & \int_{\mathbb{S}^{d-1}} \int_0^1 \int_0^x x^{-\frac{2}{p}} \frac{1}{p} t^{\frac{2}{p}-1} \mathrm{d}t Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) + \int_{\mathbb{S}^{d-1}} \int_1^\infty \int_0^x x^{-\frac{2}{p}} \frac{1}{p} t^{\frac{2}{p}-1} \mathrm{d}t Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\
= & \int_{\mathbb{S}^{d-1}} \int_0^1 \int_0^x x^{-\frac{2}{p}} \frac{1}{p} t^{\frac{2}{p}-1} \mathrm{d}t Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\
& \qquad + \int_{\mathbb{S}^{d-1}} \int_1^\infty \left(\int_0^1 x^{-\frac{2}{p}} \frac{1}{p} t^{\frac{2}{p}-1} \mathrm{d}t + \int_1^x x^{-\frac{2}{p}} \frac{1}{p} t^{\frac{2}{p}-1} \mathrm{d}t \right) Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\
\leq & \int_{\mathbb{S}^{d-1}} \int_0^1 \int_0^1 x^{-\frac{2}{p}} \frac{1}{p} t^{\frac{2}{p}-1} \mathrm{d}t Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\
& \qquad + \int_{\mathbb{S}^{d-1}} \int_1^\infty \left(x^{-\frac{2}{p}} \frac{1}{2} + \int_1^x x^{-\frac{2}{p}} \frac{1}{p} x^{\frac{2}{p}} t^{-1} \mathrm{d}t \right) Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\
= & \int_{\mathbb{S}^{d-1}} \int_0^1 \int_0^1 x^{-\frac{2}{p}} \frac{1}{2} Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) + \int_{\mathbb{S}^{d-1}} \int_1^\infty \left(x^{-\frac{2}{p}} \frac{1}{2} + \int_1^x \frac{1}{p} t^{-1} \mathrm{d}t \right) Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\
< & \int_{\mathbb{S}^{d-1}} \int_0^1 \int_0^1 x^{-\frac{2}{p}} Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) + \int_{\mathbb{S}^{d-1}} \int_1^\infty \left(x^{-\frac{2}{p}} + \frac{1}{p} \log x \right) Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\
= & \int_{\mathbb{S}^{d-1}} \int_0^1 \int_0^1 x^{-\frac{2}{p}} Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) + \int_{\mathbb{S}^{d-1}} \int_1^\infty \left(x^{-\frac{2}{p}} + \log x^{\frac{1}{p}} \right) Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\
\leq & \int_{\mathbb{S}^{d-1}} \int_0^1 \int_0^1 x^{-\frac{2}{p}} Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) + \int_{\mathbb{S}^{d-1}} \int_1^\infty \left(1 + \log x^{\frac{1}{p}} \right) Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\
= & \int_{\mathbb{S}^{d-1}} \int_0^\infty \left(x^{-\frac{2}{p}} \wedge [1 + \log x^{\frac{1}{p}}] \right) Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{S}^{d-1}}\int_0^\infty\int_1^\infty r^{-1}e^{-r^px}\mathrm{d}rQ_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi)\\
=\;&\int_{\mathbb{S}^{d-1}}\int_0^\infty\int_x^\infty \frac{1}{p}t^{-1}e^{-t}\mathrm{d}tQ_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi)\\
=\;&\int_{\mathbb{S}^{d-1}}\left(\int_0^1\int_x^\infty \frac{1}{p}t^{-1}e^{-t}\mathrm{d}tQ_\xi(\mathrm{d}x)+\int_1^\infty\int_x^\infty \frac{1}{p}t^{-1}e^{-t}\mathrm{d}tQ_\xi(\mathrm{d}x)\right)\sigma(\mathrm{d}\xi)\\
=\;&\int_{\mathbb{S}^{d-1}}\left(\int_0^1\left[\int_x^1 \frac{1}{p}t^{-1}e^{-t}\mathrm{d}t+\int_1^\infty \frac{1}{p}t^{-1}e^{-t}\mathrm{d}t\right]Q_\xi(\mathrm{d}x)\right.\\
&\qquad\left.+\int_1^\infty\int_x^\infty \frac{1}{p}t^{-1}e^{-t}\mathrm{d}tQ_\xi(\mathrm{d}x)\right)\sigma(\mathrm{d}\xi)\\
\le\;&\int_{\mathbb{S}^{d-1}}\left(\int_0^1\left[\int_x^1 \frac{1}{p}t^{-1}\mathrm{d}t+\int_1^\infty \frac{1}{p}e^{-t}\mathrm{d}t\right]Q_\xi(\mathrm{d}x)+\int_1^\infty\int_x^\infty \frac{1}{p}e^{-t}\mathrm{d}tQ_\xi(\mathrm{d}x)\right)\sigma(\mathrm{d}\xi)\\
=\;&\int_{\mathbb{S}^{d-1}}\left(\int_0^1\left[\log x^{-\frac{1}{p}}+\frac{1}{p}e^{-1}\right]Q_\xi(\mathrm{d}x)+\int_1^\infty \frac{1}{p}e^{-1}Q_\xi(\mathrm{d}x)\right)\sigma(\mathrm{d}\xi)\\
=\;&\int_{\mathbb{S}^{d-1}}\int_0^\infty\left([\log x^{-\frac{1}{p}}+\frac{1}{p}e^{-1}]\vee\frac{1}{p}e^{-1}\right)Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi)
\end{aligned}
$$

So if $\int_{\mathbb{S}^{d-1}}\int_0^\infty\left(x^{-\frac{2}{p}}\wedge[1+\log x^{\frac{1}{p}}]\right)Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi)<\infty$ and $\int_{\mathbb{S}^{d-1}}\int_0^\infty\left([\log x^{-\frac{1}{p}}+\frac{1}{p}e^{-1}]\vee\frac{1}{p}e^{-1}\right)Q_\xi(\mathrm{d}x)\sigma(\mathrm{d}\xi)<\infty$, then ν is a Lévy measure.

Proposition 9. *Let σ be a finite measure defind on $\mathbb{S}^{d-1}$ and X^ϵ be the pure jump Lévy process with generating triplet $(0,\nu^\epsilon,0)$. If, for any $\xi\in\mathbb{S}^{d-1}$, $Q(r,\xi)\to h(\xi)$ in $L^1(\sigma)$ as $r\to0$, i.e. $\int_{\mathbb{S}^{d-1}}|Q(r,\xi)-h(\xi)|\sigma(\mathrm{d}\xi)\to0$ as $r\to0$, then $\frac{X^\epsilon}{\epsilon}\xrightarrow{d}Y^1$ as $\epsilon\downarrow0$*

Proof of Proposition 9. Define $\rho(r,\xi)=Q(r,\xi)r^{-1}$, then it holds immediately following Proposition 8. ■

CHAPTER 3: Construct the Random Variable

We have studied the limit properties of multivariate Dickman distribution. Nevertheless, all these are done in the background of Lévy processes. Now, we turn to the limit property of the multivariate Dickman distribution in the view of distribution. Finally, this property can lead us to construct multivariate Dickman random variables.

Definition 13. *l is a **slowly varying at** 0 function, if for every $t > 0$, $\lim_{x \to 0^+} \frac{l(xt)}{l(x)} = 1$.*

Proposition 10. *Suppose σ is a probability measure defined on $\mathbb{S}^{d-1}$ and G is a probability measure such that $1 - G(x) = (1-x)^\alpha l(1-x)$ where l is a slowly varying at 0 function and $\alpha > 0$. Assume $T_i \overset{\text{iid}}{\sim} \sigma$, $X_i \overset{\text{iid}}{\sim} G$ and $0 \leq X_i \leq 1$, and T_i and X_i are independent for $i = 1, 2, 3, \cdots$. Define $S_n = \sum_{i=1}^{N_n} T_i X_i^n$, where N_n is an integer depending on n and $N_n \to \infty$ as $n \to \infty$. Let Y be a random variable that has the infinitely divisible distribution with Lévy measure $\nu_y((0, x] \times C) = 1 - \sigma(C) a \left(\ln \left(\frac{1}{x} \right) \right)^\alpha$. If $\frac{N_n}{n^\alpha} l(\frac{1}{n}) \to a$ where $a \in \mathbb{R}^+$, then $S_n \xrightarrow{d} Y$ as $n \to \infty$.*

Proof of Proposition 10. Let P be the probability measure, for any $A = (s, 1] \times C$,

$$\begin{aligned}
\lim_{n\to\infty} N_n \mathrm{P}(A) &= \lim_{n\to\infty} N_n \mathrm{P}\left(|T_1 X_1^n| > s, \frac{T_1 X_1^n}{|T_1 X_1^n|} \in C\right) \\
&= \lim_{n\to\infty} N_n \mathrm{P}(X_1^n > s, T_1 \in C) \\
&= \lim_{n\to\infty} N_n \mathrm{P}(X_1^n > s) \mathrm{P}(T_1 \in C) \\
&= \sigma(C) \lim_{n\to\infty} N_n \mathrm{P}(X_1 > s^{\frac{1}{n}}) \\
&= \sigma(C) \lim_{n\to\infty} N_n (1 - s^{\frac{1}{n}})^\alpha l(1 - s^{\frac{1}{n}}) \\
&= \sigma(C) \lim_{n\to\infty} N_n \frac{1}{n^\alpha} l(\frac{1}{n}) \left(\frac{1 - s^{\frac{1}{n}}}{\frac{1}{n}}\right)^\alpha \frac{l(1 - s^{\frac{1}{n}})}{l(\frac{1}{n})} \\
&= \sigma(C) a \left(\ln\left(\frac{1}{s}\right)\right)^\alpha \\
&= 1 - \nu_y(A)
\end{aligned}$$

For all $t \in \mathbb{R}^d$,

$$\begin{aligned}
& \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} N_n \mathrm{E}[\langle t, X_1^n T_1 \rangle^2 \mathbb{I}(|X_1^n T_1| < \epsilon)] \\
= \;& \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} N_n \int_0^\infty \mathrm{P}(|\langle t, T_1 X_1^n \rangle^2 \mathbb{I}(|T_1 X_1^n| < \epsilon)| > s) \mathrm{d}s \\
= \;& \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} N_n \int_0^\infty \mathrm{P}(|\langle t, T_1 X_1^n \rangle \mathbb{I}(|T_1 X_1^n| < \epsilon)|^2 > s) \mathrm{d}s \\
= \;& \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} N_n \int_0^\infty \mathrm{P}(|\langle t, T_1 X_1^n \rangle|^2 > s, X_1^n < \epsilon) \mathrm{d}s \\
\leq \;& \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} N_n \int_0^\infty \int_0^B \mathrm{P}((\frac{\sqrt{s}}{|t|})^{\frac{1}{n}} < X_1 < \epsilon^{\frac{1}{n}}) \mathrm{d}\sigma(t) \mathrm{d}s \\
= \;& \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} N_n \int_0^{|t|^2\epsilon^2} \int_0^B \left[\mathrm{P}\left(\left(\frac{\sqrt{s}}{|t|}\right)^{\frac{1}{n}} < X_1\right) - \mathrm{P}(\epsilon^{\frac{1}{n}} < X_1)\right] \mathrm{d}\sigma(t) \mathrm{d}s \\
= \;& \int_0^B \lim_{\epsilon\downarrow 0} \int_0^{|t|^2\epsilon^2} \left(a(\ln\left(\frac{|t|}{\sqrt{s}}\right))^\alpha - a(\ln(\frac{1}{\epsilon}))^\alpha\right) \mathrm{d}s \mathrm{d}\sigma(t) \\
= \;& \int_0^B \left(0 - a|t| \lim_{\epsilon\downarrow 0} \epsilon (\ln(\frac{1}{\epsilon}))^\alpha\right) \mathrm{d}\sigma(t) \\
= \;& 0
\end{aligned}$$

The second equation holds because $\langle t, T_1 X_1^n \rangle$ is a scalar. By Jensen's inequality, $\mathrm{E}^2[\langle t, X_1^n T_1\rangle \mathbb{I}(|X_1^n T_1| < \epsilon)] \leq \mathrm{E}[\langle t, X_1^n T_1\rangle^2 \mathbb{I}(|X_1^n T_1| < \epsilon)]$, then we have $\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} N_n \mathrm{E}^2[\langle t, X_1^n T_1\rangle^2 \mathbb{I}(|X_1^n T_1| < \epsilon)] = 0$. Thus, we conclude that

$$\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} N_n \left(\mathrm{E}(\langle t, X_1^n T_1\rangle^2 \mathbb{I}(|X_1^n T_1| < \epsilon)) - \mathrm{E}^2(\langle t, X_1^n T_1\rangle \mathbb{I}(|X_1^n T_1| < \epsilon))\right) = 0$$

According to Thoerem 1.2.21 and Example 1.2.22 in Meerschaert and Scheffler [14], the measure of S_n converges vaguely to Φ on $\mathscr{B}((0,1] \times \mathbb{S}^{d-1})$. Then by Theorem 3.2.2 in Meerschaert and Scheffler [14], we have $S_n \xrightarrow{d} Y$. ■

Remark 12. *If we let* $N_n = n$, $\alpha = 1$, $l(x) = 1$, *and* $X \sim Unif([0,1])$, *then* $\frac{N_n}{n^\alpha} l(\frac{1}{n}) = 1$, $\nu_y = \mathrm{D}^1$, *then by Proposition 10,* $\sum_{i=1}^n T_i X_i^n \xrightarrow{d} Y^1$ *where the distribution of* Y^1 *has the Dickman Lévy measure* D^1.

Remark 13. *Recall Equation* (2.4) *in Definition 8, Remark 12 coincides with the multivariate Dickman distribution. Thus, we can define the multivariate Dickman distribution as the summation of series, which is the same as the univariate case. A random variable* X *defined on* $\mathbb{R}^d$ *follows the* ***multivariate Dickman distribution****, if*

$$X = V_1 U_1 + V_2 U_1 U_2 + V_3 U_1 U_2 U_3 + \cdots$$

where V_i *are i.i.d. random variables defined on* $\mathbb{S}^{d-1}$, U_i *are i.i.d. uniform random variables defined on* $[0,1]$, *and* V_i *and* U_i *are independent.*

Cohen and Rosiński (Theorem 2.2) [20] provided the condition under which the transformed truncated Lévy process converges to the Brownian motion. For every $\epsilon \in (0,1]$, let $\{X_t^\epsilon : t \geq 0\}$ be a Lévy process with characteristic function

$$\hat{\mu}_{X_t^\epsilon}(z) = \exp\left\{ t \int_{\mathbb{R}^d} \left(\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle \right) \nu^\epsilon(\mathrm{d}x) \right\}$$

Let $\{W_t : t \geq 0\}$ be a standard Brownian motion and $\Sigma_\epsilon = \int_{\mathbb{R}^d} xx^T \nu^\epsilon(\mathrm{d}x)$. Assume Σ_ϵ is non-sigular. Then $\Sigma_\epsilon^{-\frac{1}{2}} X^\epsilon \xrightarrow{d} W$ as $\epsilon \downarrow 0$ if and only if $\int_{\langle \Sigma_\epsilon^{-1} x, x\rangle > h} \langle \Sigma_\epsilon^{-1} x, x\rangle \nu^\epsilon(\mathrm{d}x) \to 0$ as $\epsilon \downarrow 0$, for every $h > 0$. And then they use Brownian motion and compound Poisson process to approximate the Lévy process.

Example 1. *Assume X^ϵ is a Lévy process with generating triplet $(0, \nu^\epsilon, 0)$ where $\nu^\epsilon(\mathrm{d}x) = -\frac{\ln x}{x} \mathbb{I}_{(0,\epsilon]}(x)\mathrm{d}x$ and W is standard Brownian mothion, then*

$$\frac{X^\epsilon}{\frac{1}{2}\epsilon\sqrt{1 - \ln \epsilon^2}} \xrightarrow{d} W$$

$$\begin{aligned} \Sigma_\epsilon = \int_{\mathbb{R}^d} x^2 \left(-\frac{\ln x}{x} \mathbb{I}_{(0,\epsilon]}(x)\right)\mathrm{d}x &= \int_0^\epsilon -x \ln x \mathrm{d}x \\ &= \frac{1}{4}\epsilon^2 (1 - \ln \epsilon^2) \end{aligned}$$

For every $h > 0$,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\frac{x^2}{\frac{1}{4}\epsilon^2(1-\ln\epsilon^2)}>h}\frac{x^2}{\frac{1}{4}\epsilon^2(1-\ln\epsilon^2)}\left(-\frac{\ln x}{x}\right)\mathbb{I}_{(0,\epsilon]}(x)\mathrm{d}x\\
=\;&\lim_{\epsilon\to 0}\frac{4}{\epsilon^2(1-\ln\epsilon^2)}\int_0^\infty \mathbb{I}_{(\frac{1}{2}\epsilon\sqrt{h(1-\ln\epsilon^2)},\epsilon]}(x)(-x\ln x)\mathrm{d}x
\end{aligned}$$

Note, $\frac{1}{2}\epsilon\sqrt{h(1-\ln\epsilon^2)} < \epsilon$ *then* $\epsilon > \exp\{\frac{1}{2}-\frac{2}{h}\}$*. Since we fix h, then when ϵ goes to* 0*, this condition doesn't hold. then the indicator function is* 0*, so the limit is 0.*

However, this is not always true. If we have a Dickman-type Lévy measure, this condition does not hold.

Example 2. *Assume X^ϵ is a Lévy process with generating triplet* $(0,\nu^\epsilon,0)$*, where* $\nu^\epsilon(\mathrm{d}x) = \frac{1}{x}\mathbb{I}_{(0,\epsilon]}(x)\mathrm{d}x$*. Note that,*

$$\Sigma_\epsilon = \int_0^\infty x^2\mathbb{I}_{(0,\epsilon]}(x)\frac{1}{x}\mathrm{d}x = \int_0^\epsilon x\mathrm{d}x = \frac{\epsilon^2}{2}$$

Then, for any $2 > h > 0$*,*

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{\frac{x^2}{\frac{\epsilon^2}{2}}>h}\frac{x^2}{\frac{\epsilon^2}{2}}\mathbb{I}_{(0,\epsilon]}(x)\frac{1}{x}\mathrm{d}x &= \lim_{\epsilon\to 0}\int_{\epsilon\sqrt{\frac{h}{2}}}^{\epsilon}\frac{2x}{\epsilon^2}\mathrm{d}x\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\,x^2\Big|_{\epsilon\sqrt{\frac{h}{2}}}^{\epsilon}\\
&= \lim_{\epsilon\to 0} 1-\frac{h}{2}\\
&= 1-\frac{h}{2}\neq 0
\end{aligned}$$

So, in this case, we can not use Brownian motion to approximate the Lévy process. In the following sections, we provide an alternative way to approximate a Lévy process. By doing this, we complete the family of approximating the Lévy process in a new perspective.

4.1 Decomposition

Consider a Lévy process $\{X_t\}$ in $\mathbb{R}^d$ with the Lévy-Khinchine representation

$$\hat{\mu}_{X_t}(z) = \exp\left\{ti\langle\gamma, z\rangle + t\int_{\mathbb{R}^d}\left(\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbb{I}_{|x|\leq 1}(x)\right)\nu(\mathrm{d}x)\right\}, z \in \mathbb{R}^d \quad (4.1)$$

Assume we are given a decomposition

$$\nu = \nu^\epsilon + \tilde{\nu}^\epsilon$$

where, $\epsilon \in (0, 1]$ and for every $B \in \mathscr{B}(\mathbb{R}^d)$, $\nu^\epsilon(B) = \int_B \mathbb{I}_{|x|\leq\epsilon}(x)\nu(\mathrm{d}x)$ and $\tilde{\nu}^\epsilon(B) = \int_B \mathbb{I}_{|x|>\epsilon}(x)\nu(\mathrm{d}x)$. Also, we assume that

$$\int_{\mathbb{R}^d} |x|\nu^\epsilon(\mathrm{d}x) < \infty \text{ and } \tilde{\nu}^\epsilon(\mathbb{R}^d) < \infty$$

Then

$$\begin{aligned}
\int_{|x|>1} |x|\nu^\epsilon(\mathrm{d}x) &\leq \int_{|x|>1} |x|^2\nu^\epsilon(\mathrm{d}x) \\
&\leq \int_{|x|>1} |x|^2\nu^\epsilon(\mathrm{d}x) + \int_{|x|\leq 1} |x|^2\nu^\epsilon(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} |x|^2\nu^\epsilon(\mathrm{d}x) < \infty
\end{aligned}$$

$$\begin{aligned}
\int_{|x|\leq 1} |x|\tilde{\nu}^\epsilon(\mathrm{d}x) &\leq \int_{|x|\leq 1} \tilde{\nu}^\epsilon(\mathrm{d}x) \\
&= \tilde{\nu}^\epsilon(|x| \leq 1) \\
&\leq \tilde{\nu}^\epsilon(\mathbb{R}^d) < \infty
\end{aligned}$$

Hence, we have the following decomposition

$$\begin{aligned}
\hat{\mu}_{X_t}(z) &= \exp\left\{ti\langle\gamma, z\rangle + t\int_{\mathbb{R}^d}\left(\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle\mathbb{I}_{|x|\le 1}(x)\right)\nu(\mathrm{d}x)\right\} \\
&= \exp\left\{t\int_{\mathbb{R}^d}\left(\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle\right)\nu^{\epsilon}(\mathrm{d}x)\right\}\exp\left\{t\int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1\right)\tilde{\nu}^{\epsilon}(\mathrm{d}x)\right\} \\
&\quad \exp\left\{ti\langle\gamma_\epsilon, x\rangle\right\} \\
&= \hat{\mu}_{X_t^\epsilon}(z)\,\widehat{\mu}_{N_t^\epsilon}(z)\,\left(\hat{\delta}_{\gamma_\epsilon}(z)\right)^t
\end{aligned}$$

where

$$\gamma_\epsilon = \gamma - \int_{|x|\le 1} x\tilde{\nu}^{\epsilon}(\mathrm{d}x)$$

So

$$X \stackrel{d}{=} X^\epsilon + N^\epsilon + \gamma_\epsilon \tag{4.2}$$

where N^ϵ is a compound Poisson process with the jump measure $\tilde{\nu}^\epsilon$, and γ_ϵ is a drift.

Proposition 11. *Let $\{X_t : t \ge 0\}$ be a Lévy process in $\mathbb{R}^d$ determined by Equation (4.1) and let decomposition (4.2) be given. Suppose assumptions in Theorem 1 hold. Let Y^1, N^ϵ, and γ_ϵ be as above. Then for every $\epsilon \in (0, 1]$, there exists a cadlag process $Z^\epsilon = \{Z_t^\epsilon : t \ge 0\}$ such that*

$$X \stackrel{d}{=} \epsilon Y^1 + N^\epsilon + \gamma_\epsilon + Z^\epsilon$$

such that, for each $T > 0$,

$$\sup_{t\in[0,T]} |\epsilon^{-1} Z_t^\epsilon| \stackrel{p}{\to} 0 \text{ as } \epsilon \to 0$$

Proof of Proposition 11. By Theorem 1, $\dfrac{X^\epsilon}{\epsilon} \stackrel{d}{\to} Y^1$. Then by Theorem 15.17 of

[15], there exists Lévy process $R^\epsilon = \{R^\epsilon_t : t \geq 0\}$ such that

$$R^\epsilon \overset{d}{=} \frac{X^\epsilon}{\epsilon} \text{ and } \sup_{t\in[0,T]} \left|R^\epsilon_t - Y^1_t\right| \overset{p}{\to} 0$$

as $\epsilon \to 0$, for each $T > 0$. Let

$$Z^\epsilon = \epsilon\left(R^\epsilon - Y^1\right)$$

Then

$$\begin{aligned} X &\overset{d}{=} X^\epsilon + N^\epsilon + \gamma_\epsilon \\ &\overset{d}{=} \epsilon R^\epsilon + N^\epsilon + \gamma_\epsilon \\ &= \epsilon Y^1 + N^\epsilon + \gamma_\epsilon + Z^\epsilon \end{aligned}$$

with $\sup_{t\in[0,T]} \left|\epsilon^{-1} Z^\epsilon_t\right| \overset{p}{\to} 0$ as $\epsilon \to 0$

■

4.2 Simulation of Lévy process with Dickman-type Lévy measure

As Proposition 11 described, a Lévy process can be approximated by small jumps and large jumps. In this section, we introduce the algorithm that simulates small jumps, i.e. X^ϵ in Equation (4.2), using multivariate Dickman distribution.

The simulation of univariate Dickman distribution and the Vervaat perpetuities which are closely related to the Dickman distribution has been extensively studied in the literature, see [21], [22], [23], and [24]. Here we use an exact method in Cont [25]. We use LePage's series representation for the σ-finite measure $\mathrm{D}(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_B(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi)$ which we defined in Proposition 3. Specifically, we use the method that is the same as Example 6.17. In our situation, for all $C \in \mathscr{B}(\mathbb{S}^{d-1})$, define $\Pi(C) = \int_C \frac{\sigma(\mathrm{d}\xi)}{\sigma(\mathbb{S}^{d-1})}$, then Π is a probability measure on the unit sphere $\mathbb{S}^{d-1}$

of $\mathbb{R}^d$, since our σ is finite. We can rewrite D as the following

$$\begin{aligned}
\mathrm{D}(B) &= \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_B(r\xi) r^{-1} \mathrm{d}r \sigma(\mathrm{d}\xi) \\
&= \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_B(r\xi) r^{-1} \mathrm{d}r \frac{\sigma(\mathrm{d}\xi)}{\sigma(\mathbb{S}^{d-1})} \sigma(\mathbb{S}^{d-1}) \\
&= \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_B(r\xi) r^{-1} \mathrm{d}r \sigma(\mathbb{S}^{d-1}) \Pi(\mathrm{d}\xi) \\
&= \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_B(r\xi) \mu(\mathrm{d}r, \xi) \Pi(\mathrm{d}\xi)
\end{aligned}$$

$\mu(*, \xi)$ is a Lévy measure on $(0, \infty)$ for each $\xi \in \mathbb{S}^{d-1}$. Thus, for the Dickman Lévy measure D^1, we have $\mu(\mathrm{d}r, \xi) = \theta \mathbb{I}(r \leq 1) r^{-1} \mathrm{d}r$, where $\theta = \sigma(\mathbb{S}^{d-1})$.

Proposition 12. *Assume $\{\Gamma_i\}$ is a sequence of arrival times of a standard Poisson process, $\{V_i\}$ is an independent sequence of independent random variables having distribution Π on unit sphere $\mathbb{S}^{d-1}$, and $\{U_i\}$ is an independent sequence of independent random variables having uniform distribution on $[0, 1]$. Suppose $Y^1 \sim \mathrm{ID}(0, \mathrm{D}^1, 0)$. $\forall t \in [0, 1]$, define*

$$X_t = \sum_{i=1}^n e^{-\frac{\Gamma_i}{\theta}} V_i \mathbb{I}_{[0,t]}(U_i) \tag{4.3}$$

Then $X_t \to Y_t^1$ almost surely and uniformly as $n \to \infty$.

Proof of Proposition 12. For $0 < x < 1$, define

$$U(x, \xi) = \int_x^\infty \mu(\mathrm{d}r, \xi) = \int_x^\infty \theta \mathbb{I}_{|r\xi| \leq 1}(r\xi) r^{-1} \mathrm{d}r = \theta \int_x^\infty \mathbb{I}_{(0,1]}(r) r^{-1} \mathrm{d}r = -\ln x^\theta$$

so $U^{-1}(z, \xi) = e^{-\frac{z}{\theta}}, z > 0$. Then define

$$\sigma(r, C) = \int_{\mathbb{S}^{d-1}} \mathbb{I}_C(e^{-\frac{r}{\theta}} \xi) \Pi(\mathrm{d}\xi) = \int_{\mathbb{S}^{d-1}} \mathbb{I}_{e^{\frac{r}{\theta}} C}(\xi) \Pi(\mathrm{d}\xi)$$

Hence,

$$\begin{aligned} A(s) &= \int_0^s \int_{|x|\le 1} x\sigma(r,\mathrm{d}x)\mathrm{d}r \\ &= \int_0^s \int_{|x|\le 1} x\Pi(e^{\frac{r}{\theta}}\mathrm{d}x)\mathrm{d}r \\ &= \int_0^s e^{-\frac{r}{\theta}}\mathrm{d}r \int_{|x|\le 1} e^{\frac{r}{\theta}} x\Pi(e^{\frac{r}{\theta}}\mathrm{d}x) \\ &= \theta(1-e^{-\frac{s}{\theta}}) \int_{\mathbb{S}^{d-1}} \xi\Pi(\mathrm{d}\xi) \\ &= \theta(1-e^{-\frac{s}{\theta}})E(V) \end{aligned}$$

Then $\gamma = \lim_{s\to\infty} A(s) = \theta E(V)$. By Theorem 6.2 in Cont [25],

$$\begin{aligned} \hat{\mu}_{X_t} &\to \exp\left\{t\left(i\langle u,\gamma\rangle + \int_{\mathbb{R}^d}(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle\mathbb{I}(|x|\le 1))\mathrm{D}(\mathrm{d}x)\right)\right\} \\ &= \exp\left\{t\left(i\langle u,\theta E(V)\rangle + \int_{\mathbb{R}^d}(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle\mathbb{I}(|x|\le 1))\mathrm{D}(\mathrm{d}x)\right)\right\} \end{aligned}$$

Note,

$$\begin{aligned} \int_{\mathbb{R}^d}\langle u,x\rangle\mathbb{I}(|x|\le 1)\mathrm{D}(\mathrm{d}x) &= \int_{\mathbb{S}^{d-1}}\int_0^1 \langle u,r\nu\rangle\theta r^{-1}\mathrm{d}r\Pi(\mathrm{d}\nu) \\ &= \langle u,\theta\int_{\mathbb{S}^{d-1}}\nu\Pi(\mathrm{d}\nu)\rangle \\ &= \langle u,\theta E(V)\rangle \end{aligned}$$

Thus $\hat{\mu}_{X_t} \to \exp\left\{t\int_{\mathbb{R}^d}(e^{i\langle u,x\rangle}-1)\mathrm{D}^1(\mathrm{d}x)\right\} = \hat{\mu}_{Y_t^1}$. ■

Proposition 12 works for $t \in [0,1]$. To simulate the whole process with small jumps, we extend it to $t\ in[0,T]$ for any $T > 0$ in the next proposition.

Proposition 13. *Assume $\{\Gamma_i\}$ is a sequence of arrival times of a standard Poisson process, $\{V_i\}$ is an independent sequence of independent random variables having distribution Π on unit sphere $\mathbb{S}^{d-1}$, and $\{U_i\}$ is an independent sequence of independent* *random variables having uniform distribution on* $[0,1]$. *Suppose* $Y^1 \sim \mathrm{ID}(0, \mathrm{D}^1, 0)$. $\forall t \in [0,T]$ *for any* $T > 1$, *define*

$$X_t = \sum_{i=1}^{n} e^{-\frac{\Gamma_i}{\theta T}} V_i \mathbb{I}_{[0,\frac{t}{T}]}(U_i) \tag{4.4}$$

Then $X_t \to Y_t^1$ *almost surely and uniformly as* $n \to \infty$.

Proof of Proposition 13. First, note that, if $X_1 \sim \mathrm{ID}(0,0,\mathrm{D}^1)$, then $X_T \sim \mathrm{ID}(0,0,T\mathrm{D}^1)$. Let $\{Y_t\}$ be another Lévy process such that $Y_1 \sim \mathrm{ID}(0,0,T\mathrm{D}^1)$. The same procedure as previous, we can get $U^{-1}(z,\xi) = e^{-\frac{z}{\theta T}}$, $\sigma(r,A) = \int_{\mathbb{S}^{d-1}} \mathbb{I}_A(e^{-\frac{z}{\theta T}}\xi)\Pi(\mathrm{d}\xi) = \int_{\mathbb{S}^{d-1}} \mathbb{I}_{e^{\frac{z}{\theta T}}A}(\xi)\Pi(\mathrm{d}\xi)$, $A(i) = \theta T(1-e^{-\frac{i}{\theta T}})E(V)$, $\gamma = \theta T E(V)$, $c_i = \theta T(e^{\frac{1}{\theta T}}-1)E(V)e^{-\frac{i}{\theta T}}$. Then the series representation of a Lévy process with Lévy measure D^1 has the following form

$$Y_t \overset{d}{=} \sum_{i=1}^{\infty} e^{-\frac{\Gamma_i}{\theta T}} V_i \mathbb{I}_{[0,t]}(U_i), \text{ for } t \in [0,1] \tag{4.5}$$

Note that $X_T \overset{d}{=} Y_1$, then $X_{sT} \overset{d}{=} Y_s$, let $t = sT$, then $X_t \overset{d}{=} Y_{\frac{t}{T}}$, i.e.

$$X_t = \sum_{i=1}^{\infty} e^{-\frac{\Gamma_i}{\theta T}} V_i \mathbb{I}_{[0,\frac{t}{T}]}(U_i), \text{ for } t \in [0,T]$$

■

Example 3. *In application, we can only use a finite series to approximate the Lévy process with Dickman-type Lévy measure. Assume* $T_1, T_2, \cdots \overset{iid}{\sim} Exp(1)$, *define* $\Gamma_i = \sum_{j=1}^{i} T_j$, *then* $\Gamma_i \sim Gamma(n,1)$. *Let* $U_i \sim U[0,1]$ *and* $\{V_i\}$ *are i.i.d with mean* $\mathrm{E}(V)$.

Γ, V, U are independent. For any $n \in \mathbb{N}$, $X_{t,n} = \sum_{i=1}^{n} e^{-\frac{\Gamma_i}{\theta T}} V_i \mathbb{I}_{[0,\frac{t}{T}]}(U_i)$, for $t \in [0, T]$.

$$\begin{aligned}
\mathrm{E}(X_{t,n}) &= \mathrm{E}\left(\sum_{i=1}^{n} e^{-\frac{\Gamma_i}{\theta T}} V_i \mathbb{I}_{[0,\frac{t}{T}]}(U_i)\right) \\
&= \sum_{i=1}^{n} \mathrm{E}\left(e^{-\frac{\Gamma_i}{\theta T}}\right) E(V_i) \mathrm{E}\left(\mathbb{I}_{[0,\frac{t}{T}]}(U_i)\right) \\
&= \sum_{i=1}^{n} \mathrm{E}\left(e^{-\frac{\Gamma_i}{\theta T}}\right) \mathrm{E}(V) \frac{t}{T} \\
&= \mathrm{E}(V) \frac{t}{T} \sum_{i=1}^{n} \left(1 - \left(-\frac{1}{\theta T}\right)\right)^{-i} \\
&= \mathrm{E}(V) \frac{t}{T} \sum_{i=1}^{n} \left(1 + \frac{1}{\theta T}\right)^{-i} \\
&= \mathrm{E}(V) \frac{t}{T} \theta T [1 - \left(\frac{\theta T}{\theta T + 1}\right)^{n}] \\
&= \mathrm{E}(V) \theta t [1 - \left(\frac{\theta T}{\theta T + 1}\right)^{n}]
\end{aligned}$$

So $\mathrm{E}(X_t) = \frac{t}{T} \mathrm{E}(V) \frac{(1 + \frac{1}{\theta T})^{-1}}{1 - (1 + \frac{1}{\theta T})^{-1}} = \frac{t}{T} \mathrm{E}(V) \theta T = \theta t \mathrm{E}(V)$. *Then*

$$\mathrm{E}(X_t) - \mathrm{E}(X_{t,n}) = \mathrm{E}(V) \theta t \left(\frac{\theta T}{\theta T + 1}\right)^{n}$$

So, if we want a precision δ, then $n \geq \frac{\log(\theta t \mathrm{E}(V)) - \log \delta}{\log(\theta T + 1) - \log(\theta T)}$. We notice that n will increase at the speed that is proportional to $\log \delta$.

4.3 Simulation of Compound Poisson Process

The remaining large jumps, i.e. N^ϵ in Equation (4.2), is a compound process. Simulation of a compound Poisson process has been extensively studied in the literature. Applying the Algorithm 6.1 in [25] to our case, we can simulate the compound Poisson process at the specific time t using the following steps:

- simulate $e_i \sim \exp(1/\lambda)$, $i = 1, 2, \cdots, n$,
- let $M(t) = \left\{ \max(k) | \sum_{i=1}^{k} e_i < t \right\}$,
- simulate Z_i from the distribution $\tilde{\nu}^\epsilon/\lambda$, $i = 1, 2, \cdots, M(t)$,
- set $N^\epsilon(t) = \sum_{i=1}^{M(t)} Z_i$,

where e_is are jumping times from time 0 to t, $M(t)$ is the total number of jumps and is a Poisson process, i.e. $M(t) \sim Pois(\lambda t)$, and $\lambda = \tilde{\nu}^\epsilon(\mathbb{R}^d)$.

To simulate Z and to calculate λ, we need to know the specific form of $\tilde{\nu}^\epsilon$. The gamma distribution can be used to model heavy-tailed and asymmetric data and allow for flexibility in modeling the tail behavior of data. Thus, for the Lévy measure $\tilde{\nu}^\epsilon$, we consider the generalized gamma distribution, i.e., in Definition 12 we are interested in the case when $\alpha = 0$. Under this consideration, for any $A \in \mathscr{B}(\mathbb{R}^d)$,

$$\tilde{\nu}^\epsilon(A) \;=\; \int_{\mathbb{S}^{d-1}} \int_\epsilon^\infty \int_0^\infty \mathbb{I}_A(r\xi) r^{-1} e^{-r^p s} Q_\xi(\mathrm{d}s)\mathrm{d}r\sigma(\mathrm{d}\xi),$$

then $\lambda = \int_{\mathbb{S}^{d-1}} \int_\epsilon^\infty \int_0^\infty r^{-1} e^{-r^p s} Q_\xi(\mathrm{d}s)\mathrm{d}r\sigma(\mathrm{d}\xi)$ and $\frac{1}{\lambda}\tilde{\nu}^\epsilon$ is a probability measure.

Since we know $\tilde{\nu}^\epsilon$, we can simulate $Z \sim \tilde{\nu}^\epsilon/\lambda$. However, it is difficult to directly simulate a random variable from this distribution. Nevertheless, we notice that if we define

$$k(\xi) = \int_\epsilon^\infty \int_0^\infty r^{-1} e^{-r^p s} Q_\xi(\mathrm{d}s)\mathrm{d}r = \frac{1}{p} \int_0^\infty \Gamma(0, \epsilon^p s) Q_\xi(\mathrm{d}s), \tag{4.6}$$

where $\Gamma(.,.)$ is the upper incomplete gamma function, and

$$\sigma_p = \frac{k(\xi)}{\lambda}\sigma, \tag{4.7}$$

then σ_p is a probability measure. Let $\xi \sim \sigma_p$. Given this ξ and define

$$l(s) = \int_{\epsilon}^{\infty} r^{-1} e^{-r^p s} \mathrm{d}r = \frac{1}{p}\Gamma(0, \epsilon^p s), \tag{4.8}$$

and

$$\Pi_S(\xi, \mathrm{d}s) = \frac{l(s)}{k(\xi)} Q_\xi(\mathrm{d}s), \tag{4.9}$$

then Π_S is a probability measure. Let $S \sim \Pi_S$. Given ξ and S, define

$$\Pi_R(s, \mathrm{d}r) = \frac{1}{l(s)} r^{-1} e^{-r^p s} \mathbb{I}(r \geq \epsilon)\mathrm{d}r, \tag{4.10}$$

then Π_R is a probability distribution. Let $R \sim \Pi_R$.

Proposition 14. *Let $Z = R\xi$, then $Z \sim \tilde{\nu}^\epsilon/\lambda$.*

Proof of Proposition 14. For any $A \in \mathscr{B}(\mathbb{R}^d)$,

$$\begin{aligned}
\mathrm{P}(Z \in A) &= \mathrm{E}(\mathbb{I}_A(Z)) \\
&= \mathrm{E}(\mathrm{E}(\mathrm{E}(\mathbb{I}_A(R\xi)|S)|\xi)) \\
&= \int_{\mathbb{S}^{d-1}} \mathrm{E}(\mathrm{E}(\mathbb{I}_A(R\xi)|S)|\xi = z)\sigma_p(\mathrm{d}z) \\
&= \int_{\mathbb{S}^{d-1}} \mathrm{E}(\mathrm{E}(\mathbb{I}_A(Rz)|S))\frac{k_\xi}{\lambda}\sigma(\mathrm{d}z) \\
&= \int_{\mathbb{S}^{d-1}} \mathrm{E}(\mathbb{I}_A(Rz)|S = s)\Pi_S(\mathrm{d}s)\frac{k_\xi}{\lambda}\sigma(\mathrm{d}z) \\
&= \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathrm{E}(\mathbb{I}_A(Rz))\frac{l(s)}{k_\xi}Q_\xi(\mathrm{d}s)\frac{k_\xi}{\lambda}\sigma(\mathrm{d}z) \\
&= \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_\epsilon^\infty \mathbb{I}_A(rz)\frac{1}{l(s)}r^{-1}e^{-r^p s}\mathrm{d}r\frac{l(s)}{k_\xi}Q_\xi(\mathrm{d}s)\frac{k_\xi}{\lambda}\sigma(\mathrm{d}z) \\
&= \int_{\mathbb{S}^{d-1}} \int_\epsilon^\infty \int_0^\infty \frac{1}{l(s)}\frac{l(s)}{k_\xi}\frac{k_\xi}{\lambda}\mathbb{I}_A(rz)r^{-1}e^{-r^p s}Q_\xi(\mathrm{d}s)\mathrm{d}r\sigma(\mathrm{d}z) \\
&= \frac{1}{\lambda}\int_{\mathbb{S}^{d-1}} \int_\epsilon^\infty \int_0^\infty \mathbb{I}_A(rz)r^{-1}e^{-r^p s}Q_\xi(\mathrm{d}s)\mathrm{d}r\sigma(\mathrm{d}z)
\end{aligned}$$

■

Still, it is difficult to simulate R directly. We need to find a distribution that we can simulate random variables easier than Π_R using the acceptance-rejection method. The next lemma leads us to the distribution we want.

Lemma 4. *Let X be a random variable from the distribution having density $g(x) = \frac{1}{l(s)}x^{-1}e^{-x^p}\mathbb{I}(x \geq \epsilon s^{\frac{1}{p}})$. Define $R = X/s^{1/p}$, then R is the random variable from the distribution having the density $f(r) = \frac{1}{l(s)}r^{-1}e^{-r^p s}\mathbb{I}(r \geq \epsilon)$.*

Proof of Lemma 4. The CDF of R is

$$\begin{aligned} F_R(r) &= \mathrm{P}(R \leq r) \\ &= \mathrm{P}(\frac{X}{s^{\frac{1}{p}}} \leq r) \\ &= \mathrm{P}(X \leq rs^{\frac{1}{p}}) \\ &= \frac{1}{l(s)} \int_{\epsilon s^{\frac{1}{p}}}^{rs^{\frac{1}{p}}} x^{-1} e^{-x^p} \mathrm{d}x, \end{aligned}$$

so the pdf of R is $f(r) = \frac{1}{l(s)}(rs^{\frac{1}{p}})^{-1} e^{-(rs^{\frac{1}{p}})^p} s^{\frac{1}{p}} = \frac{1}{l(s)} r^{-1} e^{-r^p s}$ and $r \geq \epsilon$. ∎

Even for X, we are unable to simulate it directly. However, until now, we can use the acceptance-rejection method. This leads us to the next lemma.

Lemma 5. *Let X be a random variable from the distribution having density $g(x) = \frac{1}{l(s)} x^{-1} e^{-x^p} \mathbb{I}(x \geq \epsilon s^{\frac{1}{p}})$. Then, for $U \sim U(0,1)$,*

$$X = \begin{cases} [\epsilon^p s - \ln(U)]^{\frac{1}{p}}, & \epsilon s^{\frac{1}{p}} > 1 \\ \begin{cases} (\epsilon s^{\frac{1}{p}})^{(1-2U)}, & U \leq \frac{1}{2} \\ [1 - \ln(2(1-U_1))]^{\frac{1}{p}}, & U > \frac{1}{2} \end{cases}, & \epsilon s^{\frac{1}{p}} \leq 1 \end{cases}$$

Proof of Lemma 5. 1. When $\epsilon s^{\frac{1}{p}} > 1$,

$$\begin{aligned} g(x) &= \frac{1}{l(s)} x^{-1} e^{-x^p} \mathbb{I}(x \geq \epsilon s^{\frac{1}{p}}) \\ &\leq \frac{1}{l(s) p e^{\epsilon^p s}} p x^{p-1} e^{\epsilon^p s - x^p} \mathbb{I}(x \geq \epsilon s^{\frac{1}{p}}). \end{aligned}$$

Let $c_1 = \frac{1}{l(s) p e^{\epsilon^p s}}$ and $h_1(x) = p x^{p-1} e^{\epsilon^p s - x^p} \mathbb{I}(x \geq \epsilon s^{\frac{1}{p}})$ is a pdf, then $g(x) \leq c_1 h_1(x)$. The CDF is $H_1(x) = \int_{\epsilon s^{\frac{1}{p}}}^{x} p y^{p-1} e^{\epsilon^p s - y^p} \mathrm{d}y = 1 - e^{\epsilon^p s - x^p}$, so $H_1^{-1}(x) = [\epsilon^p s - \ln(1-x)]^{\frac{1}{p}}$. Thus we generate $U_1, U_2 \overset{\text{iid}}{\sim} U(0,1)$, and let $X = [\epsilon^p s - \ln(U_1)]^{\frac{1}{p}}$, if $U_2 \leq \frac{g(X)}{c_1 h_1(X)} = X^{-p}$, then we accept X and set $R = \frac{X}{s^{\frac{1}{p}}}$.

2. When $\epsilon s^{\frac{1}{p}} < 1$,

$$\begin{aligned}
g(x) &= \frac{1}{l(s)} x^{-1} e^{-x^p} \mathbb{I}(x \geq \epsilon s^{\frac{1}{p}}) \\
&= \frac{1}{l(s)} x^{-1} e^{-x^p} (\mathbb{I}(\epsilon s^{\frac{1}{p}} \leq x < 1) + \mathbb{I}(x \geq 1)) \\
&\leq \frac{-\ln(\epsilon s^{\frac{1}{p}})}{l(s)} \frac{x^{-1}}{-\ln(\epsilon s^{\frac{1}{p}})} \mathbb{I}(\epsilon s^{\frac{1}{p}} \leq x < 1) + \frac{1}{l(s)pe} p x^{p-1} e^{1-x^p} \mathbb{I}(x \geq 1) \\
&\leq 2 \max\left\{\frac{1}{l(s)pe}, \frac{-\ln(\epsilon s^{\frac{1}{p}})}{l(s)}\right\} \frac{1}{2}\Bigg(\frac{x^{-1}}{-\ln(\epsilon s^{\frac{1}{p}})} \mathbb{I}(\epsilon s^{\frac{1}{p}} \leq x < 1) \\
&\quad + p x^{p-1} e^{1-x^p} \mathbb{I}(x \geq 1)\Bigg),
\end{aligned}$$

due to the fact that $\frac{1}{l(s)} x^{-1} e^{-x^p} \mathbb{I}(x \geq 1) \leq \frac{1}{l(s)pe} p x^{p-1} e^{1-x^p} \mathbb{I}(x \geq 1)$ and $\frac{1}{l(s)} x^{-1} e^{-x^p} \mathbb{I}(\epsilon s^{\frac{1}{p}} \leq x < 1) \leq \frac{1}{l(s)} x^{-1} \mathbb{I}(\epsilon s^{\frac{1}{p}} \leq x < 1) = \frac{-\ln(\epsilon s^{\frac{1}{p}})}{l(s)} \frac{x^{-1}}{-\ln(\epsilon s^{\frac{1}{p}})} \mathbb{I}(\epsilon s^{\frac{1}{p}} \leq x < 1)$. Let $c_2 = 2 \max\left\{\frac{1}{l(s)pe}, \frac{-\ln(\epsilon s^{\frac{1}{p}})}{l(s)}\right\}$ and $h_2(x) = \frac{1}{2}\left(p x^{p-1} e^{1-x^p} \mathbb{I}(x \geq 1) + \frac{x^{-1}}{-\ln(\epsilon s^{\frac{1}{p}})} \mathbb{I}(\epsilon s^{\frac{1}{p}} \leq x < 1)\right)$ is a pdf, then $g(x) \leq c_2 h_2(x)$.

Note $\mathrm{P}(1 \leq X < \infty) = 0.5$, and when $x \geq 1$,

$$\begin{aligned}
H_2(x) &= \frac{1}{2}\left(\int_{\epsilon s^{\frac{1}{p}}}^{1} \frac{y^{-1}}{-\ln(\epsilon s^{\frac{1}{p}})} \mathrm{d}y + \int_1^x p y^{p-1} e^{1-y^p} \mathrm{d}y\right) \\
&= \frac{1}{2}\left(1 + 1 - e^{1-x^p}\right) = 1 - \frac{1}{2} e^{1-x^p},
\end{aligned}$$

then $H_2^{-1}(x) = [1 - \ln 2(1-x)]^{\frac{1}{p}}$.

Also $\mathrm{P}(\epsilon s^{\frac{1}{p}} \leq x < 1) = 0.5$, and when $\epsilon s^{\frac{1}{p}} < x < 1$,

$$\begin{aligned}
H_2(x) &= \frac{1}{2} \int_{\epsilon s^{\frac{1}{p}}}^{x} \frac{y^{-1}}{-\ln(\epsilon s^{\frac{1}{p}})} \mathrm{d}y \\
&= \frac{1}{2}\left(1 - \frac{\ln x}{\ln(\epsilon s^{\frac{1}{p}})}\right),
\end{aligned}$$

then $H_2^{-1}(x) = e^{(1-2x)\ln(\epsilon s^{\frac{1}{p}})} = (\epsilon s^{\frac{1}{p}})^{(1-2x)}$.

Thus, we generate $U_1, U_2 \overset{\text{iid}}{\sim} U(0,1)$, if $U_1 \leq \frac{1}{2}$, let $X = (\epsilon s^{\frac{1}{p}})^{(1-2U_1)}$, else let $X = [1 - \ln(2(1-U_1))]^{\frac{1}{p}}$, and if $U_2 \leq \frac{g(X)}{c_2 h_2(X)}$ then set $R = \frac{X}{s^{\frac{1}{p}}}$.

■

Remark 14. *Notice,* $\lim_{\epsilon\to 0} l(s) = \lim_{\epsilon\to 0} \frac{1}{p}\Gamma(0, \epsilon^p s) = \infty$ *and* $\lim_{\epsilon\to 0} \frac{-\ln(\epsilon s^{\frac{1}{p}})}{l(s)} = \lim_{\epsilon\to 0} \frac{-\ln(\epsilon s^{\frac{1}{p}})}{\int_\epsilon^\infty r^{-1} e^{-r^p s} \mathrm{d}r} = \lim_{\epsilon\to 0} \frac{-\epsilon^{-1}}{-\epsilon^{-1} e^{-\epsilon^p s}} = \lim_{\epsilon\to 0} e^{\epsilon^p s} = 1$, *so* $\lim_{\epsilon\to 0} c_2 = \lim_{\epsilon\to 0} 2\max\left\{\frac{1}{l(s)pe}, \frac{-\ln(\epsilon s^{\frac{1}{p}})}{l(s)}\right\} = 2$. *This indicates that the probability of acceptance is around* 0.5 *when* ϵ *is small.*

Now that we have everything, we can simulate the compound Poisson process using the following algorithm.

Algorithm 1.

1. *Simulate* $e_i \sim \exp(\lambda)$, $i = 1, 2, \cdots, n$.
2. *Let* $M(t) = \{\max(k) | \sum_{i=1}^{k} e_i < t\}$.
3. *For* $i = 1, 2, \cdots, M(t)$:
   (a) *Simulate* $\xi_i \sim \sigma_p$.
   (b) *Given* ξ_i, *simulate* $S_i \sim \Pi_S$
   (c) *Given* ξ_i *and* S_i, *simulate* $R_i \sim \Pi_R$, *and this is in two cases:*
      i. *When* $\epsilon S_i^{\frac{1}{p}} > 1$, *generate* $U_1, U_2 \overset{\text{iid}}{\sim} U(0,1)$, *and let* $X_i = [\epsilon^p S_i - \ln(1-U_1)]^{\frac{1}{p}}$. *If* $U_2 \leq \frac{g(X_i)}{c_1 h_1(X_i)} = X_i^{-p}$, *set* $R_i = \frac{X_i}{S_i^{\frac{1}{p}}}$, *otherwise repeat this step.*
      ii. *When* $\epsilon S_i^{\frac{1}{p}} < 1$, *generate* $U_1, U_2 \overset{\text{iid}}{\sim} U(0,1)$, *let* $X_i = (\epsilon S_i^{\frac{1}{p}})^{1-2U_1}\mathbb{I}(U_1 \leq \frac{1}{2}) + [1 - \ln(2U_1)]^{\frac{1}{p}}\mathbb{I}(U_1 > \frac{1}{2})$. *If* $U_2 \leq \frac{g(X_i)}{c_2 h_2(X_i)}$ *then set* $R_i = \frac{X_i}{S_i^{\frac{1}{p}}}$, *otherwise repeat this step.*
   (d) *Set* $Z_i = R_i \xi_i$.

4. set $N^\epsilon(t) = \sum_{i=1}^{M(t)} Z_i$

Example 4. *In this example, we implement the Algorithm 1 in the bivariate case, i.e.* $N^\epsilon(t) = (N_1^\epsilon(t), N_2^\epsilon(t))$. *For simplicity, we take* $p = 1$, $\theta \in [0, 2\pi)$, $\xi = (\cos\theta, \sin\theta)$, $Q_\xi = \delta_1$, *and* $\sigma(\mathbb{S}^{d-1}) = 1$. *Then* $\Pi_S(\mathrm{d}s) = \delta_1(\mathrm{d}s)$, *the random variable* $S = 1$, $\lambda = k_\xi = l(s) = \Gamma(0, \epsilon)$, $\sigma_p = \sigma$, *and* $\Pi_R(\mathrm{d}r) = \frac{1}{\Gamma(0, \epsilon s)} r^{-1} e^{-r^p s} \mathbb{I}(r \geq \epsilon)\mathrm{d}r$. *Further, we take* σ *as a uniform distribution on* $[0, 2\pi)$, *i.e.* $\theta_i = \frac{2\pi}{n}(i-1)$, $i = 1, 2, \cdots, n$, *with probability* $\frac{1}{n}$. *Then we have the following theoretical values:*

$$\begin{aligned}
\mathrm{E}(N_1^\epsilon(t)) &= te^{-\epsilon}\frac{1}{n}\sum_{i=1}^n \cos\theta_i, \qquad \mathrm{E}(N_2^\epsilon(t)) = te^{-\epsilon}\frac{1}{n}\sum_{i=1}^n \sin\theta_i, \\
\mathrm{Var}(N_1^\epsilon(t)) &= t(\epsilon+1)e^{-\epsilon}\frac{1}{n}\sum_{i=1}^n \cos^2\theta_i, \ \mathrm{Var}(N_2^\epsilon(t)) = t(\epsilon+1)e^{-\epsilon}\frac{1}{n}\sum_{i=1}^n \sin^2\theta_i, \\
\mathrm{Cov}(N_1^\epsilon(t), N_2^\epsilon(t)) &= t(\epsilon+1)e^{-\epsilon}\frac{1}{n}\sum_{i=1}^n \cos\theta_i \sin\theta_i.
\end{aligned}$$

We can see that they are all linear with t*; when we compare the theoretical value with the empirical value, the error will also be linear with* t. *To get rid of the effect of* t *and focus on the effect of distribution, we define our metric as this:*

$$\begin{aligned}
\mathit{ErrorMean}_1(t) &= \frac{|\mathrm{E}(N_1^\epsilon(t)) - m_1(t)|}{t}, \ \mathit{ErrorMean}_2(t) = \frac{|\mathrm{E}(N_2^\epsilon(t)) - m_2(t)|}{t} \\
\mathit{ErrorVar}_1(t) &= \frac{|\mathrm{Var}(N_1^\epsilon(t)) - s_1(t)|}{t}, \ \mathit{ErrorVar}_2(t) = \frac{|\mathrm{Var}(N_2^\epsilon(t)) - s_2(t)|}{t} \\
\mathit{ErrorCov}(t) &= \frac{|\mathrm{Cov}(N_1^\epsilon(t), N_2^\epsilon(t)) - s_{1,2}(t)|}{t}
\end{aligned}$$

where $m_1(t), m_2(t)$ *are sample means,* $s_1(t), s_2(t)$ *are sample variances, and* $s_{1,2}(t)$ *is the sample covariance. Then we define the total error as the mean square error of the above errors, i.e.* $\mathit{TotalError}(t) = \big\{\mathit{ErrorMean}_1(t)^2 + \mathit{ErrorMean}_2(t)^2 + \mathit{ErrorVar}_1(t)^2 + \mathit{ErrorVar}_2(t)^2 + \mathit{ErrorCov}(t)^2\big\}^{1/2}$

In our experiment, we first fix $\epsilon = 0.1$ *and compare different Monte Carlo numbers:*

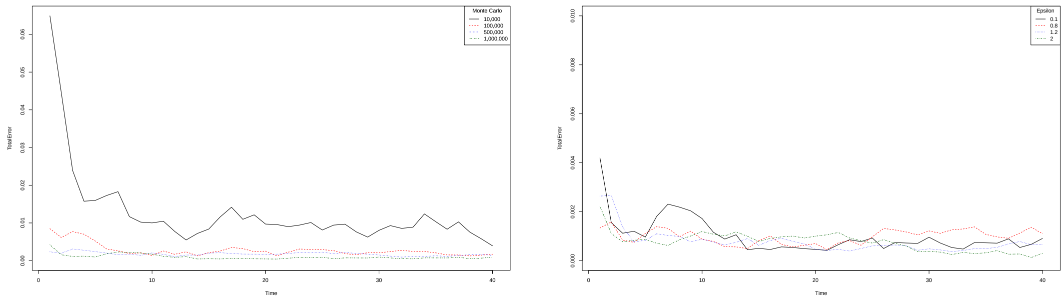

Figure 4.1: Plots of errors comparing different Monte Carlo numbers and different epsilon values. The x-axis represents t, the time, and the y-axis represents the total errors.

10000, 100000, 500000, 1000000; *Then we fix Monte Carlo number* $N = 1000000$ *and compare different epsilons:* 0.1, 0.8, 1.2, 2. *Increasing the Monte Carlo number will decrease the total error; however, if we want to decrease the total error further, we need very large numbers.*

4.4 Simulation of Multivariate Dickman Random Variable

Until now, we have three methods to simulate the multivariate Dickman random variables. Xia and Grabchak [26], introduce the discretization and simulation(DS) method using univariate random variables to simulate multivariate random variables, see Theorem 2 in detail. We can also use Equation (4.3) in Proposition 12 by letting the indicator function always be 1, and we call this the shot noise (SN) method. Last, we use the method described in Remark 12 as the third method, and we call it the triangular array (TA) method.

Example 5. *In this example, we simulate 2-d Dickman random variables. To simulate a random variable V defined on $\mathbb{S}^{d-1}$, we first generate $\theta \sim Beta(\alpha, \beta)$ on $[0, 2\pi]$, then $V = (\cos\theta, \sin\theta)$. By Remark 9, we can calculate the theoretical mean, variance, and covariance as below:*

$$
\begin{aligned}
\mathrm{E}(X_1) &= \frac{1}{B(\alpha,\beta)(2\pi)^{\alpha+\beta-1}} \int_0^{2\pi} \cos\theta\theta^{\alpha-1}(2\pi-\theta)^{\beta-1}\mathrm{d}\theta \\
\mathrm{E}(X_2) &= \frac{1}{B(\alpha,\beta)(2\pi)^{\alpha+\beta-1}} \int_0^{2\pi} \sin\theta\theta^{\alpha-1}(2\pi-\theta)^{\beta-1}\mathrm{d}\theta \\
\mathrm{Var}(X_1) &= \frac{1}{2B(\alpha,\beta)(2\pi)^{\alpha+\beta-1}} \int_0^{2\pi} \cos^2\theta\theta^{\alpha-1}(2\pi-\theta)^{\beta-1}\mathrm{d}\theta \\
\mathrm{Var}(X_2) &= \frac{1}{2B(\alpha,\beta)(2\pi)^{\alpha+\beta-1}} \int_0^{2\pi} \sin^2\theta\theta^{\alpha-1}(2\pi-\theta)^{\beta-1}\mathrm{d}\theta \\
\mathrm{Cov}(X_1,X_2) &= \frac{1}{4B(\alpha,\beta)(2\pi)^{\alpha+\beta-1}} \int_0^{2\pi} \sin(2\theta)\theta^{\alpha-1}(2\pi-\theta)^{\beta-1}\mathrm{d}\theta
\end{aligned}
$$

where $X = (X_1, X_2)$ is the 2-d Dickman random variable, $B(\alpha, \beta)$ is the beta function. We use the following metric to measure the error of the simulation:

$$
\begin{aligned}
TotalError &= \big\{(\mathrm{E}(X_1)-m_1)^2 + (\mathrm{E}(X_2)-m_2)^2 + (\mathrm{Var}(X_1)-s_1)^2 \\
&\qquad +(\mathrm{Var}(X_2)-s_2)^2 + (\mathrm{Cov}(X_1,X_2)-s_{1,2})^2\big\}^{1/2}
\end{aligned}
$$

where m_1, m_2 are sample means, s_1, s_2 are sample variances, and $s_{1,2}$ is the sample

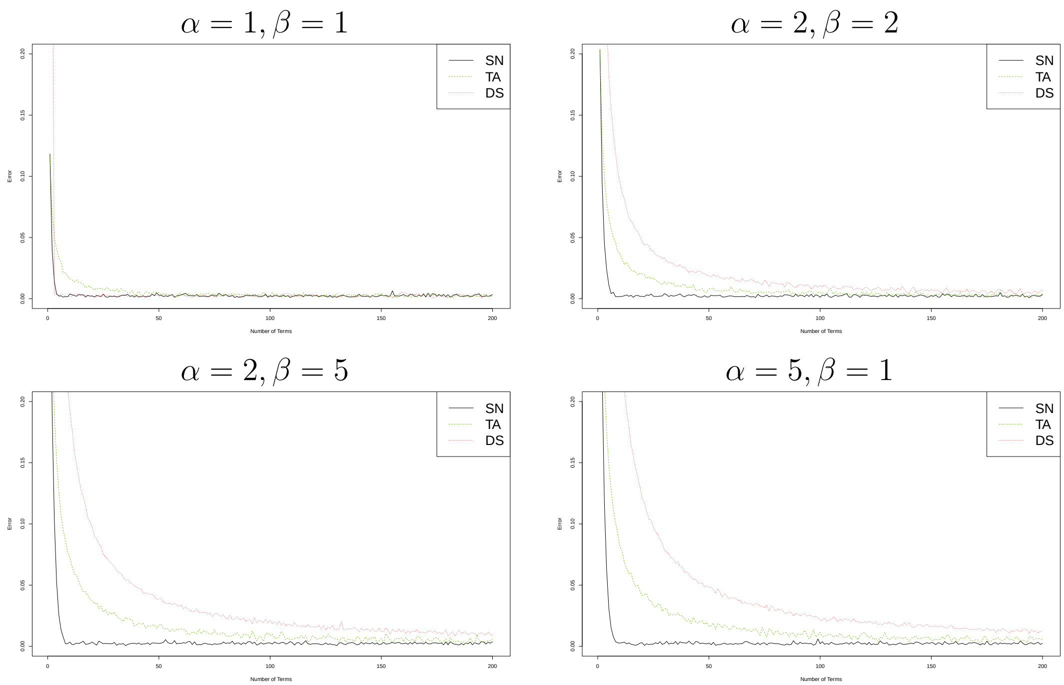

Figure 4.2: Plots of errors in the beta model with all three methods and several choices of the parameters. The x-axis represents k, the number of terms in the sum, and the y-axis represents the errors.

covariance.

In our experiment, we compare the result of the simulation under different combinations of the parameters (α, β) *of the Beta distribution:* $(1, 1)$, $(2, 2)$, $(2, 5)$, *and* $(5, 1)$. *We use Monte Carlo to generate random variables and run* $160,000$ *replications. Generally speaking, the SN method converges quickly in all the cases; the DS method and the TA method need more terms to converge. All these methods are approximations, however, in the discrete case, the DS method is an exact method instead of an approximation. In our experiment, the TA method is only a simplified version of Proposition 10. People can choose different distributions other than the uniform distribution.*

CHAPTER 5: The Density of the Truncated Subordinator

In previous sections, we studied the limit properties of the truncated Lévy process. Now, let us turn to study the distributional properties of the density.

Let $X \sim \mathrm{ID}_0(0, \nu, 0)$ with the density function f_X. The characteristic function of X_t is $\hat{\mu}_{X_t}(u) = e^{t\int_{\mathbb{S}^{d-1}}\int_0^\infty (e^{i\langle u, r\xi\rangle}-1)\nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)}$. Let X^b be the truncated Lévy process with the Lévy measure ν_b which is upper bounded by a level $b > 0$ and has the density function f_b. The characteristic function is $\hat{\mu}_{X_t^b}(u) = e^{\left\{t\int_{\mathbb{S}^{d-1}}\int_0^b (e^{i\langle u, r\xi\rangle}-1)\nu_b(\mathrm{d}r)\sigma(\mathrm{d}\xi)\right\}}$. Define $V = \displaystyle\int_{\mathbb{S}^{d-1}}\int_b^\infty \nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)$.

Proposition 15. *The density of the truncated Lévy process is*

$$
\begin{aligned}
f_b(x) \quad = \quad & e^V \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \\
& f_X(x - y_1 - y_2 - \cdots - y_n)\mathbb{I}(|y_1| > b)\cdots\mathbb{I}(|y_n| > b)\nu(\mathrm{d}y_1)\cdots\nu(\mathrm{d}y_n)
\end{aligned}
$$

Proof of Proposition 15. By Proposition 2.5 xii in [8],

$$\begin{aligned}
f_b(x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}\hat{\mu}_{Z_1}(z)\mathrm{d}z \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}e^{\int_{\mathbb{S}^{d-1}}\int_0^b(e^{i\langle z,r\xi\rangle}-1)\nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)}\mathrm{d}z \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}\Big[e^{\int_{\mathbb{S}^{d-1}}\int_0^\infty(e^{i\langle z,r\xi\rangle}-1)\nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)} \\
&\qquad e^{-\int_{\mathbb{S}^{d-1}}\int_b^\infty(e^{i\langle z,r\xi\rangle}-1)\nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)}\Big]\mathrm{d}z \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}e^{\int_{\mathbb{S}^{d-1}}\int_0^\infty(e^{i\langle z,r\xi\rangle}-1)\nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)}e^{\int_{\mathbb{S}^{d-1}}\int_b^\infty\nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)} \\
&\qquad e^{-\int_{\mathbb{S}^{d-1}}\int_b^\infty e^{i\langle z,r\xi\rangle}\nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)}\mathrm{d}z \\
&= \frac{e^V}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}e^{\int_{\mathbb{S}^{d-1}}\int_0^\infty(e^{i\langle z,r\xi\rangle}-1)\nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)} \\
&\qquad \sum_{n=0}^\infty\frac{(-1)^n}{n!}\left(\int_{\mathbb{S}^{d-1}}\int_b^\infty e^{i\langle z,r\xi\rangle}\ \nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)\right)^n\mathrm{d}z \\
&= \frac{e^V}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}\hat{\mu}_{X_1}(z)\sum_{n=0}^\infty\frac{(-1)^n}{n!}\left(\int_{\mathbb{S}^{d-1}}\int_b^\infty e^{i\langle z,r\xi\rangle}\ \nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)\right)^n\mathrm{d}z \\
&= \frac{e^V}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}\hat{\mu}_{X_1}(z)\,\Bigg(1+ \\
&\qquad \sum_{n=1}^\infty\frac{(-1)^n}{n!}\left(\int_{\mathbb{S}^{d-1}}\int_b^\infty e^{i\langle z,r\xi\rangle}\ \nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)\right)^n\Bigg)\mathrm{d}z \\
&= e^V f_X(x)+\frac{e^V}{(2\pi)^d} \\
&\qquad \int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}\hat{\mu}_{X_1}(z)\sum_{n=1}^\infty\frac{(-1)^n}{n!}\left(\int_{\mathbb{S}^{d-1}}\int_b^\infty e^{i\langle z,r\xi\rangle}\ \nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)\right)^n\mathrm{d}z
\end{aligned}$$

When $n = 1$,

$$\begin{aligned}
& -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}\hat{\mu}_{X_1}(z)\int_{\mathbb{S}^{d-1}}\int_b^\infty e^{i\langle z,r\xi\rangle}\ \nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)\mathrm{d}z \\
= & -\int_{\mathbb{S}^{d-1}}\int_b^\infty \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x-r\xi\rangle}\hat{\mu}_{X_1}(z)\mathrm{d}z\nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi) \\
= & -\int_{\mathbb{S}^{d-1}}\int_b^\infty f_X(x-r\xi)\nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi) \\
= & -\int_{\mathbb{R}^d} f_X(x-y)\mathbb{I}(|y|>b)\nu(\mathrm{d}y)
\end{aligned}$$

When $n = 2$,

$$\begin{aligned}
& \frac{1}{2}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}\hat{\mu}_{X_1}(z)\left(\int_{\mathbb{S}^{d-1}}\int_b^\infty e^{i\langle z,r\xi\rangle}\ \nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)\right)^2\mathrm{d}z \\
= & \frac{1}{2}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}\hat{\mu}_{X_1}(z)\int_{\mathbb{R}^d} e^{i\langle z,y_1\rangle}\mathbb{I}(|y_1|>b)\nu(\mathrm{d}y_1) \\
& \qquad \int_{\mathbb{R}^d} e^{i\langle z,y_2\rangle}\mathbb{I}(|y_2|>b)\nu(\mathrm{d}y_2)\mathrm{d}z \\
= & \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x-y_1-y_2\rangle}\hat{\mu}_{X_1}(z)\mathrm{d}z\mathbb{I}(|y_1|>b)\mathbb{I}(|y_2|>b)\nu(\mathrm{d}y_1)\nu(\mathrm{d}y_2) \\
= & \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_X(x-y_1-y_2)\mathbb{I}(|y_1|>b)\mathbb{I}(|y_2|>b)\nu(\mathrm{d}y_1)\nu(\mathrm{d}y_2)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \frac{(-1)^n}{n!}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}\hat{\mu}_{X_1}(z)\left(\int_{\mathbb{S}^{d-1}}\int_b^\infty e^{i\langle z,r\xi\rangle}\ \nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)\right)^n\mathrm{d}z \\
= & \frac{(-1)^n}{n!}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x\rangle}\hat{\mu}_{X_1}(z)\int_{\mathbb{R}^d} e^{i\langle z,y_1\rangle}\mathbb{I}(|y_1|>b)\nu(\mathrm{d}y_1)\cdots \\
& \qquad \int_{\mathbb{R}^d} e^{i\langle z,y_n\rangle}\mathbb{I}(|y_n|>b)\nu(\mathrm{d}y_n)\mathrm{d}z \\
= & \frac{(-1)^n}{n!}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle z,x-y_1-\cdots-y_n\rangle}\hat{\mu}_{X_1}(z)\mathrm{d}u \\
& \qquad \mathbb{I}(|y_1|>b)\cdots\mathbb{I}(|y_n|>b)\nu(\mathrm{d}y_1)\cdots\nu(\mathrm{d}y_n) \\
= & \frac{(-1)^n}{n!}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} f_X(x-y_1-y_2-\cdots-y_n) \\
& \qquad \mathbb{I}(|y_1|>b)\cdots\mathbb{I}(|y_n|>b)\nu(\mathrm{d}y_1)\cdots\nu(\mathrm{d}y_n)
\end{aligned}$$

Thus

$$\begin{aligned} f_b(x) &= e^V \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f_X(x - y_1 - y_2 - \cdots - y_n) \\ &\qquad \mathbb{I}(|y_1| > b) \cdots \mathbb{I}(|y_n| > b) \nu(\mathrm{d}y_1) \cdots \nu(\mathrm{d}y_n) \end{aligned}$$

■

Remark 15. *the density f_b of X_t within $(0, b)$ is $f_b(x)\mathbb{I}(0 < x < b) = e^V f_{X_t}(x)\mathbb{I}(0 < x < b)$*

Corollary 2. *Define $\mu_b = \frac{\nu|_{|y|>b}}{V}$, then μ_b is a probability measure. Suppose $Y_1, Y_2, \cdots, Y_n \overset{\text{iid}}{\sim} \mu_b$, then*

$$\begin{aligned} f_b(z) &= e^V \sum_{n=0}^{\infty} \frac{(-V)^n}{n!} f_{X_1+Y_1+\cdots+Y_n}(z), \text{ and} \\ f_{X_1+Y_1+\cdots+Y_n}(z) &= \frac{1}{V} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f_X(z - y_1 - y_2 - \cdots - y_n) \\ &\qquad \mathbb{I}(|y_1| > b) \cdots \mathbb{I}(|y_n| > b) \nu(\mathrm{d}y_1) \cdots \nu(\mathrm{d}y_n). \end{aligned}$$

Proof of Corollary 2. Since $Y_1, Y_2, \cdots, Y_n \overset{\text{iid}}{\sim} \mu_b$, then $\hat{\mu}_{X_1+Y_1+\cdots+Y_n}(u) = \hat{\mu}_{X_1}(u) \left(\mathrm{E}(e^{i\langle u,Y\rangle})\right)^n$. Since $\int_{\mathbb{S}^{d-1}} \int_b^{\infty} e^{i\langle u, r\xi\rangle} \ \nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi) = V \int_{\mathbb{R}^d} e^{i\langle u,y\rangle} \mu_b(\mathrm{d}y) = V\mathrm{E}(e^{i\langle u,Y\rangle})$, then, if $Y \sim \nu_b$,

$$\begin{aligned} & \int_{\mathbb{R}^d} e^{-i\langle z,u\rangle} \hat{\mu}_{X_1}(u) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int_{\mathbb{S}^{d-1}} \int_b^{\infty} e^{i\langle u,r\xi\rangle} \ \nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi) \right)^n \mathrm{d}u \\ = & \int_{\mathbb{R}^d} e^{-i\langle z,u\rangle} \hat{\mu}_{X_1}(u) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(V\mathrm{E}(e^{i\langle u,Y\rangle}) \right)^n \mathrm{d}u \\ = & \int_{\mathbb{R}^d} e^{-i\langle z,u\rangle} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} V^n \hat{\mu}_{X_1}(u) \left(\mathrm{E}(e^{i\langle u,Y\rangle}) \right)^n \mathrm{d}u \end{aligned}$$

Thus,

$$\begin{aligned}
& \int_{\mathbb{R}^d} e^{-i\langle z,u\rangle}\hat{\mu}_{X_1}(u)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\int_{\mathbb{S}^{d-1}}\int_b^{\infty} e^{i\langle u,r\xi\rangle}\ \nu_\xi(\mathrm{d}r)\sigma(\mathrm{d}\xi)\right)^n \mathrm{d}u \\
= & \int_{\mathbb{R}^d}\sum_{n=0}^{\infty}\frac{(-V)^n}{n!}e^{-i\langle z,u\rangle}\hat{\mu}_{X_1+Y_1+\cdots+Y_n}(u)\mathrm{d}u \\
= & \int_{\mathbb{R}^d}\lim_{m\to\infty}\sum_{n=0}^{m}\frac{(-V)^n}{n!}e^{-i\langle z,u\rangle}\hat{\mu}_{X_1+Y_1+\cdots+Y_n}(u)\mathrm{d}u \\
= & \lim_{m\to\infty}\int_{\mathbb{R}^d}\sum_{n=0}^{m}\frac{(-V)^n}{n!}e^{-i\langle z,u\rangle}\hat{\mu}_{X_1+Y_1+\cdots+Y_n}(u)\mathrm{d}u \\
= & \lim_{m\to\infty}\sum_{n=0}^{m}\frac{(-V)^n}{n!}\int_{\mathbb{R}^d} e^{-i\langle z,u\rangle}\hat{\mu}_{X_1+Y_1+\cdots+Y_n}(u)\mathrm{d}u \\
= & \sum_{n=0}^{\infty}\frac{(-V)^n}{n!}\int_{\mathbb{R}^d} e^{-i\langle z,u\rangle}\hat{\mu}_{X_1+Y_1+\cdots+Y_n}(u)\mathrm{d}u \\
= & (2\pi)^d\sum_{n=0}^{\infty}\frac{(-V)^n}{n!}f_{X_1+Y_1+\cdots+Y_n}(z)
\end{aligned}$$

Then, by Proposition 15, $f_b(z) = \frac{e^V}{(2\pi)^d}(2\pi)^d\sum_{n=0}^{\infty}\frac{(-V)^n}{n!}f_{X_1+Y_1+\cdots+Y_n}(z) = e^V\sum_{n=0}^{\infty}\frac{(-V)^n}{n!}f_{X_1+Y_1+\cdots+Y_n}(z)$. By comparing with the result in Proposition 15, we have

$$\begin{aligned}
f_{X_1+Y_1+\cdots+Y_n}(z) = & \frac{1}{V}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} f_X(z-y_1-y_2-\cdots-y_n) \\
& \mathbb{I}(|y_1|>b)\cdots\mathbb{I}(|y_n|>b)\nu(\mathrm{d}y_1)\cdots\nu(\mathrm{d}y_n)
\end{aligned}$$

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Remark 16. *Furthermore, we have, for $z\in\mathbb{R}^d$,*

$$\begin{aligned}
F_{X_1^b}(z) = & \int_{-\infty}^{z_1}\cdots\int_{-\infty}^{z_d} e^V\sum_{n=0}^{\infty}\frac{(-V)^n}{n!}f_{X_1+Y_1+\cdots+Y_n}(z)\mathrm{d}z_1\cdots\mathrm{d}z_d \\
= & e^V\sum_{n=0}^{\infty}\frac{(-V)^n}{n!}F_{X_1+Y_1+\cdots+Y_n}(z)
\end{aligned}$$

Definition 14. *Given a Lévy process* $\{Z_t : t \geq 0\}$ *on* $\mathbb{R}^d$ *and* $c > 0$*. Let* $\{X_t : t \geq 0\}$ *be a stochastic process such that* $\mathrm{d}X_t = -cX_t\mathrm{d}t + \mathrm{d}Z_t$*, then* $\{X_t\}$ *is a* ***OU-process****.*

The differential equation in Definition 14 has an almost surely unique solution $X_t = \int_0^t e^{-c(t-s)}\mathrm{d}Z_s$, see Proposition 2.9 in Arteaga and Sato [27]. We call $\{X_t\}$ having this integral form a stochastic integral process, for a detailed introduction of the stochastic integral process see [9] and [28]. Now, let's consider a more general stochastic integral process $X_t = \int_0^t f_t(s)\mathrm{d}Z_s$ where $\{Z_t\}$ is a Lévy process having the Lévy measure ν_z such that $\int_{\mathbb{R}^d}(1 \wedge |x|)\nu_z(\mathrm{d}x) < \infty$ and f_t is a positive real-valued bounded measurable function on finite interval depending on the time t, and we denote the Lévy measure of X_t as ν_{X_t}. The characteristic function of X_t is (see Proposition 2.2 in Arteaga and Sato [27])

$$\mathrm{E}(e^{i\langle z,X_t\rangle}) = \exp\left\{\int_0^t \psi_{Z_1}(f_t(s)z)\mathrm{d}s\right\} = \exp\left\{\int_0^t \int_{\mathbb{R}^d}(e^{i\langle f_t(s)z,x\rangle} - 1)\nu_z(\mathrm{d}x)\mathrm{d}s\right\} \tag{6.1}$$

So $\nu_{X_t}(B) = \int_0^t \int_{\mathbb{R}^d} \mathbb{I}_B(f_t(s)x)\nu_z(\mathrm{d}x)\mathrm{d}s$.

Assume $\forall\gamma > 0$, $\{S_t^\gamma\}$ is a stochastic integral process having the solution $S_t^\gamma = \int_0^t f_t(s)\mathrm{d}Y_s^\gamma$ where $\{Y_t^\gamma\}$ is a Lévy process having the Dickman-type Lévy measure $\mathrm{D}^\gamma(B) = \int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbb{I}_B(r\xi)\mathbb{I}_{(0,\gamma]}(r)r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)$. We denote the Lévy measure of S_t^γ as $\nu_{S_t^\gamma}$.

Proposition 16. *The characteristic function of* $\dfrac{S_t^\gamma}{\gamma}$ *is*

$$\mathrm{E}(e^{i\langle z,\frac{S_t^\gamma}{\gamma}\rangle}) = \exp\left\{\int_0^t \int_{\mathbb{S}^{d-1}}\int_0^\infty (e^{i\langle z,r\xi\rangle} - 1)\mathbb{I}_{(0,1\wedge f_t(s)]}(r)^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\mathrm{d}s\right\}$$

***Proof of Proposition 16*.**

$$\begin{aligned}
\mathrm{E}(e^{i\langle z,\frac{S_t^\gamma}{\gamma}\rangle}) &= \exp\left\{\int_0^t\int_{\mathbb{R}^d}(e^{i\langle f_t(s)z,\frac{x}{\gamma}\rangle}-1)\mathrm{D}^\gamma(\mathrm{d}x)\mathrm{d}s\right\}\\
&= \exp\left\{\int_0^t\int_{\mathbb{S}^{d-1}}\int_0^\infty(e^{i\langle f_t(s)z,\frac{r}{\gamma}\xi\rangle}-1)\mathbb{I}_{(0,\gamma]}(r)r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\mathrm{d}s\right\}\\
&= \exp\left\{\int_0^t\int_{\mathbb{S}^{d-1}}\int_0^\infty(e^{i\langle f_t(s)z,r\xi\rangle}-1)\mathbb{I}_{(0,1]}(r)r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\mathrm{d}s\right\}\\
&= \exp\left\{\int_0^t\int_{\mathbb{S}^{d-1}}\int_0^\infty(e^{i\langle z,r\xi\rangle}-1)\mathbb{I}_{(0,1\wedge f_t(s)]}(r)r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\mathrm{d}s\right\}
\end{aligned}$$

■

Remark 17. *It's obvious that from the third equation in the proof of Proposition 16,*

$$\frac{S_t^\gamma}{\gamma}\stackrel{d}{=}S_t^1. \tag{6.2}$$

However, S_t^γ does not have stationary increments, thus it is not a Lévy process, Proposition 4 does not apply, and $\nu_{S_t^\gamma}\neq\mathrm{D}^\gamma$.

Remark 18. *Assume M_t^γ is the Lévy measure of $\frac{S_t^\gamma}{\gamma}$, then from the proof of Proposition 16 we know that*

$$\mathrm{M}_t^\gamma(B)=\nu_{S_t^1}(B)=\int_0^t\int_{\mathbb{S}^{d-1}}\int_0^1\mathbb{I}_B(f_t(s)r\xi)r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\mathrm{d}s \tag{6.3}$$

Recall, in Theorem 1, we truncate the Lévy process and then transform it. For the stochastic integral process, we have two ways to get the same transformation: we directly truncate the stochastic process X_t or truncate the background driving Lévy process Z_t.

6.1 Truncate the Stochastic Integral Process

Suppose $\{X_t^\epsilon\}$ is a truncated stochastic process having Lévy measure $\nu_{X_t^\epsilon}(B) = \int_{\mathbb{R}^d} \mathbb{I}_{(0,\epsilon]}(|x|)\nu_{X_t}(\mathrm{d}x) = \int_0^t \int_{\mathbb{R}^d} \mathbb{I}_B(f_t(s)x)\mathbb{I}_{(0,\epsilon]}(|x|)\nu_z(\mathrm{d}x)\mathrm{d}s$ for $B \in \mathscr{B}(\mathbb{R}^d)$. Let Y_t^1 be defined as in Theorem 1, i.e. $Y^1 \sim \mathrm{ID}(0, \mathrm{D}^1, 0)$. Define $\mu_z(\epsilon, C) = \int_{\mathbb{R}^d} |x|\mathbb{I}_{(0,\epsilon]C}(x)\nu_z(\mathrm{d}x)$, for any $C \in \mathscr{B}(\mathbb{S}^{d-1})$.

Proposition 17. *If,* $\forall C \in \mathscr{B}(\mathbb{S}^{d-1})$ *with* $\sigma(\partial C) = 0$ *and* $\forall \epsilon \in (0,1]$, $\dfrac{\mu_z(\epsilon, C)}{\epsilon} \to \sigma(C)$ *as* $\epsilon \downarrow 0$, *then* $\dfrac{X_t^\epsilon}{\epsilon} \xrightarrow{d} Y_t^1$.

Proof of Proposition 17.

$$
\begin{aligned}
\frac{\mu_X(\epsilon, C)}{\epsilon} &= \frac{1}{\epsilon}\int_{\mathbb{R}^d} |x|\mathbb{I}_{(0,\epsilon]C}(x)\nu_X(\mathrm{d}x) \\
&= \frac{1}{\epsilon}\int_0^t \int_{\mathbb{R}^d} |f_t(s)x|\mathbb{I}_{(0,\epsilon]C}(f_t(s)x)\nu_z(\mathrm{d}x)\mathrm{d}s \\
&= \frac{1}{\epsilon}\int_0^t \int_{\mathbb{R}^d} f_t(s)|x|\mathbb{I}_{(0,\epsilon/f_t(s)]C}(x)\nu_z(\mathrm{d}x)\mathrm{d}s \\
&= \int_0^t \frac{\int_{\mathbb{R}^d} |x|\mathbb{I}_{(0,\epsilon/f_t(s)]C}(x)\nu_z(\mathrm{d}x)}{\epsilon/f_t(s)}\mathrm{d}s \\
&= \int_0^t \frac{\mu_X(\epsilon/f_t(s), C)}{\epsilon/f_t(s)}\mathrm{d}s \\
&= t\sigma(C)
\end{aligned}
$$

Define $\sigma^*(C) = t\sigma(C)$, obviously $\sigma^*(\partial C) = 0$, by Theorem 1, we conclude that $\dfrac{X_t^\epsilon}{\epsilon} \xrightarrow{d} Y_t^1$. ■

In the next example, we show how to use Proposition 17 and consider the background driving process as a Gamma process.

Example 6. *If* Z_t *has the Lévy measure* $\nu_z(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_B(r\xi)r^{-1}e^{-r}\mathrm{d}r\sigma(\mathrm{d}\xi)$ *for any Borel set* B, *then* $\{Z_t\}$ *is a Gamma process. Assume the stochastic integral* *process* $\{X_t\}$ *is driven by this Gamma process.*

$$\begin{aligned}
\frac{\mu_z(\epsilon, C)}{\epsilon} &= \frac{1}{\epsilon}\int_{\mathbb{R}^d} |x|\mathbb{I}_{(0,\epsilon]C}(x)\nu_z(\mathrm{d}x) \\
&= \frac{1}{\epsilon}\int_{\mathbb{S}^{d-1}}\int_0^\infty |r\xi|\mathbb{I}_{(0,\epsilon]C}(r\xi)r^{-1}e^{-r}\mathrm{d}r\sigma(\mathrm{d}\xi) \\
&= \frac{1}{\epsilon}\int_C\int_0^\epsilon e^{-r}\mathrm{d}r\sigma(\mathrm{d}\xi) \\
&= \sigma(C)\frac{1-e^{-\epsilon}}{\epsilon} \\
&\to \sigma(C)
\end{aligned}$$

Then, by Proposition 17, $\dfrac{X_t^\epsilon}{\epsilon} \xrightarrow{d} Y_t^1$.

6.2 Truncate the Driving Lévy Process

Suppose $\{X_t^\epsilon\}$ is driven by a truncated Lévy process $\{Z_t^\epsilon\}$ and we denote the Lévy measure of X_t^ϵ as $\nu_{X_t^\epsilon}$. Define $\nu_{f_t}(B) = \int_{\mathbb{R}^d} \mathbb{I}_B(f_t(s)x)\mathbb{I}_{(0,\epsilon]}(|x|)\nu_z(\mathrm{d}x)$ for any $B \in \mathscr{B}(\mathbb{R}^d)$, the characteristic function of X_t^ϵ is

$$\begin{aligned} \mathrm{E}(e^{i\langle z, X_t^\epsilon\rangle}) &= \exp\left\{\int_0^t \psi_{Z_1^\epsilon}(f_t(s)z)\mathrm{d}s\right\} \\ &= \exp\left\{\int_0^t \int_{\mathbb{R}^d} (e^{i\langle f_t(s)z, x\rangle} - 1)\mathbb{I}_{(0,\epsilon]}(|x|)\nu_z(\mathrm{d}x)\mathrm{d}s\right\} \\ &= \exp\left\{\int_0^t \int_{\mathbb{R}^d} (e^{i\langle z, x\rangle} - 1)\nu_{f_t}(\mathrm{d}x)\mathrm{d}s\right\} \end{aligned}$$

then $\nu_{X_t}^\epsilon(B) = \int_0^t \int_{\mathbb{R}^d} \mathbb{I}_B(x)\nu_{f_t}(\mathrm{d}x)\mathrm{d}s$.

Suppose, for $\gamma > 0$, $S_t^\gamma = \int_0^t f_t(s)\mathrm{d}Y_t^\gamma$ where $\{Y_t^\gamma\}$ is the truncated Lévy process having Dickman-type Lévy measure D^γ. From Remark 18, the Lévy measure of $\frac{S_t^\gamma}{\gamma}$ is $\mathrm{M}_t^\gamma(B) = \int_0^t \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_B(r\xi)\mathbb{I}_{(0,f_t(s)]}(r)r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\mathrm{d}s$. Define $\mu_z(\epsilon, C) = \int_{\mathbb{R}^d} |x|\mathbb{I}_{(0,\epsilon]C}(x)\nu_z(\mathrm{d}x)$, for any $C \in \mathscr{B}(\mathbb{S}^{d-1})$.

Theorem 2. *If,* $\forall C \in \mathscr{B}(\mathbb{S}^{d-1})$ *with* $\sigma(\partial C) = 0$, $\frac{\mu_z(\epsilon, C)}{\epsilon} \to \sigma(C)$ *as* $\epsilon \downarrow 0$*, then, for any* $t > 0$, $\frac{X_t^\epsilon}{\epsilon} \xrightarrow{d} \frac{S_t^\gamma}{\gamma}$ *as* $\epsilon \to 0$.

Proof of Theorem 2. The same as Theorem 1, it suffices to prove

1. $\mathrm{M}^\epsilon \xrightarrow{v} \mathrm{M}_t^\gamma$.
2. $\int_{h<|x|\le 1} x\mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{h<|x|\le 1} x\mathrm{M}_t^\gamma(\mathrm{d}x)$ and $\int_{|x|\le h} xx^T\mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} xx^T\mathrm{M}_t^\gamma(\mathrm{d}x)$, for every $h > 0$.

Define $\eta_\epsilon(\mathrm{d}x) = |x|\mathrm{M}_t^\epsilon(\mathrm{d}x)$ and $\eta(\mathrm{d}x) = |x|\mathrm{M}_t^\gamma(\mathrm{d}x)$. For any $0 < h$,

$$\begin{aligned}
\eta_\epsilon((0,h]C) &= \int_{(0,h]C} |x|\mathrm{M}_t^\epsilon(\mathrm{d}x) \\
&= \int_{(0,\epsilon h]C} \frac{|x|}{\epsilon}\nu_{X_t^\epsilon}(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} \frac{|x|}{\epsilon}\mathbb{I}_{(0,\epsilon h]C}(x)\nu_{X_t^\epsilon}(\mathrm{d}x) \\
&= \int_0^t \int_{\mathbb{R}^d} \frac{|x|}{\epsilon}\mathbb{I}_{(0,\epsilon h]C}(x)\nu_f(\mathrm{d}x)\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}^d} f_t(s)\frac{|x|}{\epsilon}\mathbb{I}_{(0,h\epsilon]C}(f_t(s)x)\mathbb{I}_{(0,\epsilon]}(|x|)\nu_z(\mathrm{d}x)\mathrm{d}s \\
&= \int_0^t f_t(s)\frac{1}{\epsilon}\mu((1\wedge h/f_t(s)))\epsilon, C)\mathrm{d}s
\end{aligned}$$

Thus, given $\dfrac{\mu(\epsilon, C)}{\epsilon} \to \sigma(C)$, by dominated convergence theorem,

$$\eta_\epsilon((0,h]C) \to \int_0^t f_t(s)(1\wedge h/f_t(s))\sigma(C)\mathrm{d}s = \sigma(C)\int_0^t (h \wedge f_t(s))\mathrm{d}s$$

$$\begin{aligned}
\eta((0,h]C) &= \int_{(0,h]C} |x|\mathrm{M}_t^\gamma(\mathrm{d}x) \\
&= \int_0^t \int_{\mathbb{S}^{d-1}} \int_0^\infty |r\xi|\mathbb{I}_{(0,h]C}(r\xi)\mathbb{I}_{(0,f_t(s)]}(r)r^{-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\mathrm{d}s \\
&= \sigma(C)\int_0^t \int_0^\infty \mathbb{I}_{(0,h\wedge f_t(s))}(r)\mathrm{d}r\mathrm{d}s \\
&= \sigma(C)\int_0^t (h\wedge f_t(s))\mathrm{d}s
\end{aligned}$$

Thus, $\eta_\epsilon((0,h]C) \to \eta((0,h]C)$ as $\epsilon \to 0$ for any $h > 0$. Then the same as the proof in Lemma 2, $\eta_\epsilon \xrightarrow{v} \eta$. Also, the same as the proof of Theorem 1, $\int_{h<|x|\le 1} x\mathrm{M}^\epsilon(\mathrm{d}x) \to \int_{h<|x|\le 1} x\mathrm{M}_t^\gamma(\mathrm{d}x)$. By Lemma 3, to show $\int_{|x|\le h} xx^T\mathrm{M}_t^\epsilon(\mathrm{d}x) \to \int_{|x|\le h} xx^T\mathrm{M}_t^\eta(\mathrm{d}x)$, it suffices to show $\lim_{h\to 0}\lim_{\epsilon\to 0}\int_{|x|\le h}\langle z, x\rangle^2\mathrm{M}_t^\epsilon(\mathrm{d}x) = 0$.

$$
\begin{aligned}
0 &\le \lim_{h\to 0}\limsup_{\epsilon\to 0}\int_{|x|\le h}\langle z,x\rangle^2 \mathrm{M}_t^\epsilon(\mathrm{d}x)\\
&\le \lim_{h\to 0}\limsup_{\epsilon\to 0}\int_{|x|\le h}|z|^2|x|^2 \mathrm{M}_t^\epsilon(\mathrm{d}x)\\
&\le \lim_{h\to 0}\limsup_{\epsilon\to 0}|z|^2\int_{|x|\le h}|x| \mathrm{M}_t^\epsilon(\mathrm{d}x)\\
&\le \lim_{h\to 0}\limsup_{\epsilon\to 0}|z|^2\int_{|x|\le 1}|x| \mathrm{M}_t^\epsilon(\mathrm{d}x)\\
&= |z|^2\lim_{h\to 0}\limsup_{\epsilon\to 0}\int_{|x|\le \epsilon}\frac{|x|}{\epsilon}\nu_{X_t^\epsilon}(\mathrm{d}x)\\
&= |z|^2\lim_{h\to 0}\limsup_{\epsilon\to 0}\int_0^t\int_{\mathbb{R}^d}\frac{|f_t(s)x|}{\epsilon}\mathbb{I}_{(0,\epsilon]}(|f_t(s)x|)\mathbb{I}_{(0,\epsilon]}(|x|)\nu_z(\mathrm{d}x)\mathrm{d}s\\
&= |z|^2\lim_{h\to 0}\limsup_{\epsilon\to 0}\int_0^t\int_{\mathbb{R}^d} f_t(s)\mathbb{I}_{(0,(1\wedge\epsilon/f_t(s))]\mathbb{S}^{d-1}}(x)\nu_z(\mathrm{d}x)\mathrm{d}s\\
&= |z|^2\lim_{h\to 0}\limsup_{\epsilon\to 0}\int_0^t f_t(s)\frac{1}{\epsilon}\mu((1\wedge 1/f_t(s))\epsilon,\mathbb{S}^{d-1})\mathrm{d}s\\
&= |z|^2\sigma(\mathbb{S}^{d-1})\int_0^t(1\wedge 1/f_t(s))\mathrm{d}s\lim_{h\to 0}h\\
&\le |z|^2\sigma(\mathbb{S}^{d-1})\int_0^t\mathrm{d}s\lim_{h\to 0}h\\
&= 0
\end{aligned}
$$

■

Theorem 2 tells us for any $t > 0$, $\mathrm{E}(e^{i\langle z,\frac{X_t^\epsilon}{\epsilon}\rangle}) \to \mathrm{E}(e^{i\langle z,\frac{S_t^\gamma}{\gamma}\rangle})$ as $\epsilon \to 0$. However, this one-dimensional convergence only involves random variables. It does not show properties related to the process. In the next corollary, we study the property of increments of the stochastic integral process and show that the increments are also convergent.

Corollary 3. *If,* $\forall C \in \mathscr{B}(\mathbb{S}^{d-1})$ *with* $\sigma(\partial C) = 0$, $\dfrac{\mu_z(\epsilon, C)}{\epsilon} \to \sigma(C)$ *as* $\epsilon \downarrow 0$, *then, for any* $0 < r < t$, $\dfrac{X_t^\epsilon - X_r^\epsilon}{\epsilon} \xrightarrow{d} \dfrac{S_t^\gamma - S_r^\gamma}{\gamma}$ *as* $\epsilon \to 0$.

Proof of Corollary 3.

$$\begin{aligned}
\mathrm{E}(e^{i\langle u,\frac{X_t^\epsilon}{\epsilon}\rangle}) &= \mathrm{E}(e^{\frac{i}{\epsilon}\sum_{n=1}^d u_n\int_0^t f_t(s)\mathrm{d}Z_{n,s}^\epsilon}) \\
&= \mathrm{E}(e^{\frac{i}{\epsilon}\sum_{n=1}^d u_n(\int_0^r f_t(s)\mathrm{d}Z_{n,s}^\epsilon+\int_r^t f_t(s)\mathrm{d}Z_{n,s}^\epsilon)}) \\
&= \mathrm{E}(e^{\frac{i}{\epsilon}\sum_{n=1}^d u_n\int_0^r f_t(s)\mathrm{d}Z_{n,s}^\epsilon})\mathrm{E}(e^{\frac{i}{\epsilon}\sum_{n=1}^d u_n\int_r^t f_t(s)\mathrm{d}Z_{n,s}^\epsilon})
\end{aligned}$$

The last equation holds because f is deterministic, and the increments of the Lévy process are independent. From Theorem 2, we know $\mathrm{E}(e^{i\langle u,\frac{X_t^\epsilon}{\epsilon}\rangle}) \to \mathrm{E}(e^{i\langle z,\frac{S_t^\gamma}{\gamma}\rangle})$ and $\mathrm{E}(e^{\frac{i}{\epsilon}\sum_{n=1}^d u_n\int_0^r f_t(s)\mathrm{d}Z_{n,s}^\epsilon}) \to \mathrm{E}(e^{\frac{i}{\gamma}\sum_{n=1}^d u_n\int_0^r f_t(s)\mathrm{d}Y_{n,s}^\gamma})$, thus we must have $\mathrm{E}(e^{\frac{i}{\epsilon}\sum_{n=1}^d u_n\int_r^t f_t(s)\mathrm{d}Z_{n,s}^\epsilon}) \to \mathrm{E}(e^{\frac{i}{\gamma}\sum_{n=1}^d u_n\int_r^t f_t(s)\mathrm{d}Y_{n,s}^\gamma})$.

$$\begin{aligned}
\mathrm{E}(e^{i\langle u,\frac{X_t^\epsilon-X_r^\epsilon}{\epsilon}\rangle}) &= \mathrm{E}(e^{\frac{i}{\epsilon}\sum_{n=1}^d u_n\left(\int_0^r (f_t(s)-f_r(s))\mathrm{d}Z_{n,s}^\epsilon+\int_r^t f_t(s)\mathrm{d}Z_{n,s}^\epsilon\right)}) \\
&= \mathrm{E}(e^{\frac{i}{\epsilon}\sum_{n=1}^d u_n\int_0^r (f_t(s)-f_r(s))\mathrm{d}Z_{n,s}^\epsilon})\mathrm{E}(e^{\frac{i}{\epsilon}\sum_{n=1}^d u_n\int_r^t f_t(s)\mathrm{d}Z_{n,s}^\epsilon}) \\
&\to \mathrm{E}(e^{\frac{i}{\gamma}\sum_{n=1}^d u_n\int_0^r (f_t(s)-f_r(s))\mathrm{d}Y_{n,s}^\gamma})\mathrm{E}(e^{\frac{i}{\gamma}\sum_{n=1}^d u_n\int_r^t f_t(s)\mathrm{d}Y_{n,s}^\gamma}) \\
&= \mathrm{E}(e^{\frac{i}{\gamma}\sum_{n=1}^d u_n\left(\int_0^r (f_t(s)-f_r(s))\mathrm{d}Y_{n,s}^\gamma+\int_r^t f_t(s)\mathrm{d}Y_{n,s}^\gamma\right)}) \\
&= \mathrm{E}(e^{i\langle u,\frac{S_t^\gamma-S_r^\gamma}{\gamma}\rangle})
\end{aligned}$$

■

To further study the limit property of the stochastic integral process, we provide the convergence of finite-dimensional distribution in the next proposition.

Proposition 18. *For any $k \in \mathbb{N}$, let $0 = t_0 < t_1 < t_2 < \cdots < t_k$, $X = (X_{t_1}, X_{t_2}, \cdots, X_{t_k})$, and $S = (S_{t_1}, S_{t_2}, \cdots, S_{t_k})$. Then $\dfrac{X^\epsilon}{\epsilon} \to \dfrac{S^\gamma}{\gamma}$ as $\epsilon \to 0$.*

Proof of Proposition 18. First, the characteristic function of $\dfrac{X^\epsilon}{\epsilon}$ is $\mathrm{E}(e^{i\langle u,\frac{X^\epsilon}{\epsilon}\rangle}) = \mathrm{E}(e^{\frac{i}{\epsilon}\sum_{n=1}^k\langle u_n,X_{t_n}^\epsilon\rangle})$. Notice, for any $n = 1, 2, \cdots, k$, $\langle u_n, X_{t_n}^\epsilon\rangle = \langle u_n, X_{t_n}^\epsilon - X_{t_{n-1}}^\epsilon + X_{t_{n-1}}^\epsilon\rangle = \langle u_n, X_{t_n}^\epsilon - X_{t_{n-1}}^\epsilon\rangle + \langle u_n, X_{t_{n-1}}^\epsilon\rangle$. Thus, we have $\langle u, X^\epsilon\rangle = \langle u_1 + \cdots + u_k, X_{t_1}^\epsilon\rangle + \langle u_2 + \cdots + u_k, X_{t_2}^\epsilon - X_{t_1}^\epsilon\rangle + \cdots + \langle u_{k-1} + u_k, X_{t_{k-1}}^\epsilon - X_{t_{k-2}}^\epsilon\rangle + \langle u_k, X_{t_k}^\epsilon - X_{t_{k-1}}^\epsilon\rangle$

$$\begin{aligned}
\mathrm{E}(e^{\frac{i}{\epsilon}\langle u,X^\epsilon\rangle}) &= \mathrm{E}\left(e^{\frac{i}{\epsilon}\left(\langle u_1+\cdots+u_k,X^\epsilon_{t_1}\rangle+\langle u_2+\cdots+u_k,X^\epsilon_{t_2}-X^\epsilon_{t_1}\rangle+\cdots+\langle u_k,X^\epsilon_{t_k}-X^\epsilon_{t_{k-1}}\rangle\right)}\right)\\
&= \mathrm{E}\left(e^{\frac{i}{\epsilon}\langle u_1+\cdots+u_k,X^\epsilon_{t_1}\rangle}\right)\mathrm{E}\left(e^{\frac{i}{\epsilon}\langle u_2+\cdots+u_k,X^\epsilon_{t_2}-X^\epsilon_{t_1}\rangle}\right)\cdots\mathrm{E}\left(e^{\frac{i}{\epsilon}\langle u_k,X^\epsilon_{t_k}-X^\epsilon_{t_{k-1}}\rangle}\right)\\
&\to \mathrm{E}\left(e^{\frac{i}{\gamma}\langle u_1+\cdots+u_k,S^\gamma_{t_1}\rangle}\right)\mathrm{E}\left(e^{\frac{i}{\gamma}\langle u_2+\cdots+u_k,S^\gamma_{t_2}-S^\gamma_{t_1}\rangle}\right)\cdots\mathrm{E}\left(e^{\frac{i}{\gamma}\langle u_k,S^\gamma_{t_k}-S^\gamma_{t_{k-1}}\rangle}\right)\\
&= \mathrm{E}(e^{\frac{i}{\gamma}\langle u,S^\gamma\rangle})
\end{aligned}$$

The convergence dues to Corollary 3: the increments are also convergent. ■

Now, we will give an example to show how Theorem 2 works. Let's consider the background driving Lévy process having the p-temple α-stable distribution as defined in the Definition 12.

Example 7. *Assume* $\{Z^\epsilon_t\}$ *is a Lévy process having the p-temple α-stable distribution, i.e.* $\nu_z(B) = \int_{\mathbb{S}^{d-1}}\int_0^\epsilon \mathbb{I}_B(r\xi)q(r^p,\xi)r^{-\alpha-1}\mathrm{d}r\sigma(\mathrm{d}\xi)$ *for any Borel set* $B \in \mathbb{R}^d$. *If, for any* $\xi \in \mathbb{S}^{d-1}$, $q(r^p,\xi)r^{-\alpha} \to h(\xi)$ *in* $L^1(\sigma)$ *as* $r \to 0$, *where h is any Borel function defined on* $\mathbb{S}^{d-1}$ *such that* $\sigma_1(C) = \int_C h(\xi)\sigma(\mathrm{d}\xi) < \infty$. *then* $\frac{X^\epsilon_t}{\epsilon} \xrightarrow{d} \frac{S^\gamma_t}{\gamma}$ *as* $\epsilon \to 0$

$$\begin{aligned}
\frac{\mu_z(\epsilon,C)}{\epsilon} &= \frac{1}{\epsilon}\int_{\mathbb{R}^d}|x|\mathbb{I}_{(0,\epsilon]C}(x)\nu_z(\mathrm{d}x)\\
&= \frac{1}{\epsilon}\int_{\mathbb{S}^{d-1}}\int_0^\infty r\mathbb{I}_{(0,\epsilon]C}(r\xi)q(r^p,\xi)r^{-\alpha-1}\mathrm{d}r\sigma(\mathrm{d}\xi)\\
&= \frac{1}{\epsilon}\int_C\int_0^\epsilon q(r^p,\xi)r^{-\alpha}\mathrm{d}r\sigma(\mathrm{d}\xi)
\end{aligned}$$

Note $\lim_{\epsilon\to 0}\frac{\int_0^\epsilon q(r^p,\xi)r^{-\alpha}\mathrm{d}r}{\epsilon} = \lim_{\epsilon\to 0}q(\epsilon^p,\xi)\epsilon^{-\alpha} = h(\xi)$, *so* $\frac{\mu_z(\epsilon,C)}{\epsilon} \to \int_C h(\xi)\sigma(\mathrm{d}\xi) = \sigma_1(C)$. *It's straightforward that* σ_1 *satisfies the condition in Theorem 2, then* $\frac{X^\epsilon_t}{\epsilon} \xrightarrow{d} \frac{S^\gamma_t}{\gamma}$ *as* $\epsilon \to 0$

Now let's consider the situation that the condition in Theorem 2 is not satisfied.

Example 8. *Consider X_t is driven by a Lévy process $\{Z_t\}$ having Lévy measure ν_z (see Theorem 1 in [29]), for any Borel set $B \in \mathbb{R}^d$ and $\alpha > 0$ but $\alpha \neq 1$,*

$$\nu_z(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{I}_B(r\xi)\alpha r^{-1}(-\ln(r))^{\alpha-1}\mathrm{d}r\sigma(\mathrm{d}\xi)$$

$$\begin{aligned}\mu(\epsilon, C) &= \int_{\mathbb{R}^d} |x|\mathbb{I}_{(0,\epsilon]C}(x)\nu_z(\mathrm{d}x) \\ &= \int_C \int_0^\epsilon r\alpha r^{-1}(-\ln(r))^{\alpha-1}\mathrm{d}r\sigma(\mathrm{d}\xi) \\ &= \sigma(C)\int_0^\epsilon \alpha(-\ln(r))^{\alpha-1}\mathrm{d}r\end{aligned}$$

It's obvious that when $\alpha > 1$, $\mu(\epsilon, C) \to \infty$. So the condition in Theorem 2 is not satisfied. Actually, the characteristic function of $\frac{X_t^\epsilon}{\epsilon}$ is

$$\begin{aligned}\mathrm{E}(e^{i\langle z, \frac{X_t^\epsilon}{\epsilon}\rangle}) &= \mathrm{E}(e^{i\langle \frac{z}{\epsilon}, X_t^\epsilon\rangle}) \\ &= \exp\left\{\int_0^t \int_{\mathbb{S}^{d-1}} \int_0^\infty (e^{i\langle \frac{z}{\epsilon}, u\xi\rangle} - 1)\mathbb{I}_{(0,\epsilon e^{-c(t-s)}]}(u)\alpha u^{-1}(-\ln(e^{c(t-s)}u))^{\alpha-1} \right. \\ &\qquad \left. \mathrm{d}u\sigma(\mathrm{d}\xi)\mathrm{d}s\right\} \\ &= \exp\left\{\int_0^t \int_{\mathbb{S}^{d-1}} \int_0^\infty (e^{i\langle z, r\xi\rangle} - 1)\mathbb{I}_{(0,e^{-c(t-s)}]}(r)\alpha r^{-1}(-\ln(e^{c(t-s)}r\epsilon))^{\alpha-1} \right. \\ &\qquad \left. \mathrm{d}r\sigma(\mathrm{d}\xi)\mathrm{d}s\right\} \\ &= \exp\left\{\int_0^t \int_{\mathbb{S}^{d-1}} \int_0^\infty (e^{i\langle z, r\xi\rangle} - 1)\mathbb{I}_{(0,e^{-c(t-s)}]}(r)\alpha r^{-1}(-\ln r - \ln(e^{c(t-s)}\epsilon))^{\alpha-1} \right. \\ &\qquad \left. \mathrm{d}r\sigma(\mathrm{d}\xi)\mathrm{d}s\right\}\end{aligned}$$

This can not converge to the characteristic function of $\frac{S_t^\gamma}{\gamma}$ described in Proposition 16.

CHAPTER 7: Application In Stochastic Volatility

One of the applications of the O-U process in mathematical finance is the stochastic volatility model, see [30]. Before we talk about the details of the application, let's introduce the definitions we will use later.

Definition 15. *A* ***filtration*** *is an increasing family of σ-algebras $(\mathscr{F}_t)$ such that $\forall t \geq s$, $\mathscr{F}_s \subseteq \mathscr{F}_t$.*

Let $\{W_t\}$ be the standard Brownian motion and $(\mathscr{F}_t)$ be the filtration generated by the Lévy process $\{Z_t\}$. Assume the PDE of the jump-diffusion model is

$$\begin{aligned} \mathrm{d}X_t^\epsilon &= \beta(\sigma_t^\epsilon)^2 \mathrm{d}t + \sigma_t^\epsilon \mathrm{d}W_t + \rho \mathrm{d}Z_t^\epsilon \\ \mathrm{d}(\sigma_t^\epsilon)^2 &= -\lambda(\sigma_t^\epsilon)^2 \mathrm{d}t + \mathrm{d}Z_t^\epsilon \end{aligned}$$

where $\{Z_t^\epsilon\}$ is the truncated background driving Lévy process, $\beta > 0$, $\rho > 0$, $\lambda > 0$. Note that the volatility σ_t^ϵ follows the O-U process instead of a constant process, and we assume $(\sigma_0)^2 = 0$. Define $\mathscr{E}(\lambda, t-s) = \dfrac{1-e^{-\lambda(t-s)}}{\lambda}$, then by Equation 15.27 in [25] we have the following solutions:

$$X_t^\epsilon = \beta \int_0^t (\sigma_s^\epsilon)^2 \mathrm{d}s + \int_0^t \sigma_s^\epsilon \mathrm{d}W_s + \rho Z_t^\epsilon \tag{7.1}$$

$$\int_0^t (\sigma_s^\epsilon)^2 \mathrm{d}s = \int_0^t \frac{1-e^{-\lambda(t-s)}}{\lambda} \mathrm{d}Z_s^\epsilon = \int_0^t \mathscr{E}(\lambda, t-s) \mathrm{d}Z_s^\epsilon \tag{7.2}$$

Now, considering $Y^1 \sim \mathrm{ID}(0, \mathrm{D}^1, 0)$, and

$$\begin{aligned} V_t^1 &= \beta \int_0^t (\sigma_s^1)^2 \mathrm{d}s + \int_0^t \sigma_s^1 \mathrm{d}W_s + \rho Y_t^1 \\ \int_0^t (\sigma_s^1)^2 \mathrm{d}s &= \int_0^t \frac{1 - e^{-\lambda(t-s)}}{\lambda} \mathrm{d}Y_t^1 = \int_0^t \mathscr{E}(\lambda, t-s) \mathrm{d}Y_t^1 \end{aligned}$$

In the next proposition, we consider the transformation of the stochastic integral componentwisely and then give the joint convergence condition.

Proposition 19. *Suppose* $X_t^\epsilon = \left(\frac{1}{\epsilon} \beta \int_0^t (\sigma_s^\epsilon)^2 \mathrm{d}s, \frac{1}{\sqrt{\epsilon}} \int_0^t \sigma_s^\epsilon \mathrm{d}W_s, \frac{1}{\epsilon} \rho Z_t^\epsilon \right)$ *and* $V_t^1 = \left(\beta \int_0^t (\sigma_s^1)^2 \mathrm{d}s, \int_0^t \sigma_s^1 \mathrm{d}W_s, \rho Y_t^1 \right)$. *If,* $\forall C \in \mathscr{B}(\mathbb{S}^{d-1})$ *with* $\sigma(\partial C) = 0$, $\frac{\mu_z(\epsilon, C)}{\epsilon} \to \sigma(C)$ *as* $\epsilon \downarrow 0$, *then* $X_t^\epsilon \xrightarrow{d} V_t^1$ *as* $\epsilon \to 0$.

Proof of Proposition 19. First, for any constant u,

$$\begin{aligned} \mathrm{E}\left(e^{u \frac{1}{\sqrt{\epsilon}} \int_0^t \sigma_s^\epsilon \mathrm{d}W_s} \right) &= \mathrm{E}\left(e^{\frac{u^2}{2\epsilon} \int_0^t (\sigma_s^\epsilon)^2 \mathrm{d}s} \right) \\ &= \mathrm{E}\left(e^{\frac{u^2}{2\epsilon} \int_0^t \mathscr{E}(\lambda, t-s) \mathrm{d}Z_s^\epsilon} \right) \\ &= \mathrm{E}\left(e^{\frac{1}{\epsilon} \int_0^t \frac{u^2}{2} \mathscr{E}(\lambda, t-s) \mathrm{d}Z_s^\epsilon} \right) \end{aligned}$$

$$
\begin{aligned}
\mathrm{E}\left(e^{\langle U,X_t^\epsilon\rangle}\right) &= \mathrm{E}\left(e^{u_1\frac{1}{\epsilon}\beta\int_0^t(\sigma_s^\epsilon)^2\mathrm{d}s+u_2\frac{1}{\sqrt{\epsilon}}\int_0^t\sigma_s^\epsilon\mathrm{d}W_s+u_3\frac{1}{\epsilon}\rho Z_t^\epsilon}\right)\\
&= \mathrm{E}\left\{\mathrm{E}\left(e^{u_1\frac{1}{\epsilon}\beta\int_0^t(\sigma_s^\epsilon)^2\mathrm{d}s+u_2\frac{1}{\sqrt{\epsilon}}\int_0^t\sigma_s^\epsilon\mathrm{d}W_s+u_3\frac{1}{\epsilon}\rho Z_t^\epsilon}\big|\mathscr{F}_t\right)\right\}\\
&= \mathrm{E}\left(e^{u_1\frac{1}{\epsilon}\beta\int_0^t(\sigma_s^\epsilon)^2\mathrm{d}s+u_3\frac{1}{\epsilon}\rho Z_t^\epsilon}\mathrm{E}\left(e^{u_2\frac{1}{\sqrt{\epsilon}}\int_0^t\sigma_s^\epsilon\mathrm{d}W_s}\right)\right)\\
&= \mathrm{E}\left(e^{u_1\frac{1}{\epsilon}\beta\int_0^t(\sigma_s^\epsilon)^2\mathrm{d}s+u_3\frac{1}{\epsilon}\rho Z_t^\epsilon}e^{\frac{u_2^2}{2\epsilon}\int_0^t(\sigma_s^\epsilon)^2\mathrm{d}s}\right)\\
&= \mathrm{E}\left(e^{\frac{1}{\epsilon}\int_0^t(u_1\beta+\frac{u_2^2}{2})(\sigma_s^\epsilon)^2\mathrm{d}s+u_3\frac{1}{\epsilon}\rho Z_t^\epsilon}\right)\\
&= \mathrm{E}\left(e^{\frac{1}{\epsilon}\int_0^t(u_1\beta+\frac{u_2^2}{2})\mathscr{E}(\lambda,t-s)\mathrm{d}Z_s^\epsilon+\frac{1}{\epsilon}\int_0^t u_3\rho\mathrm{d}Z_s^\epsilon}\right)\\
&= \mathrm{E}\left(e^{\frac{1}{\epsilon}\int_0^t[(u_1\beta+\frac{u_2^2}{2})\mathscr{E}(\lambda,t-s)+u_3\rho]\mathrm{d}Z_s^\epsilon}\right)
\end{aligned}
$$

For any fixed $U = (u_1, u_2, u_3)$ and t, $(u_1\beta + \frac{u_2^2}{2})\mathscr{E}(\lambda, t-s) + u_3\rho$ is a real function of s, suppose we define $f(s) = (u_1\beta + \frac{u_2^2}{2})\mathscr{E}(\lambda, t-s) + u_3\rho$ and define $O_t^\epsilon = \int_0^t f(s)\mathrm{d}Z_s^\epsilon$. Recall Theorem 2, we have proven that $\frac{O_t^\epsilon}{\epsilon} \xrightarrow{d} S_t^1$ where S_t^1 is driven by a Dickman-type Lévy process $\{Y_t^1\}$ in the same setting as X_t but with $\epsilon = 1$. So $\mathrm{E}(e^{u_1\frac{1}{\epsilon}\beta\int_0^t(\sigma_s^\epsilon)^2\mathrm{d}s+u_2\frac{1}{\sqrt{\epsilon}}\int_0^t\sigma_s^\epsilon\mathrm{d}W_s+u_3\frac{1}{\epsilon}\rho Z_t^\epsilon}) \to \mathrm{E}(e^{u_1\beta\int_0^t(\sigma_s^1)^2\mathrm{d}s+u_2\int_0^t\sigma_s^1\mathrm{d}W_s+u_3\rho Y_t^1})$. ■

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