1. [10 points] Consider the probability space $([0,1], \mathcal{B}, dx)$, where $\mathcal{B}$ is the Borel σ-algebra and dx is the Lebesgue measure. Consider the random variable (a function on [0,1]): $X(\omega) = \max(\sin(2\pi\omega), 0)$.

   (a) Find $\sigma(X)$, i.e., the minimal σ-algebra generated by the r.v. $X(\cdot)$.

   (b) For $Y(\omega) = [X(\omega)]^2$, calculate (one of the versions of) $E[Y|X] = E[Y|\sigma(X)]$.

   (c) Find the distribution function for the r.v. $X(\omega)$ and its decomposition on a.c. and discrete parts.

2. [10 points] Let $X_1, X_2, \cdots,$ be a sequence of I.I.D. random variables with pdf: $f(x) = e^{-x}, x > 0$. Show that

$$\limsup_{n\to\infty} \frac{X_n}{\ln n} = 1 \quad \text{a.s.}$$

3. [10 points] For any two random variables X and Y with the finite variance of X, show that
$$Var(X) = E\left[Var(X|Y)\right] + Var[E(X|Y)]\,.$$

4. [10 points] Let X, Y, and Z be independent $N(0,1)$ r.v. Find

$$E\left[(X+5Y+1)(X+2Y+Z+2) | X+5Y=a,\ Y-3Z=b\right].$$