Probability Qualifying Exam

Do any three of the four problems.

1. Consider tossing a fair die until the number 6 appears for the first time.
a) Construct the probability space.
b) Let X be the number of times that we saw an odd number (1,3, or 5) before we stopped. Find $\mathrm{E}[X]$.

2. a) State the two Borel-Cantelli Lemmas.
b) Let $X_1, X_2, \ldots$ be iid random variables with pdf

$$f(x) = \alpha x^{-1-\alpha}, \quad x > 1,$$

where $\alpha > 0$ is a parameter. Prove that, with probability 1,

$$\limsup_{n\to\infty} n^{-1/\alpha} X_n > 1.$$

3. Let $\{X_n\}$ be a sequence of iid random variables with pdf

$$f(x) = 1, \quad 0 < x < 1.$$

Let $M_n = \min\{X_1, X_2, \ldots, X_n\}$ be the minimum of the first n random variables. Show that

$$nM_n \to^d Y$$

for some random variable Y. What is the distribution of Y?

4. a) State the definition of the conditional expectation of a random variable X given a sub σ-field $\mathcal{G}$.
b) Let $\mathcal{G}$ is a sub σ-field of $\mathcal{F}$ and let X be a random variable on $(\Omega, \mathcal{F}, P)$ with $\mathrm{E}[X^2] < \infty$. The conditional variance of X given $\mathcal{G}$ is defined by

$$\mathrm{Var}(X|\mathcal{G}) = \mathrm{E}\left[(X - \mathrm{E}[X|\mathcal{G}])^2 \,|\mathcal{G}\right].$$

Prove that

$$\mathrm{Var}(X) = \mathrm{E}[\mathrm{Var}(X|\mathcal{G})] + \mathrm{Var}[\mathrm{E}(X|\mathcal{G})].$$