1. [10 points] Consider tossing a fair die until the number 6 appears for the first time.

   (a) Construct the probability space.

   (b) Let X be the number of times that we saw an odd number (1, 3, or 5) before we stopped. Find $\mathrm{E}[X]$.

2. [10 points] Let $\{X_n\}$ be a sequence of iid random variables with a uniform distribution on $[0, 1]$. Let $M_n = \max\{X_1, X_2, \ldots, X_n\}$. Show that there exists a deterministic sequence $\{a_n\}$ in $\mathbb{R}$ such that the sequence of random variables $\{Z_n\}$ given by $Z_n = a_n(1 - M_n)$ has a limiting (nondegenerate) distribution and find this distribution.

3. [10 points] Let $Y_1, Y_2, \dots$ be independent random variables such that

$$P(Y_n = 2^n) = P(Y_n = -2^n) = \frac{1}{3n^\alpha}$$

and

$$P(Y_n = 0) = 1 - \frac{2}{3n^\alpha}.$$

Prove that the series $\sum_{n=1}^{\infty} Y_n$ converges P-a.s. for $\alpha > 1$ and diverges P-a.s. for $\alpha \leq 1$.

4. [10 points] Let X, Y, and Z be independent $N(0,1)$ random variables. Calculate

$$\mathrm{E}\left[(1+Z)(X+Y+1)^2(Y+1)|X+Y\right].$$