1. [10 points] Let $Y_1, Y_2, \cdots$ be a sequence of iid random variables following an exponential distribution with parameter $\lambda > 0$. This means that $P(Y_1 > t) = e^{-t\lambda}$, $t > 0$. Let $S_n = \sum_{k=1}^{n} kY_k$.

   (a) Find the mean and variance of $S_n$.

   (b) Prove that there exists a real number $A$ such that

   $$\frac{S_n}{n^2} \xrightarrow{p} A$$

   and find $A$.

2. Let $X_1, X_2, \cdots$ be a sequence of iid random variables. A sequence of real numbers $\{a_n\}$ is called an upper function for $\{X_n\}$ if $P$-a.s. we have only finitely many events $[X_n > a_n]$. A sequence of real numbers $\{\tilde{a}_n\}$ is called a lower function for $\{X_n\}$ if $P$-a.s. we have infinitely many events $[X_n > \tilde{a}_n]$. Assume that $\{X_n\}$ have the Cauchy distribution with density
$$p(x) = \frac{1}{\pi(1+x^2)}.$$
Use the Borell–Cantelli Lemmas to prove that a sequence of real numbers $\{a_n\}$ with $a_n \to \infty$ is an upper function if and only if $\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty$ and that it is a lower function if and only if $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$.

3. [10 points] Let $X$ and $Y$ be independent random variables and let $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$ be a Borel function with $\mathrm{E}|\phi(X,Y)| < \infty$. Let $g(x) = \mathrm{E}[\phi(x,Y)]$. Prove that

$$\mathrm{E}[\phi(X,Y)|X] = g(X) \ a.s.$$

4. [10 points] Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $X, X_1, X_2, X_3, \cdots$ be random variables on this space. Assume that, with probability 1, we have $X_1 \le X_2 \le X_3 \le \cdots$ and $X_n \le X$ for every $n = 1, 2, \cdots$. Prove that if $X_n \xrightarrow{p} X$, then $X_n \to X$ with probability 1.