Probability Exam

1. Consider the following experiment. First, we toss a fair coin. If we see heads we stop. Otherwise, we toss a fair die until the first time we see a six.
a) Construct the probability space.
b) Let T be the number of times we toss something (either the coin or the die). Find $\phi(z) = \mathrm{E}[z^T]$, which is the generating function of T, and find $\mathrm{E}[T]$.

2. Let X and Y be independent and identically distributed random variables, each having a uniform distribution on $[0,1]$. Find the probability density function (pdf) of the random variable $Z = X + 2Y$.

3. Let $X_1, X_2, \ldots$ be iid random variables, whose distribution is absolute continuous with pdf

$$f(x) = 2xe^{-x^2} 1_{[x>0]}.$$

Show that

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{\log n}} \leq \sqrt{2}.$$

4. Let X and Y be random variables having a jointly Gaussian distribution. Assume that $\mathrm{E}[X] = \mathrm{E}[Y] = 0$, $\mathrm{E}[X^2] = \mathrm{E}[Y^2] = 1$, and $\mathrm{E}[XY] = \rho$ for some $|\rho| < 1$. Calculate

$$\mathrm{E}\left[(X+1)(X+2Y+1)|X\right]$$